Convergence to Equilibrium of Some Kinetic Models

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Abstract

We introduce in this paper a new constructive approach to the problem of the convergence to equilibrium for a large class of kinetic equations. The idea of the approach is to prove a 'weak' coercive estimate, which implies exponential or polynomial convergence rate. Our method works very well not only for hypocoercive systems in which the coercive parts are degenerate but also for the linearized Boltzmann equation.

Keyword Goldstein-Taylor model, Boltzmann equation, hard potential, soft potential, rate of convergence to equilibrium.
MSC: 76P05, 82B40, 82C40, 82D05.

1 Introduction

In [7] and [8], L. Desvillettes and C. Villani started the program about the trend to equilibrium for kinetic equations. Up to now, there are three classes of techniques to study the convergence to equilibrium. The first class of technique is the Lyapunov functional technique, which works for nonlinear equations. These techniques are developed in [5], [7], [8], [9], [12]. The second class of techniques is the pseudodifferential calculus, which works for linear hypoelliptic equations, developed in [19], [11], [18], [20], [28]. The third class of techniques is developed by Yan Guo in [16], which is in some sense an intermediate method between the two previous ones, which works for nonlinear kinetic equations in a close-to-equilibrium regime or the linearized versions of nonlinear kinetic equations. For a full discussion on

this, we refer to the note [29].
Using the techniques developed in [7], [8], L. Desvillettes and F. Salvarani have investigated the speed of relaxation to equilibrium in the case of linear collisional models where the collision frequency is not uniformly bounded away from 0. The two models that they considered are the non-homogeneous transport equation and the Goldstein-Taylor model

$$\frac{\partial f}{\partial t} + v.\nabla f = \sigma(x)(\bar{f} - f), \tag{1.1}$$

and

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \sigma(x)(v-u), \\ \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \sigma(x)(u-v). \end{cases} \tag{1.2}$$

They prove that when σ is greater than a positive polynomial and σ belongs to H^2, one can get polynomial decays of the solutions toward the equilibrium points. However, the techniques used in the paper could not be extended to consider the case where the cross section σ is 0 on a set of strictly positive measure. A conjecture in this paper is to find explicit decay rates for these systems in wider classes of σ. In the same spirit of [10], K. Aoki and F. Golse [3] have studied the case of a collisionless gas enclosed in a vessel, where the surface is kept at a constant temperature, and they have investigated the convergence to equilibrium for such a system.
We introduce a new approach to the problem of convergence toward equilibrium in the kinetic theory and use it to study the question of L. Desvillettes and F. Salvarani in [10] for Goldstein-Taylor and related models. We can relax the regularity property of σ as well as the condition that σ is greater than a positive polynomial and prove that the decay is exponential (see Theorems 2.1, 2.2). The main idea of our techniques is similar to the work of Haraux [17]: in order to prove an exponential decay for the solution of the equation

$$\begin{cases} \frac{\partial f}{\partial t} + \mathcal{A}(f) = -\mathcal{K}(f), t \in \mathbb{R}_+, \\ f(0) = f_0, \end{cases} \tag{1.3}$$

we can study the following homogeneous equation with the same initial condition

$$\begin{cases} \frac{\partial g}{\partial t} + \mathcal{A}(g) = 0, t \in \mathbb{R}_+, \\ g(0) = f_0, \end{cases} \tag{1.4}$$

and prove that the following observability inequality holds

$$\int_0^T < \mathcal{K}(g), g > dt \quad \geq C\|f_0\|^2. \tag{1.5}$$

A natural way of proving the exponential decay for the solutions of (1.3) is to prove that $\mathcal{K}$ is coercive

$$< \mathcal{K}(g), g > \geq C\|g\|^2,$$

however this is not always true, especially in the case of Goldstein-Taylor and related models. The task of proving of the observability inequality (1.5) turns out to be much easier than proving an exponential decay for solutions of (1.3) since the solutions of (1.4) are explicit. Inequality (1.5) could be considered as a 'weak' coercive inequality. The details of this technique will be explained in section 3 (see Lemmas 3.1, 3.2, 3.3, 3.4 and 3.5).
Consider the dissipative inequality for (1.1)

$$\partial_t \|f\|_{L^2}^2 = -\int_{\mathbb{T}^d \times \mathbb{R}^d} \sigma(x)|\bar{f} - f|^2 dxdv,$$

we can see that the damping $\int_{\mathbb{T}^d \times \mathbb{R}^d} \sigma(x)|\bar{f} - f|^2 dxdv$ is too strong to lead to a polynomial decay. A reasonable question is if we can get a polynomial decay with a weaker damping. We give an example where the damping is quite weak

$$\partial_t \|f\|_{L^2}^2 = -\int_{\mathbb{T}^d \times \mathbb{R}^d} |(1 - \Delta_x)^{-\epsilon/2} \sigma(x)^{1/2} (\bar{f} - f)|^2 dxdv,$$

where ϵ is a positive constant. Since the order of the pseudo-differential operator $(1 - \Delta_x)^{-\epsilon/2}$ is $-\epsilon$, it leads to a polynomial decay and this is the result of Theorem 2.3.
Another question is that: our method works well for kinetic models of collisionless particles, could it be applied to more sophisticated models? The answer is yes. We also succeed to apply our technique to study the convergence toward equilibrium for the linearized Boltzmann equation (see Theorem 2.4). In the context of the linearized Boltzmann equation, the main tool to prove the exponential and polynomial convergence toward the equilibrium is based on the spectral gap estimate for the hard potential case and the coercivity estimate for the soft potential case. Using this technique, C. Mouhot has proved exponential decays in the case of hard potential (see [4], [22], [23], [24]). For the soft potential case, R. Strain and Y. Guo have proved results about the almost exponential decay (which means that the convergence is faster than any polynomial convergence) in [25] , or some exponential decay of the type $\exp(-t^p)$, $(p < 1)$ in [26]. However, obtaining spectral gap and coercivity estimates is sometimes very hard. Using our tools, we can prove an exponential decay for the hard potential case and an

almost exponential decay for the soft potential case. Since we do not need the coercivity of the collision operator, we do not really need assumptions on the collision kernel $B(|v-v_*|, \cos\theta)$ including the smoothness, convexity,... The linearized Boltzmann collision operator is usually split into two parts

$$L[f] = \nu(v)f - Kf,$$

where $\nu(v)f$ is the dominant part. If K is good enough, the spectrum of L is included in the spectrum of $\nu(v)$, which leads to the coercivity of L. Our idea is to consider the 'weak' coercivity of L for only a small class of functions: the solutions of (1.4). For a solution g of (1.4), the integral

$$\int_0^T L(g)dt$$

is equivalent to

$$T\nu(v)g - C(T)Kg$$

in some sense, where $C(T) << T$. This means that $C(T)Kg$ is absorbed by $T\nu(v)g$ when T is large and we still have the 'weak' coercivity of L without assuming more conditions on K. The only assumption we need is that the usual dominant part in the linearized Boltzmann collision kernel remains dominant with our very general conditions (see assumptions (2.18), (2.19)). These assumptions is the least property that we could expect from the linearized Boltzmann collision operator and they cover both cases: with and without Grad cut-off assumptions. Similar as in the case of the Goldstein-Taylor and related models, our proof remains true if the collision kernel $B(|v-v_*|, \cos\theta)$ depends on the space variable, which means that the effect of the collision of particles depends also on the position where they collide; however, we have not found any real model for this.
The plan of the paper is the following: the main results of the paper is stated in Section 2 and the main tool of the proofs is studied in Section 3. Sections 3, 4, 5, 6 are devoted to the proofs of Theorems 2.1, 2.2, 2.3 and 2.4.

2 Preliminaries and Statements of the Main Results

2.1 Stabilization of the Goldstein-Taylor equation and related models

We consider the Goldstein-Taylor model

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = \sigma(x)(v-u), \\ \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = \sigma(x)(u-v), \end{cases} \tag{2.1}$$

where $u := u(t,x)$, $v := v(t,x)$, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $t \geq 0$, with the initial condition

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x). \tag{2.2}$$

Suppose that $\sigma \in L^2(\mathbb{T})$. Define the asymptotic profile of the system (2.1):

$$(u_\infty, v_\infty) = \left(\frac{1}{2}\int_{\mathbb{T}} (u_0+v_0)dx, \frac{1}{2}\int_{\mathbb{T}} (u_0+v_0)dx \right), \tag{2.3}$$

and the energy is then

$$H_u(t) = \int_{\mathbb{T}} [(u-u_\infty)^2 + (v-v_\infty)^2]dx. \tag{2.4}$$

We also consider the following non-homogeneous (in space) transport equation

$$\frac{\partial f}{\partial t} + v.\nabla f = \sigma(x)(\bar{f} - f), \tag{2.5}$$

where $f := f(t,x,v)$ is the density of particles at time t, position x and velocity v. The notation $\bar{f}$ is $\int_V f(t,x,v)$, where V is $(-1,1)^d \backslash (-\epsilon,\epsilon)^d$, $\epsilon > 0$, we can normalize the measure such that $|V| = 1$. The solutions are considered of periodic 1 or on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. We give an example where the damping is week enough to give a polynomial decay

$$\frac{\partial f}{\partial t} + v.\nabla f = \sigma(x)^{1/2}(1-\Delta_x)^{-\frac{\epsilon}{2}}\sigma(x)^{1/2}(\bar{f} - f), \tag{2.6}$$

where ϵ is a positive constant. The initial data is

$$f(0,x,v) = f_0(x,v). \tag{2.7}$$

Define the energy of (2.6)

$$E_f(t) = \int_{\mathbb{T}^d}\int_V |f - f_\infty|^2 dv dx, \tag{2.8}$$

where

$$f_\infty = \int_V \int_{\mathbb{T}^d} f_0(x,v) dx dv. \tag{2.9}$$

Our main results are

Theorem 2.1 *When $\sigma \geq 0$, $\sigma \in L^2(\mathbb{T}^d)$, $\sigma \neq 0$, $(u_0, v_0) \in L^2(\mathbb{T}^d) \times L^2(\mathbb{T}^d)$, the solution of the equation* (2.1) *decays exponentially in time towards the equilibrium state of the equation.*

Theorem 2.2 *Suppose that $\sigma \geq 0$, periodic, $\sigma \in C^\infty(\mathbb{T}^d)$, $\sigma \neq 0$, and σ satisfies: there exists constants $T_* > 0$, $C_* > 0$, such that for $T > T_*$*

$$\int_0^T \sigma(x + vt) dt \geq T C_*. \tag{2.10}$$

Suppose as well that $f_0 \in L^\infty(\mathbb{T}^d \times V)$, the solution of the equation (2.5) *decays exponentially in time towards the equilibrium state of the equation in the L^2 norm.*

Remark 2.1 *Compare to the results in [10], we do not need the condition that the cross section σ is greater than a positive polynomial.*

Theorem 2.3 *When $\sigma = 1$, $f_0 \in C^\infty(\mathbb{T}^d)$, the solution of the equation* (2.6) *decays polymonially in the following sense $\forall M > 0$, there exist positive constants $C(M)$ and $k > M$ such that*

$$H_f(t) \leq C(M)(t+1)^{-k} \|f_0 - f_\infty\|^2_{H^\epsilon}. \tag{2.11}$$

Remark 2.2 *The existence of a solution of this equation can be proved by a Picard iteration technique; however, we do not go into details of this classical proof.*

Remark 2.3 *Since the order of the pseudo-differential operator $(1 - \Delta_x)^{-\epsilon/2}$ is $-\epsilon$ in* (2.6)*, which means that the damping is quite weak, we get a polynomial decay. According to our theorem the order of the convergence is $-\infty$, or we can get an almost exponential decay with this damping.*

2.2 Stabilization of the linearized Boltzmann equation

The Boltzmann equation describes the behavior of a dilute gas when the interactions are binary (see [6], [13], [27])

$$\partial_t F + v.\nabla_x F = Q(F,F), t \geq 0, x \in \mathbb{T}^d, v \in \mathbb{R}^d. \tag{2.12}$$

In (2.12), Q is the quadratic Boltzmann collision operator, defined by

$$Q(F,F) = \int_{S^{N-1}} \int_{\mathbb{R}^N} (F'F'_* - FF_*) B(|v-v_*|, \cos\theta) d\sigma dv_*,$$

where $F = F(t,x,v)$, $F_* = F(t,x,v_*)$, $F'_*(t,x,v'_*)$, $F' = F(t,x,v')$ in which

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma; v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma, \sigma \in \mathbb{S}^{N-1}.$$

This is the so called "σ-representation" of the Boltzmann collision operator. Up to a Jacobian factor $2^{N-2}\sin^{N-2}(\theta/2)$, where $\cos\theta = (v'_* - v').(v_* - v)/|v_* - v|^2$, one can also define the alternative "ω-representation",

$$Q(F,F) = \int_{S^{N-1}} \int_{\mathbb{R}^N} (F'F'_* - FF_*) \mathcal{B}(v - v_*, \omega) dv_* d\omega,$$

with

$$v' = v + ((v_* - v).\omega)\omega, v'_* = v_* - ((v_* - v).\omega)\omega, \omega \in \mathbb{S}^{d-1},$$

and

$$\mathcal{B}(v - v_*, \omega) = 2^{N-2}\sin^{N-2}(\theta/2) B(|v-v_*|, \cos\theta).$$

The equilibrium distribution is given by the Maxwellian distribution

$$M(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{\frac{N}{2}}} \exp\left(-\frac{|u-v|^2}{2T}\right), \tag{2.13}$$

where ρ, u, T are the density, mean velocity and temperature of the gas at the point x

$$\rho = \int_{\mathbb{R}^d} f(v)dv, u = \frac{1}{\rho}\int_{\mathbb{R}^d} vf(v)dv, T = \frac{1}{N\rho}\int_{\mathbb{R}^d} |u-v|^2 f(v)dv. \tag{2.14}$$

Denote by

$$\mu(v) = (2\pi)^{-d/2}\exp(-|v|^2/2),$$

the normalized unique equilibrium with mass 1, momentum 0 and temperature 1, we consider F to be a solution of the equation near μ. Put $F = \mu + \sqrt{\mu} f$, then

$$\partial_t f + v.\nabla_x f = 2\mu^{-1/2} Q(\mu, \sqrt{\mu} f) + \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} f). \tag{2.15}$$

Define

$$\Gamma(f, f) = \mu^{-1/2} Q(\sqrt{\mu} f, \sqrt{\mu} f),$$

and

$$L[f] = 2\mu^{-1/2} Q(\mu, \sqrt{\mu} f),$$

the following equation is the linearized Boltzmann equation

$$\partial_t f + v.\nabla_x f = L[f], \tag{2.16}$$

where $L[f] =$

$$\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} 2\mathcal{B} \mu^{1/2}(v)\mu(v_*)[\mu^{1/2}(v')f(v'_*) + \mu^{1/2}(v'_*)f(v') - \mu^{1/2}(v_*)f(v) - \mu^{1/2}(v)f(v_*)]dv_* d\sigma, \tag{2.17}$$

with the initial condition

$$f(0, x, v) = f_0(x, v).$$

We assume the following conditions on the collision kernel $\mathcal{B}$
($\mathbb{B}_1$) There exist a constant $\alpha > -d + 1$ and a positive constant M_1 such that

$$\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mu(v_*)\mathcal{B}(|v - v_*|, \omega) d\omega dv_* \geq M_1(|v| + 1)^\alpha. \tag{2.18}$$

($\mathbb{B}_2$) There exist constants $1 - d < \beta < \alpha + 2/3$, and $M_2 > 0$ such that

$$\mathcal{B}(|v - v_*|, \omega) \leq M_2 |v - v_*|^\beta |v' - v|^{d-2}. \tag{2.19}$$

We impose these conditions to assure that the term

$$\int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathcal{B}(|v - v_*|, \omega)\mu(v_*)f(v) dv_* d\sigma$$

is the dominant term in the linearized Boltzmann collision operator. These assumptions cover both cases: with and without Grad cut-off.
Consider the energy of f

$$H_f(t) = \int_{\mathbb{T}^d \times \mathbb{R}^d} |f|^2 dx dv, \tag{2.20}$$

and its derivative in time

$$
\begin{aligned}
&\frac{d}{dt}H_f(t) \qquad (2.21)\\
= \;& -\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}}\mathcal{B}\mu_*\mu\times\\
&\times[f'_*\mu_*'^{-1/2}+f'\mu'^{-1/2}-f_*\mu_*^{-1/2}-f\mu^{-1/2}]^2 d\sigma dv_* dv dx\\
\leq \;& 0,
\end{aligned}
$$

where we use the notation $f'_*=f(v'_*)$, $f'=f(v')$, $f_*=f(v_*)$, $f=f(v)$, $\mu'_*=\mu(v'_*)$, $\mu'=\mu(v')$, $\mu_*=\mu(v_*)$ and $\mu=\mu(v)$.
For $\rho\in\mathbb{R}$, define

$$L^2((|v|+1)^\rho):=\{f|(|v|+1)^\rho f\in L^2(\mathbb{T}^d\times\mathbb{R}^d)\}.$$

Denote by $\mathcal{S}(t)f_0$ the solution of the linearized Boltzmann equation and suppose that f_0 is orthogonal to the kernel of the linearized Boltzmann collision kernel:

$$\int_{\mathbb{R}^d}\mu^{1/2}f_0dv=\int_{\mathbb{R}^d}\mu^{1/2}|v_i|f_0dv=\int_{\mathbb{R}^d}\mu^{1/2}|v|^2f_0dv=0,$$

for all $i\in\{1,\dots,d\}$.

Theorem 2.4 *With the assumptions* $(\mathbb{B}_1)$ *and* $(\mathbb{B}_2)$*:*

- *The 'hard potential' case* $\alpha,\beta>0$*: suppose that* $f_0\in L^2(\mathbb{T}^d\times\mathbb{R}^d)$*, there exist positive constants* M_0, δ *such that*

$$\left\|\mathcal{S}(t)\left(f_0-\int_{\mathbb{T}^d}f_0dx\right)\right\|_{L^2}\leq M_0\exp(-\delta t)\left\|f_0-\int_{\mathbb{T}^d}f_0dx\right\|_{L^2}. \qquad (2.22)$$

- *The 'soft potential' case* $-(d-1)<\alpha,\beta<0,$ *: suppose that* $f_0\in L^2((|v|+1)^\delta)$, $(\forall\delta>0)$*, for any* $M_1>0$*, there exist* $p>M_1$ *and* $M_2>0$ *such that*

$$\left\|\mathcal{S}(t)\left(f_0-\int_{\mathbb{T}^d}f_0dx\right)\right\|_{L^2}\leq M_2t^{-p}\left\|f_0-\int_{\mathbb{T}^d}f_0dx\right\|_{L^2}. \qquad (2.23)$$

Remark 2.4 *In this theorem, since we prove a 'weak' coercive estimate instead of spectral gap and coercivity estimates for the linearized Boltzmann operator, we can get exponential and almost exponential decays without requiring too much assumptions on the collision kernel including the smoothness, convexity, ... The only property that we need is that the dominant term remains dominant with our conditions* $(\mathbb{B}_1)$ *and* $(\mathbb{B}_2)$*.*

Remark 2.5 *Our proof works well also for the case where* B *depends on* x*; however, we have not found any real application for this.*

3 The main tool

Let $(H, <.,.>, \|.\|)$ be a real Hilbert space with its inner product and its norm, $\mathcal{A}$ be an operator on H satisfying $<\mathcal{A}(x), x>=0$ for all x in H and $\mathcal{K}$ be a self-ajoint linear operator. Suppose that

$$<\mathcal{K}(x), y>=<x, \mathcal{K}(y)>=<\mathcal{K}^{1/2}(x), \mathcal{K}^{1/2}(y)>.$$

Let f be the solution of the evolution equation

$$\begin{cases} \frac{\partial f}{\partial t} + \mathcal{A}(f) = -\mathcal{K}(f), t \in \mathbb{R}_+, \\ f(0) = f_0, f_0 \in H, \end{cases} \tag{3.1}$$

and let g be the solution of

$$\begin{cases} \frac{\partial g}{\partial t} + \mathcal{A}(g) = 0, t \in \mathbb{R}_+, \\ g(0) = f_0. \end{cases} \tag{3.2}$$

Lemma 3.1 *For all T in $\mathbb{R}_+$*

$$\int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt \leq \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt. \tag{3.3}$$

Proof Consider the norm of $\|f-g\|^2$

$$\begin{aligned} \|f-g\|^2(T) &= 2\int_0^T <\partial_t f - \partial_t g, f-g> dt \\ &= -\int_0^T 2 <\mathcal{K}^{1/2}(f), \mathcal{K}^{1/2}(f-g)> dt \\ &= -2\int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt + 2\int_0^T <\mathcal{K}^{1/2}(f), \mathcal{K}^{1/2}(g)> dt \\ &\leq -\int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt + \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt, \end{aligned}$$

which leads to

$$\int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt \leq \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt.$$

∎

Lemma 3.2 *If $\mathcal{K}^{1/2}$ is bounded, then for all T in $\mathbb{R}_+$*

$$M_1 \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt \leq \int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt, \tag{3.4}$$

where M_1 is a positive constant.

Proof Take the derivative in time of $\|f-g\|^2$

$$\begin{aligned}
\partial_t \|f-g\|^2 &= 2 < \partial_t f - \partial_t g, f-g > \\
&= -2 < \mathcal{K}^{1/2}(f), \mathcal{K}^{1/2}(f-g) > \\
&\leq \|\mathcal{K}^{1/2}(f)\|^2 + \|\mathcal{K}^{1/2}(f-g)\|^2 \\
&\leq \|\mathcal{K}^{1/2}(f)\|^2 + C\|f-g\|^2,
\end{aligned}$$

the last inequality follows from the boundedness of $\mathcal{K}^{1/2}(f-g)$, where C is a positive constant. Gronwall's inequality then leads to

$$\|f-g\|^2(t) \leq \int_0^t \exp(C(t-s))\|\mathcal{K}^{1/2}(f)\|^2 ds,$$

which together with the boundedness of $\mathcal{K}^{1/2}(f-g)$ leads to

$$\|\mathcal{K}^{1/2}(f-g)\|^2(t) \leq C\exp(Ct)\int_0^t \|\mathcal{K}^{1/2}(f)\|^2 ds,$$

where C is some positive constant. This deduces

$$\int_0^T \|\mathcal{K}^{1/2}(f-g)\|^2 dt \leq CT\exp(CT)\int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt.$$

The triangle inequality deduces

$$\int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt \leq 2(CT\exp(CT)+1)\int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt.$$

■

Lemma 3.3 *Let $(H', \|.\|_0)$ be a Banach subspace of H with its norm. Suppose that for any h in H', $\|h\| \leq M\|h\|_0$, where M is a positive constant and that for any solution g of* (3.2)

$$\|f_0\| = \|g(t)\|, \forall t \in \mathbb{R}_+. \tag{3.5}$$

We assume that for any positive constant ϵ, the operator $\mathcal{K}$ could be decomposed into the sum of two linear operators $\mathcal{K}_{\epsilon,1}$ and $\mathcal{K}_{\epsilon,2}$ such that

$$
\begin{aligned}
&\mathcal{K} = \mathcal{K}_{\epsilon,1} + \mathcal{K}_{\epsilon,2}, && (3.6)\\
&\|\mathcal{K}^{1/2}\|^2 = \|\mathcal{K}_{\epsilon,1}{}^{1/2}\|^2 + \|\mathcal{K}_{\epsilon,2}{}^{1/2}\|^2, && (3.7)\\
&\|\mathcal{K}_{\epsilon,1}{}^{1/2}(h)\| \leq C_1(\epsilon)\|h\|, \quad \forall h \in H', && (3.8)\\
&\|\mathcal{K}_{\epsilon,2}{}^{1/2}(h)\| \leq C_2(\epsilon)\|h\|_0, \quad \forall h \in H', && (3.9)\\
&\|\mathcal{K}^{1/2}(h)\| \leq C(\mathcal{K})\|h\|_0, \quad \forall h \in H', && (3.10)
\end{aligned}
$$

where $C_1(\epsilon)$, $C_2(\epsilon)$ and $C(\mathcal{K})$ are positive constants, $C_2(\epsilon)$ tends to 0 as ϵ tends to 0, and $\mathcal{K}_{\epsilon,i}^{1/2}$, $i \in \{1,2\}$ are defined in the following way

$$< \mathcal{K}_{\epsilon,i}(h), k >=< \mathcal{K}_{\epsilon,i}^{1/2}(h), \mathcal{K}_{\epsilon,i}^{1/2}(k) >, \quad \forall h, k \in H'.$$

Suppose that there exist positive numbers T_0 and C such that

$$\int_0^{T_0} \|\mathcal{K}^{1/2}(g)\|^2 dt \geq C\|f_0\|_0^2. \tag{3.11}$$

Then there exists a constant M_1 depending on T_0 such that

$$M_1 \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt \leq \int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt, \tag{3.12}$$

for all $T \geq T_0$.

Proof Similar as in the previous lemma

$$
\begin{aligned}
\partial_t\|f-g\|^2 &= -2 < \mathcal{K}_{\epsilon,1}^{1/2}(f), \mathcal{K}_{\epsilon,1}^{1/2}(f-g) > -2 < \mathcal{K}_{\epsilon,2}^{1/2}(f), \mathcal{K}_{\epsilon,2}^{1/2}(f-g) >\\
&\leq \|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + \|\mathcal{K}_{\epsilon,1}^{1/2}(f-g)\|^2 - 2 < \mathcal{K}_{\epsilon,2}^{1/2}(f), \mathcal{K}_{\epsilon,2}^{1/2}(f-g) >\\
&\leq \|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + C_1(\epsilon)^2\|f-g\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|,
\end{aligned}
$$

the last inequality follows from (3.8). Gronwall's inequality deduces

$$
\begin{aligned}
\|f-g\|^2(t) &\leq \int_0^t (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|)\exp(C_1(\epsilon)^2(t-s))ds\\
&\leq \exp(C_1(\epsilon)^2 t)\int_0^t (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|)ds.
\end{aligned}
$$

The previous inequality implies

$$\int_0^T \|f-g\|^2 dt \leq T\exp(C_1(\epsilon)^2 T)\int_0^T (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|)dt, \tag{3.13}$$

for any $T > T_0$. The two inequalities (3.8) and (3.13) lead to

$$\begin{aligned}
&\int_0^T \|\mathcal{K}_{\epsilon,1}^{1/2}(f-g)\|^2 dt \\
\leq\ & TC_1(\epsilon)^2 \exp(C_1(\epsilon)^2 T) \int_0^T (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|) dt.
\end{aligned}$$

Apply the triangle inequality to the previous inequality to get

$$\begin{aligned}
&\int_0^T \|\mathcal{K}_{\epsilon,1}^{1/2}(g)\|^2 dt \\
\leq\ & 2(TC_1(\epsilon)^2 \exp(C_1(\epsilon)^2 T) + 1) \int_0^T (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|) dt.
\end{aligned} \tag{3.14}$$

The three inequalities (3.9), (3.10) and (3.11) imply that for ϵ small enough

$$\int_0^T \|\mathcal{K}_{\epsilon,1}^{1/2}(g)\|^2 dt \geq C(\epsilon) \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt, \tag{3.15}$$

where $C(\epsilon)$ is a positive constant depending on ϵ.
Combine (3.14) and (3.15) to get

$$\begin{aligned}
&\frac{C(\epsilon)}{2(TC_1(\epsilon)^2 \exp(C_1(\epsilon)^2 T) + 1)} \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt \\
\leq\ & \int_0^T (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|) dt.
\end{aligned} \tag{3.16}$$

Since for any positive constant δ

$$\begin{aligned}
&\int_0^T (\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + 2\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|) dt \\
\leq\ & \int_0^T \left(\|\mathcal{K}_{\epsilon,1}^{1/2}(f)\|^2 + \frac{1}{\delta}\|\mathcal{K}_{\epsilon,2}^{1/2}(f)\|^2 + \delta\|\mathcal{K}_{\epsilon,2}^{1/2}(f-g)\|^2 \right) dt \\
\leq\ & \left(1 + 2\delta + \frac{1}{\delta}\right) \int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt + 2\delta \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt,
\end{aligned}$$

Inequality (3.16) leads to

$$\begin{aligned}
&\left(\frac{C(\epsilon)}{2(TC_1(\epsilon)^2 \exp(C_1(\epsilon)^2 T) + 1)} - 2\delta \right) \int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt \\
\leq\ & \left(1 + 2\delta + \frac{1}{\delta}\right) \int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt,
\end{aligned}$$

which implies (3.11) for δ small enough. ∎

Remark 3.1 *Lemma* 3.2 *will be used later for the case of Goldstein-Taylor and related models, while Lemma* 3.3 *will be used for the linearized Boltzmann equation.*

Lemma 3.4 *Suppose that* $\mathcal{K}$ *satisfies the conditions in Lemmas* 3.2 *or* 3.3 *and that there exist positive numbers* T_0 *and* C *such that*

$$\int_0^{T_0} \|\mathcal{K}^{1/2}(g)\|^2 dt \geq C\|f_0\|^2, \tag{3.17}$$

then there exist positive numbers T_1 *and* δ *such that for all* $t \geq T_1$

$$\|f(t)\| \leq \exp(-\delta t)\|f_0\|. \tag{3.18}$$

Moreover, (3.18) *also leads to* (3.17).

Proof
Step 1: (3.17) leads to (3.18).
Choose $T = kT_0$, where k is a positive integer. Since

$$\|f(0)\| - \|f(T)\| = \int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt,$$

there exists p in $\{0, \ldots, k-1\}$ such that

$$\frac{\|f(0)\|}{k} \geq \int_{pT_0}^{(p+1)T_0} \|\mathcal{K}^{1/2}(f)\|^2 dt.$$

Let h be the solution of

$$\partial_t h + A(h) = 0,$$

with $h(0) = f(pT_0)$. Inequality (3.17) implies that

$$\int_0^{T_0} \|\mathcal{K}^{1/2}(h)\|^2 dt \geq C\|f(pT_0)\|,$$

which together with Lemmas 3.2 and 3.3 deduces

$$\int_0^{T_0} \|\mathcal{K}^{1/2}(f)\|^2 dt \geq C\|f(pT_0)\|.$$

This leads to

$$\|f(kT_0)\| \leq \|f(pT_0)\| \leq \frac{1}{Ck}\|f(0)\|,$$

where C is some positive constant, since

$$\partial_t \|f\|^2 = -2 < \mathcal{K}^{1/2} f, \mathcal{K}^{1/2} f >,$$

or $\|f\|$ is decreasing; for k large enough. The previous inequality implies

$$\|f(T_*)\| \leq \exp(-\delta_* T_*) \|f(0)\|,$$

where $T_* = kT_0$ and $\delta_* = \frac{\ln(Ck)}{T_*}$, which means

$$\|f(nT_*)\| \leq \exp(-\delta_* T_*) \|f((n-1)T_*)\| \leq \exp(-\delta_* n T_*) \|f(0)\|, \quad \forall n \in \mathbb{N}.$$

For $t \in [nT_*, (n+1)T_*)$,

$$\|f(t)\| \leq \|f(nT_*)\| \leq \exp(-\delta_* n T_*) \|f(0)\| \leq \exp(-\frac{\delta_*}{2} t) \|f(0)\|,$$

which leads to the exponential decay (3.18) with $\delta = \frac{\delta_*}{2}$.
Step 2: (3.18) leads to (3.17).
Inequality (3.18) deduces that there exist constants $C < 1$ and $T_* > 0$ such that for $T > T_*$

$$\|f(0)\| - \|f(T)\| = \int_0^T \|\mathcal{K}^{1/2}(f)\|^2 dt \geq C \|f(0)\|^2.$$

Lemma 3.1 implies that

$$\int_0^T \|\mathcal{K}^{1/2}(g)\|^2 dt \geq C \|f(0)\|^2.$$

■

We also recall Lemma 4.4 in [2], for a proof of this lemma we refer to [1] and [21].

Lemma 3.5 *Let $\{\mathcal{E}_k\}$ be a sequence of positive real numbers satisfying*

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - C\mathcal{E}_{k+1}^{2+\zeta}, \forall k \geq 0,$$

where $C > 0$ and $\zeta > -1$ are constants. Then there exists a positive constant M, such that

$$\mathcal{E}_k \leq \frac{M}{(k+1)^{\frac{1}{1+\zeta}}}, k \geq 0.$$

4 Decay rates of the Goldstein-Taylor model

Consider the following system:

$$\begin{cases} \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} = 0, \\ \frac{\partial \phi}{\partial t} - \frac{\partial \phi}{\partial x} = 0, \end{cases} \tag{4.1}$$

where $\varphi := \varphi(t,x)$, $\phi := \phi(t,x)$, $x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, $t \geq 0$, with the initial condition

$$\varphi(0,x) = \varphi_0(x), \quad \phi(0,x) = \phi_0(x). \tag{4.2}$$

Then asymptotic profile and the energy of the system are then

$$(\varphi_\infty, \phi_\infty) = \left(\frac{1}{2}\int_{\mathbb{T}}(\varphi_0 + \phi_0)dx, \frac{1}{2}\int_{\mathbb{T}}(\varphi_0 + \phi_0)dx\right), \tag{4.3}$$

and

$$H_\varphi(t) = \int_{\mathbb{T}}[(\varphi - \varphi_\infty)^2 + (\phi - \phi_\infty)^2]dx. \tag{4.4}$$

The following proposition is a consequence of Lemmas 3.1, 3.2 and 3.4.

Proposition 4.1 *Suppose that there exist positive numbers T_0 and δ such that*

$$\forall t \geq T_0, \forall u_0, v_0 \in W^{1,1}(\mathbb{T}) : \qquad H_u(t) \leq \exp(-\delta t) H_u(0), \tag{4.5}$$

then there exist a positive number T_1 and a nonnegative number C such that

$$\int_0^{T_1}\int_{\mathbb{T}} \sigma(\varphi - \phi)^2 dxdt \geq C\int_{\mathbb{T}}[(\varphi - \varphi_\infty)^2 + (\phi - \phi_\infty)^2]dx, \tag{4.6}$$

for $\varphi_0 = u_0$ and $\phi_0 = v_0$.
Moreover, if there exist T_1 and C such that (4.6) *satisfies, then there exist T_0 and δ such that* (4.5) *is true.*

Theorem 2.1 is a direct consequence of Proposition 4.1 and the following proposition.

Proposition 4.2 *There exists a positive constant T_0 such that for $T > T_0$*

$$\int_0^{T}\int_{\mathbb{T}} \sigma(\varphi - \phi)^2 dxdt \geq C(T)\int_{\mathbb{T}}[(\varphi_0 - \varphi_\infty)^2 + (\phi_0 - \phi_\infty)^2]dx, \tag{4.7}$$

where $C(T)$ is a positive constant depending on T.

Proof Since (φ, ϕ) is the solution of the system (4.1),

$$(\varphi, \phi) = (\varphi_0(x-t), \phi_0(x+t)).$$

Write φ_0 and ϕ_0 under the form of Fourier series:

$$\begin{aligned} \varphi_0(x) &= \sum_{-\infty}^{\infty} \exp(in\pi x) a_n, \\ \phi_0(x) &= \sum_{-\infty}^{\infty} \exp(in\pi x) b_n, \end{aligned}$$

then

$$\begin{aligned} \varphi_0(x-t) &= \sum_{-\infty}^{\infty} \exp(in\pi(x-t)) a_n, \\ \phi_0(x+t) &= \sum_{-\infty}^{\infty} \exp(in\pi(x+t)) b_n. \end{aligned}$$

Choose T to be a positive integer, the previous formulas imply

$$\begin{aligned} & \int_0^T \sigma(\varphi - \phi)^2 dt \\ = & \int_0^T \sigma \left| \sum_{-\infty}^{\infty} \exp(in\pi x)(a_n \exp(-in\pi t) - b_n \exp(in\pi t)) \right|^2 dt \\ = & \lim_{M \to \infty} \left(\sum_{|n|<M} \int_0^T \sigma |a_n \exp(-in\pi t) - b_n \exp(in\pi t)|^2 dt + \right. \\ & + \sum_{|n|,|m|<M, n \neq m} \exp(i(n-m)\pi x) \times \\ & \left. \times \int_0^T [a_n \exp(-in\pi t) - b_n \exp(in\pi t)] \overline{a_m \exp(-im\pi t) - b_m \exp(im\pi t)} dt \right) \\ = & \lim_{M \to \infty} \left(\sum_{|n|<M} \int_0^T \sigma |a_n \exp(-in\pi t) - b_n \exp(in\pi t)|^2 dt + \right. \\ & + \sum_{|n|,|m|<M, n \neq m} \exp(i(n-m)\pi x) \times \\ & \left. \times \int_0^T [a_n \exp(-in\pi t) - b_n \exp(in\pi t)] \overline{a_m \exp(-im\pi t) - b_m \exp(im\pi t)} dt \right) \end{aligned}$$

$$= \lim_{M\to\infty} \sum_{|n|<M} \int_0^T \sigma |a_n \exp(-in\pi t) - b_n \exp(in\pi t)|^2 dt$$
$$= \sum_{n\in\mathbb{R}, n\neq 0} T\sigma(|a_n|^2 + |b_n|^2) + T\sigma|a_0 - b_0|^2,$$

which leads to

$$\int_{\mathbb{T}} \int_0^T \sigma(\varphi - \phi)^2 dt dx = T \int_{\mathbb{T}} \sigma dx \left(\sum_{n\in\mathbb{R}, n\neq 0} (|a_n|^2 + |b_n|^2) + |a_0 - b_0|^2 \right). \tag{4.8}$$

Moreover, the right hand side of (4.7) is equal to

$$\int_{\mathbb{T}} [(\varphi_0 - \varphi_\infty)^2 + (\phi_0 - \phi_\infty)^2] dx = \sum_{n\in\mathbb{R}, n\neq 0} (|a_n|^2 + |b_n|^2). \tag{4.9}$$

Inequality (4.7) follows by (4.8) and (4.9). ■

5 Decay rates of the non-homogeneous transport equation

We recall the conditions on σ : $\sigma \geq 0$, $\sigma \in C^\infty(\mathbb{T}^d)$, $\sigma \neq 0$, and σ satisfies: there exists constants $T_* > 0$, $C_* > 0$, such that for $T > T_*$

$$\frac{1}{T} \int_0^T \sigma(x + vt) dt \geq C_*.$$

Consider the equation

$$\frac{\partial g}{\partial t} + v.\nabla g = 0, \tag{5.1}$$

with the initial condition

$$g(0, x, v) = g_0(x, v). \tag{5.2}$$

The energy of (6.1) is then defined

$$E_g(t) = \int_{\mathbb{T}^d} |g - g_\infty|^2 dx, \tag{5.3}$$

where

$$g_\infty = \int_V \int_{\mathbb{T}^d} g_0(x, v) dx dv. \tag{5.4}$$

We suppose that $g_\infty = 0$ without loss of genarality. The following proposition is a direct consequence of Lemmas 3.1, 3.2 and 3.4.

Proposition 5.1 *Suppose that there exist positive numbers T_0 and δ such that*

$$\forall t \geq T_0, \forall f_0 \in L^\infty(\mathbb{T}^d \times V): \qquad H_f(t) \leq \exp(-\delta t) H_f(0), \tag{5.5}$$

then there exist a positive number T_1 and a nonnegative number C such that

$$\int_0^{T_1} \int_{\mathbb{T}^d} \int_V \int_V \sigma |g(t,x,v) - g(t,x,v')|^2 dv' dv dv dx dt \geq C \int_{\mathbb{T}^d} \int_V (g_0(x,v) - g_\infty)^2 dv dx, \tag{5.6}$$

for $g_0 = f_0$.
Moreover, if there exist T_1 and C such that (5.6) *satisfies, then there exist T_0 and δ such that* (5.5) *is true.*

Theorem 2.2 is a direct consequence of Proposition 5.1 and the following proposition.

Proposition 5.2 *There exists a positive constant T_0 such that for $T > T_0$*

$$\int_0^{T} \int_{\mathbb{T}^d} \int_{V\times V} \sigma(x) |g(t,x,v) - g(t,x,v')|^2 dv' dv dx dt \geq C(T) \int_{\mathbb{T}^d} \int_V (g_0(x,v) - g_\infty)^2 dv dx. \tag{5.7}$$

Proof We suppose that $g_\infty = 0$ without loss of genarality. Write g under the form of Fourier series

$$g(x,v,t) = g_0(x - vt, v) = \sum_{n\in\mathbb{Z}^d} a_n(v) \exp(i2\pi n(x - vt)).$$

Then

$$\begin{aligned}
&\int_{V\times V} |g(x,v,t) - g(x,v',t)|^2 dv' dv \\
= \ &\int_{V\times V} \left| \sum_{n\in\mathbb{Z}^d} a_n(v) \exp(i2\pi n(x - vt)) - \sum_{m\in\mathbb{Z}^d} a_m(v') \exp(i2\pi m(x - v't)) \right|^2 dv' dv \\
= \ &\int_V \left| \sum_{n\in\mathbb{Z}^d} a_n(v) \exp(i2\pi n(x - vt)) \right|^2 dv + \int_V \left| \sum_{n\in\mathbb{Z}^d} a_n(v') \exp(i2\pi n(x - v't)) \right|^2 dv' \\
&-2 \sum_{m,n\in\mathbb{Z}^d} \int_{V\times V} \overline{a_m}(v') a_n(v) \exp(i2\pi(-m+n)x) \exp(i2\pi t(mv - nv')) dv dv'
\end{aligned}$$

$$
\begin{aligned}
&= 2\int_V \left|\sum_{n\in\mathbb{Z}^d} a_n(v)\exp(i2\pi n(x-vt))\right|^2 dv \\
&\quad -2\sum_{m,n\in\mathbb{Z}^d}\int_{V\times V}\overline{a_m}(v')a_n(v)\exp(i2\pi(-m+n)x)\exp(i2\pi t(mv-nv'))dvdv' \\
&= 2\int_V |g(x,v,t)|^2\,dv \\
&\quad -2\sum_{m,n\in\mathbb{Z}^d}\int_{V\times V}\overline{a_m}(v')a_n(v)\exp(i2\pi(-m+n)x)\exp(i2\pi t(mv-nv'))dvdv'.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)|g(x,v,t)-g(x,v',t)|^2dv'dvdxdt \qquad (5.8)\\
&= 2\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\,|g(x,v,t)|^2\,dv'dvdxdt \\
&\quad -2\sum_{m,n\in\mathbb{Z}^d}\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\overline{a_m}(v')a_n(v)\exp(i2\pi(-m+n)x)\exp(i2\pi(mv-nv'))dvdv'dt.
\end{aligned}
$$

We first consider one component in the second term on the right hand side of (5.8). We drop the constant 2 for the sake of simplicity

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\overline{a_m}(v')a_n(v)\exp(i2\pi(-m+n)x)\exp(i2\pi(mv-nv'))dvdv'dxdt \qquad (5.9)\\
&= \int_{\mathbb{T}^d}\sigma(x)\exp(i2\pi(-m+n)x)dx\int_0^T\int_{V\times V}\overline{a_m}(v')a_n(v)\exp(i2\pi(mv-nv'))dvdv'dt.
\end{aligned}
$$

Consider the first component in (5.9). Suppose that $|m_k-n_k|=\max\{|m_1-n_1|,\dots,|m_d-n_d|\}$, and do the integration by part in the x_k direction, we get

$$
\begin{aligned}
\left|\int_{\mathbb{T}^d}\sigma(x)\exp(i2\pi(n-m)x)dx\right| &= \left|(-1)^p\int_{\mathbb{T}^d}\partial_k^p\sigma(x)\frac{\exp(i2\pi(n-m)x)}{(i2\pi(n_k-m_k))^p}dx\right| \\
\le \int_{\mathbb{T}^d}|\partial_k^p\sigma(x)|\frac{1}{2\pi|n_k-m_k|^p}dx &\le C\|\sigma\|_{W^{p,1}}\frac{1}{|n-m|^p}, \qquad (5.10)
\end{aligned}
$$

where C is some positive constant and p is a positive integer greater than d. Consider the second component on the right hand side of (5.8)

$$
\left|\int_0^T\int_{V\times V}\overline{a_m}(v')a_n(v)\exp(i2\pi(mv-nv')t)dvdv'dt\right| \qquad (5.11)
$$

$$
\begin{aligned}
&= \left|\int_{V\times V} \overline{a_m}(v')a_n(v)\frac{1-\exp(i2\pi(mv-nv')T)}{i2\pi(mv-nv')}dvdv'\right| \\
&\leq \|a_n\|_{L^2}\|a_m\|_{L^2}\left[\int_{V\times V}\left|\frac{1-\exp(i2\pi(mv-nv')T)}{i2\pi(mv-nv')}\right]^2 dvdv'\right]^{1/2} \\
&\leq \|a_n\|_{L^2}\|a_m\|_{L^2}\times \\
&\quad\times\left[\int_{V\times V}\frac{(1-\cos(2\pi(mv-nv')T))^2+\sin^2(2\pi(mv-nv')T)}{|2\pi(mv-nv')|^2}dvdv'\right]^{1/2} \\
&\leq \|a_n\|_{L^2}\|a_m\|_{L^2}\left[\int_{V\times V}\frac{2-2\cos(2\pi(mv-nv')T)}{|2\pi(mv-nv')|^2}dvdv'\right]^{1/2} \\
&\leq \|a_n\|_{L^2}\|a_m\|_{L^2}\left[\int_{V\times V}\frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dvdv'\right]^{1/2}.
\end{aligned}
$$

Let ϵ be a positive constant. We now try to estimate the integral

$$
\|a_n\|_{L^2}\|a_m\|_{L^2}\left[\int_{V\times V}\frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dvdv'\right]^{1/2}.
$$

For fixed v' and n, we have

$$
\begin{aligned}
&\int_V \frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dv \\
&= \int_{\{|\pi(mv-nv')|\leq\epsilon, v\in V\}}\frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dv+\int_{\{|\pi(mv-nv')|>\epsilon, v\in V\}}\frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dv \\
&\leq T^2|\{|\pi(mv-nv')|\leq\epsilon, v\in V\}|+\frac{1}{\epsilon^2}|\{|\pi(mv-nv')|>\epsilon, v\in V\}| \\
&\leq T^2\frac{C\epsilon^d}{\pi^d|m_1|\dots|m_d|}+\frac{1}{\epsilon^2}|V|,
\end{aligned}
$$

where the second inequality follows from the following fact:

$$
|\{\pi(mv-nv')|\leq\epsilon, v\in V\}|\leq|\{\max_{i=1}^{d}|\pi(m_iv_i-n_iv_i')|\leq\epsilon, v\in V\}|
$$

$$
=|\{\max_{i=1}^{d}|\pi(v_i-n_iv_i')|\leq\frac{\epsilon}{m_i}, v\in V\}|=\Pi_{i=1}^d|\{|\pi(v_i-n_iv_i')|\leq\frac{\epsilon}{m_i}, v\in V\}|=\frac{C\epsilon^d}{\pi^d|m_1|\dots|m_d|}.
$$

Optimizing over ϵ, we have that

$$
T^2\frac{C\epsilon^d}{\pi^d|m_1|\dots|m_d|}+\frac{1}{\epsilon^2}|V| \quad = \quad \frac{CT^{\frac{4}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{2}{d+2}}},
$$

we get

$$\int_V \frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dv \le \frac{CT^{\frac{4}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{2}{d+2}}},$$

which implies

$$\int_{V\times V} \frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dvdv' \le \frac{CT^{\frac{4}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{2}{d+2}}}.$$

A similar argument gives

$$\int_{V\times V} \frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dvdv' \le \frac{CT^{\frac{4}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{2}{d+2}}}.$$

Therefore, the above argument gives

$$\int_{V\times V} \frac{|\sin(\pi(mv-nv')T)|^2}{|\pi(mv-nv')|^2}dvdv' \le \frac{CT^{\frac{4}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{1}{(d+2)}}(|n_1|\dots|n_d|)^{\frac{1}{(d+2)}}} \tag{5.12}$$

Combine (5.9), (5.10), (5.11) and (5.12), we get for $m \ne n$

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\overline{a_m}(v')a_n(v)\exp(i2\pi(-m+n)x)\exp(i2\pi t(mv-nv'))dvdv'dxdt\right| \\
\le\ & C\|a_n\|_{L^2}\|a_m\|_{L^2}\frac{CT^{\frac{2}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{1}{2(d+2)}}(|n_1|\dots|n_d|)^{\frac{1}{2(d+2)}}}\|\sigma\|_{W^{p,1}}\frac{1}{|n-m|^p} \qquad (5.13)\\
\le\ & C(\|a_n\|_{L^2}^2+\|a_m\|_{L^2}^2)\frac{CT^{\frac{2}{d+2}}}{(|m_1|\dots|m_d|)^{\frac{1}{2(d+2)}}(|n_1|\dots|n_d|)^{\frac{1}{2(d+2)}}}\|\sigma\|_{W^{p,1}}\frac{1}{|n-m|^p}.
\end{aligned}$$

The case $m = n$, we have that

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\overline{a_n}(v')a_n(v)\exp(i2\pi t(nv-nv'))dvdv'dxdt\right| \\
\le\ & C\|a_n\|_{L^2}^2\frac{T^{\frac{2}{d+2}}}{(|n_1|\dots|n_d|)^{\frac{1}{d+2}}}\|\sigma\|_{L^2}. \qquad (5.14)
\end{aligned}$$

Therefore

$$\sum_{m,n\in\mathbb{Z}^d}\left|\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\overline{a_m}(v')a_n(v)\exp(i2\pi(-m+n)x)\exp(i2\pi(mv-nv'))dvdv'dxdt\right|$$

$$
\begin{aligned}
&\leq \; CT^{\frac{2}{d+2}}\|\sigma\|_{W^{p,1}}\sum_{n\in\mathbb{Z}^d}\|a_n\|_{L^2}^2\left(\sum_{m\neq n, m\in\mathbb{Z}^d}\frac{1}{(|m_1|\dots|m_d|)^{\frac{1}{2(d+2)}}(|n_1|\dots|n_d|)^{\frac{1}{2(d+2)}}}\frac{1}{|n-m|^p}\right)\\
&\quad +C\sum_{n\in\mathbb{Z}^d}\|a_n\|_{L^2}^2\frac{T^{\frac{2}{d+2}}}{(|n_1|\dots|n_d|)^{\frac{1}{d+2}}}\|\sigma\|_{L^2} \qquad (5.15)\\
&\leq \; CT^{\frac{2}{d+2}}(\|\sigma\|_{W^{p,1}}+\|\sigma\|_{L^2})\sum_{n\in\mathbb{Z}^d}\|a_n\|_{L^2}^2\\
&\leq \; CT^{\frac{2}{d+2}}(\|\sigma\|_{W^{p,1}}+\|\sigma\|_{L^2})\|g_0\|_{L^2}^2,
\end{aligned}
$$

here we use the fact that $\sum_{m\in\mathbb{Z}^d}\frac{1}{|m|^p}$ is bounded for $p>d$, which implies the boundedness of $\frac{1}{(|m_1|\dots|m_d|)^{\frac{1}{2(d+1)}}(|n_1|\dots|n_d|)^{\frac{1}{2(d+1)}}}\frac{1}{|n-m|^p}$. We now consider the first term in (5.8)

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\,|g(v)|^2\,dv'dvdxdt \qquad (5.16)\\
=\;&\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\,|g_0(x-vt)|^2\,dv'dvdxdt\\
=\;&\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)\,|g_0(x-vt)|^2\,dv'dvdxdt.
\end{aligned}
$$

Fix v and consider the integral

$$
\begin{aligned}
&\int_{\mathbb{T}^d}\int_0^T\sigma(x)\,|g_0(x-vt)|^2\,dtdx \qquad (5.17)\\
=\;&\int_{\mathbb{T}^d}\int_0^T\sigma(y+vt)\,|g_0(y)|^2\,dy\\
\geq\;& TC_*\int_{\mathbb{T}^d}|g_0(y)|^2\,dtdy\\
\geq\;& TC_*\|g_0\|_{L^2}^2,
\end{aligned}
$$

where we use the change of variable $y=x+vt$. Combining (5.16) and (5.17), we get

$$
\begin{aligned}
&\int_0^T\int_{\mathbb{T}^d}\int_{V\times V}\sigma(x)|g(v)-g(v')|^2dv'dvdxdt \qquad (5.18)\\
\geq\;& C(T-T^{2/(d+2)})\|g_0\|_{L^2}^2\\
\geq\;& CT\|g_0\|_{L^2}^2,
\end{aligned}
$$

for T large enough. ■

6 Decay rates of the special transport equation

Similar as in the previous section, consider the equation

$$\frac{\partial g}{\partial t} + v.\nabla g = 0, \tag{6.1}$$

with the initial condition

$$g(0, x, v) = g_0(x), \tag{6.2}$$

and

$$g_\infty = \int_{\mathbb{T}^d} g_0(x)dx.$$

For n in $\mathbb{Z}^d$, define

$$A_n = \int_{(0,1)^d} g_0(x)\exp(-in2\pi x)dx.$$

6.1 The Observability Inequality

Proposition 6.1 *There exist positive constants T_0 and $C(T)$ such that for $T > T_0$*

$$\int_0^T \int_{\mathbb{T}^d} \int_V |(1-\Delta_x)^{-\epsilon/2}(g-\bar{g})|^2 dvdxdt \geq C(T) \sum_{n\in\mathbb{Z}^d} \frac{\int_{\mathbb{R}^d} |A_n|^2 dv}{(1+|n|^2)^\epsilon}. \tag{6.3}$$

Proof
Write g under the form of Fourier series:

$$g(x, v, t) = g_0(x - vt) = \sum_{n\in\mathbb{Z}^d} A_n \exp(i2\pi n(x - vt)),$$

which deduces

$$\begin{aligned} &\int_{\mathbb{T}^d}\int_V\int_0^T |(1-\Delta_x)^{-\epsilon/2}(g-\bar{g})|^2 dtdx \\ = \quad &\lim_{m\to\infty}\int_{\mathbb{T}^d}\Bigg\{\sum_{n\in\mathbb{Z}^d;|n|\leq m;n\neq 0} |A_n|^2|(1-\Delta_x)^{-\frac{\epsilon}{2}}\exp(i2\pi nx)|^2 dx\times \\ &\times\int_0^T\int_V |\exp(-i2\pi nvt) - \int_V \exp(-i2\pi nvt)dv|^2 dvdt \end{aligned}$$

$$
\begin{aligned}
+ & \sum_{p,q\in\mathbb{Z}^d;|p|,|q|\le m;p\neq q;p,q\neq 0} (1-\Delta_x)^{-\frac{\epsilon}{2}}\exp(i2\pi px)A_p \times \qquad (6.4)\\
& \overline{(1-\Delta_x)^{-\frac{\epsilon}{2}}\exp(i2\pi qx)A_q}\int_0^T\int_V [(\exp(-i2\pi pvt)-\int_V \exp(-i2\pi pvt)dv)\times\\
& \overline{\times\exp(-i2\pi qvt)-\int_V \exp(-i2\pi qvt)dv}]dvdt\Big\}\, dx.
\end{aligned}
$$

Similar as in the previous section, we have

$$
\int_0^T\int_V |\exp(-i2\pi nvt)-\int_V \exp(-i2\pi nvt)dv|^2 dvdt \ge \frac{T}{2},
$$

for T large enough, and

$$
\begin{aligned}
& \Big|\int_0^T\int_V \Big((\exp(-i2\pi pvt)-\int_V \exp(-i2\pi pvt)dv)\times\\
& \overline{\exp(-i2\pi qvt)-\int_V \exp(-i2\pi qvt)dv}\Big)\, dvdt\Big|\\
= & \Big|\int_0^T\int_V \exp(i2\pi(q-p)vt)dvdt\\
& -\int_0^T(\int_V \exp(-i2\pi pvt)dv\int_V \exp(i2\pi qvt)dv)dt\Big|\\
\le & C\left(\frac{T^{1/2}}{|p-q|^{1/2}}+\frac{1}{\sqrt{|p||q|}}\right),
\end{aligned}
$$

where C is some positive constant. Consider the sum

$$
\begin{aligned}
& \sum_{n\in\mathbb{Z}^d;|n|\le m;n\neq 0} |A_n|^2\int_{\mathbb{T}^d} |(1-\Delta_x)^{-\frac{\epsilon}{2}}\sigma\exp(i2\pi nx)|^2 dx\times \qquad (6.5)\\
& \times\int_0^T\int_V |\exp(-i2\pi nvt)-\int_V \exp(-i2\pi nvt)dv|^2 dvdt\\
\ge & \sum_{n\in\mathbb{Z}^d;|n|\le m;n\neq 0} \frac{T}{2}|A_n|^2\int_{\mathbb{T}^d} |(1-\Delta_x)^{-\frac{\epsilon}{2}}\exp(i2\pi nx)|^2 dx \qquad (6.6)\\
\ge & TC\sum_{n\in\mathbb{Z}^d;|n|\le m;n\neq 0} \frac{|A_n|^2}{(1+n^2)^{\epsilon}}, \qquad (6.7)
\end{aligned}
$$

where C is a positive constant.
Now, consider the term

$$\Bigg| \sum_{p,q\in\mathbb{Z}^d;|p|,|q|\le m;p\neq q} (1-\Delta_x)^{-\frac{\epsilon}{2}}\sigma \exp(i2\pi px)A_p \times \overline{(1-\Delta_x)^{-\frac{\epsilon}{2}}\exp(i2\pi qx)A_q} \int_0^T \Big[\int_V (\exp(-i2\pi pvt) - \int_V \exp(-i2\pi pvt)dv)\times \overline{\times \exp(-i2\pi qvt) - \int_V \exp(-i2\pi qvt)dv}\Big] dvdt \Bigg| \tag{6.8}$$

$$\le \sum_{p,q\in\mathbb{Z}^d;|p|,|q|\le m;p\neq q} C\frac{|A_p|}{(1+|p|^2)^{\frac{\epsilon}{2}}}\frac{|A_q|}{(1+|q|^2)^{\frac{\epsilon}{2}}}\left(\frac{T^{\frac{1}{2}}}{|p-q|^{3/2}}+\frac{1}{|p-q|\sqrt{|p||q|}}\right)$$

$$\le \sum_{p,q\in\mathbb{Z}^d;|p|,|q|\le m;p\neq q} C\left(\frac{T^{\frac{1}{2}}}{|p-q|^{3/2}}+\frac{1}{|p-q|\sqrt{|p||q|}}\right)\left(\frac{|A_p|^2}{(1+|p|^2)^{\epsilon}}+\frac{|A_q|^2}{(1+|q|^2)^{\epsilon}}\right).$$

Combine (6.4), (6.5) and (6.8) to get

$$\int_0^T\int_{\mathbb{T}^d}\int_V |(1-\Delta_x)^{-\epsilon/2}(g-\bar{g})|^2 dvdxdt \ge C(T)\sum_{n\in\mathbb{Z}^d}\frac{\int_{\mathbb{R}^d}|A_n(v)|^2dv}{(1+|n|^2)^{\epsilon}},$$

for T large. ■

6.2 Convergence to Equilibrium: Proof of Theorem 2.3

Step 1: The boundedness of $\|\partial_x^k f\|_{L^2}$, $\forall k\in\mathbb{Z}^d$.
Derive (2.6) to get

$$\int_{\mathbb{T}^d}\partial_t\partial_x^k f+\int_{\mathbb{T}^d} v\partial_x^{k+1}f=\int_{\mathbb{T}^d}(1-\Delta_x)^{-\epsilon}\partial_x^k(\bar{f}-f).$$

This leads to

$$\partial_t\|\partial_x^k f\|_{L^2}^2\le 0,$$

which means

$$\|\partial_x^k f\|_{L^2}^2(t)\le\|\partial_x^k f_0\|_{L^2}^2.$$

Step 2: The polynomial convergence.
The previous proposition and Lemma 3.2 imply

$$H_f(0)-H_f(T)\ge C(T,\sigma)\sum_{n\in\mathbb{Z}^d}\frac{|A_n|^2}{(1+|n|^2)^{\epsilon}}. \tag{6.9}$$

Let k_1, k_2 and k_3 be positive numbers satisfying $-2\epsilon k_1+k_2k_3=0$. According to Jensen inequality

$$\begin{aligned}
&\left(\frac{\sum_{n\in\mathbb{Z}^d}\frac{|A_n|^2}{(1+|n|^2)^\epsilon}}{\sum_{n\in\mathbb{Z}^d}|A_n|^2}\right)^{k_1}\left(\frac{\sum_{n\in\mathbb{Z}^d}|A_n|^2|n|^{k_2}}{\sum_{n\in\mathbb{Z}^d}|A_n|^2}\right)^{k_3}\\
\geq\ &\left(\sum_{n\in\mathbb{Z}^d}\frac{|A_n|^2((1+|n|^2)^{-\frac{\epsilon k_1}{k_1+k_3}}|n|^{\frac{k_2k_3}{k_1+k_3}})}{\sum_{n\in\mathbb{Z}^d}|A_n|^2}\right)^{k_1+k_3}\\
\geq\ &C\left(\sum_{n\in\mathbb{Z}^d}\frac{|A_n|^2}{\sum_{n\in\mathbb{Z}^d}|A_n|^2}\right)^{k_1+k_3}\\
\geq\ &C,
\end{aligned}$$

where C is some positive constant, which yields

$$\sum_{n\in\mathbb{Z}^d}\frac{|A_n|^2}{(1+|n|^2)^\epsilon}\geq C\left(\sum_{n\in\mathbb{Z}^d}|A_n|^2\right)\left(\frac{\sum_{n\in\mathbb{Z}^d}|A_n|^2}{\sum_{n\in\mathbb{Z}^d}|A_n|^2|n|^{k_2}}\right)^{\frac{k_3}{k_1}},\tag{6.10}$$

for some positive constant C.
Denote

$$M((l-1)T)=\sum_{n\in\mathbb{Z}^d}|f(\hat{lT})(n)|^2|n|^{k_2},$$

for $l\in\mathbb{N}\{0\}$, where $f(\hat{lT})$ is the Fourier transform in x of $f(lT)$.
Inequalities (6.9) and (6.10) imply

$$H_f(0)-CH_f(0)\left(\frac{H_f(0)}{M(0)}\right)^{\frac{k_3}{k_1}}\geq H_f(T).\tag{6.11}$$

Since the energy H_f is decreasing, (6.11) deduces

$$H_f(lT)-CH_f(lT)\left(\frac{H_f(lT)}{M(lT)}\right)^{\frac{k_3}{k_1}}\geq H_f((l+1)T),\tag{6.12}$$

for all l in $\mathbb{N}\cup\{0\}$. Step 1 implies $M(lT)\leq C$, where C is a positive constant, which together with Inequality (6.12) implies

$$H_f(lT)-CH_f(lT)\left(\frac{H_f(lT)}{C}\right)^{\frac{k_3}{k_1}}\geq H_f((l+1)T).\tag{6.13}$$

Put

$$\mathcal{E}_l = \frac{H_f(lT)}{C},$$

Inequality (6.13) yields

$$\mathcal{E}_{l+1} \leq \mathcal{E}_l - C\mathcal{E}_{l+1}^{\frac{k_3}{k_1}+1},$$

where C is some positive constant. According to Lemma 3.5

$$H_f(lT) \leq C\left(\frac{1}{l+1}\right)^{\frac{k_1}{k_3}} H_f(0),$$

where C is some positive constant. Let $\frac{k_1}{k_3}$ tend to ∞ we get the theorem.

7 Decay rates of the linearized Boltzmann equation

Let g be the solution of

$$\partial_t g + v\partial_x g = 0, \tag{7.1}$$

with the initial datum

$$g(0,x,v) = f_0(x,v),$$

where $f_0(x,v)$ is the initial datum of (2.16). For the sake of simplicity, we suppose that

$$\int_{\mathbb{R}^d} f_0 dv = 0. \tag{7.2}$$

7.1 The Observability Inequality

Similar as in the previous sections, we prove

Proposition 7.1 *There exists a constant T_*, depending on the structure of the equation, such that for all $T > T_*$*

$$\begin{aligned} &\int_0^T \int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}(|v_* - v|,\omega)\mu_*\mu \times \\ &\times[g'_*\mu'^{-1/2}_* + g'\mu'^{-1/2} - g_*\mu_*^{-1/2} - g\mu^{-1/2}]^2 d\sigma dv_* dv dx \\ \geq\ & C\int_{\mathbb{T}^d\times\mathbb{R}^d} (|v|+1)^\alpha |f_0|^2 dxdv. \end{aligned} \tag{7.3}$$

Proof Since g is a solution of (7.1), it could be written under the form

$$g(t,x,v) = g_0(x-vt,v) = \sum_{n\in\mathbb{Z}^d} A_n(v)\exp(i2\pi n(x-vt)),$$

this implies

$$\begin{aligned}
& \int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}(|v_*-v|,\omega)\mu_*\mu \times \qquad (7.4)\\
& \times[g'_*{\mu'_*}^{-1/2} + g'\mu'^{-1/2} - g\mu^{-1/2} - g_*\mu_*^{-1/2}]^2 d\omega dv_* dv dx dt\\
= & \sum_{n\in\mathbb{Z}^d}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu_*\mu\Big|A'_{n*}{\mu'_*}^{-1/2}\exp(-i2\pi nv'_*t) + A_n{}'\mu'^{-1/2}\exp(-i2\pi nv't)\\
& -A_{n*}\mu_*^{-1/2}\exp(-i2\pi nv_*t) - A_n\mu^{-1/2}\exp(-i2\pi nvt)\Big|^2 d\omega dv_* dv dt\\
= & 4\sum_{n\in\mathbb{Z}^d}\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu_*\mu[-A'_{n*}{\mu'_*}^{-1/2}\overline{A_n}\mu^{-1/2}\exp(i2\pi n(v-v'_*)t) -\\
& -A_n{}'\mu'^{-1/2}\overline{A_n}\mu^{-1/2}\exp(i2\pi n(v-v')t)\\
& +A_{n*}\mu_*^{-1/2}\overline{A_n}\mu^{-1/2}\exp(i2\pi n(v-v_*)t) + |A_n\mu^{-1/2}|^2]d\omega dv_* dv dt.
\end{aligned}$$

Using the same technique as in [14], [15] and [24], we consider the components of the last integral of (7.4) separately.
Part 1: Consider the dominant component of (7.4)

$$\begin{aligned}
& \int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu_*\mu|A_n\mu^{-1/2}|^2 d\omega dv_* dv dt \qquad (7.5)\\
= & T\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu_*|A_n|^2 d\omega dv_* dv\\
\geq & TC\int_{\mathbb{R}^d}(|v|+1)^\alpha|A_n|^2 dv,
\end{aligned}$$

where C is some positive constant.
Part 2: Consider the second component of (7.4)

$$\begin{aligned}
& \left|\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu^{1/2}\mu_*^{1/2}A_{n*}\overline{A_n}\exp(i2\pi n(v-v_*)t)d\omega dv_* dv dt\right|\\
\leq & \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu^{1/2}\mu_*^{1/2}|A_{n*}||A_n|\frac{|\sin(\pi n(v-v_*)T)|}{|\pi n(v-v_*)|}d\omega dv_* dv \qquad (7.6)\\
\leq & \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu^{1/2}\mu_*^{1/2}|A_n|^2\frac{|\sin(\pi n(v-v_*)T)|}{|\pi n(v-v_*)|}d\omega dv_* dv.
\end{aligned}$$

The kernel of (7.6) could be bounded in the following way

$$\begin{aligned}
&\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\mathcal{B}\mu_*^{1/2}\mu^{1/2}\frac{|\sin(\pi n(v-v_*)T)|}{|\pi n(v-v_*)|}d\omega dv_* \qquad (7.7)\\
\leq\ & |\mathbb{S}^{d-1}|^{1/2}\left(\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\mathcal{B}^2(\mu\mu_*)^{1/2}d\omega dv_*\right)^{1/2}\times\\
&\times\left(\int_{\mathbb{R}^d}\frac{|\sin(\pi n(v-v_*)T)|^2}{|\pi n(v-v_*)|^2}(\mu\mu_*)^{1/2}dv_*\right)^{1/2}\\
\leq\ & C(|v|+1)^{\beta-2/3}\left(\int_{\mathbb{R}^d}\frac{|\sin(\pi n(v-v_*)T)|^2}{|\pi n(v-v_*)|^2}(\mu\mu_*)^{1/2}dv_*\right)^{1/2},
\end{aligned}$$

where C is some positive constant.
In order to estimate the last integral of (7.7), let ϵ be a positive constant, we consider two cases.
For $|n(v-v_*)|<\epsilon$,

$$\begin{aligned}
&\left(\int_{\{|n(v-v_*)|\}<\epsilon}\frac{|\sin(\pi n(v-v_*)T)|^2}{|\pi n(v-v_*)|^2}(\mu\mu_*)^{1/2}dv_*\right)^{1/2} \qquad (7.8)\\
\leq\ & T\left(\int_{\{|n(v-v_*)|\}<\epsilon}(\mu\mu_*)^{1/2}dv_*\right)^{1/2}\\
\leq\ & TC(\epsilon),
\end{aligned}$$

where $C(\epsilon)$ tends to 0 as ϵ tends to 0.
For $|n(v-v_*)|>\epsilon$,

$$\begin{aligned}
&\left(\int_{\{|n(v-v_*)|\}>\epsilon}\frac{|\sin(\pi n(v-v_*)T)|^2}{|\pi n(v-v_*)|^2}(\mu\mu_*)^{1/2}dv_*\right)^{1/2} \qquad (7.9)\\
\leq\ & \left(\int_{\{|n(v-v_*)|\}>\epsilon}(\mu\mu_*)^{1/2}\frac{1}{\epsilon^2}dv_*\right)^{1/2}\\
\leq\ & \frac{C}{\epsilon},
\end{aligned}$$

where C is some positive constant. Inequalities (7.7), (7.8) and (7.9) then imply

$$\left|\int_0^T\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}}\mathcal{B}\mu^{1/2}\mu_*^{1/2}A_{n*}\overline{A_n}\exp(i2\pi n(v-v_*)t)d\omega dv_*dvdt\right|$$

$$\leq \quad \min\{TC(\epsilon), \frac{C}{\epsilon}\} \int_{\mathbb{R}^d} (|v|+1)^{\beta-2/3} |A_n(v)|^2 dv. \tag{7.10}$$

Part 3: Consider the last components of (7.4), by the change of variables $\omega \to -\omega$

$$\begin{aligned}
& I := \qquad\qquad (7.11)\\
= & \left| \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}\mu_*\mu[-{A_n}'_*{\mu'_*}^{-1/2}\overline{A_n}\mu^{-1/2}\exp(i2\pi n(v-v'_*)t) - \right.\\
& \left. -{A_n}'\mu'^{-1/2}\overline{A_n}\mu^{-1/2}\exp(i2\pi n(v-v')t)]d\omega dv_* dv dt \right|\\
= & \left| \int_0^T \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} 2\mathcal{B}\mu_*\mu {A_n}'\mu'^{-1/2}\overline{A_n}\mu^{-1/2}\exp(i2\pi n(v-v')t) d\omega dv_* dv dt \right|\\
\leq & \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} 2|v-v_*|^\beta |v-v'|^{d-2}|{A_n}'||A_n| \frac{|\sin(\pi n(v-v')T)|}{|\pi n(v-v')|}\mu_*^{1/2}{\mu'_*}^{1/2} d\omega dv_* dv,
\end{aligned}$$

the last inequality is derived by taking the integral in time.
Denote

$$K^* := \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 2|v-v_*|^\beta |v-v'|^{d-2}|{A_n}'| \frac{|\sin(\pi n(v-v')T)|}{|\pi n(v-v')|}\mu_*^{1/2}{\mu'_*}^{1/2} d\omega dv_*,$$

and for ω fixed perform the following changes of variables on K^*: $v_* \to V = v_* - v$ and $V = r\omega + z$ with $z \in \omega^\perp$. Since the Jacobians of the two changes of variables are 1,

$$K^* = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 2r^{d-2}|A_n(v+r\omega)| \frac{|\sin(\pi r T n.\omega)|}{|\pi r n.\omega|} \left(\int_{\omega^\perp} (\mu_*\mu'_*)^{1/2} |r\omega + z|^\beta dz \right) d\omega dr.$$

Now, make the change of variable $(r,\omega) \to W = r\omega$. The Jacobian of this change of variables is $2r^{-d+1}$.

$$K^* = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 4|A_n(v+W)||W|^{-1} \frac{|\sin(\pi T n.W)|}{|\pi n.W|} \left(\int_{W^\perp} (\mu_*\mu'_*)^{1/2} |W + z|^\beta dz \right) dW.$$

Since

$$\begin{aligned}
|v_*|^2 + |v'_*|^2 &= |v+W+z|^2 + |v+z|^2\\
&= \frac{1}{2}|W + 2(v+z)|^2 + \frac{1}{2}|W|^2\\
&= \frac{1}{2}|W + 2(v.\omega)\omega|^2 + 2|z + v - (v.\omega)\omega|^2 + \frac{1}{2}|W|^2,
\end{aligned}$$

then

$$(\mu_*\mu_*')^{1/2} = (2\pi)^{-d/2}\exp\left(-\frac{|W|^2}{8}-\frac{|z+v-(v.\omega)\omega|^2}{2}-\frac{|W+2(v.\omega)\omega|^2}{8}\right),$$

which implies

$$\begin{aligned} K^* &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 4(2\pi)^{-d/2}|A_n(v+W)||W|^{-1}\exp(-\frac{|W|^2}{8}-\frac{|W+2(v.\omega)\omega|^2}{8}) \\ &\quad\times\frac{|\sin(\pi Tn.W)|}{|\pi n.W|}\left(\int_{W^\perp}\exp(-\frac{|z+v-(v.\omega)\omega|^2}{2})|W+z|^\beta dz\right)dW. \end{aligned}$$

Define

$$\begin{aligned} K &:= 4(2\pi)^{-d/2}|v'-v|^{-1}\exp(-\frac{|v'-v|^2}{8}-\frac{|v'-v+2(v.\omega)\omega|^2}{8})\frac{|\sin(\pi Tn.(v'-v))|}{|\pi n.(v'-v)|} \\ &\quad\times\left(\int_{\omega^\perp}\exp(-\frac{|z+v-(v.\omega)\omega|^2}{2})|v'-v+z|^\beta dz\right), \end{aligned}$$

then

$$I \le \int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} K|A_n(v')||A_n(v)|d\omega dv'dv. \tag{7.12}$$

Now, consider the integral in z in the kernel K

$$\begin{aligned} &\int_{\omega^\perp}\exp(-\frac{|z+v-(v.\omega)\omega|^2}{2})|v'-v+z|^\beta dz \qquad (7.13)\\ =& \int_{\omega^\perp}\exp(-\frac{|\overline{z}|^2}{2})|v'-v+\overline{z}-(v-(v.\omega)\omega)|^\beta d\overline{z} \\ \le& C(1+|v'-v-(v-(v.\omega)\omega)|)^\beta, \end{aligned}$$

since $\beta > -(d-1)$, the integral is well-defined. Let s be a real number, according to the inequality

$$(1+|\zeta'|)^s \le C(1+|\zeta|)^s(1+|\zeta'-\zeta|)^{|s|},$$

the following estimate follows from (7.13)

$$\begin{aligned} &\int_{\mathbb{R}^d} K(1+|v'|)^s dv' \qquad (7.14)\\ \le& C(1+|v|)^s\int_{\mathbb{R}^d}|v'-v|^{-1}\exp(-\frac{|v'-v|^2}{8}-\frac{|v'-v+2(v.\omega)\omega|^2}{8}) \\ &\times\frac{|\sin(\pi Tn.(v'-v))|}{|\pi n.(v'-v)|}(1+|v'-v-(v-(v.\omega)\omega)|)^\beta(1+|v'-v|)^{|s|}dv' \end{aligned}$$

$$
\begin{aligned}
&\leq C(1+|v|)^s\left(\int_{\mathbb{R}^d}\frac{|\sin(\pi Tn.(v'-v))|^3}{|\pi n.(v'-v)|^3}\exp(-\frac{|v'-v|^2}{8})dv'\right)^{1/3}\times\\
&\quad\times\left(\int_{\mathbb{R}^d}|v'-v|^{-3/2}\exp(-\frac{|v'-v|^2}{8}-\frac{3|v'-v+2(v.\omega)\omega|^2}{16})\right.\\
&\quad\left.\times(1+|v'-v-(v-(v.\omega)\omega)|)^{3/2\beta}(1+|v'-v|)^{3/2|s|}dv'\right)^{2/3}\\
&\leq (1+|v|)^s\min\{TC(\epsilon),\frac{C}{\epsilon}\}\left(\int_{\mathbb{R}^d}\exp(-\frac{|v'-v|^2}{8}-\frac{3|v'-v+2(v.\omega)\omega|^2}{16})\right.\\
&\quad\left.\times C|v'-v|^{-3/2}(1+|v'-v-(v-(v.\omega)\omega)|)^{3/2\beta}(1+|v'-v|)^{3/2|s|}dv'\right)^{2/3},
\end{aligned}
$$

the last inequality is obtained by the same argument that we use in Part 2. Now, we consider two cases $\beta\geq 0$ and $\beta<0$.
Case 1: $\beta\geq 0$.

$$
\begin{aligned}
&\int_{\mathbb{R}^d}K(1+|v'|)^s dv'\\
&\leq (1+|v|)^s\min\{TC(\epsilon),\frac{C}{\epsilon}\}\left(\int_{\mathbb{R}^d}\exp(-\frac{|v'-v|^2}{8}-\frac{3|v'-v+2(v.\omega)\omega|^2}{16})\right.\\
&\quad\left.\times C|v'-v|^{-3/2}(1+|v-(v.\omega)\omega|)^{3/2\beta}(1+|v'-v|)^{3/2|s|+3/2\beta}dv'\right)^{2/3}\\
&\leq (1+|v|)^{s+\beta}\min\{TC(\epsilon),\frac{C}{\epsilon}\}\left(\int_{\mathbb{R}^d}\exp(-\frac{|v'-v|^2}{8}-\frac{3|v'-v+2(v.\omega)\omega|^2}{16})\right.\\
&\quad\left.\times C|v'-v|^{-3/2}(1+|v'-v|)^{3/2|s|+3/2\beta}dv'\right)^{2/3}.
\end{aligned}
$$

Denote

$$
J_1:=\int_{\mathbb{R}^d}\exp(-\frac{|v'-v|^2}{8}-\frac{3|v'-v+2(v.\omega)\omega|^2}{16})|v'-v|^{-3/2}(1+|v'-v|)^{3/2|s|+3/2\beta}dv'.
$$

Perform the changes of variables $V\to u=v'-v$ and $u=r\omega$, $r\in\mathbb{R}_+$, $\omega\in S^{d-1}$, and choose v as the north pole vector in the angle parametrization

$$
\begin{aligned}
J_1 &= |S^{d-2}|\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2\beta}\exp(-\frac{r^2}{8})\times\\
&\quad\times\int_0^\pi\exp(-\frac{3(r+2|v|\cos\varphi)^2}{16})\sin^{d-2}(\varphi)d\varphi dr.
\end{aligned}
$$

For the case $d\geq 3$, since $\sin^{d-2}(\varphi)\leq\sin(\varphi)$,

$$
J_1 \leq |S^{d-2}|\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2\beta}\exp(-\frac{r^2}{8})\times
$$

$$\times \int_0^\pi \exp(-\frac{3(r+2|v|\cos\varphi)^2}{16})\sin(\varphi)d\varphi dr.$$

Now, make the change of variables $y = r + 2|v|\cos(\varphi)$ in the φ integral to get

$$\begin{aligned} J_1 &\leq |S^{d-2}||v|^{-1}\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2\beta}\exp(-\frac{r^2}{8})\int_{-\infty}^\infty \exp(-3\frac{y^2}{16})dydr \\ &\leq C|v|^{-1}, \end{aligned}$$

where C is a positive constant. Notice that since $\beta > 0$, the integral

$$\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2\beta}\exp(-\frac{r^2}{8})dr,$$

is well-defined.

For the case $d = 2$, we perform the same change of variables

$$\begin{aligned} J_1 &\leq |S^{d-2}||v|^{-1}\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2\beta}\exp(-\frac{r^2}{8}) \\ &\quad\times\int_{r-2|v|}^{r+2|v|}\exp(-\frac{3y^2}{16})(1-(\frac{y-r}{2|v|})^2)^{-1/2}dydr \\ &\leq C\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2\beta}\exp(-\frac{r^2}{8}) \\ &\quad\times\int_{r-2|v|}^{r+2|v|}\exp(-\frac{3y^2}{16})(4|v|^2-(y-r)^2)^{-1/2}dydr, \end{aligned}$$

where C is some positive constant.

We consider the integral in two regions $|y-r| \leq |v|$ and $|y-r| \geq |v|$. On the first region, $(4|v|^2-(y-r)^2)^{-1/2} \leq |v|^{-1}$. On the second region, either $r \geq |v|/2$ or $|y| \geq |v|/2$ gives an exponential decay. Finally, we get

$$\int_{\mathbb{R}^d} K(1+|v'|)^s dv' \leq C(1+|v|)^{\beta-2/3+s}.$$

Case 2: $\beta < 0$.

$$\begin{aligned} &\int_{\mathbb{R}^d} K(1+|v'|)^s dv' \\ \leq\ & (1+|v|)^s \min\{TC(\epsilon), \frac{C}{\epsilon}\}\left(\int_{\mathbb{R}^d}\exp(-\frac{|v'-v|^2}{8}-\frac{3|v'-v+2(v.\omega)\omega|^2}{16})\right. \end{aligned}$$

$$\times C|v'-v|^{-3/2}(1+|v-(v.\omega)\omega|)^{3/2\beta}(1+|v'-v|)^{3/2|s|+3/2|\beta|}dv'\Big)^{2/3}.$$

Again, perform the change of variables $u = r\omega$, $r \in \mathbb{R}_+$, $\omega \in \mathbb{S}^{d-1}$, choose v as the north pole vector in the angle parametrization. Denote

$$\begin{aligned}
J_2 &= |S^{d-2}|\int_0^\infty r^{d-5/2}(1+r)^{3/2|s|+3/2|\beta|}\exp(-\frac{r^2}{8})\times \\
&\quad \times\int_0^\pi (1+|v|\sin\varphi)^{3/2\beta}\exp(-\frac{3(r+2|v|\cos\varphi)^2}{16})\sin^{d-2}\varphi d\varphi dr.
\end{aligned}$$

Split the integral into two region $|\cos\varphi| \le \frac{1}{\sqrt{2}}$ and $|\cos\varphi| > \frac{1}{\sqrt{2}}$. In the first case, since $\sin\varphi \ge \frac{1}{\sqrt{2}}$, then

$$(1+|v|\sin\varphi)^{3/2\beta} \le C(1+|v|)^{3/2\beta},$$

the proof is then similar as in the case $\beta > 0$. In the second case,

$$\frac{(r+2|v|\cos\varphi^2)}{2} \ge \frac{|v|^2}{12} - \frac{r^2}{16},$$

this leads to an exponential decay in v. Finally, we get

$$\int_{\mathbb{R}^d} K(1+|v'|)^s dv' \le C(1+|v|)^{\beta-2/3+s}.$$

Combine this estimate with (7.11), (7.12), (7.13) and (7.14) to get

$$\begin{aligned}
I &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |A_n(v)||A_n(v')|K dv dv' \\
&\le \left(\int_{\mathbb{R}^d} |A_n(v)|^2(1+|v|)^{\beta-2/3}\right)^{1/2}\Big[\int_{\mathbb{R}^d}(1+|v|)^{-\beta+2/3}\int_{\mathbb{R}^d}K(v,v')dv'\times \\
&\quad (\int_{\mathbb{R}^d} K(v,v'')|A_n(v'')|^2dv'')dv\Big]^{1/2},
\end{aligned}$$

which implies

$$I \le C\min\{TC(\epsilon), \frac{C}{\epsilon}\}\int_{\mathbb{R}^d}|A_n(v)|^2(1+|v|)^{\beta-2/3}. \tag{7.15}$$

When ϵ is small and T is large enough, (7.4), (7.5), (7.10), (7.15) imply

$$\int_0^T\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathcal{B}(|v_*-v|,\omega)\mu_*\mu\times \tag{7.16}$$

$$
\begin{aligned}
&\times[g'_*\mu'^{-1/2}_* + g'\mu'^{-1/2} - g\mu^{-1/2} - g_*\mu_*^{-1/2}]^2 d\omega dv_* dv dx \\
\geq\ & CT\sum_{n\in\mathbb{Z}^d}\int_{\mathbb{R}^d}(1+|v|)^\alpha|A_n(v)|^2 dv \\
\geq\ & CT\int_{\mathbb{T}^d\times\mathbb{R}^d}|g_0|^2 dxdv.
\end{aligned}
$$

■

Proposition 7.2 *Suppose that $\alpha, \beta > 0$ and there exist positive numbers T_1 and C such that*

$$
\begin{aligned}
&\int_0^{T_1}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}}\mathcal{B}(|v_*-v|,\omega)\mu_*\mu\times \\
&\times[g'_*\mu'^{-1/2}_* + g'\mu'^{-1/2} - g\mu^{-1/2} - g_*\mu_*^{-1/2}]^2 d\sigma dv_* dv dx \\
\geq\ & C\int_{\mathbb{T}^d\times\mathbb{R}^d}(|v|+1)^\alpha|f_0|^2 dxdv.
\end{aligned}
\tag{7.17}
$$

then there exist positive numbers T_0 and δ such that $\forall\ t \geq T_0$, $\forall\ f_0 \in L^\infty(\mathbb{T}^d\times\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d, H^1(\mathbb{T}^d))$

$$
H_f(t) \leq \exp(-\delta t)H_f(0), \tag{7.18}
$$

Proof We check that L satisfies the conditions (3.6), (3.7), (3.8), (3.9). Let ϵ be any positive constant, define

$$
I_\epsilon := \chi\left(|v-v_*|\leq\frac{1}{\epsilon}\right),
$$

$$
\begin{aligned}
L_{\epsilon,1}[g] \ :=\ & -\int_0^{T_1}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} I_\epsilon\mathcal{B}(|v_*-v|,\omega)\mu_*\mu^{1/2}\times \\
&\times[g'_*\mu'^{-1/2}_* + g'\mu'^{-1/2} - g\mu^{-1/2} - g_*\mu_*^{-1/2}]d\sigma dv_* dx,
\end{aligned}
$$

$$
\begin{aligned}
L_{\epsilon,2}[g] \ :=\ & -\int_0^{T_1}\int_{\mathbb{T}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} (1-I_\epsilon)\mathcal{B}(|v_*-v|,\omega)\mu_*\mu^{1/2}\times \\
&\times[g'_*\mu'^{-1/2}_* + g'\mu'^{-1/2} - g\mu^{-1/2} - g_*\mu_*^{-1/2}]d\sigma dv_* dx.
\end{aligned}
$$

It is not difficult to see that L, $L_{\epsilon,1}$, $L_{\epsilon,2}$ satisfy (3.6), (3.7), (3.8), with $H' = L^2((1+|v|)^\alpha)$. Proceed similar as in the previous proposition to get

$$
\|L_{\epsilon,2}[g]\|^2_{L^2} \leq C(\epsilon)\int_{\mathbb{T}^d\times\mathbb{R}^d}(|v|+1)^\beta|g|^2 dxdv,
$$

which means that (3.9) is satisfied. By Lemma 3.3, the conclusion of the proposition follows. ■

7.2 Convergence to Equilibrium: Proof of Theorem 2.4

The case $\alpha, \beta > 0$ is straight forward from Proposition 7.1 and Proposition 7.2. We now prove the theorem for the case $-d+1 < \alpha, \beta < 0$. According to Proposition 7.1 and Lemma 3.2, there exist a time T and a constant C such that

$$\|f(0)\|_{L^2}^2 - \|f(T)\|_{L^2}^2 \geq C \int_{\mathbb{T}^d \times \mathbb{R}^d} (|v|+1)^\alpha |f(0)|^2 dx dv.$$

This implies that for all k in

$$\|f(kT)\|_{L^2}^2 - \|f((k+1)T)\|_{L^2}^2 \geq C \int_{\mathbb{T}^d \times \mathbb{R}^d} (|v|+1)^\alpha |f(kT)|^2 dx dv. \quad (7.19)$$

Now, for positive numbers k_1, k_2 and k_3 satisfying $\alpha k_1 + k_2 k_3 = 0$, according to the Holder inequality

$$\left(\int_{\mathbb{T}^d \times \mathbb{R}^d} (|v|+1)^\alpha |f(kT)|^2\right)^{k_1} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} (|v|+1)^{k_2} |f(kT)|^2\right)^{k_3} \geq \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |f(kT)|^2\right)^{k_1+k_3}. \quad (7.20)$$

Combine (7.19) and (7.20) to get

$$\|f((k+1)T)\|_{L^2}^2 \leq \|f(kT)\|_{L^2}^2 - C \frac{\|f(kT)\|_{L^2}^{2\frac{k_1+k_3}{k_1}}}{\left(\int_{\mathbb{T}^d \times \mathbb{R}^d} (|v|+1)^{k_2} |f(kT)|^2\right)^{\frac{k_3}{k_1}}}. \quad (7.21)$$

Now, choose $(|v|+1)^{k_2} f, (k_2 > 0)$ as a test function in the linearized Boltzmann equation to obtain

$$\begin{aligned} & \|(|v|+1)^{k_2/2} f(0)\|_{L^2}^2 - \|(|v|+1)^{k_2/2} f(kT)\|_{L^2}^2 \\ \geq & \int_{\mathbb{T}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \mathcal{B}(|v_* - v|, \omega) \mu_* \mu^{1/2} (|v|+1)^{k_2} \times \\ & \times [f \mu^{-1/2} + f_* \mu_*^{-1/2} - f'_* \mu_*'^{-1/2} - f' \mu'^{-1/2}] f d\sigma dv_* dx \\ \geq & \ 0, \end{aligned}$$

then

$$\|(|v|+1)^{k_2/2} f(kT)\|_{L^2}^2 \leq C \|f(0)\|_{L^2((1+|v|)^{k_2/2})}^2, \quad (7.22)$$

where C is some positive constant. The two inequalities (7.21) and (7.22) lead to

$$\|f((k+1)T)\|_{L^2}^2 \leq \|f(kT)\|_{L^2}^2 - C\|f(kT)\|_{L^2}^{2\frac{k_1+k_3}{k_1}}.$$

This implies

$$\|f((k+1)T)\|_{L^2}^2 \leq \|f(kT)\|_{L^2}^2 - C\left(\|f((k+1)T)\|_{L^2}^2\right)^{\frac{k_1+k_3}{k_1}}.$$

According to Lemma 3.5,

$$\|f(kT)\|^2 \leq \frac{Mk}{(k+1)^{\frac{k_1}{k_3}}}.$$

Let $\frac{k_1}{k_3} = -\frac{k_2}{\alpha}$ tend to ∞, we get the theorem.

8 Conclusion

We have presented a new approach to the problem of convergence to equilibrium of kinetic equations. The idea of our technique is to prove a 'weak' coercive estimate on the damping. The approach seems to work very well in the context of linear equations. Our technique is constructive, since the constants in the decay rates could be obtained explicitly. An reasonable question is if this technique could be extended to study the trend to equilibrium of nonlinear kinetic equations, where a typical example is the nonlinear Boltzmann equation. In an ongoing project, we are trying to analyse this method for the linearized Uehling-Uhlenbeck equation, where a spectral gap estimate is hard to obtain but a 'weak' coercive estimate is easier to get.
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