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On the numerical solution of the control problem of switched linear systems

Alessandro N. Vargas, João Y. Ishihara, and João B. R. do Val

***Abstract*— This paper presents a method to compute an epsilon-optimal solution of the control problem of switched linear systems. A difficulty that emerges in the evalution of the optimal solution is that the cardinality of the solution set increases exponentially as long as the time-horizon increases linearly, which turns the problem intractable when the horizon is sufficiently large. We propose a numerical method to overcome such difficulty, in the sense that our approach allows the evalution of epsilon-optimal solutions with corresponding sets that do not increase exponentially.**

I. Introduction

The control of switched linear systems is a subject of intensive investigation over the last few years, with many contributions spreading in the literature. For instance, there are many recent monographs and articles dedicated to the investigation of this theme, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], as a small sample.

The formal definition of the system we deal with is the following. Consider the discrete-time switched linear system represented by

$$x_{k+1} = A_{v_k} x_k + B_{v_k} u_k, \quad \forall k \geq 0,\ x_0 \in \mathbb{R}^n, \qquad (1)$$

where $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$, and $v_k \in \mathcal{V} := \{1, \ldots, \sigma\}$, $\forall k \geq 0$, denote respectively the system state, control input, and discrete switching control. For each instant of time $k \geq 0$, the matrix pair (A_{v_k}, B_{v_k}) belongs to the given finite set $\{(A_v, B_v) : v = 1, \ldots, \sigma\}$.

Let us assume that the system (1) evolves within a fixed, finite-time interval $[0, \ldots, N]$. A policy $\pi = \{\pi_0, \ldots, \pi_{N-1}\}$ represents a sequence of control laws in the form

$$\pi_k := (u_k, v_k), \quad k = 0, \ldots, N-1,$$

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and the set of all admissible policies is denoted by Π. For a given policy $\pi \in \Pi$, let us associate the corresponding system (1) with a quadratic cost

$$J_N(\pi, x_0) = \sum_{k=0}^{N-1} (x_k' Q_{v_k} x_k + u_k' R_{v_k} u_k) + x_N' Q_{v_N} x_N,$$

where $Q_v = Q_v' \geq 0$, and $R_v = R_v' > 0$, for each $v = 1, \ldots, \sigma$, are given matrices.

The linear quadratic regulator control problem of N stages is defined as

$$J_N^*(x_0) := \min_{\pi \in \Pi} J_N(\pi, x_0). \qquad (2)$$

The solution of the control problem as in (2) is well-known in the literature [8], [14], [15], and it is based on the so-called *solution set*, as we now explain. Indeed, to build the solution set, one must start with a set $\mathcal{L}_0 = \{Q_1, \ldots, Q_\sigma\}$. In a recursively manner, the argument determines that $\mathcal{L}_k$ must be generated through a combinatorial design from the elements of the set $\mathcal{L}_{k-1}$, that is, each element of $\mathcal{L}_{k-1}$ acts as a vertice of σ nodes, and each node corresponds to a Riccati-like matrix to be aggregated into $\mathcal{L}_k$. Due to this combinatorial construction, the cardinality of the set $\mathcal{L}_N$ equals σ^N, i.e., $|\mathcal{L}_N| = \sigma^N$. The optimal solution for the problem in (2) is then given by

$$J_N^*(x_0) = \min_{L \in \mathcal{L}_N} x_0' L x_0.$$

An important drawback arises, mainly from the computational point of view, when the horizon N becomes sufficiently large. Notice that the computation of the optimal solution becomes intractable when N tends to infinity. An attempt to overcome such computational difficulty is made by the authors of [8], [14], and [15]. In [8] and [14], the authors adopt the strategy of excluding from the set $\mathcal{L}_N$ the redundant matrices. The authors of [15] introduce a relaxation algorithm in the receding horizon context to evaluate suboptimal solutions for the problem. However, all of these approaches may suffer numerically because the cardinality of the respective evaluation set can increase exponentially as long as N increases linearly.

The main contribution of this paper is in the computational front. Indeed, we derive a method to compute an ε-optimal solution with $\varepsilon > 0$ chosen arbitrarily. Besides, the method determines and evaluates the corresponding ε-optimal solution set, and as numerical experiments suggest, the cardinality of the solution remains bounded as long as the horizon N increases to infinity, see Section III for an account. Thus, our approach can be seen as a feasible tool to compute the optimal solution, with a clear numerical advantage when compared with the algorithms available in the literature.

The remaining part of this paper is structured as follows. Section II introduces preliminary notation, the linear quadratic regulator problem and the proposed solution. A numerical example illustrates the effectiveness of the result in Section III.

II. Definitions and Main Results

Let $\mathbb{R}^r$ be the r-dimensional Euclidean space, $\mathbb{R}^{r,s}$ be the linear space formed by all real matrices of dimension $r \times s$, and $\mathbb{S}^{r0}$ ($\mathbb{S}^{r+}$) be the subspace of $\mathbb{R}^{r,r}$ given by all symmetric non-negative (positive) matrices such that $\{U \in \mathbb{R}^{r,r} : U = U', U \geq 0\ (> 0)\}$, where U' denotes the transpose of U.

Let us define the operators $\mathcal{G}_v : \mathbb{S}^{n0} \mapsto \mathbb{R}^{r,n}$ and $\mathcal{P}_v : \mathbb{S}^{n0} \mapsto \mathbb{S}^{n0}$, $v = 1, \ldots, \sigma$, as follows:

$$\mathcal{G}_v(L) = (R_v + B_v' L B_v)^{-1} B_v L A_v, \tag{3}$$

$$\begin{aligned} \mathcal{P}_v(L) &= Q_v + \mathcal{G}_v(L)' R_v \mathcal{G}_v(L) \\ &\quad + (A_v + B_v \mathcal{G}_v(L))' L (A_v + B_v \mathcal{G}_v(L)),\ \forall L \in \mathbb{S}^{n0}. \end{aligned} \tag{4}$$

Remark 2.1: Notice that the expressions in (3) and (4) retrieves the well-known Riccati equation when $\sigma = 1$.

It is now necessary to present a set that is resulting from the rules in (3) and (4). Let $\mathcal{M}$ be any enumerable set of matrices. Let us then define

$$\mathcal{P}(\mathcal{M}) = \{\mathcal{P}_v(M) : \forall v \in \mathcal{V}, \forall M \in \mathcal{M}\}. \tag{5}$$

Definition 2.1: (ε-optimality sets). Set $j = 1$ and $\mathcal{M}_0 = \{Q_1, \ldots, Q_\sigma\}$, and consider the following method to construct $\mathcal{M}_1, \ldots, \mathcal{M}_N$:

- *Step 1*: Pick the set $\mathcal{P}(\mathcal{M}_{j-1})$ as in (5), and enumerate it in the format $\mathcal{P}(\mathcal{M}_{j-1}) = \{U_{[1]}, \ldots, U_{[d]}\}$. Take $M_{[0]} = \emptyset$, $\ell = 0$, and $s = 0$ and go to the next step.
- *Step 2*: Set $\ell = \ell + 1$, and consider the expression:

$$P_{[\ell]} \notin \bigcup_{i=0}^{s} \mathcal{B}_\varepsilon\left(M_{[i]}\right)$$
$$\implies\ s = s + 1 \ \text{ and } \ M_{[s]} = U_{[\ell]}.$$

  Return to the beginning of *Step 2* if $\ell < d$.
- *Step 3*: Take $\mathcal{M}_j = \{M_{[1]}, \ldots, M_{[s]}\}$ and set $j = j+1$. Return to the beginning of *Step 1* if $j < N$.

Example 2.1: (Illustrative example: construction of an ε-optimality set). The aim of this example is to illustrate how the ε-optimality sets are constructed and the basic idea is depicted in Fig. 1. In this example we suppose that $\sigma = 2$ and $\mathcal{M}_0 = \{Q_1, Q_2\}$. Let us identify the elements of $\mathcal{P}(\mathcal{M}_0)$, say $\mathcal{P}(\mathcal{M}_0) = \{Z_1, Z_2, Z_3, Z_4\}$. The first element of $\mathcal{P}(\mathcal{M}_0)$ is always transferred to the new set $\mathcal{M}_1$. The next idea is that the elements from $\mathcal{P}(\mathcal{M}_0)$ must have

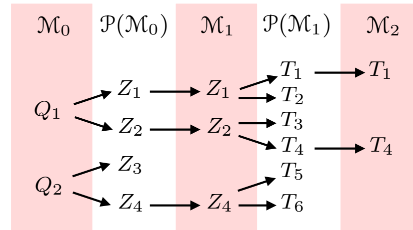

Fig. 1. Construction of the ε-optimality sets according to the Example 2.1. III.

a distance of ε from one to another, and only these elements can be transferred to $\mathcal{M}_1$. For instance, let us assume that $\|Z_1 - Z_2\| \geq \varepsilon$, hence Z_2 can be aggregated into $\mathcal{M}_1$. The next candidate is Z_3 and let us assume that it satisfies $\|Z_2 - Z_3\| < \varepsilon$; hence Z_3 fails the distance condition and can not be aggregated into $\mathcal{M}_1$. We now assume that $\|Z_1 - Z_4\| \geq \varepsilon$ and $\|Z_2 - Z_4\| \geq \varepsilon$, and hence Z_4 can be aggregated into $\mathcal{M}_1$. The argument proceeds similarly and the elements $T2, T3, T5, T6$ are excluded from $\mathcal{M}_2$ because they do not respect the distance either from T_1 or T_4.

A. Functionals based on the ε-optimal sets

Let us now define the functional

$$W_{k,N}(x) = \min_{(v,M) \in (\mathcal{V}, \mathcal{M}_{N-k-1})} x' \mathcal{P}_v(M) x, \qquad k = 0, \ldots, N-1. \tag{6}$$

Definition 2.1 and the rule in (5) assure that, for every element (v, M) taken from $(\mathcal{V}, \mathcal{M}_\ell)$, $\ell = 0, \ldots, N-1$, and applied into (3) and (4) to produce the matrix $\mathcal{P}_v(M)$, this resulting matrix in fact belongs to the set $\mathcal{P}(\mathcal{M}_\ell)$. It enables us to conclude that

$$W_{k,N}(x) = \min_{M \in \mathcal{P}(\mathcal{M}_{N-k-1})} x' M x, \quad k = 0, \ldots, N-1. \tag{7}$$

Consider now the next functional:

$$V_{k,N}(x) = \min_{M \in \mathcal{M}_{N-k}} x' M x, \quad k = 0, \ldots, N. \tag{8}$$

The next result introduces a useful inequality.

Lemma 2.1: For each $k = 0, \ldots, N-1$, there holds

$$V_{k,N}(x) - \varepsilon \|x\| < W_{k,N}(x) \leq V_{k,N}(x), \quad \forall x \in \mathbb{R}^n. \tag{9}$$

Proof: First of all, let us fix some $0 \leq \ell < N$. One can notice from Definition 2.1 that the set $\mathcal{M}_{\ell+1}$ is made up exclusively of elements extracted from $\mathcal{P}(\mathcal{M}_\ell)$. Hence,

$$\mathcal{M}_{\ell+1} \subseteq \mathcal{P}(\mathcal{M}_\ell), \tag{10}$$

and as a result we have

$$\min_{(v,M) \in (\mathcal{V}, \mathcal{M}_\ell)} x' \mathcal{P}_v(M) x \leq \min_{M \in \mathcal{M}_{\ell+1}} x' M x,$$

which shows the rigthmost inequality in (9).

Now, we show the validity of the other inequality of (9). Indeed, we have from Definition 2.1 that, for each

$U \in \mathcal{M}_{\ell+1}$, there corresponds some $Z \in \mathcal{P}(\mathcal{M}_\ell)$ such that $\|U - Z\| < \varepsilon$. Hence

$$x'Ux = x'(U - Z)x + x'Zx \geq -\varepsilon\|x\| + x'Zx \tag{11}$$

Suppose that we choose $U \in \mathcal{M}_{\ell+1}$ in such a manner that

$$\min_{M \in \mathcal{M}_{\ell+1}} x'Mx = x'Ux. \tag{12}$$

Since

$$x'Zx \geq \min_{M \in \mathcal{P}(\mathcal{M}_\ell)} x'Mx, \tag{13}$$

we can conclude from (11)–(13) that

$$\min_{M \in \mathcal{M}_{\ell+1}} x'Mx \;\geq\; -\varepsilon\|x\| + \min_{M \in \mathcal{P}(\mathcal{M}_\ell)} x'Mx,$$

which yields the desired result. ■

In the sequel, we will present an evaluation that yields the main result, and for this purpose it is required to defi the next functional:

$$S_{k,N}(x) = \min_{v \in \mathcal{V}, u \in \mathbb{R}^r} x'Q_v x + u'R_v u + V_{k+1,N}(A_v x + B_v u),$$
$$\forall x \in \mathbb{R}^n,\ k = 0, \ldots, N-1. \tag{14}$$

Lemma 2.2: For each $k = 0, \ldots, N-1$, there holds

$$S_{k,N}(x) = W_{k,N}(x), \quad \forall x \in \mathbb{R}^n. \tag{15}$$

Proof: From (8), we can write

$$S_{k,N}(x) = \min_{(v,u) \in (\mathcal{V}, \mathbb{R}^r)} \Big[x'Q_v x + u'R_v u + \min_{M \in \mathcal{M}_{N-k-1}} (A_v x + B_v u)'M(A_v x + B_v u)\Big].$$

Since the two minimizers in the last expression are interchangeable [16, Th.6S, p.54], we have

$$S_{k,N}(x) = \min_{(v,M) \in (\mathcal{V}, \mathcal{M}_{N-k-1})} \min_{u \in \mathbb{R}^r} \Big[x'Q_v x + u'R_v u + (A_v x + B_v u)'M(A_v x + B_v u)\Big]. \tag{16}$$

Taking the minimum with respect to u in the expression inside the square brackets of (16), we obtain [17, p.257]

$$S_{k,N}(x) = \min_{(v,M) \in (\mathcal{V}, \mathcal{M}_{N-k-1})} \Big[x'(Q_v + \mathcal{G}_v(M)'R_v\mathcal{G}_v(M) + (A_v + B_v\mathcal{G}_v(M))'M(A_v + B_v\mathcal{G}_v(M)))x\Big], \tag{17}$$

where $\mathcal{G}_v(\cdot)$ denotes the operator as in (3). It is then clear from (4) and (17) that

$$S_{k,N}(x) = \min_{(v,M) \in (\mathcal{V}, \mathcal{M}_{N-k-1})} x'\mathcal{P}_v(M)x,$$

which is what we wished to show. ■

Lemma 2.3: Let the control rule $\pi_k = (v_k, u_k)$, $k = 0, \ldots, N-1$, be defined as follows.

$$(v_k, M_k) = \arg \min_{(v,M) \in (\mathcal{V}, \mathcal{M}_{N-k})} x_k'\mathcal{P}_v(M)x_k$$
$$u_k = \mathcal{G}_{v_k}(M_k)x_k, \tag{18}$$

where x_k represents the trajectory corresponding to π_k as in (1). Then

$$V_{0,N}(x_0) \geq J_N(\pi, x_0), \quad \forall x_0 \in \mathbb{R}^n. \tag{19}$$

Proof: Let $x_0, x_1, \ldots, x_N$ be the trajectory as in (2) corresponding to the controls $\pi_0, \ldots, \pi_{N-1}$. Taking Lemma 2.2 with $k = 0$, we have

$$W_{0,N}(x_0) = S_{0,N}(x_0) = x_0'Q_{v_0}x_0 + u_0'R_{v_0}u_0 + V_{1,N}(x_1).$$

Since, from Lemma 2.1, the quantity $W_{0,N}(x_0)$ is bounded above by $V_{0,N}(x_0)$, we get that

$$V_{0,N}(x_0) - V_{1,N}(x_1) \geq x_0'Q_{v_0}x_0 + u_0'R_{v_0}u_0.$$

Proceeding similarly for $k = 1, \ldots, N-1$, one can show that

$$V_{k,N}(x_k) - V_{k+1,N}(x_{k+1}) \geq x_k'Q_{v_k}x_k + u_k'R_{v_k}u_k. \tag{20}$$

Summing up the elements of (20) on k, we obtain

$$V_{0,N}(x_0) - V_{N,N}(x_N) \geq \sum_{k=0}^{N-1} x_k'Q_{v_k}x_k + u_k'R_{v_k}u_k,$$

which shows the result. ■

Remark 2.2: An interesting conclusion derived from Lemma 2.3 is as follows. From optimality, we have $J_N^*(x_0) \leq J_N(\pi, x_0)$ for all policies $\pi \in \Pi$, and in particular if π obeys the rule in (18) then we can use (19) to write

$$J_N^*(x_0) \leq J_N(\pi, x_0) \leq V_{0,N}(x_0), \quad \forall x_0 \in \mathbb{R}^n. \tag{21}$$

The inequalities in (21) will be useful in the sequel.

Let us now consider the cost-to-go of the switched control problem as defined below:

$$J_{k,N}^*(x) = \min_{\pi \in \Pi} \sum_{\ell=k}^{N-1} (x_\ell'Q_{v_\ell}x_\ell + u_\ell'R_{v_\ell}u_\ell) + x_N'Q_{v_N}x_N,$$
$$\forall x_k = x \in \mathbb{R}^n,\ k = 0, \ldots, N.$$

Notice, in particular, that $J_{0,N}^*(x_0) = J_N^*(x_0)$.

Lemma 2.4: Let $\{x_k^*\}$, with $x_0^* = x \in \mathbb{R}^n$, be the optimal trajectory. Then there holds

$$V_{k,N}(x_k^*) \leq J_{k,N}^*(x_k^*) + \varepsilon \sum_{\ell=k}^{N-1} \|x_\ell^*\|, \quad 0 \leq k \leq N. \tag{22}$$

Proof: The proof follows by induction on k. Under $k = N$, the result is trivial since $V_{N,N}(x_N^*) = J_{N,N}^*(x_N^*)$. Now, take $k = N-1$ and $x = x_{N-1}^*$, and recall from the dynamic programming argument that [17, Ch. 6], [18],

$$J_{N-1,N}^*(x) = \min_{v \in \mathcal{V}, u \in \mathbb{R}^r} \Big[x'Q_v x + u'R_v u + J_{N,N}^*(A_v x + B_v u)\Big]$$
$$= \min_{v \in \mathcal{V}, u \in \mathbb{R}^r} \Big[x'Q_v x + u'R_v u + V_{N,N}(A_v x + B_v u)\Big].$$

It follows from Lemma 2.2 that

$$J^*_{N-1,N}(x) \geq V_{N-1,N}(x) - \varepsilon\|x\|,$$

which shows the result for $k = N-1$.

Let us assume that the result holds for $k = t+1$, and for sake of notational simplicity, let us set $x = x^*_t$. In this case, we have from the dynamic programming that

$$J^*_{t,N}(x) = \min_{v\in\mathcal{V}, u\in\mathbb{R}^r} x'Q_v x + u'R_v u + J^*_{t+1,N}(A_v x + B_v u). \tag{23}$$

If we let (v^*, u^*) to denote the minimizer of the rigth-hand side of (23), we can employ (22) with $k = t+1$ to write

$$\begin{aligned} J^*_{t,N}(x) &= x'Q_{v^*}x + u^{*\prime}R_{v^*}u^* + J^*_{t+1,N}(A_{v^*}x + B_{v^*}u^*) \\ &= x'Q_{v^*}x + u^{*\prime}R_{v^*}u^* + J^*_{t+1,N}(x^*_{t+1}) \\ &\geq x'Q_{v^*}x + u^{*\prime}R_{v^*}u^* \\ &\qquad + V_{t+1,N}(x^*_{t+1}) - \varepsilon \sum_{\ell=t+1}^{N-1} \|x^*_\ell\|. \end{aligned}$$

Hence,

$$\begin{aligned} & J^*_{t,N}(x) + \varepsilon \sum_{\ell=t+1}^{N-1} \|x^*_\ell\| \\ & \geq x'Q_{v^*}x + u^{*\prime}R_{v^*}u^* + V_{t+1,N}(A_{v^*}x + B_{v^*}u^*) \\ & \geq \min_{v\in\mathcal{V}, u\in\mathbb{R}^r} x'Q_v x + u'R_v u + V_{t+1,N}(A_v x + B_v u). \end{aligned} \tag{24}$$

Since Lemma 2.2 guarantees that the rightmost expression in (24) is bounded below by $V_{k,N}(x) - \varepsilon\|x\|$, and recalling that $x = x^*_t$, we can conclude that

$$J^*_{t,N}(x^*_t) + \varepsilon \sum_{\ell=t+1}^{N-1} \|x^*_\ell\| \geq V_{k,N}(x^*_t) - \varepsilon\|x^*_t\|,$$

which is what we wished to show. ■

The next concept recalls the stabilizability concept for the system (1).

Assumption 2.1: (Stabilizability). There exist a policy $\pi \in \Pi$ and a constant $\gamma > 0$ such that

$$J_N(\pi, x_0) \leq \gamma\|x_0\|, \quad \forall x_0 \in \mathbb{R}^n.$$

The next result is a consequence of the stabilizability assumption.

Lemma 2.5: Suppose that Assumption 2.1 holds, and let $\{x^*_k\}$, with $x^*_0 = x \in \mathbb{R}^n$, be the optimal trajectory. If the weighting matrices $Q_1, \ldots, Q_\sigma$, are positive definite, then

$$\sum_{k=0}^{N} \|x^*_k\| \leq \frac{\gamma\|x\|}{\lambda_{\min}(Q)},$$

where

$$\lambda_{\min}(Q) = \min\{\lambda(Q_1), \ldots, \lambda(Q_\sigma)\}.$$

Combining the inequality in (21) and the results from Lemmas 2.3, 2.4, and 2.5, we obtain the next main result.

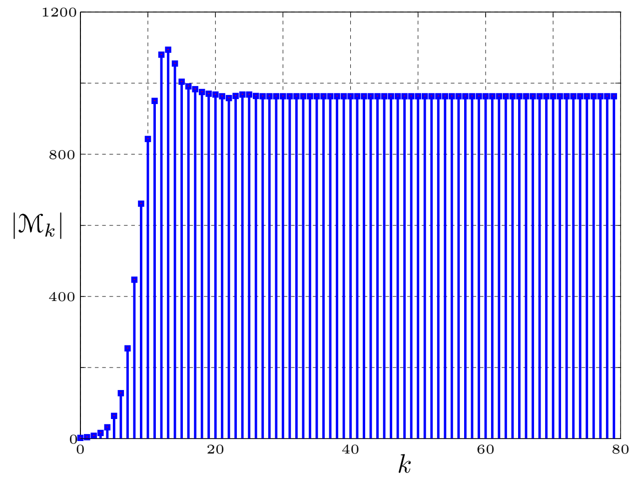

Fig. 2. Cardinality of the sets $\mathcal{M}_k$, $k = 0, \ldots, N$, according to the numerical example of Section III.

Theorem 2.1: Let $\pi = \{\pi_0, \ldots, \pi_{N-1}\} \in \Pi$ be the policy as in (18). If Assumption 2.1 holds, and if the weighting matrices $Q_1, \ldots, Q_\sigma$, are positive definite, then

$$J^*_N(x_0) \leq J_N(\pi, x_0) \leq J^*_N(x_0) + \varepsilon \frac{\gamma\|x_0\|}{\lambda_{\min}(Q)}, \quad \forall x_0 \in \mathbb{R}^n. \tag{25}$$

Remark 2.3: The policy π as announced in Theorem 2.1 depends on the choice of ε, and in view of the inequalities in (25), it is classified as an ε-optimal policy [18, Sec. 2.2].

The numerical application of the next section illustrates the advantage of using Theorem 2.1 to evaluate ε-optimal cost instead of the hard computing required by the optimal one.

III. Numerical example

This numerical example considers the switched linear system in (1) taking the parameters

$$A_v = \begin{bmatrix} a^{(v)}_{11} & a^{(v)}_{12} & a^{(v)}_{13} & a^{(v)}_{14} \\ a^{(v)}_{21} & a^{(v)}_{22} & a^{(v)}_{23} & a^{(v)}_{24} \\ a^{(v)}_{31} & a^{(v)}_{32} & a^{(v)}_{33} & a^{(v)}_{34} \\ a^{(v)}_{41} & a^{(v)}_{42} & a^{(v)}_{43} & a^{(v)}_{44} \end{bmatrix}, \; v = 1, 2,$$

whose values are described in Table II, and

$$B_1 = B_2 = \begin{bmatrix} 0.17637 & 0.326018 & 1.82362 & 1.67398 \end{bmatrix}'.$$

TABLE I

Comparison of the cardinality of the solution set required to compute the optimal and ε-optimal costs.

Method	Cardinality	Cost evaluation
[8, Th. 1]	$\lvert\mathcal{L}_N\rvert = 65536$	$J^*_N(x_0) = 0.7733823$
Theorem 2.1	$\lvert\mathcal{M}_N\rvert = 1004$	$J_N(\pi, x_0) = 0.7735249$

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TABLE II
PARAMETERS OF THE SWITCHED LINEAR SYSTEM AS IN NUMERICAL EVALUATION OF SECTION III.

Parameters	$v = 1$	$v = 2$
$a_{11}^{(v)}$	0.83912455	0.59642355
$a_{12}^{(v)}$	1.81717842	1.67398131
$a_{13}^{(v)}$	0.16087544	0.40357644
$a_{14}^{(v)}$	0.18282157	0.32601868
$a_{21}^{(v)}$	-0.14709211	-0.33025084
$a_{22}^{(v)}$	0.77375028	0.56690680
$a_{23}^{(v)}$	0.14709211	0.33025084
$a_{24}^{(v)}$	0.22624971	0.43309319
$a_{31}^{(v)}$	0.16087544	0.40357644
$a_{32}^{(v)}$	0.18282157	0.32601868
$a_{33}^{(v)}$	0.83912455	0.59642355
$a_{34}^{(v)}$	1.81717842	1.67398131
$a_{41}^{(v)}$	0.14709211	0.33025084
$a_{42}^{(v)}$	0.22624971	0.43309319
$a_{43}^{(v)}$	-0.14709211	-0.33025084
$a_{44}^{(v)}$	0.77375028	0.56690680

To perform the evaluation, we set $N = 16$, $R_1 = R_2 = 1$, $Q_1 = Q_2 = I$, $F_1 = F_2 = 5I$, and $x_0 = [0.2\ 0.3\ -0.3\ -0.2]'$.

Setting $\varepsilon = 0.01$, we obtain the numerical values as described in Table I. The cardinality of the set $\mathcal{L}_N$, required to compute the exact optimal solution [8, Th. 1], is $|\mathcal{L}_N| = 2^{N-1} = 65536$. However, the cardinality $|\mathcal{M}_N| = 1004$ is much fewer than that of $\mathcal{L}_N$, thus for sake of computational effort it is better to employ our approach from Theorem 2.1 because it also provides a good approximation of the optimal solution (Table I).

Now, we consider a new evaluation for $N = 80$. It is observed in the experiment that the cardinality of ε-optimal solution set does not depend on the time N as N increases to infinity (Fig. 2).

This example illustrates the usefulness of our approach (Theorem 2.1) to compute an ε-optimal solution of the switched linear control problem as in (2).

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