Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results

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Abstract

The aim of this paper is two folded. Firstly, we study the validity of a Pohozaev-type identity for the Schrödinger operator

$$A_\lambda := -\Delta - \frac{\lambda}{|x|^2}, \quad \lambda \in \mathbb{R},$$

in the situation where the origin is located on the boundary of a smooth domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, showing some applications to semi-linear elliptic equations. The problem we address is very much related to optimal Hardy-Poincaré inequalities with boundary singularities which have been investigated in the recent past in various papers. In view of that, the proper functional framework is described and explained. Secondly, we use the Pohozaev identity to derive the method of multipliers and we apply it to study the exact boundary controllability for the wave and Schrödinger equations corresponding to the singular operator A_λ. In particular, this complements and extends well-known results by Vancostenoble and Zuazua [38], who discussed the same issue in the case of interior singularity.

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1 Introduction

In this paper we are dealing with the Schrödinger operator $A_\lambda := -\Delta - \lambda/|x|^2$, $\lambda \in \mathbb{R}$, acting in a domain where the potential $1/|x|^2$ is singular at the boundary. Our main goal consists in studying the control properties of the corresponding wave and Schrödinger equations. Moreover, our aim is to find necessary and sufficient conditions for the existence of non-trivial solutions to semi-linear elliptic equations associated to A_λ. Operators like A_λ may arise in molecular physics [29], quantum cosmology [6], combustion models [23], linearization of critical non-linear PDE's (e.g. [9], [34]), etc. From a mathematical point of view they are interesting due to their criticality since they are homogeneous of degree -2.

The qualitative properties of evolution problems involving the operator A_λ require either positivity or coercivity of A_λ in the sense of quadratic forms in L^2. Roughly speaking, this is equivalent to making use of Hardy-type inequalities. There is a large literature concerning the study of such inequalities, especially in the context of interior singularities (e.g. see [40], [2], [20], [3], [12] and references therein). The classical Hardy inequality is stated as follows. Assume Ω is an open subset in $\mathbb{R}^N$, $N \geq 3$, containing the origin, i.e., $0 \in \Omega$. Then it holds (see [25])

$$\int_\Omega |\nabla u|^2 dx - \frac{(N-2)^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx > 0, \quad \forall u \in H_0^1(\Omega), \tag{1.1}$$

and the constant $(N-2)^2/4$ is optimal and not attained in $H_0^1(\Omega)$. We remind that the optimal Hardy constant is defined by the quotient

$$\mu(\Omega) := \inf_{u \in C_0^\infty(\Omega)} \Big(\int_\Omega |\nabla u|^2 dx \Big/ \int_\Omega \frac{u^2}{|x|^2} dx \Big).$$

In this paper, we consider Ω to be a smooth subset of $\mathbb{R}^N$, $N \geq 1$, with the origin $x = 0$ placed on its boundary Γ. Hardy inequalities with an isolated singularity on the boundary have been less investigated so far. However, in the recent past some substantial work has been developed in that direction.

It has been proved that the best constants depend both on the local geometry near the origin and the entire shape of the domain.

More precisely, starting with the work by Filippas, Tertikas and Tidblom [22], and continuing with [10], [18], [19], it has been proved that, whenever Ω is a smooth domain with the origin located on the boundary, there exists a positive constant $r_0 = r_0(\Omega, N) > 0$ such that

$$\mu(\Omega \cap B_{r_0}(0)) = \frac{N^2}{4}. \tag{1.2}$$

where $B_{r_0}(0) \subset \mathbb{R}^N$ denotes the ball of radius r_0 centered in origin. Next we recall the definition of the upper half space $\mathbb{R}^N_+$ which is given by the set

$$\mathbb{R}^N_+ = \big\{ x = (x_1, \dots, x_N) = x' + x_N e_N \in \mathbb{R}^N \mid x_N > 0 \big\}, \tag{1.3}$$

where e_N is the N-th canonical vector in $\mathbb{R}^N$ and $x' = (x_1, \ldots, x_{N-1}, 0)$. In addition, if $\Omega \subset \mathbb{R}^N_+$, $N \geq 1$, the new Hardy inequality

$$\int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx, \quad \forall u \in H^1_0(\Omega), \tag{1.4}$$

holds true and the constant $N^2/4$ is optimal, i.e. $\mu(\Omega) = N^2/4$.

Otherwise, if Ω is a smooth domain which, up to a rotation, is not supported in $\mathbb{R}^N_+$, the constant $N^2/4$ is optimal, up to lower order terms in $L^2(\Omega)$-norm as shown later in inequality (1.8). In general $\mu(\Omega) = N^2/4$ is not true for any smooth bounded domain Ω containing the origin on the boundary (e.g. [18]).

Without losing generality, since the operator A_λ is invariant under rotations, next in the paper we consider Ω such that

$$x \cdot \nu = O(|x|^2), \text{ on } \Gamma, \tag{1.5}$$

where ν stands for the outward normal vector to Γ. Moreover, since optimal inequalities have been obtained regardless of the shape of Ω, throughout the paper we discuss two main situations of geometries motivated by the remarks above.

C1. Ω is a smooth domain satisfying (1.5) and $x_N > 0$ holds for all $x \in \Omega$ (i.e. $\Omega \subset \mathbb{R}^N_+$).

C2. Ω is a smooth domain satisfying (1.5) such that x_N changes sign in Ω ($\Omega \not\subset \mathbb{R}^N_+$).

Next we need to introduce the constant

$$R_\Omega := \sup_{x \in \overline{\Omega}} |x|. \tag{1.6}$$

The following optimal Hardy-Poincaré inequalities are valid for each one of the cases above.

If Ω fulfills the case C1, then (e.g. [10]) it holds that

$$\forall\, u \in C_0^\infty(\Omega), \quad \int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + \frac{1}{4} \int_\Omega \frac{u^2}{|x|^2 \log^2(R_\Omega/|x|)} dx, \tag{1.7}$$

and $N^2/4$ is the sharp constant.

If Ω satisfies the case C2 then (e.g. [18]) there exist two constants $C_2 = C_2(\Omega) \in \mathbb{R}$ and $C_3 = C_3(\Omega, N) > 0$ such that for any $u \in C_0^\infty(\Omega)$ it holds

$$C_2 \int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx \geq \frac{N^2}{4} \int_\Omega \frac{u^2}{|x|^2} dx + C_3 \int_\Omega \frac{u^2}{|x|^2 \log^2(R_\Omega/|x|)} dx. \tag{1.8}$$

Due to the inequalities mentioned above, a new notation is used throughout the paper, namely

$$\lambda(N) := \frac{N^2}{4}. \tag{1.9}$$

In view of this, let us now describe the content of the paper.

In Section 2 we introduce the functional framework induced by the above Hardy inequalities. We refer to the Hilbert space H_λ defined in Subsection 2.1. Then we check the validity of the Pohozaev identity for the Schrödinger operator A_λ in this functional setting. For that we define the domain of A_λ as

$$D(A_\lambda) := \big\{ u \in H_\lambda \mid A_\lambda u \in L^2(\Omega) \big\}, \tag{1.10}$$

and we prove that

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = -\int_\Omega (x\cdot\nabla u) A_\lambda u dx - \frac{N-2}{2}||u||^2_{H_\lambda}, \quad \forall u \in D(A_\lambda), \tag{1.11}$$

where $||\cdot||_{H_\lambda}$ denotes the norm associated to H_λ. We refer to Theorems 2.1, 2.2 for a complete statement of this result. For the sake of clarity, we will mainly discuss the case C1 above. Nevertheless, similar results could be also extended to the case C2 in a weaker functional setting due to weaker Hardy inequalities (see Subsection 2.2).

Formally, identity (1.11) can be obtained by direct integrations. However, a rigorous justification of the integrations is needed due to the lack of regularity of A_λ at the origin where standard elliptic regularity does not apply. In addition, we need to justify the integrability of the boundary term in (1.11) which is no more obvious since the singularity is located on the boundary and standard trace regularity fails. As we mentioned before, we give a rigorous justification of these facts in Theorems 2.1, 2.2.

Pohozaev-type identities arise in many applications and mostly when studying non-linear equations (see [17], [24], [13] and references therein).

In Section 3, we apply Theorem 2.2 to characterize the existence of non-trivial solutions to a semi-linear singular elliptic PDE in star-shaped domains. We refer mainly to Theorem 3.1.

In Section 4 we present some applications of the Pohozaev identity in Theorem 2.2 to the controllability of conservative systems like wave and Schrödinger equations, for which the multiplier method plays a crucial role.

In the last few decades, most of the studies in Controllability Theory and its applications to evolution PDEs, have applied methods like *Hilbert Uniqueness Method* (HUM) introduced by J. L. Lions in [30], Carleman estimates developed by Fursikov and Imanuvilov [21], microlocal analysis due to Bardos, Lebeau and Rauch ([5], [4]), but also multiplier techniques with the pioneering papers by Komornik and Zuazua ([27], [28], [41]). In particular, the controllability properties and stabilization of the heat like equation corresponding to A_λ have been analyzed in [37], [14], [36] in the case of interior singularity using tools based on Carleman estimates.

Now, let us detail the controllability problem we are interested in Section 4. For $N \geq 1$ we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ where Γ denotes its boundary. Moreover, we denote by Γ_0 a non-empty part of the set Γ that will be specified later.

Next we consider the wave-like process

$$
\begin{cases}
u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\
u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\
u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\
u(0,x) = u_0(x), & x \in \Omega, \\
u_t(0,x) = u_1(x), & x \in \Omega.
\end{cases} \tag{1.12}
$$

To better specify the problem under consideration, we say that the system (1.12) is exactly controllable from Γ_0, in time T, if for any initial data $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ and any target $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda^{'}$, there exists a control $h \in L^2((0,T) \times \Gamma_0)$ such that the solution of (1.12) satisfies:

$$
(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \text{for all } x \in \Omega.
$$

This issue was analyzed by Vancostenoble and Zuazua [38] under the assumption that the singularity $x = 0$ is located in the interior of Ω. They proved well-posedness and exact controllability of system (1.12) for any $\lambda \leq \lambda_\star := (N-2)^2/4$ for boundary controls acting in Γ_0 defined by

$$
\Gamma_0 := \{x \in \Gamma \mid x \cdot \nu \geq 0\}. \tag{1.13}
$$

Roughly speaking, the authors showed in [38] that the parameter $\lambda_\star$ is critical when asking the well-posedness and control properties of (1.12), and the results are very much related to the best constant in the Hardy inequality with interior singularity.

In Section 4, we address the same controllability question in the case of boundary singularity. Our main result asserts that for the same geometrical setup (1.13), we can increase the range of values λ (from $\lambda_\star$ to $\lambda(N)$) for which the exact boundary controllability of system (1.12) holds. This is due to the new Hardy inequalities above.

By now classical HUM, the controllability of system (1.12) is equivalent to the so-called *Observability Inequality* for the adjoint system

$$
\begin{cases}
w_{tt} - \Delta w - \lambda \frac{w}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\
w(t,x) = 0, & (t,x) \in (0,T) \times \Gamma, \\
w(0,x) = w_0(x), & x \in \Omega, \\
w_t(0,x) = w_1(x), & x \in \Omega,
\end{cases} \tag{1.14}
$$

which formally states that for any $\lambda \leq \lambda(N)$ and $T > 0$ large enough there exists a constant $C_T > 0$ such that

$$
C_T \left(||w_1||^2_{L^2(\Omega)} + \int_\Omega \left(|\nabla w_0(x)|^2 - \lambda \frac{w_0^2(x)}{|x|^2} \right) dx \right) \leq \int_0^T \int_{\Gamma_0} (x \cdot \nu) \left(\frac{\partial w}{\partial \nu} \right)^2 d\sigma dt, \tag{1.15}
$$

holds true for w solution of (1.14). We point out that, since the weight $x \cdot \nu$ degenerates at the origin, our inequality (1.15) is stronger than the one proved in the case of interior singularity in [38] which formally states that

$$C_T\left(||w_1||^2_{L^2(\Omega)}+\int_\Omega\Big(|\nabla w_0(x)|^2-\lambda\frac{w_0^2(x)}{|x|^2}\Big)dx\right)\leq\int_0^T\int_{\Gamma_0}\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt. \quad (1.16)$$

The main tool to prove (1.15) relies on the multiplier method and compactness-uniqueness argument [30]. In view of that, the Pohozaev identity provides a direct tool to show that the solution of system (1.14) satisfies the multiplier identity which is formally given by

$$\frac{1}{2}\int_0^T\int_\Gamma(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt=\frac{T}{2}(||w_1||^2_{L^2(|\Omega)}+||w_0||^2_{H_\lambda})+\int_\Omega w_t\Big(x\cdot\nabla w+\frac{N-1}{2}w\Big)\Big|_{t=0}^{t=T}dx, \quad (1.17)$$

producing a "hidden regularity" effect for the normal derivative. We refer to Theorem 4.2 for a rigorous statement. As a consequence, the solution of system (1.14) verifies the reverse Observability inequality. Then identity (1.17) together with the sharp-Hardy inequality stated in Theorem 1.1 lead to *Observability inequality* (1.15) as emphasized in Theorem 4.3.

Theorem 1.1. *Assume Ω satisfies one of the cases C1-C2 . Then, there exists a constant $C=C(\Omega)\in\mathbb{R}$ such that*

$$\int_\Omega|x|^2|\nabla w|^2dx\leq R^2_\Omega\Big(\int_\Omega|\nabla w|^2dx-\frac{N^2}{4}\int_\Omega\frac{w^2}{|x|^2}dx\Big)+C\int_\Omega w^2dx,\quad\forall w\in C_0^\infty(\Omega). \quad (1.18)$$

Theorem 1.1 above, whose proof is given in Appendix, extends to the case of boundary singularity a similar inequality shown in [38] ,on page 2, as part of Theorem 1.1, in the context of interior singularity.

Remark 1.1. *The result of Theorem 1.1, more precisely the constant R^2_Ω which appears in inequality* (1.18)*, helps to obtain the control time $T>T_0=2R_\Omega$ in* (1.15)*, which is expected to be optimal due to the Geometric Control Condition (GCC), see e.g. [5].*

Although Theorem 1.1 is sharp for our applications to controllability, it is worth mentioning that we are able to obtain a more general result as follows.

Theorem 1.2. *Assume Ω satisfies one of the cases C1-C2. Let be $\varepsilon>0$ small enough. Then, there exists a constant $C_\varepsilon=C(\Omega,\varepsilon)\in\mathbb{R}$ such that*

$$\int_\Omega|x|^\varepsilon|\nabla w|^2dx\leq R^\varepsilon_\Omega\Big(\int_\Omega|\nabla w|^2dx-\frac{N^2}{4}\int_\Omega\frac{w^2}{|x|^2}dx\Big)+C_\varepsilon\int_\Omega w^2dx,\quad\forall w\in C_0^\infty(\Omega). \quad (1.19)$$

We have omitted including the proof of Theorem 1.2 since it applies the same steps as in the case of Theorem 1.1.

Finally in Section 4.2 we will consider the Schrödinger-like process

$$\begin{cases} iu_t - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in (0,T) \times \Omega, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{1.20}$$

where the singularity is located on the boundary, and we will briefly discuss the well-posedness and controllability properties of (1.20). In Section 5 we will deal with some open related problems.

The main results of this paper have been announced in a short presentation in [11].

2 Pohozaev identity for A_λ

In this section we rigorously justify the Pohozaev-type identity associated to A_λ. We discuss in detail the case C1. The details of the case C2 are let to the reader. In this latter case we only state the corresponding functional framework, see Subsection 2.2.

2.1 The case C1

In the following we introduce the functional framework which is used throughout the paper and we discuss some of its properties.

Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth domain which satisfies the case C1 and let $\lambda \leq \lambda(N)$, where $\lambda(N)$ was defined in (1.9). Thanks to inequality (1.7) the Hardy functional

$$B_\lambda[u] := \int_\Omega \Big(|\nabla u|^2 - \lambda \frac{u^2}{|x|^2} \Big) dx, \tag{2.1}$$

is positive and finite for all $u \in C_0^\infty(\Omega)$. For any $\lambda \leq \lambda(N)$, $B_\lambda[u]$ induces a Hilbert space H_λ, defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$||u||^2_{H_\lambda} = B_\lambda[u], \quad \forall u \in C_0^\infty(\Omega). \tag{2.2}$$

We point out that the space H_λ was firstly analyzed by Vazquez and Zuazua [40] in the case of interior singularity. As emphasized above, it may be extended to the case of boundary singularity. In the subcritical case $\lambda < \lambda(N)$, it holds that $H_0^1(\Omega) = H_\lambda$, according to the estimates

$$\Big(1 - \frac{\max\{0,\lambda\}}{\lambda(N)}\Big) ||u||_{H_0^1(\Omega)} \leq ||u||^2_{H_\lambda} \leq \Big(1 - \frac{\min\{0,\lambda\}}{\lambda(N)}\Big) ||u||^2_{H_0^1(\Omega)}, \quad \forall u \in C_0^\infty(\Omega),$$

which ensure the equivalence of the norms.

The critical space $H_{\lambda(N)}$ turns out to be slightly larger than $H_0^1(\Omega)$. We observe that $B_{\lambda(N)}[u]$ is finite for any $u \in H_0^1(\Omega)$, but it makes sense as an improper integral for more general distributions $u \in \mathcal{D}'(\Omega)$ i.e.

$$\exists \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(0)} \Big(|\nabla u|^2 - \frac{N^2}{4} \frac{u^2}{|x|^2} \Big) dx < \infty.$$

As it happens in the case of an interior singularity (see [39]), in general the meaning of $||u||_{H_{\lambda(N)}}$ does not coincide with the improper integral of $B_{\lambda(N)}[u]$. Following some ideas in [39], in the sequel we build a counterexample even in the case when the singularity is located on the boundary. We proceed by the absurd method so let us assume that

$$||u||^2_{H_{\lambda(N)}} = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B_\varepsilon(0)} \Big(|\nabla u|^2 - \frac{N^2}{4} \frac{u^2}{|x|^2} \Big) dx. \tag{2.3}$$

Indeed, we consider Ω to be the unit ball in $\mathbb{R}^N$ centered at $(0, \dots, 0, 1)$ that is

$$\Omega := \big\{ x \in \mathbb{R}^N_+ \ : \ |x'|^2 + (x_N - 1)^2 \leq 1 \big\},$$

where x', x_N were defined in (1.3). Moreover, we introduce the distribution

$$e_1 = x_N |x|^{-N/2} J_0(z_{0,1}|x|),$$

where $z_{0,1}$ is the first positive zero of the Bessel function J_0. We observe that $B_{\lambda(N)}[e_1]$ is finite as an improper integral. Indeed, for the above defined Ω and e_1 we have

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon(0)} \Big(|\nabla e_1|^2 - \frac{N^2}{4} \frac{e_1^2}{|x|^2} \Big) dx &= \int_{\Omega \setminus B_\varepsilon(0)} |\nabla (J_0(z_{0,1}|x|))|^2 e_1^2 dx \\ &+ \int_{S_\varepsilon^{N-1,+}} J_0^2(z_{0,1}|x|) x_N |x|^{-N/2} \nabla \big(x_N |x|^{-N/2} \big) \cdot \nu d\sigma, \end{aligned} \tag{2.4}$$

where $S_\varepsilon^{N-1,+} = \{ x \in \mathbb{R}^N \mid |x| = \varepsilon, x_N > 0 \}$ is the upper-half of the sphere $S_\varepsilon^{N-1} = \{ x \in \mathbb{R}^N \mid |x| = \varepsilon \}$. Switching to polar coordinates in (2.4) and using basic properties of J_0 (in particular, $|J_0'(x)| \sim |x|/2$ as $x \to 0$) we get that

$$\int_{\Omega \setminus B_\varepsilon(0)} \Big(|\nabla e_1|^2 - \frac{N^2}{4} \frac{e_1^2}{|x|^2} \Big) dx = O(1), \text{ as } \varepsilon \to 0. \tag{2.5}$$

On the other hand, let us fix $\phi \in C_0^\infty(\Omega)$ and consider the transformation

$$e_1 - \phi = x_N |x|^{-N/2} w.$$

Then we have $w = J_0(z_{0,1}|x|) - |x|^{N/2} x_N^{-1} \phi$ and in particular $w \in C_0^\infty(\Omega)$ with $w(0) = J_0(0) > 0$. As before we have that

$$\begin{aligned} \int_{\Omega \setminus B_\varepsilon(0)} \Big(|\nabla (e_1 - \phi)|^2 - \frac{N^2}{4} \frac{(e_1 - \phi)^2}{|x|^2} \Big) dx &= \int_{\Omega \setminus B_\varepsilon(0)} |\nabla w|^2 (e_1 - \phi)^2 dx \\ &+ \int_{S_\varepsilon^{N-1,+}} J_0^2(z_{0,1}|x|) x_N |x|^{-N/2} \nabla \big(x_N |x|^{-N/2} \big) \cdot \nu d\sigma \end{aligned} \tag{2.6}$$

Applying polar coordinates (see e.g. [32], page 293) we obtain

$$\int_{S_\varepsilon^{N-1,+}} J_0^2(z_{0,1}|x|)x_N|x|^{-N/2}\nabla\big(x_N|x|^{-N/2}\big)\cdot \nu d\sigma = \frac{N-2}{2}J_0^2(z_{0,1}\varepsilon)R, \quad (2.7)$$

where

$$R=\pi\Big(\int_0^\pi \cos^2\theta_1\sin^{N-2}\theta_1 d\theta_1\Big)\Big(\int_0^\pi \sin^{N-3}\theta_2 d\theta_2\Big)\dots\Big(\int_0^\pi \sin\theta_{N-2}d\theta_{N-2}\Big).$$

Therefore, due to (2.6)-(2.7) passing to the limit when $\varepsilon\to 0$ we obtain

$$||e_1-\phi||_{H_{\lambda(N)}}\geq \frac{N-2}{2}J_0^2(0)R>0, \quad \forall \phi\in C_0^\infty(\Omega),$$

provided $N>2$. This is in contradiction with the definition of $H_{\lambda(N)}$ which allows the existence of a sequence $\phi_n\in C_0^\infty(\Omega)$ converging to e_1 in $H_{\lambda(N)}$-norm ! Therefore, the assumption of considering the definition of the $H_{\lambda(N)}$-norm as an improper integral of $B_{\lambda(N)}$ is false (at least for $N>2$). In other words, there are distributions $u\in H_{\lambda(N)}$ for which

$$||u||^2_{H_{\lambda(N)}}\neq \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\Big[|\nabla u|^2-\lambda(N)\frac{u^2}{|x|^2}\Big]dx. \quad (2.8)$$

Next we propose an equivalent norm on H_λ, $\lambda\leq\lambda(N)$, which overcomes the anomalous behavior in (2.8) and perfectly describes the meaning of the H_λ-norm.

2.1.1 The meaning of the H_λ-norm

For reasonable considerations that will be specified in (2.10), we introduce the functional

$$B_{\lambda,1}[u]=\int_\Omega\Big|\nabla u+\frac{N}{2}\frac{x}{|x|^2}u-\frac{e_N}{x_N}u\Big|^2dx+(\lambda(N)-\lambda)\int_\Omega\frac{u^2}{|x|^2}dx. \quad (2.9)$$

which is positive and finite for any $u\in C_0^\infty(\Omega)$ and $\lambda\leq\lambda(N)$. Next, we observe that, for any $\lambda\leq\lambda(N)$,

$$B_\lambda[u]=B_{\lambda,1}[u], \quad \forall u\in C_0^\infty(\Omega). \quad (2.10)$$

Besides, notice that both $B_{\lambda,1}[u]$ and $B_\lambda[u]$ are norms in H_λ and they coincide on $C_0^\infty(\Omega)$. Due to definition (2.2) of H_λ, we conclude that the H_λ could be defined as the closure of $C_0^\infty(\Omega)$ in the norm induced by $B_{\lambda,1}[u]$. Therefore, the H_λ-norm is characterized by the identification

$$||u||^2_{H_\lambda}=\lim_{\varepsilon\to 0}B^\varepsilon_{\lambda,1}[u], \quad \forall u\in H_\lambda, \quad (2.11)$$

where $\lambda\leq\lambda(N)$ and

$$B^\varepsilon_{\lambda,1}[u]:=\int_{|x|\geq\varepsilon}\Big|\nabla u+\frac{N}{2}\frac{x}{|x|^2}u-\frac{e_N}{x_N}u\Big|^2dx+(\lambda(N)-\lambda)\int_{|x|\geq\varepsilon}\frac{u^2}{|x|^2}dx, \quad \forall u\in H_\lambda.$$

Next in the paper we will understand the meaning of the norm $||\cdot||_{H_\lambda}$ as in formula (2.11).

2.1.2 Main results

In what follows, $D(A_\lambda)$ stands for the domain of A_λ defined in (1.10). First of all, we note that standard elliptic estimates do not apply for A_λ if we want to obtain enough regularity for the normal derivative since the singularity $x = 0$ is located on the boundary. However, the following trace regularity result stated in Theorem 2.1 holds true.

Next, we claim the main results of Section 2.

Theorem 2.1 (Trace regularity)**.** *Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a bounded smooth domain satisfying the case C1. Let us consider $\lambda \leq \lambda(N)$ and $u \in D(A_\lambda)$. Then*

$$\Big(\frac{\partial u}{\partial \nu}\Big)^2 |x|^2 \in L^1(\Gamma), \tag{2.12}$$

and moreover, there exists a positive constant $C = C(\Omega) > 0$ such that

$$\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2 |x|^2 d\sigma \leq C(||u||^2_{H_\lambda} + ||A_\lambda u||^2_{L^2(\Omega)}), \quad \forall u \in D(A_\lambda). \tag{2.13}$$

Theorem 2.2 (Pohozaev identity)**.** *Assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain satisfying the case C1 and let $\lambda \leq \lambda(N)$. If $u \in D(A_\lambda)$ we claim that*

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = -\int_\Omega A_\lambda u (x\cdot\nabla u) dx - \frac{N-2}{2}||u||^2_{H_\lambda}. \tag{2.14}$$

The proofs of Theorems 2.1, 2.2 are quite technical, so we need to apply some preliminary lemmas which are stated below. The proofs of Lemmas 2.1, 2.3 are postponed at the end of Subsection 2.1 while Lemma 2.2 is a consequence of an abstract approximation lemma in [1].

Lemma 2.1. *Supppose $u \in D(A_\lambda)$ and denote $f := A_\lambda u \in L^2(\Omega)$. Let us also consider $\theta_\varepsilon \in C_0^\infty(\Omega)$, $\varepsilon > 0$, a family of cut-off functions such that*

$$\theta_\varepsilon(x) = \theta_\varepsilon(|x|) = \begin{cases} 0, & |x| \leq \varepsilon \\ 1, & |x| \geq 2\varepsilon. \end{cases} \tag{2.15}$$

Assume $\vec{q} := (q^1, \dots, q^n) \in (C^2(\overline{\Omega}))^N$ is a vector field such that $\vec{q} = \nu$ on Γ, where ν denotes the outward normal to the boundary Γ (such an election of $\vec{q}$ can always be done in smooth domains, see [30], Lemma 3.1, page 29). Then we have the identity

$$\begin{aligned}\frac{1}{2}\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2 |x|^2\theta_\varepsilon d\sigma = &-\int_\Omega f(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx + 2\int_\Omega (x\cdot\nabla u)(\vec{q}\cdot\nabla u)\theta_\varepsilon dx \\ &+ \sum_{i,j=1}^N \int_\Omega u_{x_i}u_{x_j}|x|^2 q^j_{x_i}\theta_\varepsilon dx - \int_\Omega |\nabla u|^2 (x\cdot\vec{q})\theta_\varepsilon dx \\ &- \frac{1}{2}\int_\Omega div\vec{q}|x|^2\Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)\theta_\varepsilon dx - \frac{1}{2}\int_\Omega |x|^2\vec{q}\cdot\nabla\theta_\varepsilon\Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)dx \\ &+ \int_\Omega |x|^2(\vec{q}\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx. \end{aligned} \tag{2.16}$$

Lemma 2.2. *Assume $f \in L^2(\Omega)$ and $\Omega \subset \mathbb{R}^N$ verifying the case C1. For any $\varepsilon > 0$ aimed to be small, we consider the following approximation problem*

$$\begin{cases} A_{\lambda(N)-\varepsilon} u_\varepsilon = f, & x \in \Omega \\ u_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{2.17}$$

Then it holds

$$u_\varepsilon \to u \quad \text{strongly in } H_{\lambda(N)}, \text{ as } \varepsilon \to 0.$$

where u verifies the limit problem

$$-\Delta u - \lambda(N)\frac{u}{|x|^2} = f, \text{ in } \mathcal{D}'(\Omega).$$

Moreover

$$\varepsilon \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \to 0. \text{ as } \varepsilon \to 0. \tag{2.18}$$

Lemma 2.3. *Assume Ω fulfills the case C1, let $\lambda \le \lambda(N)$ and fix $f \in C^\infty(\Omega)$. Moreover, we assume that u_λ solves the problem*

$$\begin{cases} A_\lambda u_\lambda = f, & x \in \Omega, \\ u_\lambda \in H_\lambda. \end{cases} \tag{2.19}$$

Then u_λ satisfies the following upper bounds: there exists $r_0 < R_\Omega$ small enough and there exist constants $C_1, C_2 > 0$, independent of λ, such that

$$|u_\lambda(x)| \le C_1 x_N |x|^{-N/2+\sqrt{\lambda(N)-\lambda}} \left| \log \frac{2R_\Omega}{|x|} \right|^{1/2}, \quad a.e.\ x \in \Omega_{r_0}, \tag{2.20}$$

$$|\nabla u_\lambda(x)| \le C_2 |x|^{-N/2+\sqrt{\lambda(N)-\lambda}} \left| \log \frac{2R_\Omega}{|x|} \right|^{1/2}, \quad a.e.\ x \in \Omega_{r_0}, \tag{2.21}$$

where $\Omega_{r_0} := \Omega \cap B_{r_0}(0)$.

Notation: In order to facilitate the computations, in the sequel, we will write " $\gtrsim$ " and " $\lesssim$ " instead of " $\ge C$" respectively " $\le C$" when we refer to universal constants C.

2.1.3 Proofs of Theorems 2.1, 2.2

Proof of Theorem 2.1. Following the proof of Theorem 1.1, as pointed out in Theorem 1.2 we are able to show that

$$\int_\Omega |x||\nabla u|^2 dx \lesssim ||u||^2_{H_\lambda}, \quad \forall u \in H_\lambda. \tag{2.22}$$

From the above estimate and the Cauchy-Schwartz inequality applied to identity (2.16) in Lemma 2.1 we obtain

$$\int_\Gamma \left(\frac{\partial u}{\partial \nu}\right)^2 |x|^2 \theta_\varepsilon d\sigma \lesssim ||u||^2_{H_\lambda} + ||f||^2_{L^2(\Omega)}, \quad \forall u \in D(A_\lambda), \quad \forall \varepsilon > 0. \tag{2.23}$$

Combining the Fatou Lemma with (2.23) we finish the proof of Theorem 2.1.

□

Proof of Theorem 2.2. We split the proof in two main steps.

Step 1. The subcritical case

We recall that $H_\lambda = H_0^1(\Omega)$. Let $u \in D(A_\lambda)$ and put $f := A_\lambda u \in L^2(\Omega)$. By standard elliptic estimates we note that $u \in H^2(\Omega \setminus B_\varepsilon(0))$, for any $\varepsilon > 0$ small enough. Moreover, the normal derivative $\partial u/\partial \nu$ belongs to $L^2_{loc}(\partial\Omega \setminus \{0\})$. We multiply $A_\lambda u$ by $x \cdot \nabla u \theta_\varepsilon$, where θ_ε was defined in (2.15). After integration we get

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2\theta_\varepsilon d\sigma = -\int_\Omega f(x\cdot\nabla u)\theta_\varepsilon dx - \frac{N-2}{2}\int_\Omega \Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)\theta_\varepsilon dx - \frac{1}{2}\int_\Omega \Big(|\nabla u|^2 - \lambda\frac{u^2}{|x|^2}\Big)x\cdot\nabla\theta_\varepsilon dx + \int_\Omega (x\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx. \tag{2.24}$$

Combining the Dominated Convergence Theorem (DCT) with Theorem 2.1 and condition (1.5), the left hand side of (2.24) converges i.e.

$$\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2\theta_\varepsilon d\sigma \to \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma, \qquad \text{as } \varepsilon \to 0.$$

In the right hand side of (2.24), we can directly pass to the limit term by term to obtain the identity (2.14) as follows. Firstly, since $x\cdot\nabla u \in L^2(\Omega)$ we have that

$$\begin{cases} |f(x\cdot\nabla u)\theta_\varepsilon| \leq |f||x\cdot\nabla u| \in L^1(\Omega), \\ \theta_\varepsilon \to 1, \text{ a.e.,} \quad \text{as } \varepsilon\to 0, \end{cases}$$

and by DCT we obtain

$$\int_\Omega f(x\cdot\nabla u)\theta_\varepsilon dx \to \int_\Omega f(x\cdot\nabla u)dx, \qquad \text{as } \varepsilon\to 0.$$

Besides, from Hardy inequality and DCT we have

$$\int_\Omega |\nabla u|^2\theta_\varepsilon dx \to \int_\Omega |\nabla u|^2 dx, \quad \int_\Omega \frac{u^2}{|x|^2}\theta_\varepsilon dx \to \int_\Omega \frac{u^2}{|x|^2}dx, \qquad \text{as } \varepsilon\to 0.$$

Using the fact that $|\nabla\theta_\varepsilon| = O(1/\varepsilon)$ it follows that

$$\Big|\int_\Omega |\nabla u|^2 x\cdot\nabla\theta_\varepsilon dx\Big| \lesssim \int_{B_{2\varepsilon}\setminus B_\varepsilon} |\nabla u|^2 dx \to 0,$$

$$\Big|\int_\Omega \frac{u^2}{|x|^2} x\cdot\nabla\theta_\varepsilon dx\Big| \lesssim \int_{B_{2\varepsilon}\setminus B_\varepsilon} \frac{u^2}{|x|^2} dx \to 0,$$

$$\Big|\int_\Omega (x\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx\Big| \lesssim \int_{B_{2\varepsilon}\setminus B_\varepsilon} |\nabla u|^2 dx \to 0,$$

as $\varepsilon \to 0$. With this we conclude the solvability of Theorem 2.2 in the subcritical case $\lambda < \lambda(N)$.

Step 2. The critical case $\lambda = \lambda(N)$

As before, let us consider $u \in D(A_{\lambda(N)})$ and define $f := A_{\lambda(N)}u \in L^2(\Omega)$. Our purpose is to show the validity of Theorem 2.2 for such u.

We proceed by approximations with subcritical values. More precisely, for $\varepsilon > 0$ small enough, we consider the problem

$$\begin{cases} A_{\lambda(N)-\varepsilon}u_\varepsilon = f, & x \in \Omega, \\ u_\varepsilon \in H_0^1(\Omega). \end{cases} \tag{2.25}$$

Applying Lemma 2.2 we obtain

$$u_\varepsilon \to u \text{ in } H_{\lambda(N)}, \quad \varepsilon \int_\Omega \frac{u_\varepsilon^2}{|x|^2} dx \to 0, \quad \text{ as } \varepsilon \to 0, \tag{2.26}$$

where u solves the limit problem. According to the Pohozaev identity applied to u_ε we get

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial \nu}\Big)^2 d\sigma = -\int_\Omega f(x\cdot\nabla u_\varepsilon)dx - \frac{N-2}{2}\Big(||u_\varepsilon||^2_{H_{\lambda(N)}} + \varepsilon\int_\Omega \frac{u_\varepsilon^2}{|x|^2}dx\Big). \tag{2.27}$$

Due to Theorem 1.1, the fact that $u_\varepsilon \to u$ in $H_{\lambda(N)}$ implies

$$x\cdot\nabla u_\varepsilon \to x\cdot\nabla u \ \text{ in } L^2(\Omega), \quad \text{as } \varepsilon \to 0.$$

Therefore, the right hand side in (2.27) converges to

$$H(u) := -\int_\Omega f(x\cdot\nabla u)dx - \frac{N-2}{2}||u||^2_{H_{\lambda(N)}},$$

and therefore, there exists

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial \nu}\Big)^2 d\sigma = H(u).$$

On the other hand, by standard elliptic regularity one can show that

$$\frac{\partial u_\varepsilon}{\partial\nu} \to \frac{\partial u}{\partial\nu} \text{ in } L^2_{\text{loc}}(\Gamma\setminus\{0\}) \text{ and } \frac{\partial u_\varepsilon}{\partial\nu} \to \frac{\partial u}{\partial\nu} \text{ a.e. on } \Gamma. \tag{2.28}$$

In the sequel, we discuss two different situations for the geometry of Ω.

Case 1. Assume Ω is flat in a neighborhood of zero (i.e. $x\cdot\nu = 0$). Then, as a consequence of DCT and (2.28) we note that

$$\lim_{\varepsilon\to 0}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2 d\sigma = \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\sigma.$$

In consequence, u satisfies the Pohozaev identity, by passing to the limit in (2.27).

Case 2. We assume Ω is not necessarily flat at origin. We distinguish two cases when discussing the smoothness of f.

The case $f \in C^\infty(\Omega)$.

Next we apply Lemma 2.3 for u_ε the solution of problem (2.25). and we obtain

$$\Big|(x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2\Big| \leq \Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2|x|^2 \leq g, \text{ a.e. on } \Gamma,$$

where $g = |x|^{2-N}\Big|\log\frac{1}{|x|}\Big| \in L^1(\Gamma)$. Applying DCT we conclude

$$\lim_{\varepsilon\to 0}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_\varepsilon}{\partial\nu}\Big)^2 d\sigma = \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\sigma.$$

The case $f \in L^2(\Omega)$.

We consider $\{f_k\}_{k\geq 1} \in C^\infty(\Omega)$ such that $f_k \to f$ in $L^2(\Omega)$, as $k \to \infty$.

Let us call u_k the solution of $A_{\lambda(N)}u_k = f_k$, for all $k \geq 1$. From the previous case, u_k satisfies

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_k}{\partial\nu}\Big)^2 d\sigma = -\int_\Omega f_k(x\cdot\nabla u_k)dx - \frac{N-2}{2}||u_k||^2_{H_{\lambda(N)}}. \qquad (2.29)$$

We know that f_k is a Cauchy sequence in $L^2(\Omega)$, and due to

$$||u_k - u_l||_{H_{\lambda(N)}} \lesssim ||f_k - f_l||_{L^2(\Omega)} \to 0, \text{ as } k, l \to \infty,$$

we deduce that $\{u_k\}_{k\geq 1}$ is Cauchy in $H_{\lambda(N)}$. Hence $u_k \to u$ in $H_{\lambda(N)}$ and

$$x\cdot\nabla u_k \to x\cdot\nabla u \text{ in } L^2(\Omega).$$

As a consequence we can pass to the limit in the right hand side of (2.29). In order to finish the proof, we need to pass to the limit in the left hand side. Indeed, in view of Theorem 2.1 we have

$$\int_\Gamma \Big(\frac{\partial(u_k - u_l)}{\partial\nu}\Big)^2|x|^2 d\sigma \lesssim ||u_k - u_l||^2_{H_\lambda} + ||f_k - f_l||^2_{L^2(\Omega)}.$$

Therefore $g_k := \frac{\partial u_k}{\partial\nu}|x|$ is a Cauchy sequence in $L^2(\Gamma)$ and g_k converges to, say, $g := \frac{\partial u}{\partial\nu}|x|$ in $L^2(\Gamma)$, as k goes to infinity. This suffices to say that

$$\lim_{k\to\infty}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u_k}{\partial\nu}\Big)^2 d\sigma = \int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial\nu}\Big)^2 d\sigma.$$

Therefore we conclude the proof of Theorem 2.2. □

2.1.4 Proofs of useful lemmas

Proof of Lemma 2.1. By standard elliptic estimates, we remark that $u \in H^2_{\mathrm{loc}}(\Omega\setminus\{0\})$. Thanks to that, when multiplying by $|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon$ we are allowed to integrate by parts on Ω. Firstly, we obtain

$$\int_\Omega \Delta u(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = \int_\Gamma \frac{\partial u}{\partial\nu}(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)d\sigma - \int_\Omega \nabla u\cdot\nabla(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx.$$

Let us now compute the boundary term above. Since u vanishes on Γ it follows that

$$\nabla u = \frac{\partial u}{\partial\nu}\nu, \quad \text{on } \Gamma, \tag{2.30}$$

and moreover, $\vec{q} = \nu$ on Γ. Thanks to these we obtain

$$\int_\Gamma \frac{\partial u}{\partial\nu}(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)d\sigma = \int_\Gamma \left(\frac{\partial u}{\partial\nu}\right)^2|x|^2\theta_\varepsilon d\sigma.$$

Therefore,

$$\begin{aligned}\int_\Omega \Delta u(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = &\int_\Gamma \left(\frac{\partial u}{\partial\nu}\right)^2|x|^2\theta_\varepsilon d\sigma - \int_\Omega \nabla u\cdot\nabla(|x|^2\vec{q}\cdot\nabla u)\theta_\varepsilon dx\\ &- \int_\Omega |x|^2(\vec{q}\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx.\end{aligned}$$

Let us compute the second term in the integration above. Doing various iterations we obtain

$$\begin{aligned}\int_\Omega \nabla u\cdot\nabla(|x|^2\vec{q}\cdot\nabla u)\theta_\varepsilon dx = 2\int_\Omega (x\cdot\nabla u)(\vec{q}\cdot\nabla u)\theta_\varepsilon dx + \sum_{i,j=1}^N \int_\Omega u_{x_i}u_{x_j}|x|^2 q^j_{x_i}\theta_\varepsilon dx\\ + \frac{1}{2}\sum_{i,j=1}^N \int_\Omega |x|^2 q^j(u^2_{x_i})_{x_j}\theta_\varepsilon dx\end{aligned} \tag{2.31}$$

For the last term in the integration above we get

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^N \int_\Omega |x|^2 q^j(u^2_{x_i})_{x_j}\theta_\varepsilon dx = &\frac{1}{2}\int_\Gamma \left(\frac{\partial u}{\partial\nu}\right)^2|x|^2\theta_\varepsilon d\sigma - \int_\Omega |\nabla u|^2(x\cdot\vec{q})\theta_\varepsilon dx\\ &- \frac{1}{2}\int_\Omega \mathrm{div}\vec{q}|x|^2|\nabla u|^2\theta_\varepsilon dx - \frac{1}{2}\int_\Omega |x|^2|\nabla u|^2\vec{q}\cdot\nabla\theta_\varepsilon dx.\end{aligned} \tag{2.32}$$

According to (2.31) and (2.32) we obtain

$$\int_\Omega \Delta u(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = \frac{1}{2}\int_\Gamma \Big(\frac{\partial u}{\partial \nu}\Big)^2|x|^2\theta_\varepsilon d\sigma - 2\int_\Omega (x\cdot\nabla u)(\vec{q}\cdot\nabla u)\theta_\varepsilon dx$$
$$- \sum_{i,j=1}^N \int_\Omega u_{x_i}u_{x_j}|x|q^j_{x_i}\theta_\varepsilon dx + \int_\Omega |\nabla u|^2(x\cdot\vec{q})\theta_\varepsilon dx$$
$$+ \frac{1}{2}\int_\Omega \mathrm{div}\vec{q}|x|^2|\nabla u|^2\theta_\varepsilon dx + \frac{1}{2}\int_\Omega |x|^2|\nabla u|^2\vec{q}\cdot\nabla\theta_\varepsilon dx$$
$$- \int_\Omega |x|^2(\vec{q}\cdot\nabla u)(\nabla u\cdot\nabla\theta_\varepsilon)dx. \tag{2.33}$$

On the other hand, it follows that

$$\int_\Omega \frac{u}{|x|^2}(|x|^2\vec{q}\cdot\nabla u\theta_\varepsilon)dx = -\frac{1}{2}\int_\Omega \mathrm{div}\vec{q}u^2\theta_\varepsilon dx - \frac{1}{2}\int_\Omega \vec{q}\cdot\nabla\theta_\varepsilon u^2 dx. \tag{2.34}$$

From (2.33) and (2.34) we finally obtain the identity of Lemma 2.1. □

Proof of Lemma 2.3. For any $\lambda \le \lambda(N)$ we fix $\phi_\lambda = x_N|x|^{-N/2+\sqrt{\lambda(N)-\lambda}}\Big|\log\frac{2R_\Omega}{|x|}\Big|^{1/2}$. Let us also consider the problem

$$\begin{cases} A_\lambda U_\lambda = |f|, & x\in\Omega,\\ U_\lambda \in H_\lambda. \end{cases} \tag{2.35}$$

The proof comprises several steps.

Step 1. Firstly let us check the validity of the Maximum Principle:

$$|u_\lambda(x)| \le U_\lambda(x) \quad \text{a.e. in } \Omega. \tag{2.36}$$

Indeed, from the equations satisfied by U_λ, u_λ we obtain

$$-\Delta(U_\lambda \pm u_\lambda) - \lambda\frac{(U_\lambda\pm u_\lambda)}{|x|^2} = |f| \pm f \ge 0, \quad \forall x\in\Omega. \tag{2.37}$$

Multiplying (2.37) by the negative part $(U_\lambda \pm u_\lambda)^-$ we get the reverse Hardy inequality

$$\int_\Omega \Big[|\nabla(U_\lambda\pm u_\lambda)^-|^2 - \lambda\frac{[(U_\lambda\pm u_\lambda)^-]^2}{|x|^2}\Big]dx \le 0. \tag{2.38}$$

From the non-attainability of the Hardy constant we necessary must have $(U_\lambda \pm u_\lambda)^- \equiv 0$ in Ω. Therefore, $U_\lambda \pm u_\lambda \ge 0$ in Ω, a fact which concludes (2.36).

Step 2. Next, we remark that there exists a positive constant $C_1 > 0$, independent of λ such that

$$-\Delta\phi_\lambda - \lambda\frac{\phi_\lambda}{|x|^2} \ge C_1, \quad \forall x\in\Omega.$$

Therefore, for some $C \geq ||f||_{L^\infty}/C_1$ we get

$$\begin{cases} -\Delta(C\phi_\lambda - U_\lambda) - \lambda\frac{(C\phi_\lambda - U_\lambda)}{|x|^2} \geq 0, & \forall x \in \Omega, \\ C\phi_\lambda - U_\lambda \geq 0, & x \in \Gamma. \end{cases} \tag{2.39}$$

Therefore, applying the Maximum Principle we obtain

$$U_\lambda \leq C\phi_\lambda, \quad \forall x \in \Omega, \quad \lambda \leq \lambda(N), \tag{2.40}$$

and the proof (2.20) is finished.

Step 3. For the estimate (2.21) we use a remark by Brezis, Marcus and Shafrir [8] as follows.

Let us first assume that Ω_{2r_0} is flat at the origin for r_0 small enough.

Fix $x \in \Omega_{r_0}$ and put $r = x_N/2$. We define then $\tilde{u}_\lambda(y) = u_\lambda(x + ry)$ where $y \in B_1(0)$. By direct computations we obtain

$$\begin{aligned} \Delta\tilde{u}_\lambda(y) = r^2\Delta u_\lambda(x+ry) = r^2\Big(- f - \lambda\frac{u_\lambda(x+ry)}{|x+ry|^2}\Big) \\ = -r^2 f - \lambda\frac{x_N^2}{4|x+ry|^2}\tilde{u}_\lambda(y). \end{aligned} \tag{2.41}$$

On the other hand, we remark that

$$\frac{4}{9} \leq \frac{|x|^2}{|x+ry|^2} \leq 4, \quad \forall y \in B_1(0).$$

By elliptic estimates it is easy to see that $\tilde{u}_\lambda \in C^1(B_1(0))$. Applying the interpolation inequality (see Evans [17]), we get that

$$\begin{aligned} |\nabla\tilde{u}_\lambda(0)| &\lesssim ||\tilde{u}_\lambda||_{L^\infty(B_1(0))} + ||\Delta\tilde{u}_\lambda||_{L^\infty(B_1(0))} \\ &\lesssim ||\tilde{u}_\lambda||_{L^\infty(B_1(0))} + ||f||_{L^\infty(\Omega)} \end{aligned} \tag{2.42}$$

Writing $\nabla\tilde{u}_\lambda$ in terms of ∇u_λ we obtain

$$|\nabla u_\lambda(x)| \lesssim \frac{1}{x_N}(||\tilde{u}_\lambda||_{L^\infty(B_1(0))} + ||f||_{L^\infty(\Omega)}) \tag{2.43}$$

In addition, from (2.36) and (2.40) we have

$$\begin{aligned} ||\tilde{u}_\lambda||_{L^\infty(B_1(0))} &= ||u_\lambda(x+ry)||_{L^\infty(B_1(0))} \\ &\lesssim \sup_{y\in B_1(0)} \Big\{(x_N + ry_N)|x+ry|^{-N/2+\sqrt{\lambda(N)-\lambda}}\Big|\log\frac{2R_\Omega}{|x+ry|}\Big|^{1/2}\Big\} \\ &\lesssim x_N|x|^{-N/2+\sqrt{\lambda(N)-\lambda}}\Big|\log\frac{2R_\Omega}{|x|}\Big|^{1/2}, \end{aligned} \tag{2.44}$$

which is verified for all $x \in \Omega_{r_0}$, $y \in B_1(0)$. From (2.43) and (2.44) we obtain the estimate (2.21).

If Ω is not flat at the origin we can consider a local parametrization of its boundary Γ given by $x_N = h(x')$, where $h(x') = \sum_{i=1}^{N-1} \alpha_i x_i^2 + o(|x'|^2)$ as $|x'| \to 0$. The numbers $\alpha_1, \dots, \alpha_{N-1} \geq 0$ (not all trivial) are the principal curvatures of Γ at the origin (or the eigenvalues of the 2nd fundamental form of Γ). For simplicity we consider Γ given by $x_N = \gamma|x'|^2$, $\gamma > 0$, close to the origin (see e.g. [24]). Then, the proof of Step 3 applies for $r = 1/2(x_N - \gamma|x'|^2)$ and $\phi_\lambda = (x_N - \gamma|x|^2)|x|^{-N/2+\sqrt{\lambda(N)-\lambda}}\left|\log\frac{2R_\Omega}{|x|}\right|^{1/2}$ in (2.40).

These yield the proof of Lemma 2.3. □

2.2 Brief presentation of the case C2

Inequalities (1.7), (1.8) can be stated in a simplified form as follows.

Assume $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain containing the origin on the boundary. For any $\lambda \leq \lambda(N)$ and any $0 < \gamma < 2$ there exists a constant $C_1(\gamma, \Omega) \geq 0$ such that

$$\forall u \in H_0^1(\Omega), \qquad \int_\Omega \frac{u^2}{|x|^\gamma}dx + \lambda\int_\Omega \frac{u^2}{|x|^2}dx \leq \int_\Omega |\nabla u|^2 + C_1(\gamma,\Omega)\int_\Omega u^2 dx. \tag{2.45}$$

2.2.1 Functional framework via Hardy inequality

Let us now define the set

$$\mathcal{C}_\gamma := \left\{ C \geq 0 \ \text{ s. t. } \ \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega \big(|\nabla u|^2 - \lambda(N)u^2/|x|^2 + Cu^2\big)dx}{\int_\Omega u^2/|x|^\gamma dx} \geq 1 \right\}. \tag{2.46}$$

Of course, $\mathcal{C}_\gamma$ is non-empty due to inequality (2.45). Next we define

$$\mathcal{C}_0^\gamma = \inf_{C \in \mathcal{C}_\gamma} C. \tag{2.47}$$

Then, for any $\lambda \leq \lambda(N)$ we introduce the Hardy functional

$$B_\lambda[u] := \int_\Omega |\nabla u|^2 dx - \lambda\int_\Omega \frac{u^2}{|x|^2}dx + \mathcal{C}_0^\gamma \int_\Omega u^2 dx, \tag{2.48}$$

which is positive for any $u \in H_0^1(\Omega)$ due to inequality (2.45) and the election of $\mathcal{C}_0^\gamma$. Then we define the corresponding Hilbert space H_λ as the closure of $C_0^\infty(\Omega)$ in the norm induced by $B_\lambda[u]$. Observe that for any $\lambda < \lambda(N)$ the identification $H_\lambda = H_0^1(\Omega)$ holds true. Indeed, if $\lambda < \lambda(N)$, we have

$$B_\lambda[u] \geq \Big(1 - \frac{\max\{0,\lambda\}}{\lambda(N)}\Big)\int_\Omega |\nabla u|^2 dx - \frac{\mathcal{C}_0^\gamma \max\{0,\lambda\}}{\lambda(N)}\int_\Omega u^2 dx. \tag{2.49}$$

On the other hand, from the definition of $\mathcal{C}_0^\gamma$ we obtain that there exists a constant $C_2 = C_2(\gamma) > 0$ such that

$$B_\lambda[u] \geq C_2 \int_\Omega u^2 dx. \tag{2.50}$$

Multiplying (2.50) by $\mathcal{C}_0^\gamma \max\{0,\lambda\}/(C_2\lambda(N))$ and summing to (2.49) we get that

$$B_\lambda[u] \geq C_\lambda \int_\Omega |\nabla u|^2 dx,$$

for some positive constant C_λ that converges to zero as λ tends to $\lambda(N)$.

Besides, in the critical case $\lambda = \lambda(N)$, H_λ is slightly larger than $H_0^1(\Omega)$. However, using cut-off arguments near the singularity (see e.g. [40]) we can show that

$$B_\lambda[u]_{\lambda(N)} \geq C_\varepsilon ||u||_{H^1(\Omega \setminus B_\varepsilon(0))}, \quad \forall u \in H_0^1(\Omega) \tag{2.51}$$

where C_ε is a constant going to zero as ε tends to zero.

Let us define the operator $A_\lambda := -\Delta - \lambda/|x|^2 + \mathcal{C}_0^\gamma I$ and define its domain as

$$D(A_\lambda) := \{u \in H_\lambda \mid A_\lambda u \in L^2(\Omega)\}. \tag{2.52}$$

The norm of the operator A_λ is given by

$$||u||_{D(A_\lambda)} = ||u||_{L^2(\Omega)} + ||A_\lambda u||_{L^2(\Omega)}. \tag{2.53}$$

2.2.2 The meaning of the H_λ-norm

First of all we remark the validity of the identity

$$\int_\Omega |\nabla u|^2 dx + \int_\Omega \frac{\Delta\Phi}{\Phi} u^2 dx = \int_\Omega \Big|\nabla u - \frac{\nabla\Phi}{\Phi} u\Big|^2 dx, \quad \forall u \in C_0^\infty(\Omega \setminus \{0\}), \tag{2.54}$$

which holds for any $\Phi \in C^2(\Omega \setminus \{0\})$ and $\Phi > 0$ in $\Omega \setminus \{0\}$. The proof of (2.54) applies direct integrations by parts.

Let us also consider $\phi(x) = \phi(|x|) \in C^\infty(\Omega)$ to be a cut-off function such that

$$\phi = \begin{cases} 1, & |x| \leq r_0/2,\ x \in \Omega \\ 0, & |x| \geq r_0,\ x \in \Omega, \end{cases} \tag{2.55}$$

where $r_0 > 0$ is meant to be small.

Case 1.Assume the points on the boundary Γ of Ω satisfy $x_N > 0$ in a neighborhood of the origin.

Next we consider $\Phi_1 = x_N|x|^{-N/2}$ which satisfies the equation

$$-\Delta\Phi_1 - \frac{N^2}{4}\frac{\Phi_1}{|x|^2} = 0, \quad \text{a. e. in } \Omega_{r_0}, \tag{2.56}$$

where $\Omega_{r_0} = \Omega \cap B_{r_0}(0)$ for some $r_0 > 0$ small enough. Applying (2.54) for $\phi = \phi_1$ from (2.56) we obtain

$$\int_{\Omega_{r_0}} |\nabla u|^2 dx - \frac{N^2}{4}\int_{\Omega_{r_0}} \frac{u^2}{|x|^2} dx = \int_{\Omega_{r_0}} \big|\nabla u - \frac{\nabla\Phi_1}{\Phi_1} u\big|^2 dx, \quad \forall u \in C_0^\infty(\Omega_{r_0}). \tag{2.57}$$

By a standard cut-off argument, due to (2.57) we remark that, there exist some weights $\rho_1, \rho_2 \in C^\infty(\Omega)$ depending on r_0, supported far from origin such that

$$B_\lambda[u] = \int_\Omega \Big|\nabla(u\phi) - \frac{\nabla\Phi_1}{\Phi_1}(u\phi)\Big|^2 dx + \int_\Omega \rho_1 |\nabla u|^2 dx$$
$$+ (\lambda(N) - \lambda)\int_\Omega \frac{u^2}{|x|^2} dx + \int_\Omega \rho_2 u^2 dx, \quad \forall u \in C_0^\infty(\Omega). \tag{2.58}$$

Then the meaning of $||\cdot||_{H_\lambda}$-norm is characterized by

$$||u||^2_{H_\lambda} = \lim_{\varepsilon\to 0} \int_{x\in\Omega, |x|>\varepsilon} \Big|\nabla(u\phi) - \frac{\nabla\Phi_1}{\Phi_1}(u\phi)\Big|^2 dx + \int_\Omega \rho_1 |\nabla u|^2 dx$$
$$+ (\lambda(N) - \lambda)\int_\Omega \frac{u^2}{|x|^2} dx + \int_\Omega \rho_2 u^2 dx, \quad \forall u \in H_\lambda, \quad \forall \lambda \le \lambda(N). \tag{2.59}$$

Case 2. Assume the points on Γ satisfy $x_N \le 0$ in a neighborhood of the origin

In this case we consider $d = d(x, \Gamma) = d(x)$ the function denoting the distance from a point $x \in \Omega$ to the boundary Γ. We remark that close enough to the origin the distribution

$$\Phi_2 = d(x) e^{(1-N)d(x)} |x|^{-N/2} \Big|\log \frac{1}{|x|}\Big|^{1/2}$$

satisfies

$$P := -\Delta\Phi_2 - \frac{N^2}{4|x|^2}\Phi_2 > 0, \quad \forall x \in \Omega_{r_0},$$

where $r_0 > 0$ is small enough. Due to this, there exist the weights $\rho_1, \rho_2 \in C^\infty(\Omega)$ depending on r_0 and supported away from origin, such that the meaning of the H_λ-norm is given by

$$||u||^2_{H_\lambda} = \lim_{\varepsilon\to 0} \int_{x\in\Omega, |x|>\varepsilon} \Big|\nabla(u\phi) - \frac{\nabla\Phi_2}{\Phi_2}(u\phi)\Big|^2 dx + (\lambda(N) - \lambda)\int_\Omega \frac{u^2}{|x|^2} dx +$$
$$+ \int_\Omega \frac{P}{\Phi_2}|u\phi|^2 dx + \int_\Omega \rho_1 |\nabla u|^2 dx + \int_\Omega \rho_2 u^2 dx, \quad \forall u \in H_\lambda, \quad \forall \lambda \le \lambda(N). \tag{2.60}$$

Case 3. Assume that x_N changes sign on Γ at the origin.

This case can be analyzed through Case 2 above.

Then, the Pohozaev identity and related results presented in case C1 might be extended to case C2 by means of the weaker functional settings introduced above.

3 Applications to semi-linear equations

Pohozaev-type identities mostly apply to show non-existence results for non-linear elliptic problems. In particular, for applications to the semi-linear Laplace equation we refer mainly to [17], page 514.

In what follows we prove a non-existence result for a non-linear elliptic equation associated to A_λ, in the case of boundary singularity. In particular, the case $\lambda = 0$ in which no singularity occurs, corresponds to the standard case analyzed in [17]. To fix the ideas, let us assume $\lambda \leq \lambda(N)$ and consider $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a domain satisfying the case C1. In the sequel we use the notation

$$\alpha_\star := \frac{N+2}{N-2}$$

which stands for the critical Sobolev exponent.

Next we claim the main result.

Theorem 3.1. *Let us consider the problem*

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2} u = |u|^{\alpha-1} u, & x \in \Omega, \\ u = 0, & x \in \Gamma. \end{cases} \tag{3.1}$$

1. *Assume $\lambda \leq \lambda(N)$. If $1 < \alpha < \alpha_\star$ the problem* (3.1) *has non-trivial solutions in H_λ. Moreover, if $1 < \alpha < \frac{N}{N-2}$ the problem* (3.1) *has non-trivial solutions in $D(A_\lambda)$.*

2. *(Non-existence.) Assume $\lambda \leq \lambda(N)$ and let Ω be a smooth star-shaped domain (i.e. $x \cdot \nu \geq 0$, for all $x \in \Gamma$). If $\alpha \geq \alpha_\star$ the problem* (3.1) *does not have non-trivial solutions in $D(A_\lambda)$.*

Proof of Theorem 3.1

Proof of item 1. The existence of non-trivial solutions for (3.1) reduces to studying the minimization problem

$$I = \inf_{u \in H_\lambda, u \neq 0} \frac{||u||^2_{H_\lambda}}{||u||^{\alpha+1}_{L^{\alpha+1}(\Omega)}}.$$

Without losing generality, we may consider the normalization

$$I = \inf_{||u||_{L^{\alpha+1}(\Omega)}=1} J(u), \tag{3.2}$$

where $J : H_\lambda \to \mathbb{R}$ is defined by $J(u) = ||u||^2_{H_\lambda}$. Next we address the question of attainability of I in (3.2).

We note that J is continuous, convex, coercive in H_λ. Let $\{u_n\}_n$ be a minimizing sequence of I, i.e.,

$$J(u_n) \searrow I, \quad ||u_n||_{L^{\alpha+1}(\Omega)} = 1.$$

By the coercivity of J we have

$$||u_n||_{H_\lambda} \leq C, \quad \forall n,$$

for some universal constant $C > 0$. Moreover, the embedding $H_\lambda \hookrightarrow L^{\alpha+1}(\Omega)$ is compact for any $\alpha < \alpha_\star$ (it can be deduced combining Theorem 1.2 and Sobolev inequality). Therefore,

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } H_\lambda, \\ u_n \to u & \text{strongly in } L^{\alpha+1}(\Omega). \end{cases} \tag{3.3}$$

According to (3.3) we get $||u||_{L^{\alpha+1}(\Omega)} = 1$. From the i.s.c. of the norm we have

$$I \le J(u) \le \liminf_{n\to\infty} J(u_n) = I,$$

and therefore $I = J(u)$ is attained by u, which, up to a constant, is a non-trivial solution of (3.1) in H_λ.

If $\alpha < N/(N-2)$ let us show that $u \in D(A_\lambda)$. Indeed, due to the compact embedding $H_\lambda \hookrightarrow L^q(\Omega)$, $q < 2N/(N-2)$, we have that $|u|^{\alpha-1}u \in L^2(\Omega)$. In consequence, $u \in D(A_\lambda)$. □

Proof of item 2. For the proof of non-existence we apply the Pohozaev identity in Theorem 2.2. In view of that we use the following lemma whose proof is postponed until the end of the section.

Lemma 3.1. *Assume $\lambda \le \lambda(N)$ and $1 < \alpha < \infty$. Then, any solution $u \in D(A_\lambda)$ of (3.1) satisfies the identity*

$$\frac{1}{2}\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = \Big(\frac{N}{1+\alpha} - \frac{N-2}{2}\Big)\int_\Omega |u|^{\alpha+1}dx. \tag{3.4}$$

The case $\alpha > \alpha_\star$.

Note that $x\cdot\nu \ge 0$ for all $x \in \Gamma$. Assuming $u \not\equiv 0$, from Lemma 3.1 we obtain $(N-2)/2 \le N/(\alpha+1)$ which is equivalent to $\alpha \le \alpha_\star$. This is in contradiction with the hypothesis on α. Therefore $u \equiv 0$ in Ω.

The case $\alpha = \alpha_\star$.

From Lemma 3.1, due to the criticality of α, u must satisfy

$$\int_\Gamma (x\cdot\nu)\Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = 0.$$

Let us consider $\Omega = \{x \in \mathbb{R}^N_+ \mid |x'|^2 + (x_N - 1)^2 \le 1\}$ which is star-shaped. Therefore,

$$\frac{\partial u}{\partial \nu} = 0, \quad \text{a.e. on } \Gamma.$$

Thus, the problem under consideration is reduced to the overdetermined system

$$\begin{cases} -\Delta u - \frac{\lambda}{|x|^2}u = |u|^{\frac{4}{N-2}}u, & x \in \Omega, \\ u = 0, & x \in \Gamma, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \Gamma. \end{cases} \tag{3.5}$$

Let us consider a compact subset $\Gamma' \subset \Gamma$ such that $x \cdot \nu > 0$ and $0 \notin \Gamma'$. Next, we extend Ω with a bounded set Ω_1 such that $\Omega_1 \cap \Omega = \emptyset$, $\partial\Omega_1 \cap \partial\Omega = \Gamma'$, $\tilde{\Omega} := \Omega \cup \Omega_1 \cup \Gamma'$.
For $\varepsilon > 0$ small enough we denote the sets $\Omega_\varepsilon := \Omega \setminus \{x \in \Omega \mid |x| < \varepsilon\}$, $\tilde{\Omega}_\varepsilon := \tilde{\Omega} \setminus \{x \in \Omega \mid |x| < \varepsilon\}$.

Next we consider the trivial prolongation of u to $\tilde{\Omega}$

$$\tilde{u} := \begin{cases} u, & x \in \Omega, \\ 0, & x \in \Omega_1. \end{cases} \tag{3.6}$$

The fact that $u \in D(A_\lambda)$ combined with the over-determined condition in (3.5), implies that $u \in H^2(\Omega_\varepsilon)$. Let us also show that $\tilde{u} \in H^2(\tilde{\Omega}_\varepsilon)$.

Indeed, thanks to (3.5) on Γ we get that

$$\int_{\tilde{\Omega}_\varepsilon} \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \phi}{\partial x_j} dx = -\int_{\tilde{\Omega}_\varepsilon} g\phi dx, \quad \forall \phi \in C_0^\infty(\tilde{\Omega}_\varepsilon), \tag{3.7}$$

where $g \in L^2(\tilde{\Omega}_\varepsilon)$ is given by

$$g = \begin{cases} \frac{\partial^2 u}{\partial x_i \partial x_j}, & x \in \Omega_\varepsilon, \\ 0, & x \in \Omega_1. \end{cases} \tag{3.8}$$

In particular we obtain that

$$\Delta \tilde{u} = \begin{cases} \Delta u, & x \in \Omega_\varepsilon, \\ 0 & x \in \Omega_1. \end{cases} \tag{3.9}$$

and $\tilde{u}$ verifies

$$-\Delta \tilde{u} - \frac{\lambda}{|x|^2} \tilde{u} = |\tilde{u}|^{\frac{4}{N-2}} \tilde{u} \text{ a.e. in } \tilde{\Omega}_\varepsilon \tag{3.10}$$

and $\tilde{u} \equiv 0$ in Ω_1. In other words we can write (3.10) as

$$-\Delta \tilde{u} = V(x)\tilde{u}, \quad x \in \tilde{\Omega}_\varepsilon,$$

where $V(x) := \frac{\lambda}{|x|^2} + |\tilde{u}|^{\frac{4}{N-2}}$. Note that $V \in L^\omega(\tilde{\Omega}_\varepsilon)$ for some $\omega > N/2$ and $\tilde{u}$ vanishes in Ω_1.

By this, we are in the hypothesis of the strong unique continuation result by Jerison and Kenig [26]. Therefore, $\tilde{u} \equiv 0$ in $\tilde{\Omega}_\varepsilon$ and in particular $u \equiv 0$ in Ω_ε, for any $\varepsilon > 0$. Hence, we conclude that $u \equiv 0$ in Ω. The proof of Theorem 3.1 is finished. □

Proof of Lemma 3.1. Since $u \in D(A_\lambda)$ we can apply the Pohozaev identity and we get

$$\frac{1}{2} \int_\Gamma (x \cdot \nu) \Big(\frac{\partial u}{\partial \nu}\Big)^2 d\sigma = \int_\Omega -|u|^{\alpha-1} u (x \cdot \nabla u) dx - \frac{N-2}{2} ||u||^2_{H_\lambda}, \tag{3.11}$$

Next we show that

$$\int_\Omega |u|^{\alpha-1}u(x\cdot\nabla u)dx = -\frac{N}{1+\alpha}\int_\Omega |u|^{\alpha+1}dx. \tag{3.12}$$

We proceed by approximation arguments. For $\varepsilon > 0$ small enough we consider

$$I_\varepsilon := \int_\Omega |u|^{\alpha-1}u(x\cdot\nabla u)\theta_\varepsilon dx,$$

where θ_ε is a cut-off function supported in $\Omega\setminus B_\varepsilon(0)$. Due to the fact that $u\in H^2(\Omega\setminus\{0\})$ we can integrate by parts as follows:

$$\begin{aligned}
I_\varepsilon &= -\frac{1}{2}\int_\Omega |u|^{\alpha-1}x\cdot\nabla(u^2)\theta_\varepsilon dx = \frac{1}{2}\int_\Omega u^2\mathrm{div}\big(|u|^{\alpha-1}x\theta_\varepsilon\big)dx\\
&= \frac{1}{2}\int_\Omega u^2\big(N|u|^{\alpha-1}\theta_\varepsilon + x\cdot\nabla\theta_\varepsilon|u|^{\alpha-1} + (\alpha-1)x\cdot\nabla u|u|^{\alpha-3}u\theta_\varepsilon\big)dx\\
&= \frac{N}{2}\int_\Omega |u|^{\alpha+1}\theta_\varepsilon dx + \frac{1}{2}\int_\Omega |u|^{\alpha+1}x\cdot\nabla\theta_\varepsilon dx - \frac{\alpha-1}{2}I_\varepsilon. \qquad (3.13)
\end{aligned}$$

Therefore we obtain

$$I_\varepsilon = \frac{N}{\alpha+1}\int_\Omega |u|^{\alpha+1}\theta_\varepsilon dx + \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1}x\cdot\nabla\theta_\varepsilon dx. \tag{3.14}$$

From the equation itself it is easy to see that $|u|^{\alpha+1}\in L^1(\Omega)$ provided $u\in D(A_\lambda)$. Therefore, by the DCT we can pass to the limit as $\varepsilon\to 0$ in (3.14) to obtain the identity (3.12). On the other hand, multiplying (3.1) by u and integrating we obtain

$$||u||^2_{H_\lambda} = \int_\Omega |u|^{\alpha+1}dx,$$

Combining this with (3.12) and (3.11) we conclude the validity of (3.4). □

4 Applications to Controllability

In this section we study the controllability of the wave and Schrödinger equations with one singularity localized on the boundary of a smooth domain. Our motivation comes from the results shown in [38] in the context of an interior singularity.

For the sake of clarity, we will discuss in a detailed manner the case C1.

4.1 The wave equation. Case C1

In the sequel, we focus upon the controllability of the wave-like system

$$\begin{cases}
u_{tt} - \Delta u - \lambda\frac{u}{|x|^2} = 0, & (t,x)\in Q_T,\\
u(t,x) = h(t,x), & (t,x)\in(0,T)\times\Gamma_0,\\
u(t,x) = 0, & (t,x)\in(0,T)\times(\Gamma\setminus\Gamma_0),\\
u(0,x) = u_0(x), & x\in\Omega,\\
u_t(0,x) = u_1(x), & x\in\Omega.
\end{cases} \tag{4.1}$$

where $Q_T = (0, T) \times \Omega$ and Γ_0 is the boundary control region defined in (1.13) where the control $h \in L^2((0, T) \times \Gamma_0)$ is acting. We also assume $\lambda \leq \lambda(N)$. In view of the time-reversibility of the equation it is enough to consider the case where the target is

$$(\overline{u_0}, \overline{u_1}) = (0, 0).$$

It is the so-called *null controllability problem*.

4.1.1 Well-posedness

Let us briefly discuss the well-posedness of system (4.1) in the corresponding functional setting.

Instead of (4.1) we firstly consider the more general system with non-homogeneous boundary conditions:

$$\begin{cases} u_{tt} - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t, x) \in Q_T, \\ u(t, x) = g(t, x), & (t, x) \in \Sigma_T, \\ u(0, x) = u_0(x), & x \in \Omega, \\ u_t(0, x) = u_1(x), & x \in \Omega. \end{cases} \tag{4.2}$$

where $\Sigma_T = (0, T) \times \Gamma$. The solution of (4.2) is defined by the transposition method (J. L. Lions [30]).

Definition 4.1. *Assume $\lambda \leq \lambda(N)$. For $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and $g \in L^2((0, T) \times \Gamma)$, we say that u is a* **weak solution** *for (4.2) if*

$$\int_0^T \int_\Omega u f dx dt = - < u_0, z'(0) >_{L^2(\Omega)} + < u_1, z(0) >_{H'_\lambda, H_\lambda} - \int_0^T \int_\Gamma g \frac{\partial z}{\partial \nu} dx dt \quad \forall f \in \mathcal{D}(\Omega), \tag{4.3}$$

where $< \cdot, \cdot >$ represents the dual product between H_λ and its dual H'_λ, and z is the solution of the non-homogeneous adjoint-backward problem

$$\begin{cases} z_{tt} - \Delta z - \lambda \frac{z}{|x|^2} = f, & (t, x) \in Q_T, \\ z(t, x) = 0, & (t, x) \in \Sigma_T, \\ z(T, x) = z'(T, x) = 0, & x \in \Omega. \end{cases} \tag{4.4}$$

Formally, (4.3) is obtained by multiplying the system (4.4) with u and integrating on Q_T. Using the Hardy inequalities above and the application of standard methods for evolution equations we obtain the following existence result.

Theorem 4.1 (well-posedness). *Assume that Ω satisfies C1. Let $T > 0$ be given and assume $\lambda \leq \lambda(N)$. For every $(u_0, u_1) \in L^2(\Omega) \times H'_\lambda$ and any $h \in L^2((0, T) \times \Gamma_0)$ there exists a unique weak solution of* (4.1) *such that*

$$u \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H'_\lambda). \tag{4.5}$$

Moreover, the solution of (4.1) satisfies

$$||(u, u_t)||_{L^\infty(0,T; L^2(\Omega) \times H'_\lambda)} \lesssim ||(u_0, u_1)||_{L^2(\Omega) \times H'_\lambda} + ||h||_{L^2((0,T) \times \Gamma_0)}. \tag{4.6}$$

The details of the proof of Theorem 4.1 are omitted since they follow the same steps as in [38].

4.1.2 Controllability and main results

It is by now classical that controllability of (4.1) is characterized through an observability inequality for the adjoint system as follows.

Given initial data $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$, a possible control $h \in L^2((0,T) \times \Gamma_0)$ must satisfy the identity

$$\int_0^T \int_{\Gamma_0} h \, \frac{\partial w}{\partial \nu} d\sigma dt - < u_t(0), w(0) >_{H_\lambda^{'}, H_\lambda} + < u(0), w_t(0) >_{L^2(\Omega)} = 0, \quad (4.7)$$

where w is the solution of the adjoint system

$$\begin{cases} w_{tt} - \Delta w - \lambda \frac{w}{|x|^2} = 0, & (t,x) \in Q_T, \\ w(t,x) = 0, & (t,x) \in \Sigma_T, \\ w(0,x) = w_0(x), & x \in \Omega, \\ w_t(0,x) = w_1(x), & x \in \Omega. \end{cases} \quad (4.8)$$

The operator $\mathcal{A}_\lambda$ defined by $\mathcal{A}_\lambda(w_0, w_1) = (w_1, \Delta w_0 + \lambda/|x|^2 w_0)$ for all $(w_0, w_1) \in D(\mathcal{A}_\lambda) = D(A_\lambda) \times H_\lambda$, generates the wave semigroup i.e. $(\mathcal{A}_\lambda, D(\mathcal{A}_\lambda))$ is m-dissipative in $H_\lambda \times L^2(\Omega)$. In view of that, due to the theory of semigroups, the adjoint system is well-posed and more precisely it holds

Proposition 4.1 (see e.g. [38]). *(1) For any initial data $(w_0, w_1) \in H_\lambda \times L^2(\Omega)$ there exists a unique solution of (4.8)*

$$w \in C([0,T]; H_\lambda) \cap C^1([0,T]; L^2(\Omega)).$$

Moreover,

$$||(w, w_t)||_{L^\infty(0,T; H_\lambda \times L^2(\Omega))} \lesssim ||w_0||_{H_\lambda} + ||w_1||_{L^2(\Omega)} \quad (4.9)$$

(2) For any initial data $(w_0, w_1) \in D(A_\lambda) \times H_\lambda$ there exists a unique solution of (4.8) such that

$$w \in C([0,T]; D(A_\lambda)) \cap C^1([0,T]; H_\lambda) \cap C^2([0,T]; L^2(\Omega)).$$

Moreover

$$||(w, w_t)||_{L^\infty(0,T; D(A_\lambda) \times H_\lambda)} \lesssim ||w_0||_{D(A_\lambda)} + ||w_1||_{H_\lambda} \quad (4.10)$$

In the sequel, we claim some "hidden regularity" effect for the system (4.8) which may not be directly deduced from the semigroup regularity but from the equation itself.

Theorem 4.2 (Hidden regularity). *Assume $\lambda \leq \lambda(N)$ and w is the solution of (4.8) corresponding to the initial data $(w_0, w_1) \in H_\lambda \times L^2(\Omega)$. Then w satisfies*

$$\int_0^T \int_\Gamma (x \cdot \nu) \Big(\frac{\partial w}{\partial \nu} \Big)^2 d\sigma dt \lesssim \int_0^T \int_\Gamma \Big(\frac{\partial w}{\partial \nu} \Big)^2 |x|^2 d\sigma dt \lesssim ||w_0||^2_{H_\lambda} + ||w_1||^2_{L^2(\Omega)}. \quad (4.11)$$

Moreover, w verifies the identity

$$\frac{1}{2}\int_0^T\int_\Gamma (x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt = \frac{T}{2}(||w_0||^2_{H_\lambda}+||w_1||^2_{L^2(\Omega)})+\int_\Omega w_t\Big(x\cdot\nabla w+\frac{N-1}{2}w\Big)\Big|_0^T dx. \tag{4.12}$$

Due to Theorem 4.2 the operator $(w_0, w_1)\mapsto \big(\int_0^T\int_{\Gamma_0}(x\cdot\nu)(\partial w/\partial\nu)^2 d\sigma dt\big)^{1/2}$ is a linear continuous map in $H_\lambda\times L^2(\Omega)$. Let $\mathcal{H}$ be the completion of this norm in $H_\lambda\times L^2(\Omega)$. We consider the functional $J:\mathcal{H}\to\mathbb{R}$ defined by

$$J(w_0,w_1)(w) := \frac{1}{2}\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt - < u_1, w_0 >_{H'_\lambda, H_\lambda} + (u_0, w_1)_{L^2(\Omega)}, \tag{4.13}$$

where w is the solution of (4.8) corresponding to initial data (w_0, w_1). Of course, $< \cdot,\cdot >_{H'_\lambda, H_\lambda}$ denotes the duality product. A control $h\in L^2((0,T)\times\Gamma_0)$ satisfying (4.7) could be chosen as $h=(x\cdot\nu)\partial w_{\min}/\partial\nu$ where $w_{\min}$ minimizes the functional J on $\mathcal{H}$ among the solutions w of (4.8) corresponding to the initial data $(u_0,u_1)\in H'_\lambda\times L^2(\Omega)$. The existence of a minimizer of J is assured by the coercivity of J, which is equivalent to the *Observability inequality* for the adjoint system (4.8):

$$||w_0||^2_{H_\lambda}+||w_1||^2_{L^2(\Omega)} \lesssim \int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt. \tag{4.14}$$

Conservation of energy.

For any $\lambda\le\lambda(N)$ and any fixed time $t\ge 0$, let us define the energy associated to (4.8):

$$E^\lambda_w(t) = \frac{1}{2}\big(||w_t(t)||^2_{L^2(\Omega)} + ||w(t)||^2_{H_\lambda}\big) \tag{4.15}$$

We note that adjoint system (4.8) is conservative and therefore

$$E^\lambda_w(t) = E^\lambda_w(0), \quad \forall\lambda\le\lambda(N), \quad \forall t\in[0,T]. \tag{4.16}$$

Next we claim our main results which answer to the controllability question.

Theorem 4.3 (Observability inequality). *For all $\lambda\le\lambda(N)$, there exists a positive constant $D_1 = D_1(\Omega,\lambda,T)$ such that for all $T\ge 2R_\Omega$ and any initial data $(w_0,w_1)\in H_\lambda\times L^2(\Omega)$, the solution of (4.8) verifies the observability inequality*

$$E^\lambda_w(0)\le D_1\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt. \tag{4.17}$$

The proof of Theorem 4.3 relies mainly on the method of multipliers (cf. [30]) and the so-called compactness-uniqueness argument (cf. [31]), combined with the new Hardy inequalities above. These results guarantee the exact controllability of (1.12) when the control acts on the boundary region Γ_0. In conclusion, we obtain

Theorem 4.4 (**Controllability**). *Assume that* Ω *satisfies the case* C1 *and* $\lambda \leq \lambda(N)$. *For any time* $T > 2R_\Omega$, $(u_0, u_1) \in L^2(\Omega) \times H_\lambda^{'}$ *and* $(\overline{u_0}, \overline{u_1}) \in L^2(\Omega) \times H_\lambda^{'}$ *there exists* $h \in L^2((0,T) \times \Gamma_0)$ *such that the solution of (4.1) satisfies*

$$(u_t(T,x), u(T,x)) = (\overline{u_1}(x), \overline{u_0}(x)) \quad \textit{for all } x \in \Omega.$$

4.1.3 Proofs of main results

First of all, we need to justify that the solution w of adjoint system (4.8) possesses enough regularity to guarantee the integrability of the boundary term in (4.17). The justification is not trivial given the presence of the singularity at the boundary.

Proof of Theorem 4.2. We will proceed straightforward from Theorem 2.2.

Firstly, we consider initial data (w_0, w_1) in $D(\mathcal{A}_\lambda) = D(A_\lambda) \times H_\lambda$. Then, according to Proposition 4.1 we have

$$w \in C([0,T]; D(A_\lambda)) \cap C^1([0,T]; H_\lambda) \cap C^2([0,T]; L^2(\Omega)).$$

For a fixed time $t \in [0,T]$ we apply identity (2.16) in Lemma 2.1 with $f = -w_{tt}$. Passing to the limit when $\varepsilon \to 0$, by DCT and Fatou lemma we obtain $(\partial w/\partial \nu)|x| \in L^2(\Omega)$ and moreover

$$\begin{aligned}\frac{1}{2}\int_\Gamma \Big(\frac{\partial w}{\partial \nu}\Big)^2 |x|^2 d\sigma = & \int_\Omega w_{tt}(|x|^2 \vec{q} \cdot \nabla w)dx + 2\int_\Omega (x \cdot \nabla w)(\vec{q} \cdot \nabla w)dx \\ & + \sum_{i,j=1}^N \int_\Omega w_{x_i} w_{x_j} |x|^2 q^j_{x_i} dx - \int_\Omega |\nabla w|^2 (x \cdot \vec{q})dx \\ & - \frac{1}{2}\int_\Omega \text{div}\vec{q}|x|^2 \Big(|\nabla w|^2 - \lambda \frac{w^2}{|x|^2}\Big)dx. \end{aligned} \tag{4.18}$$

We have the following upper bounds for the terms in the right hand side of (4.18):

$$\Big|\int_\Omega |\nabla w|^2 (x \cdot \vec{q})dx\Big|, \Big|\int_\Omega (x \cdot \nabla w)(\vec{q} \cdot \nabla w)dx\Big| \lesssim \int_\Omega |x||\nabla w|^2 dx \lesssim ||w||^2_{H_\lambda},$$

$$\Big|\sum_{i,j=1}^N \int_\Omega w_{x_i} w_{x_j} |x|^2 q^i_{x_i} dx\Big| \lesssim \int_\Omega |x|^2 |\nabla w|^2 dx \lesssim ||w||^2_{H_\lambda},$$

$$\int_\Omega \text{div}\vec{q}|x|^2 \Big(|\nabla w|^2 - \lambda \frac{w^2}{|x|^2}\Big)dx \lesssim \int_\Omega |x|^2 |\nabla w|^2 dx + \int_\Omega w^2 dx \lesssim ||w||^2_{H_\lambda},$$

which hold true due to Hardy inequality in Theorem 1.2. In consequence, integrating in time in (4.18) we obtain

$$\frac{1}{2}\int_0^T \int_\Gamma \Big(\frac{\partial w}{\partial \nu}\Big)^2 |x|^2 d\sigma dt \lesssim \Big|\iint_{Q_T} w_{tt}|x|^2 \vec{q} \cdot \nabla w dx dt\Big| + \int_0^T ||w(t)||^2_{H_\lambda} dt. \tag{4.19}$$

Integrating by parts and applying the Cauchy-Schwartz inequality we have

$$\Big|\iint\limits_{Q_T} w_{tt}|x|^2\vec{q}\cdot\nabla w dxdt\Big| = \Big|\int_\Omega w_t|x|^2\vec{q}\cdot\nabla w\Big|_{t=0}^{t=T}dx - \iint\limits_{Q_T} w_t|x|^2\vec{q}\cdot\nabla w_t dxdt\Big|$$
$$\lesssim \int_\Omega \Big(w_t^2+|x|^2|\nabla w|^2\Big)\Big|_{t=0}^{t=T}dx + \frac{1}{2}\iint\limits_{Q_T} w_t^2 \mathrm{div}(|x|^2\vec{q})dxdt \tag{4.20}$$

From Theorem 1.1, (4.20) and due to the conservation of energy we obtain

$$\Big|\iint\limits_{Q_T} w_{tt}|x|^2\vec{q}\cdot\nabla w dxdt\Big| \lesssim \int_\Omega (w_t^2(0,x)+w_t^2(T,x))dx + ||w(0)||_{H_\lambda}^2 + ||w(T)||_{H_\lambda}^2 + \iint\limits_{Q_T} w_t^2 dxdt$$
$$\lesssim 2E_w^\lambda(0) + 2E_w^\lambda(T) + 2TE_w^\lambda(0)$$
$$= (T+2)(||w_0||_{H_\lambda}^2 + ||w_1||_{L^2(\Omega)}^2). \tag{4.21}$$

Since $x\cdot\nu \lesssim |x|^2$ on Γ, from (4.19) and (4.21) we conclude the inequality (4.11).

Next, we apply the Pohozaev identity for $w(t)$, $t\in[0,T]$. Indeed, integrating in time in Theorem 2.2 for $A_\lambda w = -w_{tt}$, we get

$$\frac{1}{2}\int_0^T\int_\Gamma (x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt = \int_\Omega w_t(x\cdot\nabla w)\Big|_{t=0}^{t=T}dx - \iint\limits_{Q_T} w_t(x\cdot\nabla w_t)dxdt - \frac{N-2}{2}\int_0^T ||w(t)||_{H_\lambda}^2 dt$$
$$= \int_\Omega w_t(x\cdot\nabla w)\Big|_{t=0}^{t=T}dx + \frac{N}{2}\int_0^T ||w_t(t)||_{L^2(\Omega)}^2 dt - \frac{N-2}{2}\int_0^T ||w(t)||_{H_\lambda}^2 dt$$
$$= \int_\Omega w_t(x\cdot\nabla w)\Big|_{t=0}^{t=T}dx + \frac{1}{2}\int_0^T \big(||w_t(t)||_{L^2(\Omega)}^2 + ||w(t)||_{H_\lambda}^2\big)dt$$
$$+ \frac{N-1}{2}\int_0^T \big(||w_t(t)||_{L^2(\Omega)}^2 - ||w(t)||_{H_\lambda}^2\big)dt. \tag{4.22}$$

Multiplying the equation of (4.8) by w and integrating, the equipartition of the energy

$$\int_\Omega ww_t\Big|_{t=0}^{t=T}dx = \int_0^T (||w_t(t)||_{L^2(\Omega)}^2 - ||w(t)||_{H_\lambda}^2)dt$$

holds true. Due to the conservation of energy and from (4.22) we obtain precisely the identity (4.12). This yields the proof of Theorem 2.1 for initial data in the domain $D(\mathcal{A}_\lambda)$. Then, by density arguments, one can extend the results for less regular initial data $(w_0,w_1)\in H_\lambda\times L^2(\Omega)$. For such density arguments we refer to Lions [30], on pages 139-141.

□

Proof of Theorem 4.3. In what follows we present the proof in the critical case $\lambda=\lambda(N)$, which is of main interest. The subcritical case $\lambda<\lambda(N)$ is let to the reader.

Step 1. Firstly, from Theorem 4.2 we remark that

$$\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=0}^{t=T} dx + TE_w^{\lambda(N)}(0) \le \frac{1}{2}\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt. \tag{4.23}$$

For a fixed time $t = t_0 > 0$, by the Cauchy-Schwartz inequality we have

$$\begin{aligned}\left|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=t_0} dx\right| &\le \frac{R_\Omega}{2}\int_\Omega w_t^2 dx + \frac{1}{2R_\Omega}\int_\Omega\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)^2 dx\\ &= \frac{R_\Omega}{2}||w_t||^2_{L^2(\Omega)} + \frac{1}{2R_\Omega}\left(\Big(\frac{N-1}{2}\Big)^2||w||^2_{L^2(\Omega)} + ||x\cdot\nabla w||^2_{L^2(\Omega)}\right.\\ &\left.+ (N-1)\int_\Omega w(x\cdot\nabla w)dx\right), \end{aligned}\tag{4.24}$$

where R_Ω was defined in (1.6). On the other hand it follows

$$\int_\Omega w(x\cdot\nabla w)dx = \frac{1}{2}\int_\Omega x\cdot\nabla(w^2)dx = -\frac{1}{2}\int_\Omega \mathrm{div}(x)w^2 dx = -\frac{N}{2}\int_\Omega w^2 dx \tag{4.25}$$

Therefore from (4.24) and (4.25) we obtain

$$\left|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=t_0} dx\right| \le \frac{1}{2R_\Omega}||x\cdot\nabla w||^2_{L^2(\Omega)} + \frac{R_\Omega}{2}||w_t||^2_{L^2(\Omega)} - \frac{1}{2R_\Omega}\Big(\frac{N^2-1}{4}\Big)||w||^2_{L^2(\Omega)}$$

Applying Theorem 1.1 we deduce

$$\left|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=t_0} dx\right| \le R_\Omega E_w^{\lambda(N)}(t_0) - C||w(t_0)||^2_{L^2(\Omega)}, \tag{4.26}$$

for some constant C. Due to the conservation of energy, taking $t_0 = 0$ respectively $t_0 = T$ and summing in (4.26) we get

$$\left|\int_\Omega w_t\Big(\frac{N-1}{2}w + x\cdot\nabla w\Big)\Big|_{t=0}^{t=T} dx\right| \le 2R_\Omega E_w^{\lambda(N)}(0) - C(||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)}), \tag{4.27}$$

From (4.23) and (4.27) we obtain

$$(T-2R_\Omega)E_w^{\lambda(N)}(0) \le \frac{1}{2}\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt + C(||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)}). \tag{4.28}$$

Step 2. To get rid of the remaining term on the right hand side of (4.28) we need the following lemma.

Lemma 4.1. *There exists a positive constant $C = C(T,\Omega) > 0$ such that*

$$||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)} \le C\int_0^T\int_{\Gamma_0}(x\cdot\nu)\Big(\frac{\partial w}{\partial\nu}\Big)^2 d\sigma dt \tag{4.29}$$

for all finite energy solutions of (4.8).

Combining Lemma 4.1 with (4.28), the observability inequality is finally proved. □

Proof of Lemma 4.1. We apply a classical compactness-uniqueness argument. Suppose by contradiction that (4.29) does not hold. Then there exists a sequence (w_0^n, w_1^n) of initial data such that the corresponding solution w^n verifies

$$\frac{||w^n(0)||^2_{L^2(\Omega)} + ||w^n(T)||^2_{L^2(\Omega)}}{\int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w^n}{\partial \nu}\Big)^2 d\sigma dt} \to \infty.$$

Normalizing we may suppose that (as $n \to \infty$)

$$||w^n(0)||^2_{L^2(\Omega)} + ||w^n(T)||^2_{L^2(\Omega)} = 1, \qquad \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w^n}{\partial \nu}\Big)^2 d\sigma dt \to 0. \quad (4.30)$$

From (4.28) and (4.30) we deduce that the sequence of energies $\{E^{\lambda(N)}_{w^n}(0)\}_n$ is uniformly bounded. In particular, we deduce that w^n is uniformly bounded in

$$C([0,T]; H_{\lambda(N)}) \cap C^1([0,T]; L^2(\Omega)).$$

Therefore, by extracting a subsequence

$$w^n \rightharpoonup w \text{ in } L^\infty(0,T; H_{\lambda(N)}) \text{ weakly-}\star, \text{ as } n \to \infty, \quad (4.31)$$

$$w^n_t \rightharpoonup w_t \text{ in } L^\infty(0,T; L^2(\Omega)) \text{ weakly-}\star, \text{ as } n \to \infty. \quad (4.32)$$

From Theorem 4.2 we obtain

$$\frac{\partial w^n}{\partial \nu}\sqrt{x\cdot\nu} \rightharpoonup \frac{\partial w}{\partial \nu}\sqrt{x\cdot\nu} \text{ in } L^\infty(0,T; L^2(\Gamma_0)) \text{ weakly-}\star, \text{ as } n \to \infty.$$

Furthermore, by lower semicontinuity and (4.30) we have

$$0 \le \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial \nu}\Big)^2 d\sigma dt \le \liminf_{n\to\infty} \int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w^n}{\partial \nu}\Big)^2 d\sigma dt = 0.$$

Hence

$$\int_0^T \int_{\Gamma_0} (x\cdot\nu)\Big(\frac{\partial w}{\partial \nu}\Big)^2 d\sigma dt = 0,$$

and

$$(x\cdot\nu)\frac{\partial w}{\partial \nu} = 0, \qquad \text{a.e. on } \Gamma_0, \quad \forall t \in [0,T]. \quad (4.33)$$

On the other hand, from compactness and (4.31) we deduce that

$$w^n \to w \text{ in } L^\infty(0,T; L^2(\Omega)),$$

which combined with (4.30) yields

$$||w(0)||^2_{L^2(\Omega)} + ||w(T)||^2_{L^2(\Omega)} = 1. \quad (4.34)$$

To end the proof of Lemma 4.1 it suffices to observe that (4.33)-(4.34) lead to a contradiction. Indeed, in view of (4.33) and by Holmgreen's unique continuation we deduce that $w \equiv 0$ in Ω which is in contradiction with (4.34). □

Remark 4.1. *Unique continuation results may be applied far from the origin where the coefficient of the lower order term of the operator $-\partial_{tt} - \Delta - \lambda/|x|^2$ is analytic in time (actually, it is independent of time and bounded in space). The principal part coincides with the D'Alambertian operator, then one can apply Homlgreen's unique continuation to get $w = 0$ a.e. in $\Omega \setminus B(0, \varepsilon)$ for any $\varepsilon > 0$. In consequence, we will have $w \equiv 0$ in Ω, see e.g. [33].*

4.2 The Schrödinger equation

In this section we consider the Schrödinger-like equation

$$\begin{cases} iu_t - \Delta u - \lambda \frac{u}{|x|^2} = 0, & (t,x) \in Q_T, \\ u(t,x) = h(t,x), & (t,x) \in (0,T) \times \Gamma_0, \\ u(t,x) = 0, & (t,x) \in (0,T) \times (\Gamma \setminus \Gamma_0), \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{4.35}$$

Moreover, we assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a smooth bounded domain satisfying case C1 and $\lambda \leq \lambda(N)$. For the Schrödinger equation we define the Hilbert spaces $L^2(\Omega;\mathbb{C})$ and $H_0^1(\Omega;\mathbb{C})$ endowed with the inner products

$$< u, v >_{L^2(\Omega;\mathbb{C})} := \text{Re} \int_\Omega u(x)\overline{v(x)} dx, \quad \forall u, v \in L^2(\Omega;\mathbb{C}),$$

$$< u, v >_{H_0^1(\Omega;\mathbb{C})} := \text{Re} \int_\Omega \nabla u(x) \cdot \nabla \overline{v(x)} dx, \quad \forall u, v \in H_0^1(\Omega;\mathbb{C}).$$

For all $\lambda \leq \lambda(N)$, we also define the Hilbert space $H_\lambda(\Omega;\mathbb{C})$ as the completion of $H_0^1(\Omega;\mathbb{C})$ with respect to the norm associated with the inner product

$$< u, v >_{H_\lambda(\Omega;\mathbb{C})} := \text{Re} \int_\Omega \Big(\nabla u(x) \cdot \nabla \overline{v(x)} - \lambda \frac{u(x)\overline{v(x)}}{|x|^2} \Big) dx, \quad \forall u, v \in H_0^1(\Omega;\mathbb{C}). \tag{4.36}$$

The spaces $L^2(\Omega;\mathbb{C})$, $H_0^1(\Omega;\mathbb{C})$, $H_\lambda(\Omega;\mathbb{C})$ inherit the properties of the corresponding real spaces. In order to simplify the notations, we will write $L^2(\Omega), H_0^1(\Omega), H_\lambda$ without making confusions.

As shown for the wave equation, the system (4.35) is well posed.

Theorem 4.5 (see [38])**.** *Let $T > 0$ be given and assume $\lambda \leq \lambda(N)$. For every $u_0 \in H_\lambda'$ and any $h \in L^2((0,T) \times \Gamma_0)$ the system* (4.35) *is well-posed, i.e. there exists a unique weak solution such that*

$$u \in C([0,T]; H_\lambda').$$

Moreover, there exists constant $C > 0$ such that the solution of (4.35) *satisfies*

$$||u||_{L^\infty(0,T;H_\lambda')} \leq C(||u_0||_{H_\lambda'} + ||h||_{L^2((0,T)\times\Gamma_0)}).$$

The system (4.35) is also controllable. More precisely, the control result states as follows.

Theorem 4.6. *The system* (4.35) *is controllable for any* $\lambda \leq \lambda(N)$. *More precisely, for any time* $T > 0$, $u_0 \in H_\lambda^{'}$ *and* $\overline{u_0} \in H_\lambda^{'}$ *there exists* $h \in L^2((0,T) \times \Gamma_0)$ *such that the solution of* (4.35) *satisfies*

$$u(T,x) = \overline{u_0}(x) \quad \text{for all } x \in \Omega.$$

As discussed in Subsection 4.1, the controllability is equivalent to the Observability inequality for the solution of the adjoint system

$$\begin{cases} iw_t + \Delta w + \lambda \frac{w}{|x|^2} = 0, & (t,x) \in Q_T, \\ w(t,x) = 0, & (t,x) \in (0,T) \times \Gamma, \\ w(0,x) = w_0(x), & x \in \Omega, \end{cases} \tag{4.37}$$

More precisely, if w solves (4.37), then for any time $T > 0$, there exists a positive constant C_T such that

$$||w_0||_{H_\lambda}^2 \leq C_T \int_0^T \int_{\Gamma_0} (x \cdot \nu) \Big| \frac{\partial w}{\partial \nu} \Big|^2 d\sigma dt. \tag{4.38}$$

Observability (4.38) might be deduced directly using the multiplier identity stated in Lemma 4.2. The proof is let to the reader since it follows the same steps as in [38].

Lemma 4.2. *Assume* $\lambda \leq \lambda(N)$ *and* w *is the solution of* (4.37) *corresponding to the initial data* $w_0 \in H_\lambda$. *Then*

$$\int_0^T \int_\Gamma \Big| \frac{\partial w}{\partial \nu} \Big|^2 |x|^2 d\sigma dt \lesssim ||w_0||_{H_\lambda}^2 \tag{4.39}$$

and w *satisfies the identity*

$$\frac{1}{2} \int_0^T \int_\Gamma (x \cdot \nu) \Big| \frac{\partial w}{\partial \nu} \Big|^2 d\sigma dt = T ||w||_{H_\lambda}^2 + \frac{1}{2} Im \int_\Omega w x \cdot \nabla \overline{w} dx \Big|_{t=0}^{t=T} dx.$$

Remark 4.2. *Besides, the proof of* (4.38) *can be deduced from the result valid for the wave equation. Indeed, the general theory presented in an abstract form in [35], assures the observability of systems like* $\dot{z} = iA_0 z$ *using results available for systems of the form* $\ddot{z} = -A_0 z$.

5 Open problems

1. Geometric constraints. In this paper we have shown the role of the Pohozaev identity, in the context of boundary singularities, when studying the controllability of conservative systems like wave and Schrödinger equations. We proved that for any $\lambda \leq \lambda(N) = N^2/4$, the corresponding systems are exactly observable from Γ_0 as specified in (1.13). Our result enlarges the range of values $\lambda \leq (N-2)^2/4$ for which the control holds, proved firstly in [38] in the context of interior singularities.

The geometrical assumption for Γ_0 is really necessary, otherwise our proof does not work. Of course, scholars still have to analyze the case when the central of gravity of Γ_0 is centered in a point x_0 different from zero, i.e. $\Gamma_{x_0} = \{x \in \Gamma \mid (x-x_0)\cdot\nu \geq 0\}$. This choice of Γ_{x_0} provides some technical difficulties which have also been emphasized in [38]. A possible proof in the case of a domain such Γ_{x_0} should apply a different technique than the one we have used so far.

2. Multipolar singularities. The same Pohozaev identity and controllability issues could be addressed for more complicated operators, such as, for instance $L = -\Delta - V(x)$, where $V(x)$ denotes a multi-particle potential. To the best of our knowledge, even if there are some important works studying Hardy-type inequalities for multipolar potentials (see e.g. [7], [16], [15]), an accurate analysis is still to be done. An interesting situation refers to the case of two-particle system in which the goal is to analyze the limit process when one particle collapses into the other. We address the question of this both in the context of controllability and the diffusion heat processes when discussing the time decay of solutions.

6 Appendix: sharp gradient bounds

Proof of Theorem 1.1. Without losing generality it is enough to consider two types of geometries for Ω as follows.

G1: The points on Γ satisfy $x_N \geq 0$ in the neighborhood of the origin.

G2; The points on Γ satisfy $x_N \leq 0$ in the neighborhood of the origin.

In the other intermediate case (when x_N changes sign at the origin) the result valid for case G2 still holds true since we can prove it for test functions extended from zero up to a domain satisfying G2.

The proof comprises several steps.

Step 1. Firstly we show that Theorem 1.1 is true in a neighborhood of $x = 0$. More precisely, there exists $r_0 = r_0(\Omega, N) > 0$ small enough, and $C = C(r_0)$ such that

$$\int_{\Omega_{r_0}} |x|^2 |\nabla w|^2 dx \leq R_\Omega^2 \Big(\int_{\Omega_{r_0}} |\nabla w|^2 dx - \frac{N^2}{4} \int_{\Omega_{r_0}} \frac{w^2}{|x|^2} dx \Big) + C(r_0) \int_{\Omega_{r_0}} w^2 dx, \tag{6.1}$$

holds true for any function $w \in C_0^\infty(\Omega_{r_0})$, where $\Omega_{r_0} = \Omega \cap B_{r_0}(0)$.

Next we check the validity of Step 1. In view of that, let us consider a function ϕ which satisfies

$$-\Delta\phi \geq \frac{N^2}{4} \frac{\phi}{|x|^2}, \quad \phi > 0, \quad \forall x \in \Omega_{r_0}, \tag{6.2}$$

for some positive constant r_0. Such a function exists for each one of the cases G1-G2. Indeed, for the case G1 we may consider

$$\phi = x_N |x|^{-N/2} \tag{6.3}$$

and for case G2 we can take

$$\phi = d(x)e^{(1-N)d(x)}\Big|\log\frac{1}{|x|}\Big|^{1/2}|x|^{-N/2}. \tag{6.4}$$

With the transformation $w = \phi u$ for such ϕ as in (6.2) we get

$$|\nabla w|^2 = |\nabla\phi|^2u^2 + \phi^2|\nabla u|^2 + 2\phi u\nabla\phi\cdot\nabla u. \tag{6.5}$$

Integrating we obtain

$$\int_{\Omega_{r_0}} |\nabla w|^2dx = \int_{\Omega_{r_0}} |\nabla u|^2\phi^2dx - \int_{\Omega_{r_0}} \frac{\Delta\phi}{\phi}w^2. \tag{6.6}$$

On the other hand, multiplying in (6.5) by $|x|^2$ and integrating we obtain

$$\int_{\Omega_{r_0}} |x|^2|\nabla w|^2dx = \int_{\Omega_{r_0}} |x|^2|\nabla\phi|^2u^2dx + \int_{\Omega_{r_0}} |x|^2\phi^2|\nabla u|^2dx + \frac{1}{2}\int_{\Omega_{r_0}} |x|^2\nabla(\phi^2)\cdot\nabla(u^2)dx \tag{6.7}$$

For the last term in (6.7) we deduce

$$\frac{1}{2}\int_{\Omega_{r_0}} |x|^2\nabla(\phi^2)\cdot\nabla(u^2)dx = -\int_{\Omega_{r_0}} 2\frac{x\cdot\nabla\phi}{\phi}w^2dx - \int_{\Omega_{r_0}} |x|^2|\nabla\phi|^2u^2dx - \int_{\Omega_{r_0}} \frac{\Delta\phi}{\phi}|x|^2w^2dx. \tag{6.8}$$

According to (6.7) and (6.8) we obtain

$$\int_{\Omega_{r_0}} |x|^2|\nabla w|^2dx = \int_{\Omega_{r_0}} |x|^2\phi^2|\nabla u|^2dx - \int_{\Omega_{r_0}} \frac{2x\cdot\nabla\phi}{\phi}w^2dx - \int_{\Omega_{r_0}} \frac{\Delta\phi}{\phi}|x|^2w^2dx. \tag{6.9}$$

Taking into account the election of ϕ in (6.2) we have

$$-\frac{\Delta\phi}{\phi} = \frac{N^2}{4|x|^2} + P, \tag{6.10}$$

where $P \geq 0$ for any $x \in \Omega_{r_0}$. Then from (6.6) and (6.10) we have

$$\begin{aligned}\int_{\Omega_{r_0}} |x|^2\phi^2|\nabla u|^2dx &\leq R_\Omega^2\int_{\Omega_{r_0}} \phi^2|\nabla u|^2 = R_\Omega^2\Big(\int_{\Omega_{r_0}} |\nabla w|^2dx + \int_{\Omega_{r_0}} \frac{\Delta\phi}{\phi}w^2\Big)\\ &= R_\Omega^2\int_{\Omega_{r_0}} \Big(|\nabla w|^2 - \frac{N^2}{4}\frac{w^2}{|x|^2}\Big)dx - R_\Omega^2\int_{\Omega_{r_0}} Pw^2dx.\end{aligned} \tag{6.11}$$

From above and (6.9) it follows that

$$\int_{\Omega_{r_0}} |x|^2|\nabla w|^2dx \le R_\Omega^2\int_{\Omega_{r_0}}\Big(|\nabla w|^2-\frac{N^2}{4}\frac{w^2}{|x|^2}\Big)dx - R_\Omega^2\int_{\Omega_{r_0}} Pw^2dx$$
$$-2\int_{\Omega_{r_0}}\frac{x\cdot\nabla\phi}{\phi}w^2dx+\int_{\Omega_{r_0}}\Big(\frac{N^2}{4|x|^2}+P\Big)|x|^2w^2dx$$
$$=R_\Omega^2\int_{\Omega_{r_0}}\Big(|\nabla w|^2-\frac{N^2}{4}\frac{w^2}{|x|^2}\Big)dx+\int_{\Omega_{r_0}}(|x|^2-R_\Omega^2)Pw^2dx$$
$$-2\int_{\Omega_{r_0}}\frac{x\cdot\nabla\phi}{\phi}w^2dx+\frac{N^2}{4}\int_{\Omega_{r_0}}w^2dx. \tag{6.12}$$

In the case G1 (ϕ satisfies (6.3)) for r_0 small enough we have $P=0$ and

$$\Big|\frac{x\cdot\nabla\phi}{\phi}\Big|\le C, \quad \forall x\in\Omega_{r_0},$$

holds for some positive constant C. Thanks to (6.12) we conclude the proof of Step 1 in the case G1. In the case G2 (ϕ satisfies (6.4)), for r_0 small enough we have

$$P>0, \quad \nabla d\cdot x\ge 0, \quad \forall x\in\Omega_{r_0}$$

Then, we remark

$$\frac{x\cdot\nabla\phi}{\phi}=\frac{x\cdot\nabla d}{d}+O(1),$$

and from above we finish the proof of Step 1 in this latter case.

Step 2. This step consists in applying a cut-off argument to transfer the validity of inequality (6.1) from Ω_{r_0} to Ω. More precisely, we consider a cut-off function $\theta\in C_0^\infty(\Omega)$ such that

$$\theta(x)=\begin{cases}1, & |x|\le r_0/2,\\ 0, & |x|\ge r_0.\end{cases} \tag{6.13}$$

Then we split $w\in C_0^\infty(\Omega)$ as follows

$$w=\theta w+(1-\theta)w:=w_1+w_2. \tag{6.14}$$

Next let us firstly prove the following lemma.

Lemma 6.1. *Let us consider a weight function $\rho: C^\infty(\overline{\Omega})\to\mathbb{R}$ which is bounded and non-negative. There exists $C(\Omega,\rho)>0$ such that the following inequality holds*

$$\int_\Omega \rho(x)\nabla w_1\cdot\nabla w_2dx\ge -C(\Omega,\rho,r)\int_\Omega |w|^2dx. \tag{6.15}$$

Proof of Lemma 6.1. From the boundary conditions, integrating by parts we have

$$\int_\Omega \rho\nabla w_1\cdot\nabla w_2 dx = \int \rho\theta(1-\theta)|\nabla w|^2dx + \int_\Omega \rho w\nabla w\cdot\nabla\rho(1-2\theta)dx - \int \theta|\nabla\theta|^2|w|^2dx$$
$$\geq \frac{1}{2}\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} \nabla(|w|^2)\cdot\nabla\theta(1-2\theta)\rho dx - ||\rho||_\infty||D\theta||_\infty^2\int_\Omega |w|^2dx$$
$$= -\frac{1}{2}\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} \mathrm{div}((1-2\theta)\rho\nabla\theta)|w|^2dx - ||\rho||_\infty||D\theta||_\infty^2\int_\Omega |w|^2dx$$
$$\geq -C(||\rho||_{W^{1,\infty}}, ||\theta||_{W^{2,\infty}})\int_\Omega |w|^2dx. \tag{6.16}$$

□

Now we are able to finalize Step 2. Indeed, splitting w as before we get

$$\int_\Omega |x|^2|\nabla w|^2dx = \int_{\Omega_{r_0}} |x|^2|\nabla w_1|^2dx + \int_{\Omega\setminus\Omega_{r_0/2}} |x|^2|\nabla w_2|^2dx + 2\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} |x|^2\nabla w_1\cdot\nabla w_2dx$$

Applying (6.1) to w_1 in (6.14) we obtain

$$\int_\Omega |x|^2|\nabla w|^2dx \leq R_\Omega^2\Big(\int_\Omega |\nabla w|^2dx - \frac{N^2}{4}\int_\Omega \frac{w_1^2}{|x|^2}dx\Big) + C\int_\Omega w^2dx -$$
$$-\int_{\Omega_{r_0}\setminus\Omega_{r_0/2}} 2(R_\Omega^2-|x|^2)\nabla w_1\cdot\nabla w_2dx. \tag{6.17}$$

Adding $\rho = 2(R_\Omega^2-|x|^2)$ in Lemma 6.1, from (6.17) we get

$$\int_\Omega |x|^2|\nabla w|^2dx \leq R_\Omega^2\Big(\int_\Omega |\nabla w|^2dx - \frac{N^2}{4}\int_{\Omega_{r_0}} \frac{w_1^2}{|x|^2}dx\Big) + C(\Omega, r_0)\int_\Omega w^2dx. \tag{6.18}$$

On the other hand we have

$$\int_{\Omega_{r_0}} \frac{w_1^2}{|x|^2} \geq \int_\Omega \frac{w^2}{|x|^2}dx - C(r_0)\int_\Omega w^2dx. \tag{6.19}$$

From (6.18) and (6.19) the conclusion of Theorem 1.1 yields choosing r_0 small enough, $r_0 \leq R_\Omega$. □

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