Local exact controllability for the 1-D compressible Navier-Stokes equation

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November 15, 2011

1 Introduction

In this article, we consider the compressible Navier-Stokes equation in one space dimension in a bounded domain $(0, L)$:

$$\begin{cases} \partial_t \rho_{\mathcal{S}} + \partial_x(\rho_{\mathcal{S}} u_{\mathcal{S}}) = 0 \text{ in } (0,T) \times (0,L), \\ \rho_{\mathcal{S}}(\partial_t u_{\mathcal{S}} + u_{\mathcal{S}} \partial_x u_{\mathcal{S}}) - \nu \partial_{xx} u_{\mathcal{S}} + \partial_x p(\rho_{\mathcal{S}}) = 0 \text{ in } (0,T) \times (0,L). \end{cases} \tag{1.1}$$

Here $\rho_{\mathcal{S}}$ is the density, $u_{\mathcal{S}}$ the velocity and p denotes the pressure, which follows the standard law:

$$p(\rho_{\mathcal{S}}) = \rho_{\mathcal{S}}^{\gamma}, \tag{1.2}$$

for some $\gamma \geq 1$. This law is the classical one when considering isentropic flow ($\gamma = 1.4$ for perfect gazes) or isothermal flow ($\gamma = 1$). We also impose the initial data:

$$(\rho_{\mathcal{S}}, u_{\mathcal{S}})_{|t=0} = (\rho_0, u_0) \text{ in } (0, L), \tag{1.3}$$

Let us emphasize that the boundary conditions do not appear in the equation (1.1). They will be used as the controls on the system. Remark that, as frequently used when controlling hyperbolic equations, the controls will never appear explicitly. Our goal is to understand the local exact controllability to constant states when the velocity part of the target does not vanish. To be more precise, given $(\overline{\rho}, \overline{u}) \in \mathbb{R}_+^* \times \mathbb{R}^*$, we want to prove that, for (ρ_0, u_0) close enough to $(\overline{\rho}, \overline{u})$, one can find a solution of (1.1) with initial data (1.3) connecting the initial state to the target in some time T.

Precisely, the goal of that article is to prove the following result.

Theorem 1.1. *Let $\overline{u} \in \mathbb{R}^*$ and $\overline{\rho} \in \mathbb{R}_+^*$. Set $T > 0$ satisfying*

$$T > \frac{L}{|\overline{u}|}. \tag{1.4}$$

Then there exists $\kappa > 0$ such that, for any $u_0 \in H^3(0, L)$ and $\rho_0 \in H^3(0, L)$ satisfying

$$\|u_0 - \overline{u}\|_{H^3(0,L)} + \|\rho_0 - \overline{\rho}\|_{H^3(0,L)} < \kappa, \tag{1.5}$$

there exists a solution $(\rho_{\mathcal{S}}, u_{\mathcal{S}}) \in L^\infty(0, T; H^3(0, L))$ of (1.1)–(1.3) satisfying

$$(\rho_{\mathcal{S}}, u_{\mathcal{S}})(T) = (\overline{\rho}, \overline{u}) \text{ in } (0, L). \tag{1.6}$$

Remark 1.2. *It is likely that we can reduce the regularity asked on the initial data. However, as can be seen in the proof, in our method we need informations on the second derivative of ρ, which can be obtained using third derivatives of u.*

Theorem 1.1 appears to be the first controllability result concerning a compressible and viscous fluid except for the recent result by Amosova [1], which deals with a controllability problem concerning compressible viscous fluids in dimension 1. In this paper, the author considers the equation in Lagrangian coordinates, with zero boundary condition for the velocity on the boundaries of the interval and an interior control on the velocity equation. She proves a result of local exact controllability to trajectories for the velocity, provided that the initial density is already on the "targeted trajectory". Our result differs because:

- We consider boundary controls for both equations, but have no assumption on the initial density,
- We suppose $\overline{u} \neq 0$ and obtain a local exact controllability to $(\overline{\rho}, \overline{u})$,
- The change of variable between Lagrangian and Eulerian coordinates (which consists in taking a primitive of the density as a new space variable) does not leave the domain (or the control zone) invariant.

Let us comment about the condition $\overline{u} \neq 0$. If we assume $\overline{u} = 0$ and if we linearize the problem around $\overline{u} = 0$, then we obtain an equation for the density which has been studied by several authors, for example [12] or indirectly by [7] (they consider an equation with a memory term which gives the same example when you take the time derivative). It is then showed that this linearized problem is not controllable as there might exist waves travelling with almost zero velocity so that they never reach the boundary in prescribed time. Controllability for incompressible viscous fluids has been studied by many authors recently. Concerning Navier-Stokes equations and related problems like Boussinesq system, local results of exact controllability to trajectories have been obtained by [5], [9], [4], [6]. Global exact controllability results have been obtained in [2] for the case of controls acting on the whole boundary and the problem is open for all other cases. The simpler model of Burger's equation has been studied in [8] in the 1-d case where they show that global exact controllability does notr occur in general. For the 2-d Burger's equation, in [10] it is shown that there are some geometries for which global controllability can be proved and other geometries for which it is not true.

The rest of the paper is devoted to the proof of Theorem 1.1. In Section 2, we describe the structure of an operator, connected to a linear controllability problem, whose fixed point will give a solution to the controllability problem. In Section 3, we describe how we solve the part of the linear controllability problem concerning the velocity. In Section 4, we decribe how we solve the part concerning the density. In Section 5, we prove that the operator that we constructed admits a fixed point, proving Theorem 1.1. Finally, Section 6 is an appendix where we put some technical lemmas.

For the rest of the paper, we will assume, without loss of generality, that

$$\overline{u} > 0.$$

It is just a matter of using the change of coordinates $x \to L - x$.

2 Main steps of the proof of Theorem 1.1

2.1 Reformulation

The general idea of the construction is to build an operator whose fixed points will give a solution of the controllability problem. It is based on the resolution of controllability problems for suitable linear approximations of equation (1.1) near the trajectory $(\overline{\rho}, \overline{u})$.

In our fixed point argument, it will be convenient to work within a class of functions vanishing at time $t = 0$. Therefore, to take the initial data into account, we extend (ρ_0, u_0) into smooth functions on $\mathbb{R}$, still denoted the same, such that $(\rho_0 - \overline{\rho}, u_0 - \overline{u})$ vanish outside $(-\varepsilon, L + \varepsilon)$ for some $\varepsilon > 0$ that will be fixed at the end, in such a way that we still have

$$\|\rho_0 - \overline{\rho}\|_{H^3(\mathbb{R})} + \|u_0 - \overline{u}\|_{H^3(\mathbb{R})} < C\kappa, \tag{2.1}$$

for some constant $C > 0$ depending on L only.

We then define (ρ_{in}, u_{in}) as the solution of

$$\begin{cases} \partial_t \rho_{in} + \partial_x((\overline{\rho} + \rho_{in})(\overline{u} + u_{in})) = 0 \text{ in } [0,T] \times \mathbb{R}, \\ (\overline{\rho} + \rho_{in})(\partial_t u_{in} + (\overline{u} + u_{in})\partial_x u_{in}) - \nu \partial_{xx} u_{in} + p'(\overline{\rho} + \rho_{in})\partial_x \rho_{in} = 0 \text{ in } [0,T] \times \mathbb{R}, \end{cases} \tag{2.2}$$

with initial data

$$\rho_{in}(0) = \rho_0 - \overline{\rho} \text{ and } u_{in}(0) = u_0 - \overline{u} \quad \text{on } \mathbb{R}. \tag{2.3}$$

The existence of (ρ_{in}, u_{in}) is given in the next proposition, which is a direct consequence of a paper by Matsumura and Nishida [11].

Proposition 2.1. *Set $(\overline{\rho}, \overline{u}) \in \mathbb{R}_+^* \times \mathbb{R}$ and $T > 0$. There exists $\kappa, K > 0$ such that, for any $u_0 \in \overline{u} + H^3(\mathbb{R})$ and $\rho_0 \in \overline{\rho} + H^3(\mathbb{R})$ satisfying (2.1), there exists a solution (ρ_{in}, u_{in}) in $L^\infty(0,T; H^3(\mathbb{R})) \cap W^{1,\infty}(0,T; H^2(\mathbb{R})) \times L^\infty(0,T; H^3(\mathbb{R})) \cap W^{1,\infty}(0,T; H^1(\mathbb{R}))$ of (2.2)-(2.3), satisfying:*

$$\|\rho_{in}\|_{L^\infty(0,T;H^3(\mathbb{R}))\cap W^{1,\infty}(0,T;H^2(\mathbb{R}))} + \|u_{in}\|_{L^\infty(0,T;H^3(\mathbb{R}))\cap W^{1,\infty}(0,T;H^1(\mathbb{R}))} \leq K \left(\|\rho_0 - \overline{\rho}\|_{H^3(\mathbb{R})} + \|u_0 - \overline{u}\|_{H^3(\mathbb{R})}\right). \quad (2.4)$$

We give some explanations on Proposition 2.1 in the appendix.

As a consequence of Proposition 2.1, we will be able to suppose that ρ_{in} and u_{in} are suitably small by choosing initial data (ρ_0, u_0) sufficiently close to $(\overline{\rho}, \overline{u})$. To express this in a convenient manner, we introduce

$$R_{in} := \|\rho_{in}\|_{L^\infty(0,T;W^{2,\infty}(0,L))\cap W^{1,\infty}(0,T;L^\infty(0,L))} + \|u_{in}\|_{L^\infty(0,T;W^{2,\infty}(\mathbb{R}))\cap W^{1,\infty}(0,T;L^\infty(\mathbb{R}))}, \quad (2.5)$$

which we will be able to consider small. when taking κ small enough in (1.5). In particular it will be systematically supposed to satisfy:

$$R_{in} \leq \min\left\{\frac{|\overline{u}|}{4}, \frac{\overline{\rho}}{4}\right\}. \quad (2.6)$$

We can now reformulate the problem as follows. First, recall that T has been chosen large enough so that (1.4) holds. We can thus introduce $T_0 > 0$ such that

$$T_0 \in \left(0, \frac{1}{4}\right) \quad \text{and} \quad 10T_0 < T - \frac{L}{\overline{u}}. \quad (2.7)$$

Now we choose a smooth cut-off function Λ such that

$$\Lambda : [0,T] \to [0,1], \quad \Lambda(t) = \begin{cases} 1 & \text{for } t \in [0, T_0], \\ 0 & \text{for } t \in [2T_0, T], \end{cases} \quad (2.8)$$

and set

$$\rho = \rho_{\mathcal{S}} - \overline{\rho} - \Lambda\rho_{in} \quad \text{and} \quad u = u_{\mathcal{S}} - \overline{u} - \Lambda u_{in}. \quad (2.9)$$

Then our goal is to show that there exists a solution (ρ, u) of

$$\begin{cases} \partial_t\rho + (\overline{u} + u + \Lambda u_{in})\partial_x\rho + \overline{\rho}\partial_x u + \frac{\overline{\rho}}{\nu}p'(\overline{\rho})\rho = f(\rho, u) \text{ in } [0,T] \times (0,L), \\ (\overline{\rho} + \Lambda\rho_{in})(\partial_t u + \overline{u}\partial_x u) - \nu\partial_{xx}u = g(\rho, u) \text{ in } [0,T] \times (0,L), \end{cases} \quad (2.10)$$

where $f(\rho, u)$ and $g(\rho, u)$ are given as follows:

$$f(\rho, u) = -\Lambda'\rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in}) - \Lambda\partial_x(\rho_{in}u) - \Lambda\rho\partial_x u_{in} - \rho\partial_x u + \frac{\overline{\rho}}{\nu}p'(\overline{\rho})\rho \quad (2.11)$$

and

$$\begin{aligned} g(\rho, u) = &- (\overline{\rho} + \Lambda\rho_{in})\Lambda' u_{in} - (p'(\overline{\rho} + \Lambda\rho_{in}) - p'(\overline{\rho} + \rho_{in}))\Lambda\partial_x\rho_{in} \\ &+ \rho_{in}\partial_t u_{in}(\Lambda - \Lambda^2) + \rho_{in}\overline{u}\partial_x u_{in}(\Lambda - \Lambda^2) + \overline{\rho}u_{in}\partial_x u_{in}(\Lambda - \Lambda^2) + \rho_{in}u_{in}\partial_x u_{in}(\Lambda - \Lambda^3) \quad (2.12) \\ &- \Lambda(\overline{\rho} + \Lambda\rho_{in})\partial_x(uu_{in}) - (\overline{\rho} + \Lambda\rho_{in})u\partial_x u \\ &- \rho(\partial_t(\Lambda u_{in} + u) + (\overline{u} + \Lambda u_{in} + u)\partial_x(\Lambda u_{in} + u)) \\ &- (p'(\overline{\rho} + \Lambda\rho_{in} + \rho) - p'(\overline{\rho} + \Lambda\rho_{in}))\partial_x(\Lambda\rho_{in} + \rho) - p'(\overline{\rho} + \Lambda\rho_{in})\partial_x\rho, \end{aligned}$$

satisfying

$$\rho(0, \cdot) = \rho(T, \cdot) = 0 \text{ and } u(0, \cdot) = u(T, \cdot) = 0. \quad (2.13)$$

The lengthy computations leading to the expressions of f and g are detailed in the appendix.

Now to obtain a solution of (2.10)-(2.13), the idea is to find a fixed point to the application

$$F(\hat{\rho}, \hat{u}) = (\rho, u), \tag{2.14}$$

where (ρ, u) is a suitable solution of

$$\partial_t \rho + (\overline{u} + u + \Lambda u_{in})\partial_x \rho + \overline{\rho}\partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\rho = f(\hat{\rho}, \hat{u}) \text{ in } [0, T] \times (0, L), \tag{2.15}$$

$$(\overline{\rho} + \Lambda \rho_{in})(\partial_t u + \overline{u}\partial_x u) - \nu \partial_{xx} u = g(\hat{\rho}, \hat{u}) \text{ in } [0, T] \times (0, L), \tag{2.16}$$

satisfying

$$\rho(0) = \rho(T) = 0 \text{ and } u(0) = u(T) = 0. \tag{2.17}$$

Of course, for this map to be well-defined, we need to make precise in which spaces the map F is defined and how the solution (ρ, u) is constructed. Indeed, the existence of such (ρ, u) is not obvious since it is a solution of a control problem that involves a heat type equation for the equation of the velocity and a transport equation for the density. Details on the construction of F will be given afterwards.

Besides, to complete the proof of Theorem 1.1, we will have to construct a convex set which is stable by F. This will be the main difficulty of the proof.

2.2 Construction of the fixed point map

The map F is constructed in two steps that will be detailed in the sections afterwards:

Step 1. Controlling u. For this to be done, we shall use a global Carleman estimate involving a weight function that will "travel" at velocity $\overline{u}$. This is the object of Section 3. The idea is very close to the control of the classical heat equation, except that one should be cautious about the fact that the weight functions travels along the characteristics.

Step 2. Constructing ρ. The idea is to use a backward solution vanishing at time T and a forward solution vanishing at time 0 and to glue them along the characteristics of the flow. This construction is very naive and natural, but the main difficulty is then to estimate the obtained ρ.

We finally end this section by giving a description of the fixed point space.

2.3 Description of the fixed point space

The space where F is to be defined is a weighted space connected to the aforementioned Carleman estimate. Let us first describe the weight function that we use. Set $\psi \in C^\infty(\mathbb{R}; \mathbb{R})$ such that

$$3 \leq \min_{[-5\overline{u}T, L]} \psi \leq \max_{[-5\overline{u}T, L]} \psi \leq 4, \quad \max_{[-3\overline{u}T, L]} \psi' < 0 \quad \text{and} \quad \min_{[-5\overline{u}T, -4\overline{u}T]} \psi' > 0. \tag{2.18}$$

Then, let $\theta = \theta(t) \in C^2([0, T]; \mathbb{R}_+)$ defined by

$$\theta(t) = \begin{cases} t & \text{in } [0, 2T_0] \\ 1 & \text{in } [3T_0, T - 3T_0] \\ T - t & \text{in } [T - 2T_0, T], \end{cases} \tag{2.19}$$

and being such that θ is increasing on $[0, 3T_0]$, respectively decreasing on $[T - 3T_0, T]$.

We then define the weight function $\varphi(t, x)$, depending on a positive parameter λ as follows

$$\varphi(t, x) = \frac{1}{\theta(t)} \left(e^{5\lambda} - e^{\lambda \psi(x - \overline{u}t)} \right). \tag{2.20}$$

To this weight we associate the time-dependent function

$$\check{\varphi}(t) := \min_{x \in [0, L]} \varphi(t, x) = \varphi(t, 0). \tag{2.21}$$

We also denote

$$\xi(t, x) = \frac{1}{\theta(t)} e^{\lambda \psi(x - \overline{u}t)}, \tag{2.22}$$

Note in particular that

$$\xi \geq 1. \tag{2.23}$$

The parameter λ used in the above definition of φ in (2.20) will always be assumed to be positive and larger than one, as well as the second parameter, called s, of the Carleman estimates:

$$s \geq 1 \quad \text{and} \quad \lambda \geq 1. \tag{2.24}$$

We can now define the set on which F is to be defined. It depends on two constants

$$R_u \in \big(0, \min\big[1, |\overline{u}|/4\big]\big) \quad \text{and} \quad R_\rho \in (0,1). \tag{2.25}$$

Given R_ρ and R_u, we define the spaces X_{s,λ,R_ρ} and Y_{s,λ,R_u} as follows:

$$\begin{aligned} X_{s,\lambda,R_\rho} = \Big\{\rho \ \text{ s. t. } \ & \xi^{-1} e^{s\varphi}\rho,\ \xi^{-3/2} e^{s\varphi}\partial_x \rho,\ \partial_t \rho \in L^2((0,T)\times(0,L)), \\ & e^{s\check{\varphi}/2}\rho \in L^\infty((0,T)\times(0,L)), \quad e^{s\check{\varphi}/2}\partial_x\rho \in L^\infty((0,T);L^2(0,L)), \\ & [\xi^{-3/2}e^{s\varphi}\rho](\cdot,0),\ [\xi^{-3/2}e^{s\varphi}\rho](\cdot,L) \in L^2(0,T), \\ \text{with } \ & \|\xi^{-1}e^{s\varphi}\rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho, \quad \|\xi^{-3/2}e^{s\varphi}\partial_x\rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho, \\ & \|\partial_t\rho\|_{L^2((0,T)\times(0,L))} \leq R_\rho, \\ & \|e^{s\check{\varphi}/2}\rho\|_{L^\infty((0,T)\times(0,L))} \leq R_\rho, \quad \|e^{s\check{\varphi}/2}\partial_x\rho\|_{L^\infty((0,T);L^2(0,L))} \leq R_\rho, \\ & \|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot,0)\|_{L^2(0,T)} \leq R_\rho, \quad \|\lambda^{1/2}[\xi^{-3/2}e^{s\varphi}\rho](\cdot,L)\|_{L^2(0,T)} \leq R_\rho \Big\} \end{aligned}$$

$$\begin{aligned} Y_{s,\lambda,R_u} = \Big\{u \ \text{ such that } \ & e^{s\varphi}u,\ \xi^{-1}e^{s\varphi}\partial_x u,\ \xi^{-2}e^{s\varphi}\partial_{xx}u,\ \xi^{-2}e^{s\varphi}\partial_t u \in L^2((0,T)\times(0,L)), \\ \text{with } \ & \|s^{3/2}\lambda^2 e^{s\varphi}u\|_{L^2((0,T)\times(0,L))} \leq R_u, \quad \|s^{1/2}\lambda\xi^{-1}e^{s\varphi}\partial_x u\|_{L^2((0,T)\times(0,L))} \leq R_u, \\ & \|s^{-1/2}\xi^{-2}e^{s\varphi}\partial_{xx}u\|_{L^2((0,T)\times(0,L))} \leq R_u, \quad \|s^{-1/2}\xi^{-2}e^{s\varphi}\partial_t u\|_{L^2((0,T)\times(0,L))} \leq R_u \Big\}. \end{aligned}$$

Let us remark that both sets are convex and compact for the topology of $L^2((0,T)\times(0,L))$. Therefore, if one shows that the map F maps $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ into itself for convenient choices of parameters $s, \lambda \geq 1$ and R_ρ, R_u small enough, we are in position to prove the existence of a fixed point by Schauder's fixed point theorem, provided the continuity of F on $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ endowed with the $(L^2((0,T)\times(0,L)))^2$-topology is proved.

3 Controlling the velocity

In this section, we study the controllability problem attached to the parabolic equation (2.16). The term $\hat{g}$ is considered as a source term. We are then in a familiar framework which can be handled using Carleman estimates and duality arguments.

3.1 Construction of u

For sake of simplicity, let us introduce the following general heat equation:

$$a\partial_t u + b\partial_x u - \nu\partial_{xx}u = g \text{ in } (0,T)\times(0,L), \quad u(t,L) = 0, \text{ in } (0,T) \tag{3.1}$$

where $a(t,x) \in W^{1,\infty}(0,T;L^\infty(0,L))$, $b(t,x) \in L^\infty(0,T;W^{1,\infty}(0,L))$ and

$$\inf_{(t,x)\in(0,T)\times(0,L)} \{a(t,x)\} > 0. \tag{3.2}$$

The source term g is assumed to be given.

We also introduce the following control problem: find a trajectory u of (3.1) such that

$$u(0,\cdot) = u(T,\cdot) = 0 \text{ in}(0,L). \tag{3.3}$$

Here again, the control is hidden in the lack of boundary condition in $x = 0$ in (3.1).

To be more precise, we shall look for conditions on the source term g that guarantee the existence of a controlled trajectory of (3.1) satisfying (3.3).

Of course, this corresponds to the construction of the u-part of $F(\hat{\rho}, \hat{u})$ with

$$a(t,x) := \overline{\rho} + \Lambda \rho_{in}(t,x), \quad b(t,x) := (\overline{\rho} + \Lambda \rho_{in}(t,x))\overline{u} \quad \text{and} \quad g := g(\hat{\rho}, \hat{u}), \tag{3.4}$$

provided that R_{in} is small enough to guarantee that $a(t,x) := \overline{\rho} + \Lambda \rho_{in}(t,x))$ satisfies (3.2).

To solve this control problem, we first extend (3.7) on a larger domain, for instance $(-4\overline{u}T, L)$ and extend a and b on $(0,T) \times (-4\overline{u}T, L)$ such that the extensions, still denoted by a and b, satisfy:

$$a(t,x) \in W^{1,\infty}(0,T; L^\infty(-4\overline{u}T, L)), \quad b(t,x) \in L^\infty(0,T; W^{1,\infty}(-4\overline{u}T, L)), \tag{3.5}$$

and

$$\inf_{(t,x) \in (0,T) \times (-4\overline{u}T, L)} \{a(t,x)\} > 0. \tag{3.6}$$

Note that, when constructing the u-part of $F(\hat{\rho}, \hat{u})$, the coefficients a and b given (3.4) are naturally defined on $(0,T) \times \mathbb{R}$ and then this extension argument is not really needed.

We shall also consider the extension of g by 0 in $(0,T) \times (-4\overline{u}T, 0)$, that we still denote the same for sake of simplicity.

We then consider the following control problem: find a control v so that the solution u of

$$\begin{cases} a\partial_t u + b\partial_x u - \nu \partial_{xx} u = g + v \mathbf{1}_{(0,T)\times(-4\overline{u}T, -\overline{u}T)} \text{ in } (0,T) \times (-4\overline{u}T, L), \\ u(t, -4\overline{u}T) = u(t, L) = 0, \text{ in } (0,T), \\ u(0, \cdot) = 0 \text{ in } (-4\overline{u}T, L), \end{cases} \tag{3.7}$$

satisfies

$$u(T, \cdot) = 0 \text{ in } (-4\overline{u}T, L). \tag{3.8}$$

By restriction, solving (3.7)–(3.8) for some v yields a controlled trajectory u of (3.1) satisfying (3.3).

As it is classical now from the work of Fursikov-Imanuvilov [5], this issue can be addressed by proving a Carleman estimate for the adjoint of the heat operator under consideration.

Hence, setting

$$P_{a,b} := a\partial_t + b\partial_x - \nu \partial_{xx} \text{ on } (0,T) \times (-4\overline{u}T, L), \\ \text{with Dirichlet boundary conditions at } x = -4\overline{u}T \text{ and } x = L, \tag{3.9}$$

we are going to derive a Carleman estimate for the operator

$$P_{a,b}^* = -\partial_t(a \cdot) - \partial_x(b \cdot) - \nu \partial_{xx} \text{ on } (0,T) \times (-4\overline{u}T, L), \\ \text{with Dirichlet boundary conditions at } x = -4\overline{u}T \text{ and } x = L, \tag{3.10}$$

with observation on $(0,T) \times (-4\overline{u}T, -\overline{u}T)$.

We are now in position to state the following Carleman estimate:

Theorem 3.1. *Assume that a and b satisfy condition* (3.5) *and* (3.6).

There exist $s_0 \geq 1$, $\lambda_0 \geq 1$ and $C > 0$ such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$, any smooth function $z : [0,T] \times [-4\overline{u}T, L] \to \mathbb{R}$ satisfying $z(t, L) = 0$ and $z(t, -4\overline{u}T) = 0$ satisfies

$$\begin{aligned} s^3 \lambda^4 \iint_{(0,T)\times(-4\overline{u}T, L)} \xi^3 e^{-2s\varphi} |z|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T, L)} \xi e^{-2s\varphi} |\partial_x z|^2 \\ + \frac{1}{s} \iint_{(0,T)\times(-4\overline{u}T, L)} \frac{1}{\xi} e^{-2s\varphi} \left(|\partial_{xx} z|^2 + |\partial_t z|^2 \right) \\ \leq C \iint_{(0,T)\times(-4\overline{u}T, L)} \left| P_{a,b}^* z \right|^2 e^{-2s\varphi} + C s^3 \lambda^4 \iint_{(0,T)\times(-4\overline{u}T, -\overline{u}T)} \xi^3 e^{-2s\varphi} |z|^2. \end{aligned} \tag{3.11}$$

The proof of Theorem 3.1 is given in Subsection 3.2. It is mainly classical (see Fursikov and Imanuvilov [5]), except for what concerns the Carleman weight. Indeed, the classical Carleman weight usually takes the form

$$\tilde{\varphi}(t,x) = \frac{1}{t(T-t)}\left(e^{5\lambda} - e^{\lambda\psi(x)}\right).$$

The differences between the weight (2.20) and the classical one are then the following: the weight function θ (see (2.19)) is constant during a certain interval of time and the variable in the function ψ is $x - \overline{u}t$ instead of x. This latest point somehow reflects the hyperbolic nature of the equation of ρ and the fact that it is important to take into account the transport at velocity $\overline{u}$.

As we shall see later, this particular form of the weight function will allow us to estimate the controlled density in weighted functional spaces, a crucial step to use the fixed point argument.

Relying on this Carleman estimate, we develop a duality argument using Theorem 3.1 and the method developed by Fursikov and Imanuvilov. Let us assume that $g : (0,T) \times (0,L) \to \mathbb{R}$ satisfies

$$\iint_{(0,T)\times(0,L)} \frac{1}{\xi^3}|\hat{g}|^2 e^{2s\varphi} < \infty. \tag{3.12}$$

We then introduce the functional J defined by

$$J(z) = \frac{1}{2}\iint_{(0,T)\times(-4\overline{u}T,L)} |P_{a,b}^* z|^2 e^{-2s\varphi} + \frac{s^3\lambda^4}{2}\iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3|z|^2 e^{-2s\varphi} - \iint_{(0,T)\times(-4\overline{u}T,L)} \hat{g}z, \tag{3.13}$$

among all z belonging to the space $\overline{\mathcal{Y}}$ defined as the completion with respect to the norm

$$\|z\|_{obs}^2 = \iint_{(0,T)\times(-4\overline{u}T,L)} |P_{a,b}^* z|^2 e^{-2s\varphi} + s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|z|^2 e^{-2s\varphi}$$

of the space of functions in $C^\infty((0,T)\times[-4\overline{u}T,L])$ vanishing at $x = L$ and $x = -4\overline{u}T$. Note that the fact that $\|\cdot\|_{obs}$ is a norm is a consequence of the Carleman estimate (3.11).

Observe that thanks to (3.11) the last term in (3.13) is well-defined (recall that (3.12) is satisfied). Moreover, one easily checks that J is strictly convex and coercive on the space $\overline{\mathcal{Y}}$ endowed with the norm $\|\cdot\|_{obs}$.

Therefore, it has a unique minimizer Z, for which, due to the coercivity of J, we have

$$\|Z\|_{obs}^2 \le C\frac{1}{s^3\lambda^4}\iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3}|g|^2 e^{2s\varphi}. \tag{3.14}$$

Besides, as a minimizer of J, Z satisfies, for all $z \in \overline{\mathcal{Y}}$,

$$\iint_{(0,T)\times(-4\overline{u}T,L)} P_{a,b}^* z P_{a,b}^* Z e^{-2s\varphi} + s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 z Z e^{-2s\varphi} - \iint_{(0,T)\times(-4\overline{u}T,L)} gz = 0. \tag{3.15}$$

Consequently, if we set

$$u := P_{a,b}^* Z e^{-2s\varphi}, \qquad v := -s^3\lambda^4\xi^3 Z e^{-2s\varphi} \tag{3.16}$$

it is not difficult to see that u satisfies, in the transposition sense,

$$a\partial_t u + b\partial_x u - \nu\partial_{xx} u = g + v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)} \text{ in } (0,T)\times(-4\overline{u}T,L), \quad u(\cdot,-4\overline{u}T) = 0 = u(\cdot,L), \tag{3.17}$$

Besides, due to (3.14), we get:

$$\iint_{(0,T)\times(-4\overline{u}T,L)} |u|^2 e^{2s\varphi} + \frac{1}{s^3\lambda^4}\iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \frac{1}{\xi^3}|v|^2 e^{2s\varphi} \le C\frac{1}{s^3\lambda^4}\iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3}|g|^2 e^{2s\varphi}. \tag{3.18}$$

Of course, thanks to the exponential blow up of the weight function φ as $t \to 0$ and as $t \to T$, (see (2.20)), this implies that $u(0,\cdot) = u(T,\cdot) = 0$ in $(-4\overline{u}T,L)$.

Besides, by uniqueness of the solution in the transposition sense, since the source term belongs to $L^2((0,T)\times(-4\overline{u}T,L))$, u is a strong solution of (3.17).

With all these ingredients, we can obtain the following (the detailed proof is available in Section 3.3):

Theorem 3.2. *Given $g \in L^2((0,T)\times(0,L))$ satisfying (3.12) and a, b satisfying (3.5)–(3.6), there exists a constant C such that for all $s \geq s_0$ and $\lambda \geq \lambda_0$, there exists a solution u of (3.7)-(3.8) and such that*

$$s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,L)} |u|^2 e^{2s\varphi} + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^2}|\partial_x u|^2 e^{2s\varphi} + \frac{1}{s}\iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^4}\left(|\partial_t u|^2 + |\partial_{xx}u|^2\right) e^{2s\varphi} \\ \leq C \iint_{(0,T)\times(-4\overline{u}T,L)} \frac{1}{\xi^3}|g|^2 e^{2s\varphi}. \quad (3.19)$$

Remark 3.3. *In this theorem and in the sequel, s_0 and λ_0 stand for two sufficiently large constants which may change from line to line.*

The u-part of $F(\hat{u}, \hat{\rho})$ is given by this u for a, b, g as indicated above:

$$a(t,x) := \overline{\rho} + \Lambda\rho_{in}(t,x) \text{ in } (0,T)\times(-4\overline{u}T,L), \quad b(t,x) := (\overline{\rho}+\Lambda\rho_{in}(t,x))\overline{u} \text{ in } (0,T)\times(-4\overline{u}T,L) \quad (3.20)$$

with source term

$$g := \begin{cases} g(\hat{\rho},\hat{u}) \text{ in } (0,T)\times(0,L), \\ 0 \text{ in } (0,T)\times(-4\overline{u}T,0). \end{cases} \quad (3.21)$$

In Section 3.2 we prove of Theorem 3.1 and we establish Theorem 3.2 in Section 3.3. For later use, in Section 3.4, we also prove interpolation estimates to get estimates on the boundary, i.e. at $x = 0$ and $x = L$.

To simplify notations, in the following, we set $Q_T = (0,T)\times(-4\overline{u}T,L)$.

3.2 Proof of Theorem 3.1

Let us begin this section by giving some properties of the Carleman weights. Thanks to the structure of φ (see (2.20)–(2.22)) simple computations give

$$\partial_x\varphi(t,x) = -\lambda\psi'(x-\overline{u}t)\xi(t,x), \quad \partial_{xx}\varphi(t,x) = -\lambda^2(\psi'(x-\overline{u}t))^2\xi - \lambda\psi''(x-\overline{u}t)\xi(t,x).$$

In particular, the boundary term in the last identity is positive. Moreover, due to (2.18) for some $\lambda_0 > 0$, there exists a constant $c_* > 0$ such that for $\lambda \geq \lambda_0$, ,

$$\begin{cases} -\partial_{xx}\varphi(t,x) \geq c_*\lambda^2\xi(t,x), \\ -\partial_x((\partial_x\varphi)^3)(t,x) \geq c_*\lambda^4\xi^3(t,x), \end{cases} \quad \forall (t,x) \in [0,T]\times[-2\overline{u}T,L], \quad (3.22)$$

whereas we obviously have for some constant C independent of λ

$$\begin{cases} |-\partial_{xx}\varphi(t,x)| \leq C\lambda^2\xi(t,x), \\ |-\partial_x((\partial_x\varphi)^3)(t,x)| \leq C\lambda^4\xi^3(t,x), \end{cases} \quad \forall (t,x) \in [0,T]\times[-4\overline{u}T,L]. \quad (3.23)$$

Besides

$$\partial_t\varphi = -\frac{\theta'}{\theta}\varphi + \lambda\overline{u}\psi'(x-\overline{u}t)\xi.$$

But $\varphi \leq \theta\xi^2$ and $\lambda \leq C\xi$ for some C independent of $\lambda > 0$. We thus obtain the bound

$$|\partial_t\varphi| \leq C\xi^2 \quad (3.24)$$

and, similarly,

$$|\partial_{tx}\varphi| \leq C\lambda\xi^2, \quad |\partial_{tt}\varphi| \leq C\xi^3. \quad (3.25)$$

for some constant C independent of λ. In the following, we shall always assume that $\lambda \geq \lambda_0$ so that formulas (3.22)–(3.25) hold.

Proof of Theorem 3.1. Let z be a smooth function on $(0,T)\times(-4\overline{u}T,L)$ satisfying $z(t,-4\overline{u}T) = z(t,L) = 0$ and set $h = a\partial_t z + \nu\partial_{xx}z$.

We then introduce the function $w = e^{-s\varphi}z$. Due to the blow up of the function φ as $t \to 0$ and $t \to T$, w satisfies

$$w(0,x) = w(T,x) = 0, \quad x \in (-4\overline{u}T,L), \qquad \partial_t w(0,x) = \partial_t w(T,x) = 0, \quad x \in (-4\overline{u}T,L),$$

still with the boundary conditions $w(t, -4\overline{u}T) = w(t, L) = 0$.

Then, setting

$$P_0 w = e^{-s\varphi}((a\partial_t + \nu\partial_{xx})(e^{s\varphi} w)),$$

we have that $Pw = he^{-s\varphi}$. We then compute the operator Pw:

$$P_0 w = P_1 w + P_2 w + Rw,$$

where

$$\begin{cases} P_1 w = a\partial_t w + 2\nu s\partial_x\varphi\partial_x w, \\ P_2 w = \nu\partial_{xx} w + sa\partial_t\varphi w + \nu s^2(\partial_x\varphi)^2 w, \\ Rw = \nu s\partial_{xx}\varphi w. \end{cases}$$

In the following of the proof, all the integrals will be on $(0, T) \times (-4\overline{u}T, L)$ if no further precision is added.

Let us now compute the mean value of $P_1 w P_2 w$. First, some integration by parts in space and time yield

$$\nu \iint_{Q_T} a\partial_t w\partial_{xx} w = \frac{1}{2}\nu \iint_{Q_T} \partial_t a|\partial_x w|^2 - \nu \iint_{Q_T} \partial_x a\partial_t w\partial_x w$$

and

$$\iint_{Q_T} a\partial_t w sa\partial_t\varphi w + \nu s^2(\partial_x\varphi)^2 w) = -\frac{s}{2} \iint_{Q_T} \partial_t(a^2\partial_t\varphi)|w|^2 - \nu\frac{s^2}{2} \iint_{Q_T} \partial_t(a(\partial_x\varphi)^2)|w|^2.$$

Then, we integrate by parts in space and we obtain

$$2\nu^2 s \iint_{Q_T} \partial_x\varphi\partial_x w\partial_{xx} w = -\nu^2 s \iint_{Q_T} \partial_{xx}\varphi|\partial_x w|^2 + \nu^2 s \int_0^T \partial_x\varphi(t, x)|\partial_x w(t, x)|^2 \Big|_{x=-4\overline{u}T}^{x=L},$$

and

$$2\nu s \iint_{Q_T} \partial\varphi\partial_x w(sa\partial_t\varphi w + \nu s^2(\partial_x\varphi)^2 w) = -\nu s^2 \iint_{Q_T} \partial_x(a\partial_x\varphi\partial_t\varphi)|w|^2 - \nu^2 s^3 \iint_{Q_T} \partial_x((\partial_x\varphi)^3)|w|^2.$$

Combining all these computations, we get

$$\begin{aligned} &\iint_{Q_T} P_1 w P_2 w \\ = \ &\frac{1}{2}\nu \iint_{Q_T} \partial_t a|\partial_x w|^2 - \nu \iint_{Q_T} \partial_x a\partial_t w\partial_x w \\ &-\frac{s}{2} \iint_{Q_T} \partial_t(a^2\partial_t\varphi)|w|^2 - \nu\frac{s^2}{2} \iint_{Q_T} \partial_t(a(\partial_x\varphi)^2)|w|^2 \\ &-\nu^2 s \iint_{Q_T} \partial_{xx}\varphi|\partial_x w|^2 - \nu s^2 \iint_{Q_T} \partial_x(a\partial_x\varphi\partial_t\varphi)|w|^2 \\ &-\nu^2 s^3 \iint_{Q_T} \partial_x((\partial_x\varphi)^3)|w|^2 \\ &+\nu^2 s \int_0^T \partial_x\varphi(t, x)|\partial_x w(t, x)|^2 \Big|_{x=-4\overline{u}T}^{x=L}. \end{aligned}$$

Recalling the fact that $a \in W^{1,\infty}((0, T) \times (0, L))$ and the formulas (3.22)–(3.25), we obtain, for λ and s

large enough,

$$
\begin{aligned}
&\iint_{Q_T} P_1 w P_2 w \\
&\geq c_* s^3\lambda^4 \iint_{Q_T} \xi^3 |w|^2 + c_* s\lambda^2 \iint_{Q_T} \xi |\partial_x w|^2 - C \iint_{Q_T} |\partial_t w||\partial_x w| \\
&\quad - C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2 \right) \\
&\geq c_* s^3\lambda^4 \iint_{Q_T} \xi^3 |w|^2 + \frac{c_*}{2} s\lambda^2 \iint_{Q_T} \xi |\partial_x w|^2 - \frac{C}{s\lambda^2} \iint_{Q_T} \frac{1}{\xi} |\partial_t w|^2 \\
&\quad - C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2 \right),
\end{aligned}
\tag{3.26}
$$

for some $c_* > 0$ and $C > 0$, both independent of $s \geq s_1$ and $\lambda \geq \lambda_1$.

Now, we estimate the $L^2(L^2)$-norm of $\partial_t w$. In order to do that, we observe that

$$
|\partial_t w| \leq C|P_1 w| + C s\lambda\xi |\partial_x w|.
$$

Therefore

$$
\frac{1}{s} \iint_{Q_T} \frac{1}{\xi} |\partial_t w|^2 \leq C \iint_{Q_T} |P_1 w|^2 + C s\lambda^2 \iint_{Q_T} \xi |\partial_x w|^2, \tag{3.27}
$$

for s large enough.

Similarly, from the definition of P_2 we get

$$
\frac{1}{s} \iint_{Q_T} \frac{1}{\xi} |\partial_{xx} w|^2 \leq C \iint_{Q_T} |P_2 w|^2 + C s^3\lambda^4 \iint_{Q_T} \xi^3 |w|^2, \tag{3.28}
$$

for s large enough.

But, using the fact that $P_1 w + P_2 w = h e^{-s\varphi} - Rw$,

$$
\iint_{Q_T} |P_1 w|^2 + \iint_{Q_T} |P_2 w|^2 + \iint_{Q_T} P_1 w P_2 w \leq 2 \iint_{Q_T} |h|^2 e^{-2s\varphi} + 2 \iint_{Q_T} |Rw|^2,
$$

and therefore estimates (3.26)–(3.27)–(3.28) yield, for $s \geq s_2$ and $\lambda \geq \lambda_2$ and for some constant $C > 0$ independent of s and λ

$$
\begin{aligned}
&s^3\lambda^4 \iint_{Q_T} \xi^3 |w|^2 + s\lambda^2 \iint_{Q_T} \xi |\partial_x w|^2 + \frac{1}{s} \iint_{Q_T} \frac{1}{\xi} \left(|\partial_{xx} w|^2 + |\partial_t w|^2 \right) + \iint_{Q_T} \left(|P_1 w|^2 + |P_2 w|^2 \right) \\
&\leq C \iint_{Q_T} |h|^2 e^{-2s\varphi} + C \iint_{Q_T} |Rw|^2 + C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2 \right).
\end{aligned}
$$

Of course, $|Rw| \leq C s\lambda^2 \xi |w|$ and thus can be easily absorbed by the left hand side: for some constant C independent of s and λ, for $s \geq s_3$ and $\lambda \geq \lambda_3$,

$$
\begin{aligned}
&s^3\lambda^4 \iint_{Q_T} \xi^3 |w|^2 + s\lambda^2 \iint_{Q_T} \xi |\partial_x w|^2 + \frac{1}{s} \iint_{Q_T} \frac{1}{\xi} \left(|\partial_{xx} w|^2 + |\partial_t w|^2 \right) + \iint_{Q_T} \left(|P_1 w|^2 + |P_2 w|^2 \right) \\
&\quad \leq C \iint_{Q_T} |h|^2 e^{-2s\varphi} + C\left(s^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi^3|w|^2 + s\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-2\overline{u}T)} \xi|\partial_x w|^2 \right).
\end{aligned}
\tag{3.29}
$$

Now, we introduce a nonnegative function χ that vanishes identically on $(-\overline{u}T, L)$ and that takes value one on $(-4\overline{u}T, -2\overline{u}T)$ and we compute $P_2 w\, \xi\, \chi^2\, w$:

$$
\iint_{Q_T} P_2 w \xi \chi^2 w = \nu \iint_{Q_T} \partial_{xx} w \xi \chi^2 w + \iint_{Q_T} s a \partial_t \varphi \xi \chi^2 |w|^2 + \nu s^2 \iint_{Q_T} (\partial_x \varphi)^2 \xi \chi^2 |w|^2.
$$

But

$$\nu \iint_{Q_T} \partial_{xx} w \xi \chi^2 w \quad = \quad -\nu \iint_{Q_T} |\partial_x w|^2 \xi \chi^2 + \frac{\nu}{2} \iint_{Q_T} |w|^2 \partial_{xx}(\xi\chi^2),$$

and therefore,

$$\nu \iint_{Q_T} |\partial_x w|^2 \xi \chi^2 = -\iint_{Q_T} P_2 w \xi \chi^2 w + \frac{\nu}{2} \iint_{Q_T} |w|^2 \partial_{xx}(\xi\chi^2) + \iint_{Q_T} sa\partial_t \varphi \xi \chi^2 |w|^2 + \nu s^2 \iint_{Q_T} (\partial_x \varphi)^2 \xi \chi^2 |w|^2. \quad (3.30)$$

Using

$$\left| \iint_{Q_T} P_2 w \xi \chi^2 w \right| \leq \frac{C}{s^{3/2}\lambda^2} \left(\iint_{Q_T} |P_2 w|^2 + s^3 \lambda^4 \iint_{Q_T} \xi^2 \chi^4 |w|^2 \right),$$

we thus obtain

$$\begin{aligned} \nu \iint_{(-4\overline{u}T,-2\overline{u}T)} |\partial_x w|^2 \xi \chi^2 &\leq \frac{C}{s^{3/2}\lambda^2} \iint_{Q_T} |P_2 w|^2 + \left(Cs^{3/2}\lambda^2 + C\lambda^2 + Cs + Cs^2\lambda^2 \right) \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 |w|^2 \\ &\leq \frac{C}{s^{3/2}\lambda^2} \iint_{Q_T} |P_2 w|^2 + Cs^2\lambda^2 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 |w|^2, \end{aligned}$$

for $s, \lambda \geq 1$.

From (3.29), we then obtain

$$s^3\lambda^4 \iint_{Q_T} \xi^3 |w|^2 + s\lambda^2 \iint_{Q_T} \xi |\partial_x w|^2 + \frac{1}{s} \iint_{Q_T} \frac{1}{\xi} \left(|\partial_{xx} w|^2 + |\partial_t w|^2 \right) + \iint_{Q_T} \left(|P_1 w|^2 + |P_2 w|^2 \right) \leq C \iint_{Q_T} |h|^2 e^{-2s\varphi} + Cs^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 |w|^2. \quad (3.31)$$

We now recall that $z = we^{s\varphi}$, and thus $|z|e^{-s\varphi} \leq |w|$, $|\partial_x z|e^{-s\varphi} \leq C(|\partial_x w| + s\lambda\xi|w|)$, $|\partial_t z|e^{-s\varphi} \leq C(|\partial_t w| + s\xi^2|w|)$, $|\partial_{xx} z|e^{-s\varphi} \leq C(|\partial_{xx} w| + s\lambda\xi|\partial_x w| + s^2\lambda^2\xi^2|w|)$.

Of course, this immediately yields

$$s^3\lambda^4 \iint_{Q_T} \xi^3 e^{-2s\varphi} |z|^2 + s\lambda^2 \iint_{Q_T} \xi e^{-2s\varphi} |\partial_x z|^2 + \frac{1}{s} \iint_{Q_T} \frac{1}{\xi} e^{-2s\varphi} \left(|\partial_{xx} z|^2 + |\partial_t z|^2 \right) \leq C \iint_{Q_T} |a\partial_t z + \nu \partial_{xx} z|^2 e^{-2s\varphi} + Cs^3\lambda^4 \iint_{(0,T)\times(-4\overline{u}T,-\overline{u}T)} \xi^3 |z|^2 e^{-2s\varphi}.$$

Taking s large enough, the lower order terms $(\partial_t a)z + \partial_x(bz)$ can be absorbed by the left hand side, thus yielding (3.11). □

3.3 Proof of Theorem 3.2.

Proof of Theorem 3.2. Let us multiply the equation (3.17) by $u\xi^{-2}e^{2s\varphi}$:

$$\iint_{Q_T} (a\partial_t u + b\partial_x u) - \nu \partial_{xx} u) u e^{2s\varphi} \frac{1}{\xi^2} = \iint_{Q_T} (g + v\mathbf{1}_{(-4\overline{u}T,-2\overline{u}T)}) u e^{2s\varphi} \frac{1}{\xi^2}. \quad (3.32)$$

Note that this computation and the ones afterwards are mainly formal since the weight function $\theta(t)$ vanishes at time $t = 0$ and $t = T$. To make these computations rigorous, one could introduce, for $\varepsilon > 0$

$$\theta_\varepsilon(t) = \begin{cases} \theta(t+\varepsilon) \text{ for } t \in (0, 3T_0), \\ 1 \text{ for } t \in (3T_0, T - 3T_0) \\ \theta(t-\varepsilon) \text{ for } t \in (T - 3T_0, T), \end{cases} \quad \text{and} \quad \varphi_\varepsilon(t,x) = \frac{1}{\theta_\varepsilon(t)} \left(e^{5\lambda} - e^{\lambda\psi(x-\overline{u}t)} \right). \quad (3.33)$$

Then, all the computations below can be done with φ_ε instead of φ and passing to the limit $\varepsilon \to 0$, we recover the desired estimates.

But

$$\left|\iint_{Q_T} a\partial_t u u e^{2s\varphi}\frac{1}{\xi^2}\right| = \left|-\frac{1}{2}\iint_{Q_T} |u|^2\partial_t\left(ae^{2s\varphi}\frac{1}{\xi^2}\right)\right| \le Cs\iint_{Q_T}|u|^2e^{2s\varphi},$$

$$\left|\iint_{Q_T} b\partial_x u u e^{2s\varphi}\frac{1}{\xi^2}\right| = \left|-\frac{1}{2}\iint_{Q_T} |u|^2\partial_x\left(be^{2s\varphi}\frac{1}{\xi^2}\right)\right| \le Cs\lambda\iint_{Q_T}\frac{1}{\xi}|u|^2e^{2s\varphi},$$

and

$$\begin{aligned}&\left|\iint_{Q_T}(g+v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)})ue^{2s\varphi}\frac{1}{\xi^2}\right|\\ &\le \frac{C}{s^{3/2}\lambda^2}\iint_{Q_T}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^3}+Cs^{3/2}\lambda^2\iint_{Q_T}\frac{1}{\xi}|u|^2e^{2s\varphi},\end{aligned}$$

for $s,\lambda\ge 1$. Therefore we focus on the term

$$-\nu\iint_{Q_T}\partial_{xx}uue^{2s\varphi}\frac{1}{\xi^2} = \nu\iint_{Q_T}|\partial_x u|^2e^{2s\varphi}\frac{1}{\xi^2}-\frac{\nu}{2}\iint_{Q_T}|u|^2\partial_{xx}\left(e^{2s\varphi}\frac{1}{\xi^2}\right),$$

which yields

$$\nu\iint_{Q_T}|\partial_x u|^2e^{2s\varphi}\frac{1}{\xi^2}\le\left|\nu\iint_{Q_T}\partial_{xx}uue^{2s\varphi}\frac{1}{\xi^2}\right|+Cs^2\lambda^2\iint_{Q_T}|u|^2e^{2s\varphi},$$

for $s,\lambda\ge 1$. Combining the above estimates and the identity (3.32), we obtain

$$\iint_{Q_T}|\partial_x u|^2e^{2s\varphi}\frac{1}{\xi^2}\le Cs^2\lambda^2\iint_{Q_T}|u|^2e^{2s\varphi}+\frac{C}{s^{3/2}\lambda^2}\iint_{Q_T}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^3}, \tag{3.34}$$

and, according to (3.18),

$$s\lambda^2\iint_{Q_T}|\partial_x u|^2e^{2s\varphi}\frac{1}{\xi^2}\le C\iint_{Q_T}|g|^2e^{2s\varphi}\frac{1}{\xi^3}, \tag{3.35}$$

Now, multiply (3.17) by $\partial_t ue^{2s\varphi}/\xi^4$:

$$\begin{aligned}&\iint_{Q_T}a|\partial_t u|^2e^{2s\varphi}\frac{1}{\xi^4}+\iint_{Q_T}b\partial_x u\partial_t ue^{2s\varphi}\frac{1}{\xi^4}-\nu\iint_{Q_T}\partial_{xx}u\partial_t ue^{2s\varphi}\frac{1}{\xi^4}\\ &=\iint_{Q_T}(g+v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)})\partial_t ue^{2s\varphi}\frac{1}{\xi^4}.\end{aligned} \tag{3.36}$$

But due to the assumption $a\in W^{1,\infty}((0,T)\times(4\overline{u}T,L))$,

$$\inf_{(t,x)}\{a\}\iint_{Q_T}|\partial_t u|^2e^{2s\varphi}\frac{1}{\xi^4}\le\iint_{Q_T}a|\partial_t u|^2e^{2s\varphi}\frac{1}{\xi^4},$$

whereas the second and the last terms can be handled as follows:

$$\left|\iint_{Q_T}b\partial_x u\partial_t ue^{2s\varphi}\frac{1}{\xi^4}\right|\le C\left(s\iint_{Q_T}\frac{1}{\xi^3}|\partial_x u|^2e^{2s\varphi}+\frac{1}{s}\iint_{Q_T}\frac{1}{\xi}|\partial_t u|^2e^{2s\varphi}\right),$$

$$\begin{aligned}&\left|\iint_{Q_T}(g+v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)})\partial_t ue^{2s\varphi}\frac{1}{\xi^4}\right|\\ &\le Cs\iint_{Q_T}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^3}+\frac{C}{s}\iint_{Q_T}|\partial_t u|^2e^{2s\varphi}\frac{1}{\xi^5}.\end{aligned}$$

We then focus on the cross term:

$$-\nu\iint_{Q_T}\partial_{xx}u\partial_t ue^{2s\varphi}\frac{1}{\xi^4}=-\frac{\nu}{2}\iint_{Q_T}|\partial_x u|^2\partial_t\left(\frac{e^{2s\varphi}}{\xi^4}\right)+\nu\iint_{Q_T}\partial_x u\partial_t u\partial_x\left(\frac{e^{2s\varphi}}{\xi^4}\right),$$

which implies that

$$\begin{aligned}\left|-\nu \iint_{Q_T} \partial_{xx}u\partial_t u e^{2s\varphi}\frac{1}{\xi^4}\right| &\leq Cs\iint_{Q_T}|\partial_x u|^2\frac{e^{2s\varphi}}{\xi^2}+Cs\lambda\iint_{Q_T}|\partial_x u||\partial_t u|\frac{e^{2s\varphi}}{\xi^3}\\ &\leq C(s+s^2\lambda^2)\iint_{Q_T}|\partial_x u|^2\frac{e^{2s\varphi}}{\xi^2}+\frac{\inf_{(t,x)}\{a\}}{2}\iint_{Q_T}|\partial_t u|^2\frac{e^{2s\varphi}}{\xi^4}. \qquad (3.37)\end{aligned}$$

Putting the above estimates in (3.36) and choosing s large enough, we obtain

$$\inf_{(t,x)}\{a\}\iint_{Q_T}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq Cs\iint_{Q_T}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^3}+Cs^2\lambda^2\iint_{Q_T}|\partial_x u|^2\frac{e^{2s\varphi}}{\xi^2},$$

which, due to (3.35), implies

$$\frac{1}{s}\iint_{Q_T}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq C\iint_{Q_T}|g|^2 e^{2s\varphi}\frac{1}{\xi^3}. \tag{3.38}$$

Finally, to obtain an estimate on $\partial_{xx}u$, we use the equation (3.17):

$$\frac{1}{s}|\partial_{xx}u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq \frac{C}{s}|\partial_t u|^2 e^{2s\varphi}\frac{1}{\xi^4}+\frac{C}{s}(|g|^2+|v\mathbf{1}_{(-4\overline{u}T,-\overline{u}T)}|^2)e^{2s\varphi}\frac{1}{\xi^4}.$$

Integrating this estimate and using (3.18) and (3.38), we easily obtain

$$\frac{1}{s}\iint_{Q_T}|\partial_{xx}u|^2 e^{2s\varphi}\frac{1}{\xi^4}\leq C\iint_{Q_T}|g|^2 e^{2s\varphi}\frac{1}{\xi^3}. \tag{3.39}$$

This concludes the proof of Theorem 3.2. □

3.4 Interpolation estimates

In the sequel, it will be important to have estimates on the value of U and $\partial_x U$ at $x=0$. In order to do this, we will use the following result:

Proposition 3.4. *Let w be a function satisfying*

$$\iint_{Q_T}|w|^2 e^{2s\varphi}+\iint_{Q_T}|\partial_x w|^2 e^{2s\varphi}\frac{1}{\xi^2}+\iint_{Q_T}|\partial_{xx}w|^2 e^{2s\varphi}\frac{1}{\xi^4}<\infty.$$

Then, we have

$$\begin{aligned}&s^2\lambda^3\left(\int_0^T|w(t,0)|^2 e^{2s\varphi(t,0)}\frac{1}{\xi(t,0)}\,dt+\int_0^T|w(t,L)|^2 e^{2s\varphi(t,L)}\frac{1}{\xi(t,L)}\,dt\right)\\ &\leq C\left(s^3\lambda^4\iint_{Q_T}|w|^2 e^{2s\varphi}+s\lambda^2\iint_{Q_T}|\partial_x w|^2 e^{2s\varphi}\frac{1}{\xi^2}\right)\end{aligned} \tag{3.40}$$

and

$$\begin{aligned}&\lambda\left(\int_0^T|\partial_x w(t,0)|^2 e^{2s\varphi(t,0)}\frac{1}{\xi^3(t,0)}\,dt+\int_0^T|\partial_x w(t,L)|^2 e^{2s\varphi(t,L)}\frac{1}{\xi^3(t,L)}\,dt\right)\\ &\leq C\left(s\lambda^2\iint_{Q_T}|\partial_x w|^2 e^{2s\varphi}\frac{1}{\xi^2}+\frac{1}{s}\iint_{Q_T}|\partial_{xx}w|^2 e^{2s\varphi}\frac{1}{\xi^4}\right),\end{aligned} \tag{3.41}$$

for $s,\lambda\geq 1$.

Proof of Proposition 3.4. We focus on the estimate of w at $x=0$, the other ones being completely similar.

Let $\eta = \eta(x)$ be a smooth positive function on $(0, L)$ that takes value 1 close to $x = 0$ and vanishing at $x = 1$. Then

$$
\begin{aligned}
s^2\lambda^3 \int_0^T |w(t,0)|^2 e^{2s\varphi(t,0)} \frac{1}{\xi} &= -s^2\lambda^3 \iint_{Q_T} \partial_x \left(\eta |w|^2 e^{2s\varphi} \frac{1}{\xi} \right) \\
&= -s^2\lambda^3 \iint_{Q_T} |w|^2 \partial_x \left(\eta e^{2s\varphi} \frac{1}{\xi} \right) - 2s^2\lambda^3 \iint_{Q_T} w \partial_x w \eta e^{2s\varphi} \frac{1}{\xi} \\
&\leq C s^3\lambda^4 \iint_{Q_T} |w|^2 e^{2s\varphi} + C s^2\lambda^3 \iint_{Q_T} |w| |\partial_x w| e^{2s\varphi} \frac{1}{\xi} \\
&\leq C s^3\lambda^4 \iint_{Q_T} |w|^2 e^{2s\varphi} + C s\lambda^2 \iint_{Q_T} |\partial_x w|^2 e^{2s\varphi} \frac{1}{\xi^2},
\end{aligned}
$$

for $s, \lambda \geq 1$.

Details of the proof are left to the reader. □

4 Controlling ρ

In this section, we construct a solution of the controllability problem attached to the ρ-part of the map F defined in (2.15).

4.1 Constructing ρ

As we will see below, the construction of the controlled density ρ is very natural. Indeed, the main remark consists in the fact that the density is transported among the flow of velocity $\overline{u} + u + \Lambda u_{in}$, which is close to $\overline{u}$. Hence, we will construct a forward solution ρ_f of (2.15), a backward solution ρ_b of (2.15) and glue these two solutions according to the characteristics of the flow. To be more precise, we introduce ρ_f defined by

$$
\begin{cases}
\partial_t \rho_f + (\overline{u} + u + \Lambda u_{in}) \partial_x \rho_f + \overline{\rho} \partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho}) \rho_f = f(\hat{\rho}, \hat{u}) \text{ in } [0,T] \times (0,L), \\
\rho_f(0,x) = 0 \text{ in } (0,L), \\
\rho_f(t,0) = 0 \text{ in } (0,T),
\end{cases}
\tag{4.1}
$$

and ρ_b defined by

$$
\begin{cases}
\partial_t \rho_b + (\overline{u} + u + \Lambda u_{in}) \partial_x \rho_b + \overline{\rho} \partial_x u + \frac{\overline{\rho}}{\nu} p'(\overline{\rho}) \rho_b = f(\hat{\rho}, \hat{u}) \text{ in } [0,T] \times (0,L), \\
\rho_b(T,x) = 0 \text{ in } (0,L), \\
\rho_b(t,L) = 0 \text{ in } (0,T).
\end{cases}
\tag{4.2}
$$

For equations (4.1) and (4.2) to be well-posed, first remark that $\overline{u} + u + \Lambda u_{in}$ is in $L^\infty(0,T; W^{1,\infty}(0,L))$ so the transport equation is easily solvable by characteristics. But one should also guarantee that $\overline{u} + u + \Lambda u_{in}$ is positive on the space boundaries $(0,T) \times \{0, L\}$. Actually, we will need an even more restrictive condition on that quantity.

In this section, we will assume that u belongs to Y_{s,λ,R_u} for some parameter s, λ, R_u to be determined. And we will also assume that R_u and R_{in} are small enough so that the $L^\infty((0,T) \times (0,L))$-bound of $u e^{s\check{\varphi}/2}$ in the definition of Y_{s,λ,R_u} and the smallness of u_{in} (coming from (2.5)–(2.6)) imply

$$
\overline{u} + u + \Lambda u_{in} \geq \frac{L}{T - 8T_0} \text{ in } [0,T] \times [0,L],
\tag{4.3}
$$

where T_0 is defined in (2.7).

Then we introduce the flow associated to the transport equation of ρ, given by

$$
\partial_t X(t,\tau,a) = \overline{u} + u(t, X(t,\tau,a)) + \Lambda u_{in}(t, X(t,\tau,a)), \quad X(s,s,a) = a.
\tag{4.4}
$$

For later use, it is convenient to introduce extensions of u and Λu_{in} to $(t,x) \in [0,T] \times \mathbb{R}$ (with comparable norms), so that we can consider the flow $X(t,s,a)$ to be defined on $[0,T] \times [0,T] \times \mathbb{R}$.

Due to (4.3), it is easy to check that there exists

$$[a_0, b_0] \subset (-\infty, 0),$$

such that

$$X(T, 0, a_0) > L, \; X(\cdot, 0, a_0)^{-1}(L) \leq T - 3T_0 \;\text{ and }\; X(\cdot, 0, b_0)^{-1}(0) \geq 3T_0,$$

see Figure 1.

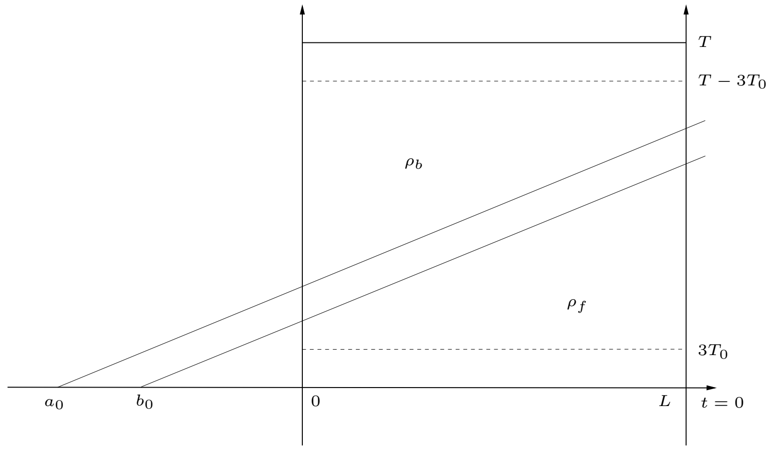

Figure 1: Geometric setting on a_0, b_0. The straight lines represent the lines $t \mapsto (t, a_0 + \overline{u}t)$ and $t \mapsto (t, b_0 + \overline{u}t)$, which approximate the flow X.

We take $\eta \in C^\infty(\mathbb{R}; \mathbb{R})$ such that

$$\eta(a) = 1 \text{ for } a < a_0 \;\text{ and }\; \eta(a) = 0 \text{ for } a > b_0.$$

We set

$$\rho(t, x) = \rho_f(t, x)(1 - \eta(X(0, t, x))) + \rho_b(t, x)\eta(X(0, t, x)). \tag{4.5}$$

Easy computations then show that ρ solves the equation of conservation of mass (2.15), and that

$$\rho(0, x) = \rho(T, x) = 0 \text{ in } [0, L],$$

due to the time boundary conditions on ρ_f and ρ_b.

Since this ρ is admissible for the control problem corresponding to the ρ-part of F, we choose this ρ.

4.2 A new variable μ

An important argument concerning the control of ρ consists in introducing a new quantity which we will denote by μ. This quantity will be easier to handle in the estimates.

Differentiating (2.15) with respect to x, multiplying it by the constant $\nu/\overline{\rho}$, we have

$$\overline{\rho}\left(\partial_t \left(\frac{\nu}{\overline{\rho}^2}\partial_x \rho\right) + (\overline{u} + u + \Lambda u_{in})\partial_x \left(\frac{\nu}{\overline{\rho}^2}\partial_x \rho\right)\right) + \overline{\rho}\left(p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + \partial_x(u + \Lambda u_{in})\right)\left(\frac{\nu}{\overline{\rho}^2}\partial_x \rho\right) + \nu \partial_{xx} u = \frac{\nu}{\overline{\rho}}\partial_x(f(\hat{\rho}, \hat{u})). \tag{4.6}$$

Of course, since both ρ_f and ρ_b satisfy (2.15), they also satisfy equation (4.6)

Besides, adding it to the equation of u, one easily obtains that

$$\mu_f(t,x) = u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_f, \quad \mu_b(t,x) = u + \frac{\nu}{\overline{\rho}^2}\partial_x\rho_b. \tag{4.7}$$

both solve the equation

$$\begin{aligned}\overline{\rho}(\partial_t\mu + (\overline{u}+u+\Lambda u_{in})\partial_x\mu) + \overline{\rho}\left(p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + \partial_x(u+\Lambda u_{in})\right)\mu \\ = \frac{\nu}{\overline{\rho}}\partial_x(f(\hat{\rho},\hat{u})) + g(\hat{\rho},\hat{u}) - \Lambda\rho_{in}(\partial_t u + \overline{u}\partial_x u) + \overline{\rho}\partial_x\left[u(u+\Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u.\end{aligned} \tag{4.8}$$

or, equivalently

$$\partial_t\mu + (\overline{u}+u+\Lambda u_{in})\partial_x\mu + k\mu = h, \tag{4.9}$$

where the source term h is defined by

$$\overline{\rho}h := \frac{\nu}{\overline{\rho}}\partial_x(f(\hat{\rho},\hat{u})) + g(\hat{\rho},\hat{u}) - \Lambda\rho_{in}(\partial_t u + \overline{u}\partial_x u) + \overline{\rho}\partial_x\left[u(u+\Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u,$$

and the potential term k is

$$k := p'(\overline{\rho})\frac{\overline{\rho}}{\nu} + \partial_x(u+\Lambda u_{in}). \tag{4.10}$$

Note that, to complete the equations (4.9), one should further introduce boundary conditions in space and time. From the definition of μ_f and μ_b in (4.7), one easily checks that the boundary conditions in time simply are

$$\mu_f(0,x) = 0 \text{ for } x \in (0,L), \qquad \mu_b(T,x) = 0 \text{ for } x \in (0,L), \tag{4.11}$$

whereas the boundary conditions in space are given by the equations (4.1)–(4.2) satisfied by ρ_f and ρ_b respectively:

$$\mu_f(t,0) = u(t,0) + \frac{\nu}{\overline{\rho}^2}\left(\frac{1}{\overline{u}+u(t,0)+\Lambda u_{in}(t,0)}\right)(f(\hat{\rho},\hat{u}) - \overline{\rho}\partial_x u(t,0)) \tag{4.12}$$

$$\mu_b(t,L) = \frac{\nu}{\overline{\rho}^2}\left(\frac{1}{\overline{u}+\Lambda u_{in}(t,L)}\right)(f(\hat{\rho},\hat{u}) - \overline{\rho}\partial_x u(t,0)), \tag{4.13}$$

where we have used in (4.13) that the function u constructed in Section 3 vanishes at $x = L$.

Note this construction also guarantees the following identities

$$\rho_f(t,x) := \frac{\overline{\rho}^2}{\nu}\int_0^x (\mu_f - u)(t,y)\,dy, \quad \rho_b(t,x) := -\frac{\overline{\rho}^2}{\nu}\int_x^L (\mu_b - u)(t,y)\,dy, \tag{4.14}$$

which will be used in the sequel.

Remark 4.1. *Note that μ_f, μ_b correspond to primitives of ρ_f and ρ_b respectively according to the formula (4.7). However, $\mu = u + \nu\partial_x\rho/\overline{\rho}^2$ is a priori different from $\mu_f(t,x)(1-\eta(X(0,t,x))) + \mu_b(t,x)\eta(X(0,t,x))$.*

Our goal in the next subsections is to obtain suitable estimates on the functions μ_f, μ_b, ρ_f, ρ_b that we constructed.

In order to do that, we will regroup the various terms in h as follows.

First, we introduce $\tilde{f}$ and $\tilde{g}$ defined by:

$$f(\hat{\rho},\hat{u}) = \tilde{f}(\hat{\rho},\hat{u}) + \frac{\overline{\rho}}{\nu}p'(\overline{\rho})\hat{\rho} - \Lambda\rho_{in}\partial_x\hat{u}, \tag{4.15}$$

$$g(\hat{\rho},\hat{u}) = \tilde{g}(\hat{\rho},\hat{u}) - p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\hat{\rho}, \tag{4.16}$$

and h_1 and h_2 as follows:

$$\overline{\rho} h_1 = -\frac{\nu}{\overline{\rho}} \Lambda \rho_{in} \partial_{xx} \hat{u} - \Lambda \rho_{in} \partial_t u, \tag{4.17}$$

$$\begin{aligned} \overline{\rho} h_2 = {} & \left(p'(\overline{\rho}) - p'(\overline{\rho} + \Lambda \rho_{in})\right) \partial_x \hat{\rho} + \frac{\nu}{\overline{\rho}} \partial_x (\tilde{f}(\hat{\rho}, \hat{u})) + \tilde{g}(\hat{\rho}, \hat{u}) - \frac{\nu}{\overline{\rho}} \Lambda \partial_x \rho_{in} \partial_x \hat{u} \\ & + \overline{\rho} \partial_x \left[u(u + \Lambda u_{in}) \right] + p'(\overline{\rho}) \frac{\overline{\rho}^2}{\nu} u - \Lambda \rho_{in} \overline{u} \partial_x u. \end{aligned} \tag{4.18}$$

The source term h can then be rewritten as

$$h = h_1 + h_2. \tag{4.19}$$

As we will see later, we grouped in h_1 the terms which have a bit less "integrability" near $t = 0$ and $t = T$ (see (4.20) below). This will be compensated by the fact that these terms can be made small by taking the initial data sufficiently close to $(\overline{\rho}, \overline{u})$ or, equivalently, R_{in} small. Namely, we shall assume

$$\iint_{(0,T)\times(0,L)} \frac{1}{\xi^4} e^{2s\varphi} |h_1|^2 + \iint_{(0,T)\times(0,L)} \frac{1}{\xi^3} e^{2s\varphi} |h_2|^2 < \infty. \tag{4.20}$$

We shall assume that $k \in L^1(0, T; L^\infty(0, L))$ and set

$$C_k := \exp(\|k\|_{L^1(L^\infty)}). \tag{4.21}$$

Of course, both assumptions (4.20) and $k \in L^1(0, T; L^\infty(0, L))$ shall be verified when getting estimates for the fixed point argument. These will be checked in Section 5.

4.3 Preliminaries: estimates on the flow

In order to estimate ρ, we will first need estimates on the flow X. In particular, the estimates measure how close X is to $(t, x) \mapsto x + t\overline{u}$ when R_{in} and R_u are small, and give consequences on the weight functions of Section 2.3 (since the Carleman weight is calibrated with respect to the straight flow $(t, x) \mapsto x + t\overline{u}$).

Lemma 4.2. *For all $(t, x) \in (0, T) \times (0, L)$ and $\tau \in (0, T)$ such that $X(\tau, t, x) \in (0, L)$,*

$$|(X(\tau, t, x) - \tau \overline{u}) - (x - t\overline{u})| \leq C|\tau - t| \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \tag{4.22}$$

Proof of Lemma 4.2. Let us define

$$\Gamma(\tau, t, x) = (X(\tau, t, x) - \tau \overline{u}) - (x - t\overline{u}).$$

As one immediately checks, $\Gamma(t, t, x) = 0$. Besides, $\Gamma(\tau, t, x)$ satisfies the equation

$$\begin{aligned} \frac{d\Gamma(\tau, t, x)}{d\tau} &= (\overline{u} + u(\tau, X(\tau, t, x)) + \Lambda u_{in}(\tau, X(\tau, t, , x))) - \overline{u} \\ &= u(\tau, X(\tau, t, x)) + \Lambda u_{in}(\tau, X(\tau, t, , x)) \\ \Gamma(t, t, x) &= 0. \end{aligned}$$

Therefore,

$$\left| \frac{d\Gamma(\tau, t, x)}{d\tau} \right| \leq \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))},$$

and estimate (4.22) immediately follows. □

In the following, we shall use the following simple identity on the Carleman weight, which comes from the design of the weight function in (2.20):

$$\varphi(\tau, x - (t - \tau)u) \begin{cases} \geq \varphi(t, x) \text{ for all } (t, \tau) \text{ satisfying } 0 < \tau \leq t \leq T - 3T_0, \\ = \varphi(t, x) \text{ for all } (t, \tau) \text{ satisfying } 3T_0 < \tau \leq t \leq T - 3T_0. \end{cases} \tag{4.23}$$

Of course, when following the characteristic flow associated to $\overline{u} + u + \Lambda u_{in}$, these formula are not true anymore but we still obtain the following approximation lemma:

Lemma 4.3. *For all $q > 0$, there exist constants $C > 0$, $\lambda_0 > 0$, $s_0 > 0$ such that for all $p \in [-q, q]$, for all $(t, x) \in (0, T - 3T_0) \times (0, L)$, for all $\tau \leq t$ such that $X(\tau, t, x) \in (0, L)$, for all $\lambda \geq \lambda_0$ and $s \geq s_0$,*

$$\begin{aligned} p \log(\xi(\tau, X(\tau, t, x))) - 2s\varphi(\tau, X(\tau, t, x)) \leq p \log(\xi(t, x)) - 2s\varphi(t, x) \\ + Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \end{aligned} \tag{4.24}$$

Proof of Lemma 4.3. This follows from an explicit computation of the difference and we shall prove the following equivalent form of (4.24):

$$2s(\varphi(\tau, X(\tau, t, x)) - \varphi(t, x)) + p \log\left(\frac{\xi(t, x)}{\xi(\tau, X(\tau, t, x))}\right) \geq -Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \tag{4.25}$$

First, for all $\tau \in (0, t)$ and $t \leq T - 3T_0$,

$$\begin{aligned} \varphi(\tau, X(\tau, t, x)) - \varphi(t, x) &= \frac{1}{\theta(\tau)}\left(e^{5\lambda} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right) - \frac{1}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(x-\overline{u}t)}\right) \\ &= \left(\frac{1}{\theta(\tau)} - \frac{1}{\theta(t)}\right)\left(e^{5\lambda} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right) \\ &\quad + \frac{1}{\theta(t)}\left(e^{\lambda\psi(x-\overline{u}t)} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right) \\ &= \left(\frac{\theta(t)}{\theta(\tau)} - 1\right)\frac{1}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right) \\ &\quad + \frac{1}{\theta(t)} e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\left(e^{\lambda(\psi(x-\overline{u}t)-\psi(X(\tau,t,x)-\overline{u}\tau))} - 1\right). \end{aligned}$$

Using (2.18), Lemma 4.2, $\tau \leq t$ and $\exp(y) - 1 \geq y$, we thus obtain

$$\begin{aligned} \varphi(\tau, X(\tau, t, x)) - \varphi(t, x) &\geq \left(\frac{\theta(t)}{\theta(\tau)} - 1\right)\frac{1}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right) \\ &\quad - \frac{C}{\theta(t)} e^{4\lambda}\lambda t \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \end{aligned}$$

Since $t \leq T - 3T_0$, $t/\theta(t)$ is bounded:

$$\begin{aligned} \varphi(\tau, X(\tau, t, x)) - \varphi(t, x) &\geq \left(\frac{\theta(t)}{\theta(\tau)} - 1\right)\frac{1}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right) \\ &\quad - Ce^{4\lambda}\lambda \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}. \end{aligned} \tag{4.26}$$

Let us emphasize that the first term in the right-hand side is positive for $\tau < t$.

We now focus on the estimate of $\log(\xi(t, x)/\xi(\tau, X(\tau, t, x)))$. According to the definition of ξ in (2.22),

$$\log\left(\frac{\xi(t, x)}{(\xi(\tau, X(\tau, t, x))}\right) = \log\left(\frac{\theta(\tau)}{\theta(t)}\right) + \lambda\left(\psi(x - \overline{u}t) - \psi(X(\tau, t, x) - \overline{u}\tau)\right).$$

Of course, from (4.22), we immediately deduce that, for $p \in [-q, q]$ and $\tau \leq t$,

$$\left| p \log\left(\frac{\xi(t, x)}{(\xi(\tau, X(\tau, t, x))}\right)\right| \leq q\left(\frac{\theta(t)}{\theta(\tau)} - 1\right) + Cq\lambda\tau \|u + \Lambda u_{in}\|_{L^\infty((0,T)\times(0,L))}, \tag{4.27}$$

where we used

$$\left|\log\left(\frac{\theta(\tau)}{\theta(t)}\right)\right| = \log\left(\frac{\theta(t)}{\theta(\tau)}\right) \leq \frac{\theta(t)}{\theta(\tau)} - 1.$$

Then we deduce (4.25) as soon as

$$|q| \leq \inf_{t,x}\left\{\frac{s}{\theta(t)}\left(e^{5\lambda} - e^{\lambda\psi(X(\tau,t,x)-\overline{u}\tau)}\right)\right\}.$$

which can be done by taking for instance $s_0(q) = 2q$ et λ_0 such that $e^{5\lambda_0} - e^{4\lambda_0} \geq 1$. □

Lemma 4.4. *For all $q > 0$, there exist constants $C > 0$, $\lambda_0 \geq 1$, $s_0 \geq 1$ such that for all $p \in [-q, q]$, for all $(t, x) \in (0, T - 3T_0) \times (0, L)$, for all $\lambda \geq \lambda_0$ and $s \geq s_0$,*

$$\int_{t^*(t,x)}^{t} \xi^p(\tau, X(\tau, t, x)) e^{-2s\varphi(\tau, X(\tau,t,x))} d\tau \leq t\xi^p(t, x) e^{-2s\varphi(t,x)} e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}}, \tag{4.28}$$

where $t^*(t, x)$ is defined as follows:

$$t^*(t, x) := \inf \left\{\tau_0 \in (0, t) \text{ such that } \forall \tau \in (\tau_0, t),\ X(\tau, t, x) \in (0, L)\right\}. \tag{4.29}$$

We also have

$$\xi^p(t^*(t, x), X(t^*(t, x), t, x)) e^{-2s\varphi(t^*(t,x), X(t^*(t,x),t,x))} \leq \xi^p(t, x) e^{-2s\varphi(t,x)} e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}}, \tag{4.30}$$

The time $t^*(t, x)$ corresponds to the entrance time in $(0, T) \times (0, L)$ of the line of the characteristic through (t, x). Accordingly,

$$t^*(t, x) = \begin{cases} 0 & \text{if } x \geq X(t, 0, 0), \\ X(\cdot, t, x)^{-1}(0) & \text{if } x \leq X(t, 0, 0). \end{cases}$$

Proof of Lemma 4.4. Taking the exponential of (4.24), we obtain, for all $\tau \leq t$,

$$\xi^p(\tau, X(\tau, t, x)) e^{-2s\varphi(\tau, X(\tau,t,x))} \leq \xi^p(t, x) e^{-2s\varphi(t,x)} \exp(Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}).$$

This immediately yields (4.30) by taking $\tau = t^*(t, x)$ and (4.28) by integration between $t^*(t, x)$ and t. □

4.4 Estimates on μ

We are now in position to estimate μ. We estimate μ_f and μ_b separately.

We first focus on μ_f, solution of (4.9), and in that section only, we remove the subscript f since all the estimates below apply to μ_b as well:

$$\begin{cases} \partial_t \mu + (\overline{u} + u + \Lambda u_{in}) \partial_x \mu + k\mu = h \text{ in } (0, T) \times (0, L), \\ \mu(t, 0) = m(t), \quad \mu(0, \cdot) = 0. \end{cases} \tag{4.31}$$

where

$$m(t) := u(t, 0) + \frac{\nu}{\overline{\rho}^2(\overline{u} + u(t, 0) + \Lambda u_{in}(t, 0))} \left(\hat{f}(t, 0) - \overline{\rho}\partial_x u(t, 0)\right).$$

Using the characteristics $X(t, s, a)$ defined in (4.4), one easily checks that, for $(t, s, a) \in [0, T] \times [0, T] \times [0, L]$, such that $X(t, s, a) \in [0, L]$,

$$\mu(t, X(t, s, a)) = \mu(s, a) e^{-\int_s^t k(\tau, X(\tau,s,a))\, d\tau} + \int_s^t h(\tilde{\tau}, X(\tilde{\tau}, s, a)) e^{-\int_s^{\tilde{\tau}} k(\tau, X(\tau,s,a))\, d\tau} d\tilde{\tau}.$$

This is due to the fact that the characteristics go from left to right, see (4.3).

Of course, for $x \in [0, L]$ and $t \in [0, T]$, we have two cases, depending on the position of x with respect to the characteristic $X(t, 0, 0)$:

- $x \geq X(t, 0, 0)$: in this case, we use the above formula to get:

$$\mu(t, x) = \int_0^t h(\tilde{\tau}, X(\tilde{\tau}, t, x)) e^{-\int_0^{\tilde{\tau}} k(\tau, X(\tau,s,a))\, d\tau} d\tilde{\tau}. \tag{4.32}$$

- $x \leq X(t, 0, 0)$: in this case, the characteristic through (t, x) lies outside $(0, L)$ at time $t = 0$. We shall therefore take s in the above formula to be the unique solution (uniqueness comes from the fact that $\partial_t X$ never vanishes) of $X(s, t, x) = 0$, or in other words $s = t^*(t, x)$ and $a = 0$:

$$\mu(t, x) = m(t^*(t, x)) e^{-\int_s^t k(\tau, X(\tau,t,x))\, d\tau} + \int_s^t h(\tilde{\tau}, X(\tilde{\tau}, t, x)) e^{-\int_s^{\tilde{\tau}} k(\tau, X(\tau,t,x))\, d\tau} d\tilde{\tau}. \tag{4.33}$$

Recall that k is supposed to be in $L^1(0,T;L^\infty(0,L))$, see (4.21) so that in particular

$$\left| e^{-\int_s^t k(\tau,X(\tau,t,x))\,d\tau} \right| \le C_k. \tag{4.34}$$

Let us begin with the estimates in the zone "below the diagonal"

$$BD := \{(t,x) \in (0,T-3T_0)\times(0,L), x \ge X(t,0,0)\}.$$

Now we can write, using (4.28) for $p=3$ and $p=4$, for $(t,x)\in BD$,

$$\begin{aligned}
|\mu(t,x)|^2 &\le CC_k \left(\int_0^t |h(\tau,X(\tau,t,x))|d\tau\right)^2 \\
&\le CC_k \left(\int_0^t |h_1(\tau,X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^4(\tau,X(\tau,t,x))}d\tau\right)\left(\int_0^t \xi^4(\tau,X(\tau,t,x))e^{-2s\varphi(\tau,X(\tau,t,x))}d\tau\right) \\
&\quad + CC_k \left(\int_0^t |h_2(\tau,X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^3(\tau,X(\tau,t,x))}d\tau\right)\left(\int_0^t \xi^3(\tau,X(\tau,t,x))e^{-2s\varphi(\tau,X(\tau,t,x))}d\tau\right) \\
&\le CC_k \left(\int_0^t |h_1(\tau,X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^4(\tau,X(\tau,t,x))}d\tau\right) t\xi^4(t,x)e^{-2s\varphi(t,x)}e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \\
&\quad + CC_k \left(\int_0^t |h_2(\tau,X(\tau,t,x))|^2 \frac{e^{2s\varphi(\tau,X(\tau,t,x))}}{\xi^3(\tau,X(\tau,t,x))}d\tau\right) \xi^3(t,x)e^{-2s\varphi(t,x)}e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}},
\end{aligned}$$

In particular, this implies that, for all $t \le T-3T_0$ such that $X(t,0,0)\le L$,

$$\begin{aligned}
\int_{X(t,0,0)}^L \xi^{-3}|\mu(t,x)|^2 e^{2s\varphi}\,dx \le{}& CC_k e^{4\lambda} e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t\int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)}\,d\tau dy \\
&+ CC_k e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t\int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)}\,d\tau dy.
\end{aligned} \tag{4.35}$$

Of course, similar estimates can be done in the zone "above the diagonal":

$$AD := \{(t,x)\in(0,T-3T_0)\times(0,L),\ x \le X(t,0,0)\},$$

except for what concerns the boundary term. This term can be handled using (4.30) with $p=3$ as follows:

$$\begin{aligned}
&\left| m(t^*(t,x)) e^{-\int_s^t k(\tau,X(\tau,t,x))\,d\tau}\right|^2 \\
&\qquad \le C_k |m(t^*(t,x))|^2 \left(\frac{e^{2s\varphi(t^*(t,x),0)}}{\xi^3(t^*(t,x),0)}\right)\left(\xi^3(t,x)e^{-2s\varphi(t,x)}\right) e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}}.
\end{aligned}$$

In particular, this implies that, for all $t \le T-3T_0$,

$$\begin{aligned}
\int_0^{\min\{X(t,0,0),L\}} \xi^{-3}|\mu(t,x)|^2 e^{2s\varphi}\,dx \le{}& CC_k e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t |m(\tau)|^2 \xi^{-3}(\tau,0) e^{2s\varphi}(\tau,0)\,d\tau \\
&+ CC_k e^{4\lambda} e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t\int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\
&+ CC_k e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^t\int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy
\end{aligned} \tag{4.36}$$

where we have used that the map $x \mapsto t^*(t,x)$ defines a change of variable of bounded jacobian.

Therefore, combining (4.35)–(4.36), for all $t\in[0,T-3T_0]$, we have

$$\begin{aligned}
\int_0^L \xi^{-3}|\mu(t,x)|^2 e^{2s\varphi}\,dx \le{}& CC_k e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} |m(\tau)|^2 \,\frac{e^{2s\varphi}(\tau,0)}{\xi^3(\tau,0)}\,d\tau \\
&+ CC_k e^{4\lambda} e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0}\int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\
&+ CC_k e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0}\int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy.
\end{aligned} \tag{4.37}$$

We can now estimate μ_f.

Lemma 4.5 (Estimates on μ_f)**.** *For $s \geq s_0$ and $\lambda \geq \lambda_0$,*

$$\begin{aligned}
&\sup_{[0,T-3T_0]} \int_0^L \xi^{-3} |\mu_f(t,x)|^2 e^{2s\varphi}\, dx + \int_0^{T-3T_0} \int_0^L \xi^{-3} |\mu_f(t,x)|^2 e^{2s\varphi}\, dtdx \\
\leq & C C_k e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} |\mu_f(\tau,0)|^2 \; \frac{e^{2s\varphi}(\tau,0)}{\xi^3(\tau,0)}\, d\tau \\
& + C C_k e^{4\lambda} e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |h_1(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^4(\tau,y)} d\tau dy \\
& + C C_k e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy,
\end{aligned} \tag{4.38}$$

where h_1, h_2 are as in (4.17)*–*(4.18)*.*

Proof of Lemma 4.5. The proof follows directly from (4.37) and the fact that

$$\int_0^{T-3T_0} \int_0^L \xi^{-3} |\mu(t,x)|^2 e^{2s\varphi}\, dtdx \leq C \sup_{[0,T-3T_0]} \int_0^L \xi^{-3} |\mu(t,x)|^2 e^{2s\varphi}\, dx.$$

□

Similarly, one can derive estimates on μ_b:

Lemma 4.6 (Estimates on μ_b)**.** *For $s \geq s_0$ and $\lambda \geq \lambda_0$.*

$$\begin{aligned}
&\sup_{[3T_0,T]} \int_0^L \xi^{-3} |\mu_b(t,x)|^2 e^{2s\varphi}\, dtdx + \int_{3T_0}^T \int_0^L \xi^{-3} |\mu_b(t,x)|^2 e^{2s\varphi}\, dtdx \\
\leq & C C_k e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_{3T_0}^T |\mu_b(\tau,L)|^2 \; \frac{e^{2s\varphi}(\tau,L)}{\xi^3(\tau,L)}\, d\tau \\
& + C C_k e^{Cs\lambda e^{4\lambda} \|u+\Lambda u_{in}\|_{L^\infty}} \int_{3T_0}^T \int_0^L |h_2(\tau,y)|^2 \frac{e^{2s\varphi(\tau,y)}}{\xi^3(\tau,y)} d\tau dy,
\end{aligned} \tag{4.39}$$

where h_2 is as in (4.18)*.*

Proof of Lemma 4.6. Set $\mu(t,x) = \mu_b(T-t, L-x)$. Then μ solves an equation of the form (4.9) ($k(t,x)$ replaced by $-k(T-t, L-x)$), where h_1 can be taken to be 0 since it vanishes outside $(0, 3T_0) \times (0, L)$, and thus estimate (4.37) applies since we never use the sign of the derivative of ψ (which has changed doing this transform), but only the direction of monotonicity of θ. Undoing the change of variables, we obtain (4.39). □

4.5 Estimates on $\partial_x \rho$

Having obtained estimates on μ_f and μ_b, we can deduce estimates on $\partial_x \rho_f$ and $\partial_x \rho_b$.

By construction, we have

$$\partial_x \rho_f = \frac{\overline{\rho}^2}{\nu} (\mu_f - u).$$

Thus estimates on $\partial_x \rho_f$ can be immediately deduced from the ones on μ_f and u:

$$\begin{aligned}
\int_0^{T-3T_0} \int_0^L \xi^{-3} |\partial_x \rho_f(t,x)|^2 e^{2s\varphi}\, dtdx \leq C \int_0^{T-3T_0} \int_0^L \xi^{-3} |\mu_f(t,x)|^2 e^{2s\varphi}\, dtdx \\
+ C \int_0^{T-3T_0} \int_0^L \xi^{-3} |u(t,x)|^2 e^{2s\varphi}\, dtdx
\end{aligned} \tag{4.40}$$

Similarly, estimates on $\partial_x \rho_b$ follows from the ones on μ_b and u:

$$\int_{3T_0}^T \int_0^L \xi^{-3} |\partial_x \rho_b(t,x)|^2 e^{2s\varphi} \, dt dx \leq \int_{3T_0}^T \int_0^L \xi^{-3} |\mu_b(t,x)|^2 e^{2s\varphi} \, dt dx + \int_{3T_0}^T \int_0^L \xi^{-3} |u(t,x)|^2 e^{2s\varphi} \, dt dx. \quad (4.41)$$

Remark that $u e^{s\varphi} \xi^{-2} \in H^1(0,T; L^2(0,L))$, (recall Theorem 3.2 in Section 3) hence it is $L^\infty(0,T; L^2(0,L))$. Therefore, using the $L^\infty(0, T-3T_0; L^2(0,L))$ estimates on μ_f in (4.38), we deduce that $\partial_x \rho_f e^{s\varphi} \xi^{-2} \in L^\infty(0, T-3T_0; L^2(0,L))$. Similarly, $\partial_x \rho_b e^{s\varphi} \xi^{-2} \in L^\infty(3T_0, T; L^2(0,L))$ and we have the estimates:

$$\|\partial_x \rho_f e^{s\varphi} \xi^{-2}\|_{L^\infty(0,T-3T_0;L^2(0,L))} \leq \|\mu_f e^{s\varphi} \xi^{-3/2}\|_{L^\infty(0,T-3T_0;L^2(0,L))} + \|u e^{s\varphi} \xi^{-2}\|_{H^1(0,T;L^2(0,L))}, \quad (4.42)$$

$$\|\partial_x \rho_b e^{s\varphi} \xi^{-2}\|_{L^\infty(3T_0,T;L^2(0,L))} \leq \|\mu_b e^{s\varphi} \xi^{-3/2}\|_{L^\infty(3T_0,T;L^2(0,L))} + \|u e^{s\varphi} \xi^{-2}\|_{H^1(0,T;L^2(0,L))}. \quad (4.43)$$

4.6 Estimates on ρ

We can now deduce estimates on ρ.

- *Step 1. Estimates on $\rho_f(t,L)$.*

Note that ρ_f solves equation (2.15) with $\rho_f(0,x) = 0$ and $\rho_f(t,0) = 0$ by construction. Therefore, for t such that $X(t,0,0) \leq L$, $\rho_f(t,L)$ is given by

$$\rho_f(t,L) = \int_0^t (\hat{f} - \overline{\rho} \partial_x u)(\tau, X(\tau,t,L)) \exp\left(-\frac{\overline{\rho}}{\nu} p'(\overline{\rho})(t-\tau)\right) d\tau,$$

whereas, for t such that $X(t,0,0) \geq L$, $\rho_f(t,L)$ is given by

$$\rho_f(t,L) = \int_{t^*(t,L)}^t (\hat{f} - \overline{\rho} \partial_x u)(\tau, X(\tau,t,L)) \exp\left(-\frac{\overline{\rho}}{\nu} p'(\overline{\rho})(t-\tau)\right) d\tau.$$

Therefore, following the proof of Lemma 4.5, we get

Lemma 4.7. *For all $s \geq s_0$ and $\lambda \geq \lambda_0$,*

$$\int_0^{T-3T_0} |\rho_f(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)} d\tau \leq C e^{Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |\hat{f}|^2 \frac{e^{2s\varphi}}{\xi^2} d\tau dy + C e^{Cs\lambda e^{4\lambda} \|u + \Lambda u_{in}\|_{L^\infty}} \int_0^{T-3T_0} \int_0^L |\partial_x u|^2 \frac{e^{2s\varphi}}{\xi^2} d\tau dy. \quad (4.44)$$

Proof of Lemma 4.7. The proof follows line to line the one of Lemma 4.5 and is left to the reader. □

- *Step 2. Global estimates on ρ.*

Here is a key lemma that will allow us to obtain global estimates on ρ directly from the ones on $\partial_x \rho_f$, $\partial_x \rho_b$ and the one of $\rho_f(t,L)$ above:

Lemma 4.8. *There exists a constant $C > 0$ such that for all $a \in H^1(0,L)$, for all $t \in (0,T)$*

$$|a(0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t,0)} + s\lambda \int_0^L |a|^2 \frac{e^{2s\varphi(t,x)}}{\xi(t,x)} dx \leq \frac{C}{s\lambda} \int_0^L |\partial_x a|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)} dx + |a(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)}. \quad (4.45)$$

Proof of Lemma 4.8. The proof is based on the following identity:

$$|a(t,L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t,L)} - |a(0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t,0)} = \int_0^L \partial_x \left(|a|^2 \frac{e^{2s\varphi(t,x)}}{\xi^2(t,x)}\right) dx$$
$$= 2\int_0^L a \partial_x a \frac{e^{2s\varphi(t,x)}}{\xi^2(t,x)} dx - 2s\lambda \int_0^L |a|^2 \partial_x \psi(x - \overline{u}t) \frac{e^{2s\varphi(t,x)}}{\xi(t,x)} dx.$$

Since $\partial_x \psi(x - \overline{u}t)$ is negative on $(0, L)$ for $t \in (0, T)$ by construction (see (2.18)), there exists $c_* > 0$ such that

$$\partial_x \psi(x - \overline{u}t) \le -c_*, \quad (t, x) \in (0, L) \times (0, T).$$

But on the other side,

$$\left| 2 \int_0^L a \partial_x a \frac{e^{2s\varphi(t,x)}}{\xi^2(t,x)} \, dx \right| \le c_* s\lambda \int_0^L |a|^2 \frac{e^{2s\varphi(t,x)}}{\xi(t,x)} \, dx + \frac{1}{c_* s\lambda} \int_0^L |\partial_x a|^2 \frac{e^{2s\varphi(t,x)}}{\xi^3(t,x)} \, dx,$$

which yields the result. $\square$

Using Lemma 4.8, we immediately obtain:

Lemma 4.9. *For $s \ge s_0$ and $\lambda \ge \lambda_0$,*

$$s\lambda \int_0^{T-3T_0} \int_0^L |\rho_f|^2 \frac{e^{2s\varphi}}{\xi} \, dtdx \le \frac{C}{s\lambda} \int_0^{T-3T_0} \int_0^L |\partial_x \rho_f|^2 \frac{e^{2s\varphi}}{\xi^3} \, dtdx + C \int_0^{T-3T_0} |\rho_f(t, L)|^2 \frac{e^{2s\varphi}}{\xi^2} \, dt \quad (4.46)$$

and

$$\int_{3T_0}^T |\rho_b(t, 0)|^2 \frac{e^{2s\varphi}}{\xi^2} \, dt + s\lambda \int_{3T_0}^T \int_0^L |\rho_b|^2 \frac{e^{2s\varphi}}{\xi} \, dtdx \le \frac{C}{s\lambda} \int_{3T_0}^T \int_0^L |\partial_x \rho_b|^2 \frac{e^{2s\varphi}}{\xi^3} \, dtdx. \quad (4.47)$$

Using Lemma 4.9 and the definition of ρ, we obtain the following estimates on ρ:

$$\begin{aligned}\int_0^T \int_0^L |\rho|^2 \frac{e^{2s\varphi}}{\xi} \, dtdx &\le \frac{C}{s^2\lambda^2} \int_{3T_0}^T \int_0^L |\partial_x \rho_b|^2 \frac{e^{2s\varphi}}{\xi^3} \, dtdx \\ &+ \frac{C}{s^2\lambda^2} \int_0^{T-3T_0} \int_0^L |\partial_x \rho_f|^2 \frac{e^{2s\varphi}}{\xi^3} \, dtdx + \frac{C}{s\lambda} \int_0^{T-3T_0} |\rho_f(t, L)|^2 \frac{e^{2s\varphi}}{\xi^2} \, dt \quad (4.48)\end{aligned}$$

Using (4.47) and since $\rho_f(t, 0) = 0$ by construction, we deduce

$$\int_0^T |\rho(t, 0)|^2 \frac{e^{2s\varphi(t,0)}}{\xi^2(t, 0)} \, dt \le \frac{C}{s\lambda} \int_{3T_0}^T \int_0^L |\partial_x \rho_b|^2 \frac{e^{2s\varphi}}{\xi^3} \, dtdx. \quad (4.49)$$

Similarly, $\rho_b(t, L) = 0$, and then

$$\int_0^T |\rho(t, L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t, L)} \, dt = \int_0^T |\rho_f(t, L)|^2 \frac{e^{2s\varphi(t,L)}}{\xi^2(t, L)} \, dt \quad (4.50)$$

Finally, let us explain how to obtain $L^\infty((0, T) \times (0, L))$ bounds on ρ. We do it independently for ρ_f and ρ_b. Using that $\rho_f(t, 0) = 0$ and (4.42), we immediately get by Sobolev embedding that $\rho_f e^{s\check{\varphi}/2} \in L^\infty((0, T - 3T_0) \times (0, L))$. Similarly, $\rho_b e^{s\check{\varphi}/2} \in L^\infty((3T_0, T) \times (0, L))$. Thus, $\rho e^{s\check{\varphi}/2} \in L^\infty((0, T) \times (0, L))$.

To get an estimate on $\partial_t \rho$ in $L^2((0, T) \times (0, L))$, we then use (2.15).

5 The fixed point argument

In this section we prove that the operator described in Section 2 admits a fixed point provided that the initial data is chosen suitably small and that the parameters s and λ are chosen suitably large. This fixed point is obtained via Schauder's fixed point theorem. Hence we prove that the operator $F : (\hat{\rho}, \hat{u}) \mapsto (\rho, u)$ maps the set $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ into itself, that F is continuous and that $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ is compact, when equipped with the proper topology.

5.1 Estimates on u

To get estimates on u, we shall use Theorem 3.2 and Proposition 3.4. Therefore, we shall first derive an estimate on the $L^2((0,T)\times(0,L))$-norm of $e^{s\varphi}g(\hat\rho,\hat u)\xi^{-3/2}$:

Lemma 5.1. *For all $(\hat\rho,\hat u)\in X_{s,\lambda,R_\rho}\times Y_{s,\lambda,R_u}$, we have the following estimates on $g(\hat\rho,\hat u)$ and $\tilde g(\hat\rho,\hat u)$:*

$$\|g(\hat\rho,\hat u)e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))}\le C\left(R_\rho+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right). \tag{5.1}$$

$$\|\tilde g(\hat\rho,\hat u)e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))}\le C\left(\mathcal{O}_{s,\lambda}(R_{in})+R_u^2+R_\rho^2,\right). \tag{5.2}$$

where the constant C is independent of s, λ and R_ρ, R_u and R_{in} .

Proof of Lemma 5.1. It is a matter of estimating the different terms in $g(\hat\rho,\hat u)$ and $\tilde g(\hat\rho,\hat u)$ by using the estimates on $\hat\rho$ and $\hat u$ in X_{s,λ,R_ρ} and Y_{s,λ,R_u}. We regroup the terms that are treated likewise. The various constants C below are independent of s, λ and R_ρ, R_u and R_{in}.

- First using the uniform bound on ρ_{in} and the definition of X_{s,λ,R_ρ} one has immediately

$$\|p'(\overline\rho+\Lambda\rho_{in})\partial_x\hat\rho e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))}\le CR_\rho.$$

It is the only term estimated by R_ρ; it appears in $g(\hat\rho,\hat u)$ but not in $\tilde g(\hat\rho,\hat u)$.

- Now one has also, using that the following terms are compactly supported in time in $(T_0,2T_0)$

$$\Bigg\|\Bigg(-(\overline\rho+\Lambda\rho_{in})\Lambda' u_{in}-(p'(\overline\rho+\Lambda\rho_{in})-p'(\overline\rho+\rho_{in}))\Lambda\partial_x\rho_{in} + \rho_{in}\partial_t u_{in}(\Lambda-\Lambda^2)+\rho_{in}\overline u\partial_x u_{in}(\Lambda-\Lambda^2)+\overline\rho u_{in}\partial_x u_{in}(\Lambda-\Lambda^2) + \rho_{in}u_{in}\partial_x u_{in}(\Lambda-\Lambda^3)\Bigg)e^{s\varphi}\xi^{-3/2}\Bigg\|_{L^2((0,T)\times(0,L))}\le\mathcal{O}_{s,\lambda}(R_{in}).$$

- Next, one obtains by using the estimates in the definition of Y_{s,λ,R_u} and (2.23) that

$$\big\|\big(\Lambda(\overline\rho+\Lambda\rho_{in})\partial_x(\hat u u_{in})\big)e^{s\varphi}\xi^{-3/2}\big\|_{L^2((0,T)\times(0,L))}\le\mathcal{O}_{s,\lambda}(R_{in})R_u.$$

- We obtain the following estimate by using the definition of X_{s,λ,R_u}, (2.23) and the rough estimate $e^{2s\varphi}\ge e^{s\varphi}$:

$$\big\|\big((\overline\rho+\Lambda\rho_{in})\hat u\partial_x\hat u\big)e^{s\varphi}\xi^{-3/2}\big\|_{L^2((0,T)\times(0,L))}\le CR_u^2.$$

- Next one obtains in the same way (remark that since $\hat u\in L^2(0,T;H^2(0,L))\cap H^1(0,T;L^2(0,L))$ hence $\hat u\in L^\infty((0,T)\times(0,L))$)

$$\Bigg\|\hat\rho\Bigg(\partial_t(\Lambda u_{in})+(\overline u+\Lambda u_{in}+\hat u)\partial_x(\Lambda u_{in})\Bigg)e^{s\varphi}\xi^{-3/2}\Bigg\|_{L^2((0,T)\times(0,L))}\le CR_\rho R_{in}.$$

- Using that for some constant c independent of $s,\lambda\ge1$ one has $e^{2s\varphi}\ge c\xi^{1/2}e^{s\varphi}$, one obtains:

$$\Bigg\|\hat\rho\Bigg(\partial_t\hat u+(\overline u+\Lambda u_{in}+\hat u)\partial_x\hat u\Bigg)e^{s\varphi}\xi^{-3/2}\Bigg\|_{L^2((0,T)\times(0,L))}\le CR_\rho R_u.$$

- Using the regularity of p and the boundedness of ρ and $\hat\rho$, we get that pointwise

$$\big|p'(\overline\rho+\Lambda\rho_{in}+\hat\rho)-p'(\overline\rho+\Lambda\rho_{in})\big|\le C|\hat\rho|,$$

and reasoning as above we have that

$$\Bigg\|\Bigg(\big[p'(\overline\rho+\Lambda\rho_{in}+\hat\rho)-p'(\overline\rho+\Lambda\rho_{in})\big]\partial_x(\Lambda\rho_{in}+\hat\rho)\Bigg)e^{s\varphi}\xi^{-3/2}\Bigg\|_{L^2((0,T)\times(0,L))}\le CR_\rho(R_{in}+R_\rho).$$

Gathering all the estimates above, we reach the conclusion. □

Using the estimates of Lemma 5.1, according to Theorem 3.2, we obtain

$$s^{3/2}\lambda^2\|ue^{s\varphi}\|_{L^2((0,T)\times(0,L))}+s^{1/2}\lambda\|\partial_x ue^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))}$$
$$+s^{-1/2}\|\partial_{xx}ue^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))}+s^{-1/2}\|\partial_t ue^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))}$$
$$\leq C\left(R_\rho+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right). \quad (5.3)$$

Hence we arrive to the following statement.

Corollary 5.2. *There exists $c_1>0$, $R_1>0$ independent of s, λ, R_u, R_ρ and R_{in} such that the following holds. If*

$$R_u\leq R_1, \quad (5.4)$$

and

$$R_\rho\leq c_1R_u, \quad (5.5)$$

then for any $s\geq s_0$, $\lambda\geq\lambda_0$, there exists $c(s,\lambda)>0$ such that if

$$R_{in}<c(s,\lambda),$$

then the u-part of $F(\hat{u},\hat{\rho})$ belongs to Y_{s,λ,R_u} for any $(\hat{\rho},\hat{u})$ in $X_{s,\lambda,R_\rho}\times Y_{s,\lambda,R_u}$.

We will suppose that R_ρ and R_u satisfy these conditions in the sequel, so that we have $\hat{u}\in Y_{s,\lambda,R_u}$.

5.2 Estimates on ρ

To get estimates on ρ, we shall use the estimates given in Section 4. They will be based on estimates on μ_f, μ_b. Of course, these first require to get estimates on the source terms h_1, h_2, and the boundary terms given by (4.12)–(4.13).

Lemma 5.3. *We have the following estimates:*

$$\|k\|_{L^1(0,T;L^\infty(0,L))}\leq C, \quad (5.6)$$
$$e^{Cs\lambda e^{4\lambda}\|u+\Lambda u_{in}\|_{L^\infty}}\leq C(1+\mathcal{O}_{s,\lambda}(R_{in})), \quad (5.7)$$
$$\|\tilde{f}(\hat{\rho},\hat{u})e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))}\leq C\left(\mathcal{O}_{s,\lambda}(R_{in})+R_u^2+R_\rho^2\right), \quad (5.8)$$
$$\|\hat{f}(\hat{\rho},\hat{u})e^{s\varphi}\xi^{-1}\|_{L^2((0,T)\times(0,L))}\leq C\left(\mathcal{O}_{s,\lambda}(R_{in})+R_\rho+R_u^2\right), \quad (5.9)$$
$$\|h_1e^{s\varphi}\xi^{-2}\|_{L^2((0,T)\times(0,L))}\leq Ce^{4\lambda}sR_{in}^2+e^{-4\lambda}R_u^2, \quad (5.10)$$
$$\|h_2e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T)\times(0,L))}\leq C\left(\mathcal{O}_{s,\lambda}(R_{in})+\frac{1}{s^{3/2}\lambda^2}R_u+R_\rho^2+R_u^2\right), \quad (5.11)$$
$$\|\mu_f(\cdot,0)e^{s\varphi}\xi^{-3}\|_{L^2(0,T)}\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})\right), \quad (5.12)$$
$$\|\mu_b(\cdot,L)e^{s\varphi}\xi^{-3}\|_{L^2(0,T)}\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})\right). \quad (5.13)$$

We recall that k is the stretching term given by (4.10) and that h_1 and h_2 are given in (4.17)-(4.18).

Proof of Lemma 5.3. All these estimates are obtained independently and we prove them one by one.

- *Proof of* (5.6). We have the following estimate:

$$\|k\|_{L^1(L^\infty)}\leq C+\|\partial_x u\|_{L^1(L^\infty)}+\|\Lambda\partial_x u_{in}\|_{L^1(L^\infty)}\leq C+CR_u+R_{in},$$

where we have used that $s^{-1/2}\partial_x ue^{s\varphi}\xi^{-2}\in L^2(0,T;H^1(0,L))$ since $u\in Y_{s,\lambda,R_u}$.

- *Proof of* (5.7). Using the definition of the space Y_{s,λ,R_u},

$$\begin{aligned}\exp(Cs\lambda e^{4\lambda}\|u\|_{L^\infty}) &\leq \exp(Cs\lambda e^{4\lambda}\exp(-s\check{\varphi}(t))R_u)\\ &\leq \exp(Cs\lambda e^{4\lambda}\exp(-s(e^{5\lambda}-e^{3\lambda}))R_u)\leq C,\end{aligned}$$

since $\lambda \geq 1$. On the other hand,

$$\exp(Cs\lambda e^{4\lambda}\|u_{in}\|_{L^\infty}) = (1 + \mathcal{O}_{s,\lambda}(R_{in})).$$

These estimates yield (5.7).

• *Proof of* (5.8). The function $\tilde{f}$ is defined by (4.15): using the definition of f in (**??**), we get:

$$\tilde{f}(\hat{\rho}, \hat{u}) = -\Lambda' \rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in} u_{in}) - \Lambda(\partial_x \rho_{in})u - \Lambda \rho \partial_x u_{in} - \rho \partial_x u, \tag{5.14}$$

The first two terms $-\Lambda'\rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})$ are compactly supported in time away from $t = 0$ and $t = T$ (in $(T_0, 2T_0)$)and depend only on $\rho_{in}u_{in}$, so

$$\| \left(-\Lambda'\rho_{in} + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})\right) e^{2s\varphi}\xi^{-1}\|_{L^2(L^2)} \leq \mathcal{O}_{s,\lambda}(R_{in}).$$

Next, using the L^∞ norm of $\partial_x \rho_{in}$, we infer

$$\| - \Lambda(\partial_x \rho_{in}) u e^{s\varphi}\xi^{-1}\|_{L^2(L^2)} \leq CR_{in}R_u \leq CR_{in}^2 + CR_u^2.$$

Similarly,

$$\| - \Lambda \rho \partial_x u_{in} e^{s\varphi}\xi^{-1}\|_{L^2(L^2)} \leq CR_\rho R_{in} \leq CR_\rho^2 + CR_{in}^2.$$

Finally, the term $\rho\partial_x u$ is quadratic:

$$\|\rho \partial_x u e^{s\varphi}\xi^{-1}\|_{L^2(L^2)} \leq CR_\rho R_u \leq CR_\rho^2 + CR_u^2.$$

This concludes the estimate (5.8) on $\tilde{f}$.

• *Proof of* (5.9). Of course, we already have the estimate (5.8), so we only need to estimate

$$f(\hat{\rho}, \hat{u}) - \tilde{f}(\hat{\rho}, \hat{u}) = \frac{\overline{\rho}}{\nu} p'(\overline{\rho})\hat{\rho} - \Lambda \rho_{in}\partial_x \hat{u}.$$

By definition,

$$\|\hat{\rho} e^{s\varphi}\xi^{-1}\|_{L^2(L^2)} \leq R_\rho.$$

The last term satisfies

$$\|\Lambda \rho_{in}\partial_x \hat{u} e^{s\varphi}\xi^{-1}\|_{L^2(L^2)} \leq CR_{in}R_u \leq CR_{in}^2 + CR_u^2.$$

This concludes the proof of (5.9), since due to (2.25), $R_\rho^2 \leq R_\rho$.

• *Proof of* (5.10). We have, using the definition of Y_{s,λ,R_u}, that

$$\|\Lambda \rho_{in}\partial_{xx}\hat{u}\, e^{s\varphi}\xi^{-2}\|_{L^2(L^2)} \leq C\sqrt{s}R_{in}R_u \leq Cse^{4\lambda}R_{in}^2 + e^{-4\lambda}R_u^2.$$

and using Corollary (5.2), that

$$\|\Lambda \rho_{in}\partial_t u\, e^{s\varphi}\xi^{-2}\|_{L^2(L^2)} \leq C\sqrt{s}R_{in}R_u \leq Cse^{4\lambda}R_{in}^2 + e^{-4\lambda}R_u^2. \tag{5.15}$$

According to the definition of h_1 in (4.17), we thus obtain (5.10).

• *Proof of* (5.11). Recall the definition of h_2 in (4.18):

$$\begin{aligned}\overline{\rho}h_2 = (p'(\overline{\rho}) - p'(\overline{\rho} + \Lambda\rho_{in}))\,\partial_x\hat{\rho} + \frac{\nu}{\overline{\rho}}\partial_x(\tilde{f}(\hat{\rho}, \hat{u}) + \tilde{g}(\hat{\rho}, \hat{u}) - \frac{\nu}{\overline{\rho}}\Lambda\partial_x\rho_{in}\partial_x\hat{u} \\ + \overline{\rho}\partial_x\left[u(u + \Lambda u_{in})\right] + p'(\overline{\rho})\frac{\overline{\rho}^2}{\nu}u - \Lambda\rho_{in}\overline{u}\partial_x u. \end{aligned} \tag{5.16}$$

We shall estimate each term separately.

$\star$ Using the fact that p' is Lipschitz (for meaningful values of ρ), we deduce

$$\| \left(p'(\overline{\rho}) - p'(\overline{\rho} + \Lambda\rho_{in})\right) \partial_x \hat{\rho} e^{s\varphi} \xi^{-3/2} \|_{L^2(L^2)} \leq C R_{in} R_\rho \leq C R_{in}^2 + R_\rho^2.$$

$\star$ *Estimates on* $\partial_x(\tilde{f}(\hat{\rho}, \hat{u}))$. To estimate the second term $\nu\partial_x(\tilde{f}(\hat{\rho}, \hat{u}))/\overline{\rho}$ we develop it. Differentiating $\partial_x \tilde{f}$, we have

$$\partial_x(\tilde{f}(\hat{\rho}, \hat{u}) = -\Lambda' \partial_x \rho_{in} + (\Lambda - \Lambda^2)\partial_{xx}(\rho_{in} u_{in}) - \Lambda\partial_x((\partial_x \rho_{in})u) - \Lambda\partial_x(\rho\partial_x u_{in}) - \partial_x \rho \partial_x u - \rho\partial_{xx} u. \quad (5.17)$$

The first two terms are compactly supported in time away from $t = 0$ and $t = T$ and depend only on (ρ_{in}, u_{in})

$$\| \left(-\Lambda' \partial_x \rho_{in} + (\Lambda - \Lambda^2)\partial_{xx}(\rho_{in} u_{in})\right) e^{s\varphi} \xi^{-3/2} \|_{L^2(L^2)} \leq \mathcal{O}_{s,\lambda}(R_{in}).$$

The third one is estimated as follows

$$\|\Lambda\partial_x((\partial_x \rho_{in})u) e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq C R_{in} R_u \leq C R_{in}^2 + C R_u^2.$$

Similarly,

$$\|\Lambda\partial_x(\rho\partial_x u_{in}) e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq C R_{in} R_\rho \leq C R_{in}^2 + C R_\rho^2.$$

Finally, the last terms are quadratic:

$$\|\partial_x \rho \, \partial_x u \, e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq \|\partial_x \rho\|_{L^\infty(L^2)} \|\partial_x u e^{s\varphi} \xi^{-1}\|_{L^2(L^\infty)} \leq C R_\rho R_u \leq C R_\rho^2 + C R_u^2.$$

$$\|\rho \, \partial_{xx} u \, e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq \|\rho \xi^{1/2}\|_{L^\infty(L^\infty)} \|\partial_{xx} u e^{s\varphi} \xi^{-2}\|_{L^2(L^2)} \leq C R_\rho R_u \leq C R_\rho^2 + C R_u^2.$$

To sum up, we have obtained the following estimate on $\partial_x \tilde{f}$

$$\|\partial_x(\tilde{f}(\hat{\rho}, \hat{u}) \, e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq C \left(\mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right).$$

$\star$ Let us now come back to the estimates of the terms of h_2. We already have an estimate on $\tilde{g}$, which is the one given by Lemma 5.1:

$$\|\Lambda\partial_x \rho_{in} \partial_x \hat{u} \, e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq C R_{in} R_u \leq C R_{in}^2 + R_u^2.$$

Similarly,

$$\|\Lambda\rho_{in} \partial_x u \, e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq C R_{in} R_u \leq C R_{in}^2 + R_u^2.$$

Then we have again a quadratic term

$$\begin{aligned} \|\partial_x \left[u(u + \Lambda u_{in})\right] e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \quad &\leq \quad \|\partial_x u e^{s\varphi} \xi^{-1}\|_{L^2(L^2)} \|u\|_{L^\infty(L^\infty)} \\ &\quad\quad + C R_{in} \left(\|\partial_x u e^{s\varphi} \xi^{-1}\|_{L^2(L^2)} + \|u e^{s\varphi} \xi^{-1}\|_{L^2(L^2)}\right) \\ &\leq \quad C R_u^2 + C R_{in}^2 \end{aligned}$$

Finally, there is a linear term in u:

$$\|u e^{s\varphi} \xi^{-3/2}\|_{L^2(L^2)} \leq \frac{C}{s^{3/2} \lambda^2} R_u.$$

These estimates all together yield (5.11).

• *Proof of* (5.12). Thanks to (4.3), we have

$$\left\| \frac{1}{\overline{u} + u(t,0) + \Lambda u_{in}(t,0)} \right\|_{L^\infty(0,T)} \leq \frac{2}{\overline{u}} \leq C.$$

It follows that

$$|\mu_f(t,0)| \leq |u(t,0)| + C|f(\hat{\rho}, \hat{u})(t,0)| + C|\partial_x u(t,0)|.$$

The difficult part consists in the estimate of $f(\hat{\rho}, \hat{u})(t, 0)$: for all $t \in (0, T)$,

$$f(\hat{\rho}, \hat{u})(t, 0) = -\Lambda' \rho_{in}(t, 0) + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})(t, 0) - \Lambda\partial_x(\rho_{in}\hat{u})(t, 0) - \Lambda\hat{\rho}(t, 0)\partial_x u_{in}(t, 0) \\ - \hat{\rho}(t, 0)\partial_x\hat{u}(t, 0) + \frac{\overline{\rho}}{\nu}p'(\overline{\rho})\hat{\rho}(t, 0).$$

Hence

$$|\mu_f(t, 0)| \leq |u(t, 0)| + C|\partial_x u(t, 0)| + C| - \Lambda'\rho_{in}(t, 0) + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})(t, 0)| + C|\hat{\rho}(t, 0)|.$$

Using the interpolation results of Proposition 3.4, we have that

$$\|u(t, 0)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} + \|\partial_x u(t, 0)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} \leq \frac{C}{s\lambda^{3/2}}R_u + \frac{C}{\lambda^{1/2}}R_u \leq \frac{C}{\sqrt{\lambda}}R_u.$$

Then, since $\hat{\rho} \in Y_{s,\lambda,R_\rho}$,

$$\|\hat{\rho}(t, 0)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} \leq \frac{C}{\sqrt{\lambda}}R_\rho.$$

Finally, since $\rho_{in}(t, 0), \partial_x\rho_{in}(t, 0), \partial_x u_{in}(t, 0)$ all are in $L^\infty(0, T)$, we have

$$\| \left(-\Lambda'\rho_{in}(t, 0) + (\Lambda - \Lambda^2)\partial_x(\rho_{in}u_{in})(t, 0)\right) e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} \leq \mathcal{O}_{s,\lambda}(R_{in}),$$

which proves (5.12).

• *Proof of* (5.13). This is the same proof as the one of (5.12). Actually, it is even easier to get (5.13) since $u(t, L) = 0$.

The proof of Lemma 5.3 is complete. □

We can now turn to the proof that the ρ-part of F is sent into X_{s,λ,R_ρ} for a proper choice of the parameters.

• First remark that, according to (5.6), the stretching term k in (4.10) indeed is such that C_k (defined in (4.21)) is bounded independently of s, λ and R_u, R_ρ, R_{in} in $(0, 1)$.

• Using then the estimates of Lemmas 4.5 and 4.6, we obtain, for $s \geq s_0$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}\|\mu_f e^{s\varphi}\xi^{-3/2}\|_{L^\infty((0,T-3T_0);L^2(0,L))} &+ \|\mu_f e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T-3T_0);L^2(0,L))} \\ &\leq C(1 + \mathcal{O}_{s,\lambda}(R_{in})) \left[\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in})\right)\right. \\ &\quad \left. + \left(e^{8\lambda}sR_{in}^2 + R_u^2\right) + \left(\mathcal{O}_{s,\lambda}(R_{in}) + \frac{1}{s^{3/2}\lambda^2}R_u + R_\rho^2 + R_u^2\right)\right] \\ &\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right),\end{aligned}$$

provided that R_{in} is sufficiently small depending on s and λ. Similarly,

$$\|\mu_b e^{s\varphi}\xi^{-3/2}\|_{L^\infty((3T_0,T);L^2(0,L))} + \|\mu_b e^{s\varphi}\xi^{-3/2}\|_{L^2((3T_0,T);L^2(0,L))} \\ \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right).$$

From estimates (4.40)-(4.41), we deduce

$$\begin{aligned}\|\partial_x\rho_f e^{s\varphi}\xi^{-3/2}\|_{L^2((0,T-3T_0);L^2(0,L))} &\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right) + \frac{C}{s^{3/2}\lambda^2}R_u \\ &\leq C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right), \qquad (5.18)\end{aligned}$$

and, similarly,

$$\|\partial_x \rho_b e^{s\varphi}\xi^{-3/2}\|_{L^2((3T_0,T);L^2(0,L))} \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right). \tag{5.19}$$

Then, using estimates (4.42),

$$\|\partial_x \rho_f e^{s\varphi}\xi^{-2}\|_{L^\infty((0,T-3T_0);L^2(0,L))}+\|\partial_x \rho_b e^{s\varphi}\xi^{-2}\|_{L^\infty((3T_0,T);L^2(0,L))} \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right)+\sqrt{s}R_u.$$

Hence we have

$$\|\partial_x \rho_f e^{s\check{\varphi}/2}\|_{L^\infty((0,T-3T_0);L^2(0,L))}+\|\partial_x \rho_b e^{s\check{\varphi}/2}\|_{L^\infty((3T_0,T);L^2(0,L))} \leq Ce^{-s\varphi(T/2,0)/4}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\sqrt{s}R_u+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right), \tag{5.20}$$

since $\varphi(T/2,0)$ is the minimum of φ on $(0,T)\times(0,L)$.

According to (5.9), Lemma 4.7 then yields, for all $s \geq s_0$ and $\lambda \geq \lambda_0$,

$$\begin{aligned}\|\rho_f(t,L)e^{s\varphi}\xi^{-1}\|_{L^2(0,T-3T_0)} &\leq C(1+\mathcal{O}_{s,\lambda}(R_{in}))\left(\mathcal{O}_{s,\lambda}(R_{in})+R_\rho+R_u^2\right)+C(1+\mathcal{O}_{s,\lambda}(R_{in}))\frac{R_u}{s^{1/2}\lambda}\\ &\leq C\left(\mathcal{O}_{s,\lambda}(R_{in})+R_\rho+R_u^2+\frac{R_u}{s^{1/2}\lambda}\right).\end{aligned} \tag{5.21}$$

Note that this last estimate and the fact that $\rho_b(t,L)=0$ by construction (see (4.50)), imply in particular that

$$\begin{aligned}\sqrt{\lambda}\|\rho(t,L)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} &\leq \sqrt{\lambda}\|\rho_f(t,L)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T-3T_0)}\\ &\leq C\sqrt{\frac{\lambda}{e^{5\lambda}-e^{4\lambda}}}\left(\mathcal{O}_{s,\lambda}(R_{in})+R_\rho+R_u^2+\frac{R_u}{s^{1/2}\lambda}\right).\end{aligned} \tag{5.22}$$

According to Lemma 4.9 and using (5.18) and (5.21), we thus have

$$\begin{aligned}\|\rho_f e^{s\varphi}\xi^{-1/2}\|_{L^2((0,T-3T_0)\times(0,L))} \leq&\frac{C}{s\lambda}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right)\\ &+\frac{C}{\sqrt{s\lambda}}\left(\mathcal{O}_{s,\lambda}(R_{in})+R_\rho+R_u^2+\frac{R_u}{s^{1/2}\lambda}\right)\\ \leq& C\left(\frac{1}{\sqrt{s\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right).\end{aligned} \tag{5.23}$$

Using Lemma 4.9 estimate (4.47) and the fact that $\rho_f(t,0)=0$ by construction (see (4.49)), we also have

$$\begin{aligned}\sqrt{\lambda}\|\rho(t,0)e^{s\varphi}\xi^{-3/2}\|_{L^2(0,T)} &\leq \sqrt{\lambda}\|\rho_b(t,0)e^{s\varphi}\xi^{-3/2}\|_{L^2(3T_0,T)} \leq \sqrt{\frac{\lambda}{e^{5\lambda}-e^{4\lambda}}}\|\rho_b(t,0)e^{s\varphi}\xi^{-1}\|_{L^2(3T_0,T)}\\ &\leq C\sqrt{\frac{1}{s(e^{5\lambda}-e^{4\lambda})}}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right).\end{aligned} \tag{5.24}$$

and

$$\|\rho_b e^{s\varphi}\xi^{-1/2}\|_{L^2((3T_0,T)\times(0,L))} \leq \frac{C}{s\lambda}\left(\frac{1}{\sqrt{\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right). \tag{5.25}$$

Combining (5.23) and (5.25), we obtain

$$\|\rho e^{s\varphi}\xi^{-1/2}\|_{L^2((0,T)\times(0,L))} \leq C\left(\frac{1}{\sqrt{s\lambda}}(R_u+R_\rho)+\mathcal{O}_{s,\lambda}(R_{in})+R_\rho^2+R_u^2\right). \tag{5.26}$$

Similarly, combining (5.23), (5.25) and estimates (5.18), (5.19), we obtain

$$\|\partial_x \rho e^{s\varphi}\, \xi^{-3/2}\|_{L^2((0,T)\times(0,L))} \leq C\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.27}$$

Finally, to get an $L^\infty((0,T)\times(0,L))$-bound on ρ, we first obtain $L^\infty((0,T)\times(0,L))$-bounds on ρ_f, ρ_b, using the fact that $\rho_f(t,0) = 0$ and $\rho_b(t,L) = 0$. Therefore, since $\partial_x \rho_f e^{s\check{\varphi}/2} \in L^\infty((0,T-3T_0); L^2(0,L))$ and $\partial_x \rho_b e^{s\check{\varphi}/2} \in L^\infty((3T_0,T); L^2(0,L))$, we can use Poincaré estimate:

$$\begin{aligned}\|\rho_f e^{s\check{\varphi}/2}\|_{L^\infty((0,T-3T_0)\times(0,L))} + \|\rho_b e^{s\check{\varphi}/2}\|_{L^\infty((0,T-3T_0)\times(0,L))}& \\ \leq C e^{-s\varphi(T/2,0)/4}\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \sqrt{s}R_u + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right)&,\end{aligned}$$

according to the estimates (5.20).

Thus, gluing these estimates, we obtain

$$\|\rho e^{s\check{\varphi}/2}\|_{L^\infty((0,T)\times(0,L))} \leq C e^{-s\varphi(T/2,0)/4}\left(\frac{1}{\sqrt{\lambda}}(R_u + R_\rho) + \sqrt{s}R_u + \mathcal{O}_{s,\lambda}(R_{in}) + R_\rho^2 + R_u^2\right). \tag{5.28}$$

We have obtained the following.

Proposition 5.4. *There exists $R_2 > 0$ independent of s, λ, R_u, R_ρ and R_{in} such that the following holds. If R_ρ satisfies* (5.4)-(5.5),

$$R_u \leq R_2, \tag{5.29}$$

and $s \geq s_0$, $\lambda \geq \lambda_0$ are large enough (depending on R_ρ), there exists $c_2 > 0$ such that if

$$R_{in} \leq c_2,$$

then the ρ-part of $F(\hat{u}, \hat{\rho})$ belongs to X_{s,λ,R_ρ} for any $(\hat{\rho}, \hat{u})$ in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$.

Proof of Proposition 5.4. Using (5.20), (5.22), (5.24), (5.26), (5.27), (5.28) we obtain the statement except for what concerns the estimates on $\partial_t \rho$. But to treat this term, we only use the equation satisfied by ρ and the estimates already known on ρ and u. □

5.3 Conclusion

Proof of Theorem 1.1. We begin with the topological aspects. We equip $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ with the $L^2((0,T)\times(0,L))$ topology.

Let us first check that $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ is compact. It is closed under the $L^2((0,T)\times(0,L))$ convergence, because clearly the uniform inequalities defining it are stable under a passage to the limit in the sense of distributions. Now that it is relatively compact is a consequence of the uniform estimate defining $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$. Let (ρ_n, u_n) a sequence in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$. Then (u_n) is bounded in $L^2(0,T; H^2(0,L))$ and in in $H^1(0,T; L^2(0,L))$, hence is it relatively compact in $L^2((0,T)\times(0,L))$ by interpolation and Rellich's theorem. All the same, (ρ_n) is bounded in $L^2(0,T; H^1(0,L))$ and in $H^1(0,T; L^2(0,L))$, so the compactness follows easily.

Now, we choose the parameters R_ρ, R_u, R_{in}, s and λ as to satisfy the assumptions of Corollary 5.2 and Proposition 5.4. For instance, we choose $R_u = R_2$, $R_\rho = \min(R_1, c_1 R_2)$, then deduce suitable s_0 and λ_0, and finally determine $R_{in} = c_2$ in terms of the previous parameters. Hence the map F maps $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ into itself.

Let us now turn to the continuity of the operator F described above under the L^2 topology. Consider (ρ_n, u_n) a sequence in $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$, converging to (ρ, u) in $L^2((0,T)\times(0,L))^2$, and consequently in any topology stronger than $L^2((0,T)\times(0,L))^2$ for which $X_{s,\lambda,R_\rho} \times Y_{s,\lambda,R_u}$ is still relatively compact: (u_n) also converges in the sense of the weak $L^2(0,T; H^2(0,L))$ and $H^1(0,T; L^2(0,L))$ topologies and the strong $L^\infty((0,T)\times(0,L))$ and $L^2((0,T); W^{1,\infty}(0,L))$ ones; (ρ_n) also converges in the sense of the weak $L^2(0,T; H^1(0,L))$ and $H^1(0,T; L^2(0,L))$ topologies and the strong $L^\infty((0,T); L^2(0,L))$ and $L^2((0,T); L^\infty(0,L))$ ones.

Let us prove that the images under F converge correspondingly. By the compactness of $X_{s,\lambda,R_u} \times Y_{s,\lambda,R_\rho}$, we only have to prove that $F(\rho,u)$ is the unique limit point of the sequence $(F(\rho_n,u_n))$. Hence we suppose (relabeling the subsequence) that $F(\rho_n,u_n)$ converges to (ρ_∞,u_∞) and have to prove that $(\rho_\infty,u_\infty) = F(\rho,u)$. Then it is clear using the convergences above one can check that each term in $g(\rho_n,u_n)$ converges in the sense of distributions to its counterpart in $g(\rho_\infty,u_\infty)$. Due to the uniform estimates of $(g(\rho_n,u_n))_n)$ in $L^2((0,T)\times(0,L); e^{s\varphi}\xi^{-3/2}\,dx\,dt)$ (see Subsection 5.1), one has the weak $L^2((0,T)\times(0,L))$ convergence of $e^{s\varphi}\xi^{-3/2}g(\rho_n,u_n)$ towards $e^{s\varphi}\xi^{-3/2}g(\rho,u)$. Hence one sees that we can pass to the limit in the variational formulation (3.15), so by uniqueness in Lax-Milgram's theorem, the u-part of $F(\rho,u)$ coincides with u_∞. Reasoning in the same way, using the uniqueness of the solution of the transport equations (4.1)-(4.2), we obtain $F(\rho,u) = (\rho_\infty,u_\infty)$.

In that case, all the assumptions of Schauder's fixed point theorem are fulfilled. Consequently, F admits a fixed point (ρ,u) in $X_{s,\lambda,R_\rho}\times Y_{s,\lambda,R_u}$. That it satisfies the equation comes from the construction. That $(\rho,u)(T)=0$ comes from the definition of the space $X_{s,\lambda,R_\rho}\times Y_{s,\lambda,R_u}$ and of the weight function φ.

This concludes the proof of Theorem 1.1. □

6 Appendix

6.1 Computation of f

To compute f in (2.10), we use that

$$\partial_t\rho_{\mathcal{S}} + \partial_x(\rho_{\mathcal{S}}u_{\mathcal{S}}) = 0 \text{ in } (0,T)\times(0,L).$$

Thus, setting $\rho = \rho_{\mathcal{S}} - \overline{\rho} - \Lambda\rho_{in}$, we have

$$\begin{aligned}
0 =& \partial_t(\rho+\Lambda\rho_{in}) + \partial_x\left((\overline{\rho}+\rho+\Lambda\rho_{in})(\overline{u}+u+\Lambda u_{in})\right)\\
=& \partial_t\rho + \Lambda'\rho_{in} + \Lambda\partial_t\rho_{in} + \partial_x\left((\overline{\rho}+\Lambda\rho_{in})(\overline{u}+\Lambda u_{in})\right) + \partial_x\left(\rho(\overline{u}+u+\Lambda u_{in})\right) + \partial_x\left((\overline{\rho}+\Lambda\rho_{in})u\right)\\
=& \partial_t\rho + (\overline{u}+u+\Lambda u_{in}))\,\partial_x\rho + \overline{\rho}\partial_x u\\
&+ \Lambda'\rho_{in} - \Lambda\partial_x\left((\overline{\rho}+\rho_{in})(\overline{u}+u_{in})\right) + \partial_x\left((\overline{\rho}+\Lambda\rho_{in})(\overline{u}+\Lambda u_{in})\right) + \rho\partial_x(u+\Lambda u_{in}) + \Lambda\partial_x(\rho_{in}u),
\end{aligned}$$

where we used (2.2) in the last identity.

This yields to f as in (2.11) once we have remarked that:

$$-\Lambda\partial_x\left((\overline{\rho}+\rho_{in})(\overline{u}+u_{in})\right) + \partial_x\left((\overline{\rho}+\Lambda\rho_{in})(\overline{u}+\Lambda u_{in})\right) = \partial_x(\rho_{in}u_{in})\left(\Lambda^2-\Lambda\right).$$

6.2 Computation of g

We start by using the equation of $u_{\mathcal{S}}$ (see the second equation in (1.1)) as well as the expressions of $u_{\mathcal{S}}$ and $\rho_{\mathcal{S}}$ (see (2.9) :

$$\begin{aligned}
&(\rho+\overline{\rho}+\Lambda\rho_{in})[\partial_t u + \partial_t(\Lambda u_{in}) + (u+\overline{u}+\Lambda u_{in})(\partial_x u + \Lambda\partial_x u_{in})]\\
&-\nu\partial_{xx}u - \nu\Lambda\partial_{xx}u_{in} + p'(\rho+\overline{\rho}+\Lambda\rho_{in})(\partial_x\rho+\Lambda\partial_x\rho_{in}) = 0
\end{aligned}$$

Since we look for the equation of u written in (2.10), we regroup the previous expression in the following way:

$$\begin{aligned}
&(\overline{\rho}+\Lambda\rho_{in})(\partial_t u + \overline{u}\partial_x u) + \Lambda(\overline{\rho}+\Lambda\rho_{in})(\partial_t u_{in} + (\overline{u}+\Lambda u_{in})\partial_x u_{in})\\
&+(\overline{\rho}+\Lambda\rho_{in})(\Lambda' u_{in} + \Lambda\partial_x(uu_{in}) + u\partial_x u)\\
&+\rho[\partial_t u + \partial_t(\Lambda u_{in}) + (u+\overline{u}+\Lambda u_{in})(\partial_x u + \Lambda\partial_x u_{in})]\\
&-\nu\partial_{xx}u - \nu\Lambda\partial_{xx}u_{in} + p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\rho + \Lambda p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\rho_{in}\\
&+(p'(\rho+\overline{\rho}+\Lambda\rho_{in}) - p'(\overline{\rho}+\Lambda\rho_{in}))(\partial_x\rho+\Lambda\partial_x\rho_{in}) = 0.
\end{aligned}$$

Next, we replace $(\overline{\rho}+\Lambda\rho_{in})(\partial_t u+\overline{u}\partial_x u)-\nu\partial_{xx}u+p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\rho$ by g. This yields

$$\begin{aligned}
&g(\rho,u)=-\Lambda((\overline{\rho}+\Lambda\rho_{in})(\partial_t u_{in}+(\overline{u}+\Lambda u_{in})\partial_x u_{in})-\nu\partial_{xx}u_{in}+p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\rho_{in})\\
&-(\overline{\rho}+\Lambda\rho_{in})(\Lambda' u_{in}+\Lambda\partial_x(uu_{in})+u\partial_x u)-p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\rho\\
&-\rho[\partial_t u+\partial_t(\Lambda u_{in})+(u+\overline{u}+\Lambda u_{in})(\partial_x u+\Lambda\partial_x u_{in})]\\
&-(p'(\rho+\overline{\rho}+\Lambda\rho_{in})-p'(\overline{\rho}+\Lambda\rho_{in}))(\partial_x\rho+\Lambda\partial_x\rho_{in}).
\end{aligned}\tag{6.1}$$

The last two lines in this expression are exactly the two last lines in (2.12). In the second line of (6.1), the second term is the first one in the first line of (2.12) while the second and third terms correspond to the third line of (2.12), and the fourth one is the first one in the first line of (2.12).

We still have to work with the first line of (6.1). For this, we make the difference between the first line of (6.1) and the equation of u_{in} (see (2.2)) :

$$(\overline{\rho}+\rho_{in})(\partial_t u_{in}+(\overline{u}+u_{in})\partial_x u_{in})-\nu\partial_{xx}u_{in}+p'(\overline{\rho}+\rho_{in})\partial_x\rho_{in}=0.$$

We obtain

$$\begin{aligned}
&-\Lambda[(\overline{\rho}+\Lambda\rho_{in})(\partial_t u_{in}+(\overline{u}+\Lambda u_{in})\partial_x u_{in})-\nu\partial_{xx}u_{in}+p'(\overline{\rho}+\Lambda\rho_{in})\partial_x\rho_{in}]\\
&+\Lambda[(\overline{\rho}+\rho_{in})(\partial_t u_{in}+(\overline{u}+u_{in})\partial_x u_{in})-\nu\partial_{xx}u_{in}+p'(\overline{\rho}+\rho_{in})\partial_x\rho_{in}]\\
&=-\Lambda[(\overline{\rho}+\Lambda\rho_{in})(\partial_t u_{in}+(\overline{u}+\Lambda u_{in})\partial_x u_{in})-(\overline{\rho}+\rho_{in})(\partial_t u_{in}+(\overline{u}+u_{in})\partial_x u_{in})]\\
&-\Lambda\partial_x\rho_{in}(p'(\overline{\rho}+\Lambda\rho_{in})-p'(\overline{\rho}+\rho_{in})).
\end{aligned}$$

In this last identity, the last term is the second term in the first line of (2.12) while, by a simple computation, the first term equals

$$\rho_{in}\partial_t u_{in}(\Lambda-\Lambda^2)+\rho_{in}\overline{u}\partial_x u_{in}(\Lambda-\Lambda^2)+\overline{\rho}u_{in}\partial_x u_{in}(\Lambda-\Lambda^2)+\rho_{in}u_{in}\partial_x u_{in}(\Lambda-\Lambda^3),$$

which constitutes exactly the second line of (2.12).

6.3 Remarks of Proposition 2.1

Actually, Matsumura and Nishida [11, Theorem 7.1] prove a much stronger result (see also [3]):

Theorem 6.1. *Let $\overline{\rho}$ be such that $p'(\overline{(\rho)})>0$. Then there exists a constant $c>0$ such that, if $(\rho_0-\overline{\rho})\in H^3(R^3)$, $u_0\in H^3(R^3)$ and*

$$\|\rho_0-\overline{\rho}\|_{H^3(\mathbb{R})}+\|u_0\|_{H^3(\mathbb{R})}\le c,$$

then the three-dimensional isentropic compressible Navier-Stokes equation:

$$\begin{cases}\partial_t\rho+div(\rho u)=0,\\ \partial_t(\rho u)+div(\rho u\otimes u)-\mu\Delta u-(\lambda+\mu)\nabla div u+\nabla P(\rho)=0,\\ (\rho,u)_{|t=0}=(\rho_0,u_0),\end{cases}$$

has a unique global solution (ρ,u) such that $\rho-\overline{\rho}\in C(\mathbb{R}^+;H^3)\cap C^1(\mathbb{R}^+;H^2)$, $\nabla\rho\in L^2(\mathbb{R}^+;H^2)$, $u\in C(\mathbb{R}^+;H^3)\cap C^1(\mathbb{R}^+;H^1)$, $\nabla u\in L^2(\mathbb{R}^+;H^3)$. Moreover for some $C>0$:

$$\|(\rho-\overline{\rho},u)\|_{L^\infty(\mathbb{R}^+;H^3(\mathbb{R}))}+\|\nabla u\|_{L^\infty(\mathbb{R}^+;H^3(\mathbb{R}))}\le C\|(\rho_0-\overline{\rho},u)\|_{H^3(\mathbb{R})}.$$

Let us add several comments on this result.

- Mastumura and Nishida's result give global in time solutions. We merely need the local result.
- In fact Mastumura and Nishida consider even the more general system, non isentropic, with the equation of temperature. The isentropic case is actually simpler and still contained in their analysis (see the end of [11, Section 1]).
- Mastumura and Nishida's result is three-dimensional, but their analysis (relying only on energy estimates and characteristics for the density equation) applies in the one dimensional setting. Actually, the one dimensional case would be much simpler, since the Morrey-Sobolev injections are better, and the energy estimates way simplified.
- In the above result, the reference velocity $\overline{u}$ is not taken into account as in Proposition 2.1. But it is just a matter of taking the Galilean invariance of the equation into account to deduce this statement.

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