Existence of strong solutions in critical spaces for barotropic viscous fluids in larger spaces

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Abstract

This paper is dedicated to the study of viscous compressible barotropic fluids in dimension $N \geq 2$. We address the question of well-posedness for *large* data having critical Besov regularity. Our result improves the analysis of R. Danchin in [13] and of the author in [15, 16] inasmuch as we may take initial density in $B^{\frac{N}{p}}_{p,1}$ with $1 \leq p < +\infty$. Our result relies on a new a priori estimate for the velocity, where we introduce a new unknown called *effective velocity* to weaken one the coupling between the density and the velocity. In particular our result is the first where we obtain uniqueness without imposing hypothesis on the gradient of the density.

1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu(\rho) D(u)) - \nabla(\lambda(\rho)\mathrm{div} u) + \nabla P(\rho) = \rho f, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density. The pressure P is a suitable smooth function of ρ. We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$ (in the sequel to simplify the calculus we will assume the viscosity coefficients are constant functions except in theorem 1.3). Such a condition ensures ellipticity for the momentum equation and is satisfied in the physical cases where $\lambda + \frac{2\mu}{N} > 0$. We supplement the problem with initial condition (ρ_0, u_0) and an outer force f. Throughout the paper, we assume that the space variable x is in $\mathbb{R}^N$ or in the periodic box $\mathbb{T}^N_a$ with period a_i, in the i-th direction. We restrict ourselves to the case $N \geq 2$.
The problem of existence of global time solutions for Navier-Stokes equations was addressed in one dimension for smooth enough data by Kazhikov and Shelukin in [29], and for discontinuous ones, but still with densities away from zero, by Serre in [36] and Hoff in [20]. Those results have been generalized to higher dimension by Hoff in [23, 26]. The

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existence and uniqueness of local classical solutions for (1.1) with smooth initial data such that the density ρ_0 is bounded and bounded away from zero has been stated by Nash in [34]. Let us emphasize that no stability condition was required there. On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [32]. Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [37], [39], [22], [28]. Concerning the existence of global strong solutions in two dimension with large initial data and specific choice on the viscosity coefficients, we refer to the pioneering works of Vaigant and Kazhikhov in [38]. For results of weak-strong uniqueness, we would like to mention the recent works of P. Germain [14].
Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see e.g [33]):

$$(NS) \qquad \begin{cases} \partial_t v + v\cdot\nabla v - \mu\Delta v + \nabla\Pi = 0, \\ \mathrm{div} v = 0, \end{cases}$$

we aim at solving (1.1) in the case where the data (ρ_0, u_0, f) have *critical* regularity.
By critical, we mean that we want to solve the system in functional spaces with norm independent of the changes of scales which leave (1.1) invariant. In the case of barotropic fluids, it is easy to see that the transformations:

$$(\rho(t,x), u(t,x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)), \quad l \in \mathbb{R}, \tag{1.2}$$

have that property, provided that the pressure term has been changed accordingly.
The use of critical functional frameworks led to several new well-posedness results for compressible fluids (see [10, 11, 13, 15, 16]). In addition to have a norm invariant by (1.2), appropriate functional space for solving (1.1) must provide a control on the L^∞ norm of the density (in order to avoid vacuum and loss of parabolicity). For that reason, we restricted our study to the case where the initial data (ρ_0, u_0) and external force f are such that, for some positive constant $\bar{\rho} > 0$:

$$(\rho_0 - \bar{\rho}) \in B^{\frac{N}{p}}_{p,1}, \; u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1} \;\text{ and }\; f \in L^1_{loc}(\mathbb{R}^+, \in B^{\frac{N}{p_1}-1}_{p_1,1})$$

for suitable choice of $(p, p_1) \in [1, +\infty[$.
In [13], however, we hand to have $p = p_1$ with the limitation $p < 2N$ for the existence of solutions and $p \leq N$ for the uniqueness, indeed in this article there exists a very strong coupling between the pressure and the velocity. To be more precise, the pressure term is considered as a remainder for the parabolic operator in the momentum equation of (1.1). This present paper improves the results of R. Danchin in [10, 13], in the sense that the initial density belongs to larger spaces $B^{\frac{N}{p}}_{p,1}$ with larger value $p \in [1, +\infty[$. To be more precised we extend the results of [10, 13] to the case where the Lebesgue index of Besov spaces are not the same for the density and the velocity. The main idea of this paper is to introduce a new variable than the velocity in the goal to *cancel out* the relation of coupling between the velocity and the density. This work may be considered as an extension of [1] and [17] (where the authors are working with different Lebesgue index for the velocity and

the density) where the studied system is the dependent density incompressible Navier-Stokes system. However one of the main difference concerns the fact that in this case the velocity and the density are naturally decoupled. It is unfortunately not the case for the barotropic Navier-Stokes system (1.1) that is why we will introduce this new unknown that we call *effective velocity*.
To simplify the notation, we assume from now on that $\bar{\rho}=1$. Hence as long as ρ does not vanish, the equations for ($a=\rho^{-1}-1$,u) read:

$$\begin{cases} \partial_t a + u\cdot\nabla a = (1+a)\mathrm{div}u, \\ \partial_t u + u\cdot\nabla u - (1+a)\mathcal{A}u + \nabla(g(a)) = f, \end{cases} \tag{1.3}$$

In the sequel we will note $\mathcal{A}=\mu\Delta+(\lambda+\mu)\nabla\mathrm{div}$ and g a smooth function which may be computed from the pressure function P. One can now state our main result.

Theorem 1.1 *Let P be a suitably smooth function of the density such that $P'(1)>0$ and $1\leq p_1\leq p<+\infty$ such that $\frac{1}{p_1}\leq\frac{1}{N}+\frac{1}{p}$ and $\frac{1}{p}+\frac{1}{p_1}>\frac{1}{N}$. Let $u_0\in B^{\frac{N}{p_1}-1}_{p_1,1}$, $f\in L^1_{loc}(\mathbb{R}^+,B^{\frac{N}{p_1}-1}_{p_1,1})$ and $a_0\in B^{\frac{N}{p}}_{p,1}$ with $1+a_0$ bounded away from zero.*
There exists then a positive time T such that system (1.1) has a solution (a,u) with $1+a$ bounded away from zero and:

$$a\in\widetilde{C}([0,T],B^{\frac{N}{p}}_{p,1}),\;\; u\in\widetilde{C}([0,T];B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})\cap L^1([0,T],B^{\frac{N}{p}+1}_{p,1}).$$

Moreover this solution is unique if:

$$\frac{2}{N}\leq\frac{1}{p}+\frac{1}{p_1}. \tag{1.4}$$

Remark 1 *We refer the reader for the notation of $\widetilde{L}^\rho(B^s_{p,r})$ (with $s\in\mathbb{R}$, $(p,r,\rho)\in[1,+\infty]^3$) to the definition 2.2.*

Remark 2 *We can observe that when p goes to infinity we are close from getting solution with initial data (a_0,u_0) in $B^0_{\infty,1}\times B^1_{N,1}$. These spaces are absolutely critical for compressible Navier-Stokes system in the sense that $B^0_{N,1}$ is close to L^N which is critical for incompressible Navier-Stokes. Furtheremore in this case we do not ask any information on the derivatives of the initial density when a_0 is in $B^0_{\infty,1}$ (this is really new compared with the different previous results existing in the literature of the topic). In passing we can remark that $B^0_{\infty,1}$ is not far of L^∞ (L^∞ being in some sense the most general space where we can hope to get solutions, indeed it is necessary to have $\rho\in L^\infty$ in order to control the non linearities appearing on the density, for example the pressure but also for some reasons related with the notion of multiplier). In this sense, we can consider that our result is quite optimal.*

Remark 3 *It seems possible to improve the theorem 1.1 by choosing initial data a_0 in $B^{\frac{N}{p}}_{p,\infty}\cap B^0_{\infty,1}$. For this we could use some arguments of density to deal with the variable coefficients of the heat equation . However some supplementary conditions appear on p_1 in this case, in particular for treating some non linear terms requiring additional conditions to use the paraproduct.*

The key of the theorem 1.1 is to introduce a new unknown v_1 to avoid the coupling between the density and the velocity, we analyze by a new way the pressure term. More precisely we write the gradient of the pressure as a Laplacian of some vector-field v, and we include this term in the linear part of the momentum equation (in other words, $v = \mathcal{G}P(\rho)$ where $\mathcal{G}P(\rho)$ stands for some pseudo-differential operator of order -1). We then introduce the effective velocity $v_1 = u - v$. By this way, we have canceled out the coupling between v_1 and the density. We next verify easily that we have a Lipschitz control on the gradient of u (it is crucial to estimate the density by the mass equation).

In [21], D. Hoff shows a very strong theorem of uniqueness for weak solutions when the pressure is of the specific form $P(\rho) = K\rho$ with $K > 0$. Similarly in [23], [26], [22], D. Hoff gets global weak solutions and point out regularizing effects on the velocity when the initial data are small. In particular when the pressure has this form, he does not need any estimates on the gradient of the initial density, he considers only $\rho_0 \in L^\infty$. In the following theorem, we will observe that this type of pressure ensures a specific structure and avoid to impose some extra conditions for the uniqueness as (1.4). We obtain in this particular case the following result which extends the analysis of theorem 1.1 for the uniqueness.

Theorem 1.2 *Assume that $P(\rho) = K\rho$ with $K > 0$. Let $1 \leq p_1 \leq p \leq +\infty$ such that $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{p_1} > \frac{1}{N}$. Assume that $u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1}$, $f \in L^1_{loc}(\mathbb{R}^+, B^{\frac{N}{p_1}-1}_{p_1,1})$ and $a_0 \in B^{\frac{N}{p}}_{p,1}$ with $1 + a_0$ bounded away from zero.*

- *There exists a positive time T such that system (1.1) has a solution (a, u) with $1+a$ bounded away from zero,*

$$a \in \widetilde{C}([0,T], B^{\frac{N}{p}}_{p,1}), \;\; u \in \widetilde{C}([0,T]; B^{\frac{N}{p_1}-1}_{p_1,1} + B^{\frac{N}{p}+1}_{p,1}) \cap L^1([0,T], B^{\frac{N}{p}+1}_{p,1}).$$

- *If moreover we assume that $\sqrt{\rho_0}u_0 \in L^2$, $\rho_0 - \bar{\rho} \in L^1_2$, $u_0 \in H^s$ with $s > 0$ if $N = 2$ and $s > \frac{1}{2}$ if $N = 3$. Finally we need to assume that u_0 belongs to $L^{2+\varepsilon}$ if $N = 2$ and to $L^{6+\varepsilon}$ if $N = 3$ with $\varepsilon > 0$. Furthermore we assume that $0 < \lambda < \frac{5}{4}\mu$. Then the solution (a, u) is unique.*

Remark 4 *In the previous theorem we did not want strive with generalities which may hide the main functional spaces used on the initial data. But in fact we need of additional regularity on the source term f when $N = 2, 3$ to obtain the previous corollary, we refer to the conditions (1.13) and (1.14) of [24].*

Remark 5 *Here L^1_2 defines the corresponding Orlicz space (see definition in [30]).*

Remark 6 *This theorem improves theorem 1.1 inasmuch as we do not need of the condition $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{p_1}$ to get uniqueness as in the theorem 1.1.*

Remark 7 *Up to my knowledge, it seems that it is the first time that we get strong solution without any control on the gradient of the initial density $\nabla\rho_0$. Indeed in [12], we have $\nabla\rho_0 \in B^0_{1,N}$. In our case $\nabla\rho_0$ has a negative index of regularity, more precisely $\nabla\rho_0 \in B^{\frac{N}{p}-1}_{p,1}$ with $\frac{N}{p} - 1 < 0$ when $p > N$.*

Remark 8 *Furthermore we can observe that with this type of pressure we are very close to have existence of strong solution in finite time for initial data* (a_0, u_0) *in* $B^0_{\infty,1} \times B^{\frac{N}{2}-1}_{2,1}$. *It means that this theorem bridges the gap between the result of D. Hoff (see [21]) where the initial density is assumed* L^∞ *but where we have no uniqueness in dimension* $N = 3$ *and the results of R. Danchin in [13] where the initial density is far from being only* L^∞. *However we are slightly subcritical on the initial velocity as we need an additional condition of type* $u_0 \in L^{6+\varepsilon}$ *with* $\varepsilon > 0$ *in dimension* $N = 3$. *It remains that it is the first result of strong solution where we can reach the critical case* $a_0 \in B^0_{\infty,1}$.

We finally treat the case of variable viscosity coefficients. More particularly we are interested in considering the specific case of the so-called *BD viscosity coefficients* (see [5]). Indeed with this choice, we naturally obtain some informations on q in $B^{\frac{N}{2}}_{2,2}$ when $N = 2$. In this context, the hypothesis of theorem 1.1 on $q_0 \in \widetilde{B}^{\frac{N}{2}}_{2,1}$ becomes natural when $p = 2$ (for more explanations see remarks 9).
We obtain then a natural extension in the case where the viscosity coefficients are variable.

Theorem 1.3 *Let* P *be a suitably smooth function of the density such that* $P'(1) > 0$, μ *and* λ *are regular functions such that* $\mu > 0$ *and* $2\mu + \lambda > 0$. *Let* $1 \le p_1 \le p < +\infty$ *such that* $\frac{1}{p_1} \le \frac{1}{N} + \frac{1}{p}$ *and* $\frac{1}{p} + \frac{1}{p_1} > \frac{1}{N}$. *Let* $u_0 \in B^{\frac{N}{p_1}-1}_{p_1,1}$, $f \in L^1_{loc}(\mathbb{R}^+, B^{\frac{N}{p_1}-1}_{p_1,1})$ *and* $a_0 \in B^{\frac{N}{p}}_{p,1}$ *with* $1 + a_0$ *bounded away from zero.*
There exists then a positive time T *such that system (1.1) has a solution* (a, u) *with* $1+a$ *bounded away from zero and:*

$$a \in \widetilde{C}([0,T], B^{\frac{N}{p}}_{p,1}), \;\; u \in \widetilde{C}([0,T]; B^{\frac{N}{p_1}-1}_{p_1,1} + B^{\frac{N}{p}+1}_{p,1}) \cap L^1([0,T], B^{\frac{N}{p}+1}_{p,1}).$$

Moreover this solution is unique if $\frac{2}{N} \le \frac{1}{p} + \frac{1}{p_1}$.

Remark 9 *This result is very interesting in the context of the BD viscosity coefficients. In this case our result is very close of the energy initial data with the optimal condition for the scaling* $(q_0, u_0) \in B^0_{\infty,1} \times B^0_{N,1}$. *In particular it applies to the shallow-water system. Indeed in [5] Bresch and Desjardins have discovered a new entropy inequality whenever the density-dependent viscosity coefficients satisfy the algebraic relation:*

$$\lambda(\rho) = \rho\mu'(\rho) - \mu(\rho).$$

In this case they show that we can control $\sqrt{\rho}\nabla\varphi(\rho)$ *in* $L^\infty(L^2)$ *where* $\varphi'(\rho) = \frac{\mu'(\rho)}{\rho}$. *Roughly it means that we control the density* ρ *in* $L^\infty(H^1)$. *It is very close in dimension* $N = 2$ *from the initial data that we need. Indeed we ask that* a_0 *belongs to* $B^{\frac{N}{p}}_{p,1}$, *and when* $p \ge 2$, *we have* $B^{\frac{N}{p}}_{p,1} \hookrightarrow B^{\frac{N}{2}}_{2,1}$. *By this way, the theorem 1.3 seems extremely critical in the case of initial data verifying the BD entropy.*

Remark 10 *Our method is more flexible than the proofs of D. Hoff in [23], [26], [22] as these works are based crucially on the notion of effective pressure and on a gain of integrability on the velocity which works only in the case of constant viscosity coefficients. By working in Besov space our technique of effective velocity appears more robust.*

We now are interested in showing a blow-up result for the solutions constructed in theorem 1.1. For this we will see as in [18] that only a control on the density is necessary to extend the strong solution of theorem 1.1. It is a crucial difference with the results on incompressible Navier-Stokes inasmuch as we need enough regularity on the velocity u for avoiding any blow-up effects. In our case, we get more precisely the following theorem.

Theorem 1.4 *Let P be a suitably smooth function of the density such that $P'(1) > 0$ and $1 \leq p_1 \leq p < +\infty$ such that $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p}$, $\frac{1}{p} + \frac{1}{p_1} > \frac{1}{N}$ and $p_1 = N + \varepsilon$ (where $\varepsilon > 0$ arbitrary small). Let $u_0 \in B^{\frac{N}{p_1}-1+\varepsilon}_{p_1,1}$, $f \in L^1_{loc}(\mathbb{R}^+, B^{\frac{N}{p_1}-1+\varepsilon}_{p_1,1})$ and $a_0 \in B^{\frac{N}{p}+\varepsilon}_{p,1}$ with $1 + a_0$ bounded away from zero. In addition we assume that $\rho_0^{\frac{1}{p_1}} u_0 \in L^{p_1}$, $u_0 \in B^0_{N,1}$ and $a_0 \in B^1_{N,1}$. Furthermore we assume the following conditions on the viscosity coefficients:*

$$\lambda \leq \frac{4\mu}{N^2(p_1 - 1)}, \tag{1.5}$$

Now, we assume that the solution constructed in theorem 1.1 satisfies on the time interval $[0, T)$ the following conditions:

- *the function a belongs to $L^\infty(0, T; B^{\frac{N}{p}+\varepsilon}_{p,\infty})$, with $\varepsilon > 0$ arbitrary small.*

Then (a, u) may be continued beyond T.

Remark 11 *As in [18], the main argument of the proof will be to obtain a gain of integrability on the velocity at the condition that we have enough integrability on the pressure.*

Remark 12 *We can observe that in this theorem, our assumption are subcritical on the initial data. In particular as we assume that $\rho_0 \geq c > 0$, we have $u_0 \in L^{p_1} \hookrightarrow B^{\frac{N}{p_1}-1}_{p_1,1}$ by Besov embedding. In fact we need of this additional assertion on the regularity of the initial data in order to prove that the time of existence T of the strong solution in theorem 1.1 depends on the initial data. More precisely we have:*

$$T \geq \frac{C}{(\|u_0\|_{L^{p_1}} + \|a_0\|_{B^{\frac{N}{p}+\varepsilon}_{p,1}} + \|\frac{1}{\rho_0}\|_{L^\infty} + \|f\|_{\widetilde{L}^1(B^{\frac{N}{p_1}-1+\varepsilon})})^\alpha},$$

with $\alpha > 0$. Here C and α depend on N and on the viscosity coefficients.

Remark 13 *Here we also assume that u_0 is in $B^0_{N,1}$ and a_0 in $B^1_{N,1}$. Indeed without these assumptions we are unable to prove some results of uniqueness. In fact we will have similar estimates on (a, u) than in theorem 1.1 and in addition by persistency results using similar techniques than for incompressible Navier-Stokes, we will show that:*

$$a \in \widetilde{L}^\infty_T(B^1_{N,1}) \;\; \text{and} \;\; u_0 \in \widetilde{L}^\infty_T(B^0_{N,1}) \cap \widetilde{L}^1_T(B^2_{N,1}).$$

It will be then enough to use some arguments of uniqueness.

Remark 14 *We would like to point out that we do not need to assert a control on the vacuum. Indeed as in [18] to control the norm L^∞ on the density, it is enough to control the norm of $\frac{1}{\rho}$ in L^∞.*

Remark 15 *As in [18], we obtain a criterion of blow-up for strong solution for compressible Navier-Stokes system without imposing a control Lipschitz on the norm ∇u as in [13]. In fact it improves [18] inasmuch as we are working with slightly subcritical initial data on the density and the velocity. In the case of [18], we need to control the velocity in dimension 3 in L^6 which is far from being critical.*

Our paper is structured as follows. In section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. Sections 3 and 4 are devoted to the proof of key estimates for the linearized system (1.1). In section 5.5, we prove the theorem 1.1 , theorem 1.2 and 1.3. whereas section 6 is devoted to the proof of continuation criterions of theorem 1.4.

2 Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X, we denote by $C([0,T],X)$ the set of continuous functions on $[0,T]$ with values in X. For $p \in [1,+\infty]$, the notation $L^p(0,T,X)$ or $L^p_T(X)$ stands for the set of measurable functions on $(0,T)$ with values in X such that $t \to \|f(t)\|_X$ belongs to $L^p(0,T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\alpha > 1$ and (φ, χ) be a couple of smooth functions valued in $[0,1]$, such that φ is supported in the shell supported in $\{\xi \in \mathbb{R}^N / \alpha^{-1} \leq |\xi| \leq 2\alpha\}$, χ is supported in the ball $\{\xi \in \mathbb{R}^N / |\xi| \leq \alpha\}$ such that:

$$\forall \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\begin{aligned}
&\Delta_{-1} u = \chi(D)u = \widetilde{h} * u \quad \text{with} \;\; \widetilde{h} = \mathcal{F}^{-1}\chi, \\
&\Delta_l u = \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y)u(x-y)dy \quad \text{with} \;\; h = \mathcal{F}^{-1}\chi, \;\; \text{if} \;\; l \geq 0, \\
&S_l u = \sum_{k \leq l-1} \Delta_k u \, .
\end{aligned}$$

Formally, one can write that: $u = \sum_{k \geq -1} \Delta_k u$. This decomposition is called nonhomogeneous Littlewood-Paley decomposition.

2.1 Nonhomogeneous Besov spaces and first properties

Definition 2.1 *For $s \in \mathbb{R}$, $p \in [1,+\infty]$, $q \in [1,+\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:*

$$\|u\|_{B^s_{p,q}} = (\sum_{l \geq -1} (2^{ls} \|\Delta_l u\|_{L^p})^q)^{\frac{1}{q}}.$$

The Besov space $B^s_{p,q}$ is the set of temperate distribution u such that $\|u\|_{B^s_{p,q}} < +\infty$.

Proposition 2.1 *The following properties hold:*

1. *there exists a constant universal C such that:* $C^{-1}\|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}$.

2. *If $p_1 < p_2$ and $r_1 \leq r_2$ then $B^s_{p_1,r_1} \hookrightarrow B^{s-N(1/p_1-1/p_2)}_{p_2,r_2}$.*

3. $B^{s'}_{p,r_1} \hookrightarrow B^s_{p,r}$ *if $s' > s$ or if $s = s'$ and $r_1 \leq r$.*

Before going further into the paraproduct for Besov spaces, let us state an important proposition.

Proposition 2.2 *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. Let $(u_q)_{q\geq -1}$ be a sequence of functions such that*

$$(\sum_{q\geq -1} 2^{qsr}\|u_q\|^r_{L^p})^{\frac{1}{r}} < +\infty.$$

If supp$\hat{u}_1 \subset \mathcal{C}(0, 2^q R_1, 2^q R_2)$ for some $0 < R_1 < R_2$ then $u = \sum_{q\geq -1} u_q$ belongs to $B^s_{p,r}$ and there exists a universal constant C such that:

$$\|u\|_{B^s_{p,r}} \leq C^{1+|s|}\big(\sum_{q\geq -1} (2^{qs}\|u_q\|_{L^p})^r\big)^{\frac{1}{r}}.$$

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [4]) and rewrite on a generalized form in [1] by H. Abidi and M. Paicu (in this article the results are written in the case of homogeneous spaces but it can easily generalize for the nonhomogeneous Besov spaces).

Proposition 2.3 *We have the following laws of product:*

- *For all $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ we have:*

$$\|uv\|_{B^s_{p,r}} \leq C(\|u\|_{L^\infty}\|v\|_{B^s_{p,r}} + \|v\|_{L^\infty}\|u\|_{B^s_{p,r}}). \tag{2.6}$$

- *Let $(p, p_1, p_2, r, \lambda_1, \lambda_2) \in [1, +\infty]^2$ such that:$\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:*
  if $s_1 + s_2 + N\inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,r}} \lesssim \|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}, \tag{2.7}$$

  when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B^{s_1}_{p_1,r}}\|v\|_{B^{s_2}_{p_2,\infty}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}}$) by $\|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,r}}$ (resp $\|v\|_{B^{s_2}_{p_2,\infty}\cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r = 1$.
  If $s_1 + s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$ and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ then:

$$\|uv\|_{B^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}_{p,\infty}} \lesssim \|u\|_{B^{s_1}_{p_1,1}}\|v\|_{B^{s_2}_{p_2,\infty}}. \tag{2.8}$$

  If $|s| < \frac{N}{p}$ for $p \geq 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:

$$\|uv\|_{B^s_{p,r}} \leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p}}_{p,\infty}\cap L^\infty}. \tag{2.9}$$

Remark 16 *In the sequel* p *will be either* p_1 *or* p_2 *and in this case* $\frac{1}{\lambda}=\frac{1}{p_1}-\frac{1}{p_2}$ *if* $p_1\leq p_2$ *or* $\frac{1}{\lambda}=\frac{1}{p_2}-\frac{1}{p_1}$ *if* $p_2\leq p_1$.

Corollary 1 *Let* $r\in[1,+\infty]$, $1\leq p\leq p_1\leq+\infty$ *and* s *such that:*

- $s\in(-\frac{N}{p_1},\frac{N}{p_1})$ *if* $\frac{1}{p}+\frac{1}{p_1}\leq 1$,
- $s\in(-\frac{N}{p_1}+N(\frac{1}{p}+\frac{1}{p_1}-1),\frac{N}{p_1})$ *if* $\frac{1}{p}+\frac{1}{p_1}>1$,

then we have if $u\in B^s_{p,r}$ *and* $v\in B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty$*:*

$$\|uv\|_{B^s_{p,r}}\leq C\|u\|_{B^s_{p,r}}\|v\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}.$$

We recall now a result concerning the composition for Besov spaces:

Proposition 2.4 *Let* I *be an open interval of* $\mathbb{R}$. *Let* $s>0$ *and* σ *be the smallest integer such that* $\sigma\geq s$. *Let* $F:I\to\mathbb{R}$ *satisfy* $F(0)=0$ *and* $F'\in W^{\sigma,\infty}(I;\mathbb{R})$. *Assume that* $v\in B^s_{p,r}$ *has values in* $J\subset\subset I$. *Then* $F(v)\in B^s_{p,r}$ *and there exists a constant* C *depending only on* s, I, J, *and* N, *and such that*

$$\|F(v)\|_{B^s_{p,r}}\leq C\big(1+\|v\|_{L^\infty}\big)^\sigma\|F'\|_{W^{\sigma,\infty}}\|v\|_{B^s_{p,r}}.$$

The study of non stationary PDE's requires spaces of type $L^\rho(0,T,X)$ for appropriate Banach spaces X. In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Paley decomposition. But, by doing so, we obtain bounds in spaces which are not type $L^\rho(0,T,X)$ (except if $r=p$). We are now going to define the spaces of Chemin-Lerner (see [8]) in which we will work, which are a refinement of the spaces $L^\rho_T(B^s_{p,r})$.

Definition 2.2 *Let* $\rho\in[1,+\infty]$, $T\in[1,+\infty]$ *and* $s_1\in\mathbb{R}$. *We set:*

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}=\big(\sum_{l\geq-1}2^{lrs_1}\|\Delta_l u(t)\|^r_{L^\rho(L^p)}\big)^{\frac{1}{r}}.$$

We then define the space $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *as the set of temperate distribution* u *over* $(0,T)\times\mathbb{R}^N$ *such that* $\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}<+\infty$.

We set $\widetilde{C}_T(\widetilde{B}^{s_1}_{p,r})=\widetilde{L}^\infty_T(\widetilde{B}^{s_1}_{p,r})\cap\mathcal{C}([0,T],B^{s_1}_{p,r})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\leq\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{if}\ r\geq\rho,\quad\|u\|_{\widetilde{L}^\rho_T(B^{s_1}_{p,r})}\geq\|u\|_{L^\rho_T(B^{s_1}_{p,r})}\ \text{if}\ r\leq\rho.$$

Remark 17 *It is easy to generalize proposition 2.3, to* $\widetilde{L}^\rho_T(B^{s_1}_{p,r})$ *spaces. The indices* s_1, p, r *behave just as in the stationary case whereas the time exponent* ρ *behaves according to Hölder inequality.*

Here we recall a result of interpolation which explains the link between the space $B^s_{p,1}$ and the space $B^s_{p,\infty}$, see [9].

Proposition 2.5 *There exists a constant C such that for all $s\in\mathbb{R}$, $\varepsilon>0$ and $1\le p<+\infty$,*

$$\|u\|_{\widetilde{L}^{\rho}_T(B^s_{p,1})}\le C\frac{1+\varepsilon}{\varepsilon}\|u\|_{\widetilde{L}^{\rho}_T(B^s_{p,\infty})}\Big(1+\log\frac{\|u\|_{\widetilde{L}^{\rho}_T(B^{s+\varepsilon}_{p,\infty})}}{\|u\|_{\widetilde{L}^{\rho}_T(B^s_{p,\infty})}}\Big).$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [9]).

Definition 2.3 *Let $m\in\mathbb{R}$. A smooth function function $f:\mathbb{R}^N\to\mathbb{R}$ is said to be a $\mathcal{S}^m$ multiplier if for all muti-index α, there exists a constant C_α such that:*

$$\forall\xi\in\mathbb{R}^N,\ \ |\partial^\alpha f(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

Proposition 2.6 *Let $m\in\mathbb{R}$ and f be a $\mathcal{S}^m$ multiplier. Then for all $s\in\mathbb{R}$ and $1\le p,r\le+\infty$ the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

We conclude this section by giving two lemma of commutators which will be useful in section 3 and 4. For a proof, we refer the reader to [3].

Lemma 1 *Let $1\le p_1\le p\le+\infty$ and $\sigma\in(-\min(\frac{N}{p},\frac{N}{p_1'}),\frac{N}{p}+1]$. There exists a sequence $c_q\in l^1(\mathbb{Z})$ such that $\|c_q\|_{l^1}=1$ and a constant C depending only on N and σ such that:*

$$\forall q\in\mathbb{Z},\ \ \|[v\cdot\nabla,\Delta_q]a\|_{L^{p_1}}\le Cc_q2^{-q\sigma}\|\nabla v\|_{B^{\frac{N}{p}}_{p,1}}\|a\|_{B^{\sigma}_{p_1,1}}. \tag{2.10}$$

In the limit case $\sigma=-\min(\frac{N}{p},\frac{N}{p_1'})$, we have:

$$\forall q\in\mathbb{Z},\ \ \|[v\cdot\nabla,\Delta_q]a\|_{L^{p_1}}\le Cc_q2^{q\frac{N}{p}}\|\nabla v\|_{B^{\frac{N}{p}}_{p,1}}\|a\|_{B^{-\frac{N}{p_1}}_{p,\infty}}. \tag{2.11}$$

Finally, for all $\sigma>0$ and $\frac{1}{p_2}=\frac{1}{p_1}-\frac{1}{p}$, there exists a constant C depending only on N and on σ and a sequence $c_q\in l^1(\mathbb{Z})$ with norm 1 such that:

$$\forall q\in\mathbb{Z},\ \ \|[v\cdot\nabla,\Delta_q]v\|_{L^p}\le Cc_q2^{-q\sigma}(\|\nabla v\|_{L^\infty}\|v\|_{B^{\sigma}_{p_1,1}}+\|\nabla v\|_{L^{p_2}}\|\nabla v\|_{B^{\sigma-1}_{p,1}}). \tag{2.12}$$

Lemma 2 *Let $1\le p_1\le p\le+\infty$ and $\alpha\in(1-\frac{N}{p},1]$, $k\in\{1,\cdots,N\}$ and $R_q=\Delta_q(a\partial_k w)-\partial_k(a\Delta_q w)$. There exists $c=c(\alpha,N,\sigma)$ such that:*

$$\sum_q 2^{q\sigma}\|R_q\|_{L^{p_1}}\le C\|a\|_{B^{\frac{N}{p}+\alpha}_{p,1}}\|w\|_{B^{\sigma+1-\alpha}_{p_1,1}} \tag{2.13}$$

whenever $-\frac{N}{p}<\sigma\le\alpha+\frac{N}{p}$.
In the limit case $\sigma=-\frac{N}{p}$, we have for some constant $C=C(\alpha,N)$:

$$\sup_q 2^{-q\frac{N}{p}}\|R_q\|_{L^{p_1}}\le C\|a\|_{B^{\frac{N}{p}+\alpha}_{p,1}}\|w\|_{B^{-\frac{N}{p_1}+1-\alpha}_{p_1,\infty}}. \tag{2.14}$$

Remark 18 *For proving proposition 3.9, we shall actually use the following non-stationary version of inequality (2.14):*

$$\sup_q 2^{-q\frac{N}{p}} \|R_q\|_{L^1_T(L^{p_1})} \leq C\|a\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\alpha}_{p,1})} \|w\|_{\widetilde{L}^1_T(B^{-\frac{N}{p_1}+1-\alpha}_{p_1,\infty})},$$

which may be easily proved by following the computations of the previous proof, dealing with the time dependence according to Hölder inequality. For a proof, we refer to [3].

3 Estimates for a parabolic system with variable coefficients

Let us first state estimates for the following constant coefficient parabolic system (see [3]):

$$\begin{cases} \partial_t u - \mu\Delta u - (\lambda+\mu)\nabla \mathrm{div} u = f, \\ u_{/t=0} = u_0. \end{cases} \tag{3.15}$$

Proposition 3.7 *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. Assume that $\mu > 0$ and that $\lambda + 2\mu > 0$. Then there exists a universal constant κ such that for all $s \in \mathbb{Z}$ and $T \in \mathbb{R}^+$,*

$$\|u\|_{\widetilde{L}^\infty_T(B^s_{p_1,1})} \leq C(\|u_0\|_{B^s_{p_1,1}} + \|f\|_{L^1_T(B^s_{p_1,1})}),$$

$$\kappa\nu\|u\|_{L^1_T(B^{s+2}_{p,r})} \leq \sum_{l\geq 0} 2^{ls}(1-e^{-\kappa\nu 2^{2l}T})(\|\Delta_l u_0\|_{L^{p_1}} + \|\Delta_l f\|_{L^1_T(L^{p_1})})$$

$$+ T\big(\|u_0\|_{B^s_{p,r}} + \|f\|_{L^1_T(B^s_{p,r})}\big),$$

with $\nu = \min(\mu, \lambda+2\mu)$.

We now consider the following parabolic system which is obtained by linearizing the momentum equation:

$$\begin{cases} \partial_t u - b\big(\mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u\big) = f + g, \\ u_{/t=0} = u_0 = u_0^1 +^2 u_0. \end{cases} \tag{3.16}$$

Above u is the unknown function and $b = 1 + a$. We assume that $u_0 \in B^s_{p_1,1}$, $f \in L^1(0,T; B^s_{p_1,1})$, $g \in L^1(0,T; B^s_{p_2,1})$, that b is bounded by below by a positive constant $\underline{b}$ and a belongs to $L^\infty(0,T; B^{\frac{N}{p}}_{p,1})$ with $p \in [1, +\infty]$.

Proposition 3.8 *Let $g = 0$. Let $\underline{\nu} = \underline{b}\min(\mu, \lambda+2\mu)$ and $\bar{\nu} = \mu + |\lambda+\mu|$. Assume that $s \in (-\frac{N}{p}, N\inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$ and $s \in (-\frac{N}{p_1'}, N\inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \geq 1$. Let $m \in \mathbb{Z}$ be such that $b_m = 1 + S_m a$ satisfies:*

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}^N} b_m(t,x) \geq \frac{\underline{b}}{2}. \tag{3.17}$$

There exist three constants c, C and κ (with c, C, depending only on N and on s, and κ universal) such that if in addition we have:

$$\|a - S_m a\|_{L^\infty(0,T; B^{\frac{N}{p}}_{p,1})} \leq c\frac{\underline{\nu}}{\bar{\nu}} \tag{3.18}$$

then setting:

$$Z_m(t) = 2^{2m}\bar{\nu}^2\underline{\nu}^{-1}\int_0^t \|a(\tau,\cdot)\|^2_{B^{\frac{N}{p}}_{p_1,1}}\,d\tau,$$

we have for all $T>0$,

$$\|u\|_{\widetilde{L}^\infty((0,T)\times B^s_{p_1,1})} + \kappa\underline{\nu}\|u\|_{\widetilde{L}^1((0,T)\times B^{s+2}_{p_1,1})} \leq e^{C(1+T)Z_m(T)}\big((1+T)\|u_0\|_{B^s_{p_1,1}} + \int_0^T e^{-C(1+\tau)Z_m(\tau)}\|f(\tau)\|_{B^s_{p_1,1}}d\tau\big).$$

Remark 19 *Let us stress the fact that if $a\in\widetilde{L}^\infty((0,T)\times B^{\frac{N}{p}}_{p,1})$ then assumption (3.17) and (3.18) are satisfied for m large enough. This will be used in the proof of theorem 1.1. Indeed, according to Bernstein inequality, we have:*

$$\|a-S_ma\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq \sum_{q\geq m}\|\Delta_q a\|_{L^\infty((0,T)\times\mathbb{R}^N)} \lesssim \sum_{q\geq m} 2^{q\frac{N}{p}}\|\Delta_q a\|_{L^\infty(L^p)}.$$

Because $a\in\widetilde{L}^\infty((0,T)\times B^{\frac{N}{p}}_{p,1})$, the right-hand side is the remainder of a convergent series hence goes to zero when m goes to infinity. For a similar reason, (3.18) is satisfied for m large enough.

Proof: In the sequel, we will treat only the case $p_1\leq p$, the other case is similar. Let us first rewrite (3) as follows:

$$\partial_t u - b_m\big(\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u\big) = f + E_m, \tag{3.19}$$

with $E_m = (\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u)(\mathrm{Id}-S_m)a$. Note that, because $s\in(-\frac{N}{p}, N\inf(\frac{1}{p},\frac{1}{p_1})]$ if $\frac{1}{p}+\frac{1}{p_1}\leq 1$ and $s\in(-\frac{N}{p_1'}, N\inf(\frac{1}{p},\frac{1}{p_1})]$ if $\frac{1}{p}+\frac{1}{p_1}\geq 1$, the error term E_m may be estimated by:

$$\|E_m\|_{B^s_{p_1,1}} \lesssim \|a-S_ma\|_{B^{\frac{N}{p}}_{p,1}}\|D^2u\|_{B^s_{p_1,1}}, \tag{3.20}$$

Now applying Δ_q to equation (3.19) yields for $q\geq 0$:

$$\frac{d}{dt}u_q - \mu\mathrm{div}(b_m\nabla u_q) - (\lambda+\mu)\nabla(b_m\mathrm{div}u_q) = f_q + E_{m,q} + \widetilde{R}_q. \tag{3.21}$$

where we denote by $u_q=\Delta_q u$ and with:

$$\widetilde{R}_q = \mu\big(\Delta_q(b_m\Delta u) - \mathrm{div}(b_m\nabla u_q)\big) + (\lambda+\mu)\big(\Delta_q(b_m\nabla\mathrm{div}u) - \nabla(b_m\mathrm{div}u_q)\big).$$

Next multiplying both sides by $|u_q|^{p_1-2}u_q$, and integrating by parts in the second, third and last term in the left-hand side, we get:

$$\frac{1}{p_1}\frac{d}{dt}\|u_q\|^{p_1}_{L^{p_1}} - \frac{1}{p_1}\int\big(|u_q|^{p_1}\mathrm{div}v + \mu\mathrm{div}(b_m\nabla u_q)|u_q|^{p_1-2}u_q + \xi\nabla\big(b_m\mathrm{div}u_q\big)|u_q|^{p_1-2}u_q\big)\big)dx \leq \|u_q\|^{p_1-1}_{L^{p_1}}(\|f_q\|_{L^{p_1}} + \|\Delta_q E_m\|_{L^{p_1}} + \|\widetilde{R}_q\|_{L^{p_1}}),$$

where we have denoted $\xi=\mu+\lambda$, $\nu=\min(\mu,\lambda+2\mu)$. Now by using (3.17), lemma [A5] of [10] and Young's inequalities we get:

$$\frac{1}{p_1}\frac{d}{dt}\|u_q\|_{L^{p_1}}^{p_1}+\frac{\nu\underline{b}(p_1-1)}{p_1^2}2^{2q}\|u_q\|_{L^{p_1}}^{p_1}\leq\|u_q\|_{L^{p_1}}^{p_1-1}\big(\|f_q\|_{L^{p_1}}+\|E_{m,q}\|_{L^{p_1}}+\|\widetilde{R}_q\|_{L^{p_1}}\big),$$

which leads, after time integration to:

$$\|u_q\|_{L^{p_1}}+\frac{\nu\underline{b}(p_1-1)}{p_1}2^{2q}\int_0^t\|u_q\|_{L^{p_1}}d\tau\leq\|\Delta_q u_0\|_{L^{p_1}}+\int_0^t\big(\|f_q\|_{L^{p_1}}+\|E_{m,q}\|_{L^{p_1}}+\|\widetilde{R}_q\|_{L^{p_1}}\big)d\tau,\tag{3.22}$$

where $\underline{\nu}=\underline{b}\nu$. For commutator $\widetilde{R}_q$, we have the following estimate (see lemma 2):

$$\|\widetilde{R}_q\|_{L^{p_1}}\lesssim c_q\bar{\nu}2^{-qs}\|S_m a\|_{B^{\frac{N}{p_1}+1}_{p_1,1}}\|Du\|_{B^s_{p_1,1}},\tag{3.23}$$

where $(c_q)_{q\in\mathbb{Z}}$ is a positive sequence such that $\sum_{q\in\mathbb{Z}}c_q=1$, and $\bar{\nu}=\mu+|\lambda+\mu|$. Note that, owing to Bernstein inequality, we have:

$$\|S_m a\|_{B^{\frac{N}{p}+1}_{p,1}}\lesssim 2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}.$$

Hence, plugging these latter estimates and (3.20) in (3.22), then multiplying by 2^{qs} and summing up on $q\geq 0$, we discover that, for all $t\in[0,T]$:

$$\begin{aligned}\|u-\Delta_1 u\|_{L^\infty_t(B^s_{p_1,1})}+\frac{\nu\underline{b}(p_1-1)}{p}\|u-\Delta_1 u\|_{L^1_t(B^{s+2}_{p_1,1})}\leq\|u_0\|_{B^s_{p_1,1}}+\|f\|_{L^1_t(B^s_{p_1,1})}\\+C\bar{\nu}\int_0^t(\|a-S_m a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+2}_{p_1,1}}+2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+1}_{p_1,1}})d\tau.\end{aligned}\tag{3.24}$$

We now need to control the block $\Delta_1 u$ corresponding to the low frequencies. To treat the term $\Delta_1 u$ similarly we apply the operator Δ_{-1} to the equation and by energy inequalities, we get:

$$\begin{aligned}\|\Delta_{-1}u(t)\|_{L^{p_1}}\leq\|\Delta_{-1}u_0\|_{L^{p_1}}+\|f\|_{L^1(L^{p_1})}+\int_0^t(\|a-S_m a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+2}_{p_1,1}}\\+2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+1}_{p_1,1}})ds,\end{aligned}$$

and:

$$\begin{aligned}\|\Delta_{-1}u(t)\|_{L^1_t(L^{p_1})}\leq t\big(\|\Delta_{-1}u_0\|_{L^{p_1}}+\|f\|_{L^1(L^{p_1})}+\int_0^t(\|a-S_m a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+2}_{p_1,1}}\\+2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+1}_{p_1,1}})ds\big).\end{aligned}$$

So we have by the two previous inequalities and (3.24):

$$\begin{aligned}\|u\|_{L^\infty_t(B^s_{p_1,1})}+\frac{\nu\underline{b}(p_1-1)}{p}\|u\|_{L^1_t(B^{s+2}_{p_1,1})}\leq C(1+t)\big(\|u_0\|_{B^s_{p_1,1}}+\|f\|_{L^1_t(B^s_{p_1,1})}\\+\bar{\nu}\int_0^t(\|a-S_m a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+2}_{p_1,1}}+2^m\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+1}_{p_1,1}})d\tau\big),\end{aligned}$$

for a constant C depending only on N and s. Let $X(t)=\|u\|_{L^\infty_t(B^s_{p_1,1})}+\nu\underline{b}\|u\|_{L^1_t(B^{s+2}_{p_1,1})}$. Assuming that m has been chosen so large as to satisfy:

$$C\bar{\nu}\|a-S_m a\|_{L^\infty_T(B^{\frac{N}{p}}_{p,1})}\leq\underline{\nu},$$

and using that by interpolation, we have:

$$C\bar{\nu}\|a\|_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^{s+2}_{p_1,1}}\leq\kappa\underline{\nu}+\frac{C^2\bar{\nu}^2 2^{2m}}{4\kappa\underline{\nu}}\|a\|^2_{B^{\frac{N}{p}}_{p,1}}\|u\|_{B^s_{p_1,1}}.$$

We end up with:

$$X(t)\leq C(1+t)\big(\|u_0\|_{B^s_{p_1,1}}+\|f\|_{L^1_t(B^s_{p_1,1})}+C\frac{\bar{\nu}^2}{\underline{\nu}}\int_0^t 2^{2m}\|a\|^2_{B^{\frac{N}{2}}_{p,1}}X(\tau)d\tau\big).$$

Grönwall lemma then leads to the desired inequality □

In the following corollary, we generalize proposition 3.9 when $g\neq 0$ and $g\in\widetilde{L}^1(B^{s'}_{p_2,1})$. Moreover here $u_0=u_1+u_2$ with $u_1\in B^s_{p_1,1}$ and $u_2\in B^{s'}_{p_2,1}$.

Corollary 2 *Let $\underline{\nu}=\underline{b}\min(\mu,\lambda+2\mu)$ and $\bar{\nu}=\mu+|\lambda+\mu|$. Assume that $s\in(-\frac{N}{p},N\inf(\frac{1}{p},\frac{1}{p_1})]$ if $\frac{1}{p}+\frac{1}{p_1}\leq 1$ and $s\in(-\frac{N}{p_1'},N\inf(\frac{1}{p},\frac{1}{p_1})]$ if $\frac{1}{p}+\frac{1}{p_1}\geq 1$. Moreover we assume that: $s'\in(-\frac{N}{p},N\inf(\frac{1}{p},\frac{1}{p_2})]$ if $\frac{1}{p}+\frac{1}{p_2}\leq 1$ and $s'\in(-\frac{N}{p_2'},N\inf(\frac{1}{p},\frac{1}{p_2})]$ if $\frac{1}{p}+\frac{1}{p_2}\geq 1$. Let $m\in\mathbb{Z}$ be such that $b_m=1+S_m a$ satisfies:*

$$\inf_{(t,x)\in[0,T)\times\mathbb{R}^N}b_m(t,x)\geq\frac{\underline{b}}{2}. \tag{3.25}$$

There exist three constants c, C and κ (with c, C, depending only on N and on s, s' and κ universal) such that if in addition we have:

$$\|a-S_m a\|_{L^\infty(0,T;B^{\frac{N}{p}}_{p,1})}\leq c\frac{\underline{\nu}}{\bar{\nu}} \tag{3.26}$$

then setting:

$$Z_m(t)=2^{2m}\bar{\nu}^2\underline{\nu}^{-1}\int_0^t\|a(\tau,\cdot)\|^2_{B^{\frac{N}{p}}_{p,1}}d\tau,$$

We have for all $T>0$,

$$\|u\|_{\widetilde{L}^\infty_T(B^s_{p_1,1}+B^{s'}_{p_2,1})}+\kappa\underline{\nu}\|u\|_{\widetilde{L}^1_T(B^{s+2}_{p_1,1}+B^{s'+2}_{p_2,1})}\leq e^{C(1+T)Z_m(T)}\big((1+T)(\|u_0^1\|_{B^s_{p_1,1}}+$$
$$\|u_0^2\|_{B^{s'}_{p_2,1}})+\int_0^T e^{-C(1+\tau)Z_m(\tau)}(\|f(\tau)\|_{B^s_{p_1,1}}+\|g(\tau)\|_{B^{s'}_{p_2,1}})d\tau\big).$$

Proof: We split the solution u in two parts u_1 and u_2 which verify the following equations:

$$\begin{cases}\partial_t u_1 + v\cdot\nabla u_1 + u_1\cdot\nabla w - b(\mu\Delta u_1 + (\lambda+\mu)\nabla\text{div}u_1 = f,\\ u_{/t=0} = u_1^0,\end{cases}$$

and:

$$\begin{cases}\partial_t u_2 + v\cdot\nabla u_2 + u_2\cdot\nabla w - b(\mu\Delta u_2 + (\lambda+\mu)\nabla\text{div}u_2 = g,\\ u_{/t=0} = u_2^0.\end{cases}$$

We have then $u = u_1 + u_2$ and we conclude by applying proposition 3.8. □

Proposition 3.8 fails in the limit case $s = -\frac{N}{p}$. The reason why is that proposition 2.3 cannot be applied any longer. One can however state the following result which will be the key to the proof of uniqueness in dimension two.

Proposition 3.9 *Under condition (3.17), there exists three constants c, C and κ (with c, C, depending only on N, and κ universal) such that if:*

$$\|a - S_m a\|_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \le c\frac{\underline{\nu}}{\bar{\nu}}, \tag{3.27}$$

then we have:

$$\|u\|_{L_t^\infty(B_{p_1,\infty}^{-\frac{N}{p_1}})} + \kappa\underline{\nu}\|u\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})} \le C(1+t)(\|u_0\|_{B_{p_1,\infty}^{-\frac{N}{p_1}}} + \|f\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{\frac{N}{p_1}})}),$$

whenever $t\in[0,T]$ satisfies:

$$\bar{\nu}^2 t(1+t)\|a\|^2_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \le c2^{-2m}\underline{\nu}. \tag{3.28}$$

Proof: We just point out the changes that have to be be done compare to the proof of proposition 3.8. The first one is that instead of (3.20), we have in accordance with proposition 2.3:

$$\|E_m\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})} \lesssim \|a - S_m a\|_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})}\|D^2 u\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})}, \tag{3.29}$$

The second change concerns the estimate of commutator $\widetilde{R}_q$. According remark 18, we now have for all $q\in\mathbb{Z}$:

$$\|\widetilde{R}_q\| \lesssim \bar{\nu}2^{q\frac{N}{p_1}}\|S_m a\|_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}+1})}\|Du\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})}. \tag{3.30}$$

Plugging all these estimates in (3.22) then taking the supremum over $q\in\mathbb{Z}$, we get:

$$\begin{aligned}\|u\|_{L_t^\infty(B_{p_1,\infty}^{-\frac{N}{p_1}})} + 2\underline{\nu}\|u\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})} \le (1+t)\big(\|u_0\|_{B_{p_1,1}^{-\frac{N}{p_1}}} + C\bar{\nu}\|a - S_m a\|_{\widetilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})}\|u\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})}\\ + 2^m\|a\|_{L_t^\infty(B_{p,1}^{\frac{N}{p}})}\|u\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{1-\frac{N}{p_1}})} + \|f\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})}\big).\end{aligned}$$

Using that:

$$\|u\|_{\widetilde{L}_t^1(B_{p_1,\infty}^{1-\frac{N}{p_1}})} \le \sqrt{t}\|u\|^{\frac{1}{2}}_{\widetilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})}\|u\|^{\frac{1}{2}}_{L_t^\infty(B_{p_1,\infty}^{\frac{N}{p_1}})},$$

and taking advantage of assumption (3.27) and (3.28), it is now easy to complete the proof. □

4 The mass conservation equation

Let us first recall standard estimates in Besov spaces for the following linear transport equation:

$$(\mathcal{H}) \qquad \begin{cases} \partial_t a + u\cdot\nabla a = g,\\ a_{/t=0} = a_0. \end{cases}$$

Proposition 4.10 *Let $1\leq p_1\leq p\leq +\infty$, $r\in[1,+\infty]$ and $s\in\mathbb{R}$ be such that:*

$$-N\min(\frac{1}{p_1},\frac{1}{p'}) < s < 1+\frac{N}{p_1}.$$

There exists a constant C depending only on N, p, p_1, r and s such that for all $a\in L^\infty([0,T],B^s_{p,r})$ of $(\mathcal{H})$ with initial data a_0 in $B^s_{p,r}$ and $g\in L^1([0,T],B^s_{p,r})$, we have for a.e $t\in[0,T]$:

$$\|f\|_{\widetilde{L}^\infty_t(B^s_{p,r})} \leq e^{CU(t)}\big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{B^s_{p_1,r}}d\tau\big), \tag{4.31}$$

with: $U(t)=\int_0^t\|\nabla u(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,\infty}\cap L^\infty}d\tau$.

For the proof of proposition 4.10, see [3]. We now focus on the mass equation associated to (1.3):

$$\begin{cases} \partial_t a + v\cdot\nabla a = (1+a)\mathrm{div} v,\\ a_{/t=0} = a_0. \end{cases} \tag{4.32}$$

Here we generalize a proof of R. Danchin in [13].

Proposition 4.11 *Let $r\in 1,+\infty$, $1\leq p_1\leq p\leq+\infty$ and $s\in(-\min(\frac{N}{p_1},\frac{N}{p'}),\frac{N}{p_1})$ if $r<+\infty$ and $s\in(-\min(\frac{N}{p_1},\frac{N}{p'}),\frac{N}{p_1})$ if $r=1$. Assume that $a_0\in B^s_{p,r}\cap L^\infty$, $v\in L^1(0,T;B^{\frac{N}{p_1}+1}_{p_1,1})$ and that $a\in\widetilde{L}^\infty_T(B^s_{p,r})\cap L^\infty_T$ satisfies (4.32). Let $V(t)=\int_0^t\|\nabla v(\tau)\|_{B^{\frac{N}{p_1}}_{p_1,1}}d\tau$. There exists a constant C depending only on N such that for all $t\in[0,T]$ and $m\in\mathbb{Z}$, we have:*

$$\|a\|_{\widetilde{L}^\infty_t(B^s_{p,r}\cap L^\infty)} \leq e^{2CV(t)}\|a_0\|_{B^s_{p,r}\cap L^\infty} + e^{2CV(t)}-1, \tag{4.33}$$

$$\|a-S_m a\|_{B^{\frac{N}{p}}_{p,1}} \leq \|a_0-S_m a_0\|_{B^{\frac{N}{p}}_{p,1}} + \frac{1}{2}(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}\cap L^\infty})(e^{2CV(t)}-1)+C\|a\|_{L^\infty}V(t), \tag{4.34}$$

$$\begin{aligned}\big(\sum_{l\leq m}2^{l\frac{N}{p}}\|\Delta_l(a-a_0)\|_{L^\infty_t(L^p)}\big) &\leq (1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})(e^{CV(t)}-1)\\ &\quad + C2^m\|a_0\|_{B^{\frac{N}{p}}_{p,1}}\int_0^t\|v\|_{B^{\frac{N}{p_1}}_{p_1,1}}d\tau.\end{aligned} \tag{4.35}$$

Proof: Applying Δ_l to (4.32) yields:

$$\partial_t \Delta_l a + v\cdot\nabla\Delta_l a = R_l + \Delta_l((1+a)\mathrm{div} v) \quad \text{with} \;\; R_l = [v\cdot\nabla, \Delta_l]a.$$

Multiplying by $\Delta_l a|\Delta_l a|^{p-2}$ then performing a time integration, we easily get:

$$\|\Delta_l a(t)\|_{L^p} \lesssim \|\Delta_l a_0\|_{L^p} + \int_0^t \big(\|R_l\|_{L^p} + \|\mathrm{div} v\|_{L^\infty}\|\Delta_l a\|_{L^p} + \|\Delta_l((1+a)\mathrm{div} v)\|_{L^p}\big)d\tau.$$

According to proposition 2.3 and interpolation, there exists a constant C and a positive sequence $(c_l)_{l\in\mathbb{N}}$ in l^r with norm 1 such that:

$$\|\Delta_l((1+a)\mathrm{div} v)\|_{L^p} \leq C c_l 2^{-ls}(1+\|a\|_{B^s_{p,r}\cap L^\infty})\|\mathrm{div} v\|_{B^{\frac{N}{p_1}}_{p_1,1}}.$$

Next the term $\|R_l\|_{L^p}$ may be bounded according to lemma 1. We end up with:

$$\forall t\in[0,T],\ \forall l\in\mathbb{Z},\ \ 2^{ls}\|\Delta_l a(t)\|_{L^p} \leq 2^{ls}\|\Delta_l a_0\|_{L^p} + C\int_0^t c_l(1+\|a\|_{B^s_{p,r}\cap L^\infty})V^{'}d\tau, \quad (4.36)$$

hence, summing up on $\mathbb{Z}$ in l^r,

$$\forall t\in[0,T],\ \forall l\in\mathbb{Z},\ \ \|a(t)\|_{B^s_{p,r}} \leq \|a_0\|_{B^s_{p,r}} + \int_0^t CV^{'}\|a(\tau)\|_{B^s_{p,r}}d\tau + \int_0^t C(1+\|a\|_{L^\infty_T})V^{'}d\tau.$$

Next we have:

$$\|a\|_{L^\infty_t} \leq \int_0^t (1+\|a(\tau)\|_{L^\infty})V^{'}(\tau)d\tau.$$

By summing the two previous inequalities, applying Gronwall lemma and proposition 2.2 yields inequality (4.33). Let us now prove inequality (4.34). Starting from (4.36) and summing up over $l\geq m$ in l^r, we get:

$$\begin{aligned}(\sum_{l\geq m} 2^{lsr}\|\Delta_l a\|^r_{L^\infty_t(L^p)})^{\frac{1}{r}} \leq (\sum_{l\geq m} 2^{lsr}\|\Delta_l a_0\|^r_{L^p})^{\frac{1}{r}} + C\int_0^t V^{'}(e^{2CV}\|a_0\|_{B^s_{p,r}\cap L^\infty} + e^{2CV}-1)d\tau \\ + \int_0^t C(1+\|a\|_{L^\infty})V^{'}d\tau.\end{aligned}$$

Straightforward calculations then leads to (4.34). In order to prove (4.35), we use the fact that $\widetilde{a} = a - a_0$ satisfies:

$$\begin{cases} \partial_t\widetilde{a} + v\cdot\nabla\widetilde{a} = (1+\widetilde{a})\mathrm{div} v + a_0\mathrm{div} v - v\cdot\nabla a_0, \\ \widetilde{a}_{/t=0} = 0. \end{cases}$$

Therefore, arguing as for proving (4.36), we get for all $t\in[0,T]$ and $l\in\mathbb{Z}$,

$$\begin{aligned}2^{l\frac{N}{p}}\|\Delta_l\widetilde{a}\|_{L^p} \leq \int_0^t 2^{l\frac{N}{p}}\big(\|\Delta_l(a_0\mathrm{div} v)\|_{L^p} + \|\Delta_l(v\cdot\nabla a_0)\|_{L^p}\big)d\tau \\ + C\int_0^t c_l(1+\|a\|_{B^{\frac{N}{p}}_{p,1}})V^{'}d\tau.\end{aligned}$$

Since $B^{\frac{N}{p}}_{p,1}$ is an algebra and the product maps $B^{\frac{N}{p}}_{p,1}\times B^{\frac{N}{p}-1}_{p,1}$ in $B^{\frac{N}{p}-1}_{p,1}$, we discover that:

$$2^{l\frac{N}{p}}\|\Delta_l\widetilde{a}\|_{L^\infty(L^p)}\leq C\big(\int_0^t 2^l c_l\|a_0\|_{B^{\frac{N}{p}}_{p,1}}\|v\|_{B^{\frac{N}{p}}_{p,1}}d\tau+\int_0^t c_l(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}}+\|a\|_{B^{\frac{N}{p}}_{p,1}})V^{'}d\tau\big),$$

hence, summing up on $l\leq m$,

$$\sum_{l\leq m}2^{l\frac{N}{p}}\|\Delta_l\widetilde{a}\|_{L^\infty(L^p)}\leq C\big(\int_0^t 2^m\|a_0\|_{B^{\frac{N}{p}}_{p,1}}\|v\|_{B^{\frac{N}{p}}_{p,1}}d\tau+\int_0^t (1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}}+\|a\|_{B^{\frac{N}{p}}_{p,1}})V^{'}d\tau\big),$$

Plugging (4.33) in the right-hand side yields (4.35).

5 The proof of theorem 1.1

5.1 Strategy of the proof

To improve the results of R. Danchin in [10], [13] and of the author [15, 16], it is crucial to *kill* the coupling between the velocity and the pressure. To achieve it, we need to include the pressure term in the study of the linearized equation of the momentum equation. For that, we will try to express the gradient of the pressure as a Laplacian term, so we have to solve for $\bar{\rho}=1$ a constant state:

$$\Delta v=\nabla P(\rho).$$

Let $\mathcal{E}$ be the fundamental solution of the Laplace operator. We will set in the sequel: $v=\nabla\mathcal{E}*\big(P(\rho)-P(\bar{\rho})\big)=\nabla\big(\mathcal{E}*[P(\rho)-P(\bar{\rho})]\big)$ ($*$ here means the operator of convolution). We verify next that:

$$\nabla\text{div}v=\nabla\Delta\big(\mathcal{E}*[P(\rho)-P(\bar{\rho})]\big)=\Delta\nabla\big(\mathcal{E}*[P(\rho)-P(\bar{\rho})]\big)=\Delta v=\nabla P(\rho).$$

By this way we can now rewrite the momentum equation of (1.3) as:

$$\partial_t u+u\cdot\nabla u-\frac{\mu}{\rho}\Delta\big(u-\frac{1}{\nu}v\big)-\frac{\lambda+\mu}{\rho}\nabla\text{div}\big(u-\frac{1}{\nu}v\big)=f,$$

with $\nu=2\mu+\lambda$. We now want to calculate $\partial_t v$, by the transport equation we get:

$$\partial_t v=\nabla\mathcal{E}*\partial_t P(\rho)=-\nabla\mathcal{E}*\big(P^{'}(\rho)\text{div}(\rho u)\big).$$

Notation 1 *To simplify the notation, we will note in the sequel*

$$\nabla\mathcal{E}*\big(P^{'}(\rho)\text{div}(\rho u)\big)=\nabla(\Delta)^{-1}\big(P^{'}(\rho)\text{div}(\rho u)\big).$$

Finally we can now rewrite the system (1.3) as follows:

$$\begin{cases}\partial_t q+(v_1+\frac{1}{\nu}v)\cdot\nabla q+\frac{1}{\nu}P^{'}(1)q=-(1+q)\text{div}v_1\\ \qquad\qquad -\frac{1}{\nu}(P(\rho)-P(1)-P^{'}(1))-\frac{1}{\nu}q(P(\rho)-P(1)),\\ \partial_t v_1-\frac{1}{1+q}\mathcal{A}v_1=f-u\cdot\nabla u+\frac{1}{\nu}\nabla(\Delta)^{-1}\big(P^{'}(\rho)\text{div}(\rho u)\big),\\ q_{/t=0}=a_0,\ (v_1)_{/t=0}=(v_1)_0,\end{cases}\tag{5.37}$$

where $v_1 = u - \frac{1}{\nu} v$ is called the effective velocity. In the sequel we will study this system by exhibiting some uniform bounds in Besov spaces on (q, v_1). The advantage of the system (5.37) is that we have canceled out the coupling between v_1 and a term of pressure. Indeed in the works [10, 13, 15, 16], the pressure was included in the study of the linear system, thus entailing a coupling between the density and the velocity. In particular it was impossible to prescribe different index of integration in Besov spaces for the velocity and the density.

5.2 Proof of the existence

Construction of approximate solutions

We use a standard scheme:

1. We smooth out the data and get a sequence of smooth solutions $(a^n, u^n)_{n\in\mathbb{N}}$ to (1.3) on a bounded interval $[0, T^n]$ which may depend on n. We set $v_1^n = u^n - v^n$ where $\mathrm{div} v^n = P(\rho^n) - P(\bar{\rho})$ with $v^n = \nabla\mathcal{E} * \big(P(\rho^n) - P(\bar{\rho})\big)$.

2. We exhibit a positive lower bound T for T^n (which does not depend on n), and prove uniform estimates on (a^n, v_1^n) in the space:
$$E_T = \widetilde{C}_T(B^{\frac{N}{p}}_{p,1}) \times \big(\widetilde{C}_T(B^{\frac{N}{p_1}-1}_{p_1,1} + B^{\frac{N}{p}+1}_{p,1}) \cap \widetilde{L}^1_T(B^{\frac{N}{p_1}+1}_{p_1,1} + B^{\frac{N}{p}+2}_{p,1})\big).$$
We will deduce then that (a^n, u^n) belong to the space:
$$F_T = \widetilde{C}_T(B^{\frac{N}{p}}_{p,1}) \times \big(\widetilde{C}_T(B^{\frac{N}{p_1}-1}_{p_1,1} + B^{\frac{N}{p}+1}_{p,1}) \cap \widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,1})\big).$$

3. We use compactness to prove that the sequence (a^n, u^n) converges, up to extraction, to a solution of (5.37).

Througout the proof, we denote $\underline{\nu} = \underline{b}\min(\mu, \lambda + 2\mu)$ and $\bar{\nu} = \mu + |\mu + \lambda|$, and we assume (with no loss of generality) that f belongs to $\widetilde{L}^1_T(B^{\frac{N}{p_1}-1}_{p_1,1})$.

First step

We smooth out the data as follows:
$$a_0^n = S_n a_0, \;\; u_0^n = S_n u_0 \quad \text{and} \quad f^n = S_n f.$$
Note that we have:
$$\forall l \in \mathbb{Z}, \;\; \|\Delta_l a_0^n\|_{L^p} \leq \|\Delta_l a_0\|_{L^p} \quad \text{and} \quad \|a_0^n\|_{B^{\frac{N}{p}}_{p,\infty}} \leq \|a_0\|_{B^{\frac{N}{p}}_{p,\infty}},$$
and similar properties for u_0^n and f^n, a fact which will be used repeatedly during the next steps. Now, according [13], one can solve (1.3) with the smooth data (a_0^n, u_0^n, f^n). We get a solution (a^n, u^n) on a non trivial time interval $[0, T_n]$ such that:
$$a^n \in \widetilde{C}([0, T_n), B^N_{2,1}) \;\text{ and }\; u^n \in \widetilde{C}([0, T_n), B^{\frac{N}{2}-1}_{2,1}) \cap \widetilde{L}^1_{T_n}(B^{\frac{N}{2}+1}_{2,1}). \tag{5.38}$$

Uniform bounds

Let T_n be the lifespan of (a_n,u_n), that is the supremum of all $T>0$ such that (1.1) with initial data (a_0^n,u_0^n) has a solution which satisfies (5.38). Let T be in $(0,T_n)$. We aim at getting uniform estimates in E_T for T small enough. For that, we need to introduce the solution $(v_1^n)_L$ to the linear system:

$$\begin{aligned}&\partial_t(v_1^n)_L-\mathcal{A}(v_1^n)_L=f^n,\\&(v_1^n)_L(0)=(v_1^n)_0.\end{aligned}$$

Now, we set $u^n=v_1^n+\frac{1}{\nu}v^n$ with $v^n=\nabla\mathcal{E}*\big(P(\rho^n)-P(\bar\rho)\big)$ (in particular we have $\operatorname{div}v^n=P(\rho^n)-P(1)$). Finally we set $\widetilde{v}_1^n=v_1^n-(v_1^n)_L$ and we can check that $\widetilde{v}_1^n$ satisfies the parabolic system:

$$\begin{cases}\partial_t\widetilde{v}_1^n-(1+a^n)\mathcal{A}\widetilde{v}_1^n=-((v_1^n)_L+\dfrac{1}{\nu}v^n)\cdot\nabla\widetilde{v}_1^n-\widetilde{v}_1^n\cdot\nabla u^n+a^n\mathcal{A}(v_1^n)_L-\dfrac{1}{\nu}((v_1^n)_L\cdot\nabla v^n\\ \qquad\qquad+v^n\cdot\nabla(v_1^n)_L+\dfrac{1}{\nu}v^n\cdot\nabla v^n)-(v_1^n)_L\cdot\nabla(v_1^n)_L+\dfrac{1}{\nu}\nabla(\Delta)^{-1}(P^{'}(\rho^n)\operatorname{div}(\rho^nu^n)),\\(\widetilde{v}_1^n)_{\ t=0}=0.\end{cases}\tag{5.39}$$

which has been studied in proposition 3.8. Define $m\in\mathbb{Z}$ by:

$$m=\inf\{p\in\mathbb{N}/\ 2\bar\nu\sum_{l\geq p}2^{l\frac{N}{p}}\|\Delta_l a_0^n\|_{L^p}\leq c\bar\nu\}\tag{5.40}$$

where c is small enough positive constant (depending only on N) to be fixed hereafter. In the sequel we will need of a control on $a^n-S_ma^n$ small to apply proposition 3.8, so here m is enough big (we explain how in the sequel). Let:

$$\bar b=1+\sup_{x\in\mathbb{R}^N}a_0(x),\ A_0=1+2\|a_0\|_{B^{\frac{N}{p}}_{p,1}},\ U_0=\|u_0\|_{B^{\frac{N}{p_1}-1}_{p_1,1}}+\|a_0\|_{B^{\frac{N}{p}}_{p,1}}+\|f\|_{L^1_T(B^{\frac{N}{p_1}-1}_{p_1,1})},$$

and $\widetilde{U}_0=2CU_0+4C\bar\nu A_0$ (where $C^{'}$ is a constant embedding and C stands for a large enough constant depending only N which will be determined when applying proposition 2.3, 3.8 and 4.10 in the following computations.) We assume that the following inequalities are fulfilled for some $\eta>0$:

$$(\mathcal{H}_1)\qquad \|a^n-S_ma^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\leq c\underline{\nu}\bar\nu^{-1},$$

$$(\mathcal{H}_2)\qquad C\bar\nu^2T\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\leq 2^{-2m}\underline{\nu},$$

$$(\mathcal{H}_3)\qquad \frac{1}{2}\underline{b}\leq 1+a^n(t,x)\leq 2\bar b\ \text{ for all }\ (t,x)\in[0,T]\times\mathbb{R}^N,$$

$$(\mathcal{H}_4)\qquad \|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\leq A_0,$$

$$(\mathcal{H}_5)\qquad \|(v_1^n)_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\leq U_0,\ \ \|(v_1^n)_L\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+3}_{p,1})}\leq\eta,$$

$$(\mathcal{H}_6)\qquad \|\widetilde{v}_1^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}+\underline{\nu}\|\widetilde{v}_1^n\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})}\leq \widetilde{U}_0\eta,$$

$$(\mathcal{H}_7)\qquad \|v^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+1}_{p,1})}\leq C^{'}A_0,$$

$$(\mathcal{H}_8)\qquad \|\nabla u^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}\leq (\underline{\nu}^{-1}\widetilde{U}_0+1)\eta$$

Remark that since:

$$1+S_m a^n=1+a^n+(S_m a^n-a^n),$$

assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_3)$ combined with the embedding $B^{\frac{N}{p}}_{p,1}\hookrightarrow L^\infty$ insure that:

$$\inf_{(t,x)\in[0,T]\times\mathbb{R}^N}(1+S_m a^n)(t,x)\geq \frac{1}{4}\underline{b}, \tag{5.41}$$

provided c has been chosen small enough (note that $\frac{\underline{\nu}}{\bar{\nu}}\leq \bar{b}$).
We are going to prove that under suitable assumptions on T and η (to be specified below) if condition $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$ are satisfied, then they are actually satisfied with strict inequalities. Since all those conditions depend continuously on the time variable and are strictly satisfied initially, a basic bootstrap argument insures that $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$ are indeed satisfied for T enough small (with a T which could depend of n). In the sequel, we will see that these conditions on T do not depend on n and by a criterion of continuation we will see that our T check $T\leq T_n$.
First we shall assume that η and T satisfies:

$$C(1+\underline{\nu}^{-1}\widetilde{U}_0)\eta+\frac{C^{'}}{\nu}A_0T<\log 2 \tag{5.42}$$

so that denoting:

$$\widetilde{V}_1^n(t)=\int_0^t\|\nabla\widetilde{v}_1^n\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1}}d\tau,\ \ V^n(t)=\frac{1}{\nu}\int_0^t\|\nabla v^n\|_{B^{\frac{N}{p}}_{p,1}}d\tau\ \text{ and}$$

$$(V_1^n)_L(t)=\int_0^t\|\nabla(v_1^n)_L\|_{B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}}d\tau,\ \ U^n(t)=\widetilde{V}_1^n(t)+(V_1^n)_L(t)+V^n(t).$$

We have, according to $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$:

$$e^{C((V_1^n)_L+\widetilde{V}_1^n+\widetilde{V}^n)(T)}<2\ \text{ and }\ e^{C((V_1^n)_L+\widetilde{V}_1^n+\widetilde{V}^n)(T)}-1\leq 1. \tag{5.43}$$

In order to bound a^n in $\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})$, we apply inequality (4.33) and get:

$$\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}<1+2\|a_0\|_{B^{\frac{N}{p}}_{p,1}}=A_0. \tag{5.44}$$

Hence $(\mathcal{H}_4)$ is satisfied with a strict inequality. $(\mathcal{H}_7)$ verifies a strict inequality, it follows from proposition 2.6 and $(\mathcal{H}_4)$. Next, applying proposition 3.7 and proposition 2.6 yields:

$$\|(v_1^n)_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\leq U_0, \tag{5.45}$$

$$\kappa\nu\|(v_1^n)_L\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+3}_{p,1})}\leq\sum_{l\geq 0}2^{l(\frac{N}{p_1}-1)}(1-e^{-\kappa\nu 2^{2l}T})(\|\Delta_l u_0\|_{L^{p_1}}+\|\Delta_l a_0\|_{L^p})+TU_0. \tag{5.46}$$

Hence taking T such that:

$$\kappa\nu\|(v_1^n)_L\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+3}_{p,1})}\leq\kappa\eta\nu, \tag{5.47}$$

insures that $(\mathcal{H}_5)$ is strictly verified. Since $(\mathcal{H}_1)$, $(\mathcal{H}_2)$, $(\mathcal{H}_5)$, $(\mathcal{H}_6)$, $(\mathcal{H}_7)$ and (5.41) are satisfied, proposition 3.8 may be applied, we obtain:

$$\begin{aligned}
&\|\widetilde{v}_1^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}+\underline{\nu}\|\widetilde{v}_1^n\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})}\\
&\qquad\leq Ce^{C(1+T)}\int_0^T\big(\|a^n\mathcal{A}(v_1^n)_L\|_{B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1}}+\|(v_1^n)_L\cdot\nabla(v_1^n)_L\|_{B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1}}\\
&+\|(v_1^n)_L\cdot\nabla v^n\|_{B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1}}+\|v^n\cdot\nabla v^n\|_{B^{\frac{N}{p}}_{p,1}}+\|\nabla(\Delta)^{-1}(P^{'}(\rho^n)\mathrm{div}(\rho^n u^n))\|_{B^{\frac{N}{p}}_{p,1}}\\
&\qquad+\|v^n\cdot\nabla(v_1^n)_L\|_{B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1}}+\|((v_1^n)_L+\frac{1}{\nu}v^n)\cdot\nabla\tilde{v}_1^n\|_{B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1}}\\
&\qquad\qquad+\|\widetilde{v}_1^n\cdot\nabla u^n\|_{B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1}}\big)dt.
\end{aligned}$$

As $\frac{N}{p}+\frac{N}{p_1}-1\geq 0$ and $P^{'}(\rho^n)\mathrm{div}(\rho^n u^n)=\nabla P(\rho^n)\cdot u^n+P^{'}(\rho^n)\rho^n\mathrm{div}u^n$, we can take advantage of proposition 2.3, 2.1 and 2.6. (In passing, we would like to mention that here a crucial point is that $\Delta\widetilde{v}_1^n$ belongs to $\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1}+B^{\frac{N}{p_1}-1}_{p_1,1})$, it means that we are able to give sense to the product $a^n\mathcal{A}\widetilde{v}_1^n$ with the condition $\frac{N}{p}+\frac{N}{p_1}-1\geq 0$. It is the main novelty compared with the works of R. Danchin in [10, 13], indeed we are able to cancel out in some sense the coupling between the pressure term and the velocity. And it is exactly at this point that we can use paraproduct laws without the restrictions that it exist in [10, 13]. An other way to express this point is to say that the constraints concerning the law of paraproduct for the term $a\Delta u$ are less important. It means that we are able to ask no more than the hypothesis on p and p_1 used in the case of Navier-Stokes with dependent density (see [1] and [17]).)

We get then with h and h_1 regular function checking the conditions of proposition 2.4:

$$\begin{aligned}
&\|\nabla(\Delta)^{-1}(P^{'}(\rho^n)\mathrm{div}(\rho^n u^n))\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}\\
&\leq C(\|\nabla(\Delta)^{-1}(h(a^n)\mathrm{div}(h_1(a^n)u^n))\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}+\|\nabla(\Delta)^{-1}(\mathrm{div}(u^n))\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}),\\
&\leq C_P\big(\|u^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}(1+\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})})\big),\\
&\leq C_P\big(\sqrt{T}(\|(v_1^n)_L\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p,1}+B^{\frac{N}{p}+2}_{p,1})}+\|\widetilde{v}_1^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p,1}+B^{\frac{N}{p}+1}_{p,1})}+T\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})})\\
&\qquad\times(1+\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})})\big).
\end{aligned}$$

The next term $v^n\cdot\nabla v^n$ determines the choice of working in the space $\widetilde{L}^1_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})$ for the remainder, indeed we recall here that at the difference of the works in [10, 13], we have no coupling between the density and the velocity. So in this sense this term is crucial inasmuch as he decides on the regularity of $\widetilde{v}_1^n$ and in particular the following term $v^n\cdot\nabla v^n$ where we have:

$$\|v^n\cdot\nabla v^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}\leq C_1T\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}.$$

We proceed similarly for the other terms and we end up with:

$$\begin{aligned}
&\|\widetilde{v}_1^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}+\underline{\nu}\|\widetilde{v}_1^n\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})}\leq Ce^{C(1+T)}\Big(C_1T\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\\
&+C_P\big(\sqrt{T}(\|(v_1^n)_L\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p,1}+B^{\frac{N}{p}+2}_{p,1})}+\|\widetilde{v}_1^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p,1}+B^{\frac{N}{p}+1}_{p,1})}+T\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})})\\
&\qquad\times(1+\|a^n\|^2_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})})\big)+\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\|(v_1^n)_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}+1}_{p,1}+B^{\frac{N}{p}+3}_{p,1})}\\
&\qquad\qquad+\|(v_1^n)_L\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p,1}+B^{\frac{N}{p}+1}_{p,1})}\|(v_1^n)_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}+1}_{p,1}+B^{\frac{N}{p}+3}_{p,1})}\\
&+\sqrt{T}\|u_L^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p,1}+B^{\frac{N}{p}+2}_{p,1})}\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}+T\|\widetilde{v}_1^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\\
&+\big(\|(v_1^n)_L\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p,1}+B^{\frac{N}{p}+2}_{p,1})}+\sqrt{T}\|a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})}\big)\|\widetilde{v}_1^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\\
&+\|\widetilde{v}_1^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}\big(\|\widetilde{v}_1^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})}+\|(v_1^n)_L\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}+1}_{p,1}+B^{\frac{N}{p}+3}_{p,1})}\big)\Big)
\end{aligned}\tag{5.48}$$

with $C=C(N)$and $C_P=(N,P,\underline{b},\bar{b})$. Now, using assumptions $(\mathcal{H}_4)$, $(\mathcal{H}_5)$ and $(\mathcal{H}_6)$, and inserting (5.43) in (5.48) gives:

$$\begin{aligned}
\|\widetilde{v}_1^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}+\|\widetilde{v}_1^n\|_{L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})}&\leq 2C(\bar{\nu}A_0+\widetilde{U}_0^2\eta+U_0)\eta+C_1TA_0(1+A_0)\\
&\quad+\sqrt{T}A_0U_0,
\end{aligned}$$

hence $(\mathcal{H}_6)$ is satisfied with a strict inequality provided when T and η verifies:

$$2C(\bar{\nu}A_0+U_0+\widetilde{U}_0^2\eta)\eta+C_1TA_0(1+A_0)+\sqrt{T}A_0U_0<C\bar{\nu}\eta.\tag{5.49}$$

$(\mathcal{H}_8)$ follows from proposition $(\mathcal{H}_5)$, $(\mathcal{H}_6)$ and $(\mathcal{H}_7)$ indeed we recall that by Besov embedding as $p_1\leq p$:

$$\|\nabla u^n\|_{\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})}\leq\int_0^t\|\nabla\widetilde{v}_1^n\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1}}d\tau+\frac{1}{\nu}\int_0^t\|\nabla v^n\|_{B^{\frac{N}{p}}_{p,1}}d\tau+\int_0^t\|\nabla(v_1^n)_L\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}}d\tau.$$

We easily obtains by this previous inequality $(\mathcal{H}_8)$.
We now have to check whether $(\mathcal{H}_1)$ is satisfied with strict inequality. For that we apply proposition (4.11) which yields for all $m\in\mathbb{Z}$,

$$\sum_{l\geq m}2^{l\frac{N}{2}}\|\Delta_l a^n\|_{L^\infty_T(L^p)}\leq\sum_{l\geq m}2^{l\frac{N}{p}}\|\Delta_l a_0\|_{L^p}+(1+\|a_0\|_{B^{\frac{N}{2}}_{p,1}})\big(e^{C((V_1^n)_L+\widetilde{V}_1^n+V^n)(T)}-1\big).\tag{5.50}$$

Using (5.42) and $(\mathcal{H}_5)$, $(\mathcal{H}_6)$, we thus get:

$$\|a^n - S_m a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} \leq \sum_{l\geq m} 2^{l\frac{N}{p}}\|\Delta_l a_0\|_{L^p} + \frac{C}{\log 2}(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})(1+\underline{\nu}^{-1}\widetilde{L}_0)\eta.$$

Hence $(\mathcal{H}_1)$ is strictly satisfied provided that η further satisfies:

$$\frac{C}{\log 2}(1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})(1+\underline{\nu}^{-1}\widetilde{U}_0)\eta < \frac{c\underline{\nu}}{2\bar{\nu}}. \tag{5.51}$$

In order to check whether $(\mathcal{H}_3)$ is satisfied, we use the fact that:

$$a^n - a_0 = S_m(a^n - a_0) + (Id - S_m)(a^n - a_0) + \sum_{l>m}\Delta_l a_0,$$

whence, using $B^{\frac{N}{p}}_{p,1} \hookrightarrow L^\infty$:

$$\|a^n - a_0\|_{L^\infty((0,T)\times\mathbb{R}^N)} \leq C\big(\|S_m(a^n-a_0)\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} + \|a^n - S_m a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} + 2\sum_{l\geq m} 2^{l\frac{N}{p}}\|\Delta_l a_0\|_{L^p}\big).$$

Changing the constant c in the definition of m and in (5.51) if necessary, one can, in view of the previous computations, assume that:

$$C\big(\|a^n - S_m a^n\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} + 2\sum_{l\geq m} 2^{l\frac{N}{p}}\|\Delta_l a_0\|_{L^p}\big) \leq \frac{\underline{b}}{4}.$$

As for the term $\|S_m(a^n - a_0)\|_{L^\infty_T(B^{\frac{N}{p}}_{p,1})}$, it may be bounded according proposition 4.11:

$$\|S_m(a^n-a_0)\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} \leq (1+\|a_0\|_{B^{\frac{N}{p}}_{p,1}})(e^{C(\widetilde{V}^n_1+V^n+(V^n_1)_L)(T)} - 1) + C2^{2m}\sqrt{T}\|a_0\|_{B^{\frac{N}{p}}_{p,1}} \times \|u^n\|_{\widetilde{L}^2_T(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}}_{p,1})}.$$

Note that under assumptions $(\mathcal{H}_5)$, $(\mathcal{H}_6)$, (5.42) and (5.51) (and changing c if necessary), the first term in the right-hand side may be bounded by $\frac{\underline{b}}{8}$. Hence using interpolation, (5.45) and the assumptions (5.42) and (5.51), we end up with:

$$\|S_m(a^n-a_0)\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} \leq \frac{\underline{b}}{8} + C2^m\sqrt{T}\|a_0\|_{B^{\frac{N}{p}}_{p,1}}\sqrt{\eta(U_0+\widetilde{U}_0\eta)(1+\underline{\nu}^{-1}\widetilde{U}_0}.$$

Assuming in addition that T satisfies:

$$C2^m\sqrt{T}\|a_0\|_{B^{\frac{N}{p}}_{p,1}}\sqrt{\eta(U_0+\widetilde{U}_0\eta)(1+\underline{\nu}^{-1}\widetilde{U}_0} < \frac{\underline{b}}{8}, \tag{5.52}$$

and using the assumption $\underline{b} \leq 1 + a_0 \leq \bar{b}$ yields $(\mathcal{H}_3)$ with a strict inequality.
One can now conclude that if $T < T^n$ has been chosen so that conditions (5.47), (5.49)

and (5.52) are satisfied (with η verifying (5.42) and (5.51), and m defined in (5.40) then (a^n,u^n) satisfies $(\mathcal{H}_1)$ to $(\mathcal{H}_8)$. Thus we observe that (a^n,u^n) is bounded independently of n on $[0,T]$.
We still have to state that T^n may be bounded by below by the supremum $\bar{T}$ of all times T such that (5.47), (5.49) and (5.52) are satisfied. This is actually a consequence of the uniform bounds we have just obtained, and of remark continuations theorems (see for example [13]) and proposition 4.10. Indeed, by combining all these informations, one can prove that if $T^n<\bar{T}$ then (a^n,u^n) is actually in:

$$\widetilde{L}^{\infty}_{T^n}(B^{\frac{N}{2}}_{2,1}\cap B^{\frac{N}{p}}_{p,1})\times\Big(\widetilde{L}^{\infty}_{T^n}(B^{\frac{N}{2}}_{2,1}\cap B^{\frac{N}{p}+1}_{p,1})\cap L^1_{T^n}(B^{\frac{N}{2}+1}_{2,1}\cap(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})\Big)^N$$

hence may be continued beyond $\bar{T}$ as we control $\nabla u^n\in L^1(L^\infty)$ (see the remark on the lifespan following the statement in [13]). We thus have $T^n\geq\bar{T}$.

Compactness arguments

We now have to prove that $(a^n,u^n)_{n\in\mathbb{N}}$ tends (up to a subsequence) to some function (a,u) which belongs to F_T. Here we recall that:

$$F_T=\widetilde{C}([0,T],B^{\frac{N}{p}}_{p,1})\times\big(\widetilde{L}^{\infty}(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})\cap\widetilde{L}^1(B^{\frac{N}{p}+1}_{p,1})\big).$$

The proof is based on Ascoli's theorem and compact embedding for Besov spaces. As similar arguments have been employed in [10] or [13], we only give the outlines of the proof.

- Convergence of $(a^n)_{n\in\mathbb{N}}$:
  We use the fact that $\widetilde{a}^n=a^n-a^n_0$ satisfies:
  $$\partial_t\widetilde{a}^n=-u^n\cdot\nabla a^n+(1+a^n)\mathrm{div}u^n.$$
  Since $(u^n)_{n\in\mathbb{N}}$ is uniformly bounded in $\widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,1})\cap\widetilde{L}^{\infty}_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})$, then $(u^n)_{n\in\mathbb{N}}$ is also bounded in $\widetilde{L}^r_T(B^{\frac{N}{p}-1+\frac{2}{r}}_{p,1})$ for any $r\in[1,+\infty]$. By using the standard product laws in Besov spaces, we thus easily get that $(\partial_t\widetilde{a}^n)$ is uniformly bounded in $\widetilde{L}^2_T(B^{\frac{N}{p}-1}_{p,1})$. Hence $(\widetilde{a}^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^{\infty}_T(B^{\frac{N}{p}-1}_{p,1}\cap B^{\frac{N}{p}}_{p,1})$ and equicontinuous on $[0,T]$ with values in $B^{\frac{N}{p}-1}_{p,1}$. Since the embedding $B^{\frac{N}{p}-1}_{p,1}\cap B^{\frac{N}{p}}_{p,1}$ is (locally) compact, and $(a^n_0)_{n\in\mathbb{N}}$ tends to a_0 in $B^{\frac{N}{p}}_{p,1}$, we conclude that $(a^n)_{n\in\mathbb{N}}$ tends (up to extraction) to some distribution a. Given that $(a^n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^{\infty}_T(B^{\frac{N}{p}}_{p,1})$, we actually have $a\in\widetilde{L}^{\infty}_T(B^{\frac{N}{p}}_{p,1})$.
- Convergence of $((v^n_1)_L)_{n\in\mathbb{N}}$:
  From the definition of $(v^n_1)_L$ and proposition 3.7, it is clear that $(v^n_1)_L$ goes to a solution $(v_1)_L$ of:
  $$\begin{cases}\partial_t(v_1)_L-\mathcal{A}(v_1)_L=f,\\(v_1)_L(0)=u_0-\dfrac{1}{\nu}v_0.\end{cases}$$

in $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})\cap\widetilde{L}^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+3}_{p,1})$.

- Convergence of $(\widetilde{v}_1^n)_{n\in\mathbb{N}}$:
  By proceeding similarly, we can prove that up to extraction, $(\widetilde{v}_1^n)_{n\in\mathbb{N}}$ converges in the distributional sense to some function $\widetilde{v}_1$ such that:

$$\widetilde{v}_1\in\widetilde{L}^\infty(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})\cap\widetilde{L}^1(B^{\frac{N}{p}+1}_{p,1}). \tag{5.53}$$

By interpolating with the bounds provided by the previous step, one obtains better results of convergence so that one can pass to the limit in the mass equation and in the momentum equation. Finally by setting $u=\widetilde{v}_1+v+(v_1)_L$, we conclude that (a,u) satisfies (1.3).
In order to prove continuity in time for a it suffices to make use of proposition 4.10. Indeed, a_0 is in $B^{\frac{N}{p}}_{p,1}$, and having $a\in\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})$ and $u\in\widetilde{L}^1_T(B^{\frac{N}{p}+1}_{p,1})$ insure that $\partial_t a+u\cdot\nabla a$ belongs to $\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})$. Similarly, continuity for u may be proved by using that $(\widetilde{v}_1)_0\in B^{\frac{N}{p_1}-1}_{p_1,1}$ and that $(\partial_t v_1-\mu\Delta v_1)\in\widetilde{L}^1_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}}_{p,1})$. We conclude by using the fact that $u=v_1+\frac{1}{\nu}v$.

5.3 The proof of the uniqueness

In this section, we are interested in proving the results of uniqueness of theorem 1.1, we will use similar arguments as in [10, 13, 16].

Uniqueness when $1\leq p_1<N$, $\frac{2}{N}<\frac{1}{p}+\frac{1}{p_1}$ and $N\geq 3$

In this section, we focus on the cases $1\leq p_1<N$, $\frac{2}{N}<\frac{1}{p}+\frac{1}{p_1}$, $N\geq 3$ and postpone the analysis of the other cases (which turns out to be critical) to the next section. Throughout the proof, we assume that we are given two solutions (a^1,u^1) and (a^2,u^2) of (1.3). In the sequel we will show that $a^1=a^2$ and $v_1^1=v_1^2$ where $u^i=v_1^i+\frac{1}{\nu}v^i$ (for the notation, we conserve the same as in the previous section). It will imply in particular that $u^1=u^2$).
We know that (a^1,v_1^1) and (a^2,v_1^2) belongs to:

$$\widetilde{C}([0,T];B^{\frac{N}{p}}_{p,1})\times\big(\widetilde{C}([0,T];B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})\cap\widetilde{L}^1(0,T;B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})\big)^N.$$

Let $\delta a=a^2-a^1$, $\delta v=v^2-v^1$and $\delta v_1=v_1^2-v_1^1$. The system for $(\delta a,\delta v_1)$ reads:

$$\begin{cases}\partial_t\delta a+u^2\cdot\nabla\delta a=\delta a\mathrm{div}u^2+(\delta v_1+\dfrac{1}{\nu}\delta v)\cdot\nabla a^1+(1+a^1)\mathrm{div}(\delta v_1+\dfrac{1}{\nu}\delta v),\\ \partial_t\delta v_1+u^2\cdot\delta\nabla v_1+\delta v_1\cdot\nabla u^1-(1+a^1)\mathcal{A}\delta v_1=\delta a\mathcal{A}v_1^2-\dfrac{1}{\nu}(u^2\cdot\nabla\delta v\\ \quad-\delta v\cdot\nabla u^1)+\nabla(\Delta)^{-1}\Big((P'(\rho^2)-P'(\rho^1))\mathrm{div}(\rho^2u^2)+P'(\rho^1)\mathrm{div}(\rho^1\delta u)\\ \qquad\qquad\qquad\qquad\qquad\qquad+P'(\rho^1)\mathrm{div}((\rho^2-\rho^1)u^2)\Big).\end{cases} \tag{5.54}$$

The function δa may be estimated by taking advantage of proposition 4.10 with $s=\frac{N}{p}-1$. Denoting $U^i(t)=\|u^i\|_{\widetilde{L}^1_t(B^{\frac{N}{p}+1}_{p,1})}$ for $i=1,2$, we get for all $t\in[0,T]$,

$$\|\delta a(t)\|_{B^{\frac{N}{p}-1}_{p,1}}\leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}\|\delta a\text{div}u^2+(\delta v_1+\frac{1}{\nu}\delta v)\cdot\nabla a^1$$
$$+(1+a^1)\text{div}(\delta v_1+\frac{1}{\nu}\delta v)\|_{B^{\frac{N}{p}-1}_{p,1}}d\tau,$$

Next using proposition 2.3 and 2.6 we obtain:

$$\|\delta a(t)\|_{B^{\frac{N}{p}-1}_{p,1}}\leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}\|\delta a\|_{B^{\frac{N}{p}-1}_{p,1}}\big(\|u^2\|_{B^{\frac{N}{p}+1}_{p,1}}+(1+2\|a^1\|_{B^{\frac{N}{p}}_{p,1}})\big)$$
$$+(1+2\|a1\|_{B^{\frac{N}{p}}_{p,1}})\|\delta v_1\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1}}d\tau,$$

Hence applying Grönwall lemma, we get:

$$\|\delta a(t)\|_{B^{\frac{N}{p}-1}_{p,1}}\leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}(1+\|a^1\|_{B^{\frac{N}{p}}_{p,1}})\|\delta v_1\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1}}d\tau. \tag{5.55}$$

For bounding δv_1, we aim at applying proposition 4.8 of [16] to the second equation of (5.54). So let us fix an integer m such that:

$$1+\inf_{(t,x)\in[0,T]\times\mathbb{R}^N}S_m a^1\geq\frac{b}{2}\ \text{ and }\ \|a^1-S_m a^1\|_{L^\infty_T(B^{\frac{N}{p}}_{p,1})}\leq c\frac{\nu}{\bar{\nu}}. \tag{5.56}$$

Note since a^1 satisfies a transport equation with right-hand side in $\widetilde{L}^1_T(B^{\frac{N}{p}}_{p,1})$, proposition 4.10 guarantees that a^1 is in $\widetilde{C}_T(B^{\frac{N}{p}}_{p,1})$. Hence such an integer does exist (see remark 19). Now applying proposition 4.8 of [16] with $s=\frac{N}{p_1}-2$ and $s'=\frac{N}{p}-1$ insures that for all time $t\in[0,T]$, we have:

$$\|\delta v_1\|_{L^1_t(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\leq Ce^{C(1+t)U(t)}\int_0^t e^{-C(1+\tau)U(\tau)}\big(\|\delta a\mathcal{A}v_1^2-\frac{1}{\nu}(\delta v\cdot\nabla v_1^1+v_1^1\cdot\nabla\delta v)$$
$$-\frac{1}{\nu^2}(v^1\cdot\nabla\delta v+\delta v\cdot\nabla v^2)\|_{B^{\frac{N}{p_1}-2}_{p_1,1}+B^{\frac{N}{p}-1}_{p,1}}\big)d\tau,$$

with $U(t)=U^1(t)+U^2(t)+2^{2m}\underline{\nu}^{-1}\bar{\nu}^2\int_0^t\|a^1\|^2_{B^{\frac{N}{p}}_{p,1}}d\tau$.

Hence, applying proposition 2.3 we get:

$$\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\leq Ce^{C(1+t)U(t)}\int_0^t e^{-C(1+\tau)U(\tau)}\big(1+\|a^1\|_{B^{\frac{N}{p}}_{p,1}}+\|a^2\|_{B^{\frac{N}{p}}_{p,1}}$$
$$+\|v_1^2\|_{B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}}\big)\|\delta a\|_{B^{\frac{N}{p}-1}_{p,1}}d\tau. \tag{5.57}$$

Finally plugging (5.55) in (5.57), we get for all $t \in [0, T_1]$,

$$\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})} \leq C\int_0^t \big(1+\|a^1\|_{B^{\frac{N}{p}}_{p,1}}+\|a^2\|_{B^{\frac{N}{p}}_{p,1}}+\|v_1^2\|_{B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}}\big) \\ \times \|\delta v_1\|_{\widetilde{L}^1_\tau(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\,d\tau.$$

Since a^1 and a^2 are in $L^\infty(B^{\frac{N}{p}}_{p,1})$ and v_1^2 belongs to $L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})$, applying Grönwall lemma yields $\delta v_1 = 0$, an $[0, T]$.

Uniqueness when: $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p}$ **or** $p_1 = N$ **or** $N = 2$**.**

The above proof fails in dimension two or in the case $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p}$ or $p_1 = N$. One of the reasons why is that the product of functions does not map $B^{\frac{N}{p}-1}_{p,1} \times B^{\frac{N}{p_1}-2}_{p_1,1}$ in $B^{\frac{N}{p_1}-2}_{p_1,1}$ but only in the larger space $B^{\frac{N}{p_1}-2}_{p_1,\infty}$. This induces us to bound δa in $L^\infty_T(B^{\frac{N}{p}-1}_{p,\infty})$ and δv_1 in $L^\infty_T(B^{\frac{N}{p_1}-2}_{p_1,\infty} + B^{\frac{N}{p}}_{p,\infty}) \cap L^1_T(B^{\frac{N}{p_1}}_{p_1,\infty} + B^{\frac{N}{p}+1}_{p,\infty})$ (or rather, in the widetilde version of those spaces, see below). Yet, we are in trouble because due to $B^{\frac{N}{p_1}}_{p_1,\infty}$ is not embedded in L^∞, the term $\delta v_1 \cdot \nabla a^1$ in the right hand-side of the first equation of (5.54) cannot be estimated properly. As noticed in [12], this second difficulty may be overcome by making use of logarithmic interpolation and Osgood lemma (a substitute for Gronwall inequality). Let us now tackle the proof. Fix an integer m such that:

$$1 + \inf_{(t,x)\in[0,T]\times\mathbb{R}^N} S_m a^1 \geq \frac{b}{2} \;\text{ and }\; \|a^1 - S_m a^1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} \leq c\frac{\nu}{\bar{\nu}}, \tag{5.58}$$

and define T_1 as the supremum of all positive times t such that:

$$t \leq T \;\text{ and }\; t\bar{\nu}^2\|a^1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})} \leq c2^{-2m}\underline{\nu}. \tag{5.59}$$

Remark that the proposition 4.10 ensures that a^1 belongs to $\widetilde{C}_T(B^{\frac{N}{p}}_{p,1})$ so that the above two assumptions are satisfied if m has been chosen large enough. For bounding δa in $L^\infty_T(B^{\frac{N}{p}-1}_{p,\infty})$, we apply proposition 4.10 with $r = +\infty$ and $s = 0$. We get (with the notation of the previous section):

$$\forall t \in [0,T],\;\; \|\delta a(t)\|_{B^{\frac{N}{p}-1}_{p,\infty}} \leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}\|\delta a \text{div} u^2 + (\delta v_1 + \frac{1}{\nu}\delta v)\cdot\nabla a^1 \\ + (1+a^1)\text{div}(\delta v_1 + \frac{1}{\nu}\delta v)\|_{B^{\frac{N}{p}-1}_{p,\infty}}\,d\tau,$$

hence using that the product of two functions maps $B^{\frac{N}{p}-1}_{p,\infty} \times B^{\frac{N}{p_1}}_{p_1,1}$ in $B^{\frac{N}{p}-1}_{p,\infty}$, and applying Gronwall lemma,

$$\|\delta a(t)\|_{B^{\frac{N}{p}-1}_{p,\infty}} \leq Ce^{CU^2(t)}\int_0^t e^{-CU^2(\tau)}(1+\|a^1\|_{B^{\frac{N}{p}}_{p,1}})\|\delta v_1\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1}}\,d\tau. \tag{5.60}$$

Next, using proposition 4.8 of [16] combined with proposition 2.3 and corollary 1 in order to bound the nonlinear terms, we get for all $t\in[0,T_1]$,:

$$\|\delta v_1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}}_{p_1,\infty}+B^{\frac{N}{p}+1}_{p,\infty})}\leq Ce^{C(1+t)(U^1+U^2)(t)}\int_0^t(1+\|a^1\|_{B^{\frac{N}{p}}_{p,1}}+\|a^2\|_{B^{\frac{N}{p}}_{p,1}}+\|v_1^2\|_{B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}})\|\delta a\|_{B^{\frac{N}{p}-1}_{p,\infty}}d\tau. \tag{5.61}$$

In order to control the term $\|\delta v_1\|_{B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1}}$ which appears in the right-hand side of (5.60), we make use of the following logarithmic interpolation inequality whose proof may be found in [12], page 120:

$$\begin{aligned}\|\delta v_1\|_{L^1_t(B^{\frac{N}{p_1}}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})}\lesssim&\\ \|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}}_{p_1,\infty})}&\log\big(e+\frac{\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}-1}_{p_1,\infty})}+\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}+1}_{p_1,\infty})}}{\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}}_{p_1,\infty})}}\big)\\ &+\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p}+1}_{p,\infty})}\log\big(e+\frac{\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,\infty})}+\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p}+2}_{p,\infty})}}{\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p}}_{p,\infty})}}\big).\end{aligned} \tag{5.62}$$

Because v_1^1 and v_2^2 belong to $\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1}_{p_1,1}+B^{\frac{N}{p}+1}_{p,1})\cap L^1_T(B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1})$, the numerator in the right-hand side may be bounded by some constant C_T depending only on T and on the norms of v_1^1 and v_1^2. Therefore inserting (5.60) in (5.61) and taking advantage of (5.62), we end up for all $t\in[0,T_1]$ with:

$$\begin{aligned}\|\delta v_1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}}_{p_1,\infty}+B^{\frac{N}{p}+1}_{p,\infty})}&\leq C(1+\|a^1\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}}_{p,1})})\\ &\times\int_0^t(1+\|a^1\|_{B^{\frac{N}{p}}_{p,1}}+\|a^2\|_{B^{\frac{N}{p}}_{p,1}}+\|v_1^2\|_{B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}})\|\delta v_1\|_{\widetilde{L}^1_t(B^{\frac{N}{p_1}}_{p_1,\infty}+B^{\frac{N}{p}+1}_{p,\infty})}\\ &\qquad\times\log(e+C_T\|\delta v_1\|^{-1}_{\widetilde{L}^1_\tau(B^{\frac{N}{p_1}}_{p_1,\infty}+B^{\frac{N}{p}+1}_{p,\infty})})d\tau.\end{aligned}$$

Since the function $t\to\|a^1(t)\|_{B^{\frac{N}{p}}_{p,1}}+\|a^2(t)\|_{B^{\frac{N}{p}}_{p,1}}+\|v_1^2(t)\|_{B^{\frac{N}{p_1}+1}_{p_1,1}+B^{\frac{N}{p}+2}_{p,1}}$ is integrable on $[0,T]$, and:

$$\int_0^1\frac{dr}{r\log(e+C_Tr^{-1})}=+\infty$$

Osgood lemma yields $\|\delta v_1\|_{\widetilde{L}^1_T(B^{\frac{N}{p_1}}_{p_1,\infty}+B^{\frac{N}{p}+1}_{p,\infty})}=0$. Note that the definition of m depends only on T and that (5.56) is satisfied on $[0,T]$. Hence, the above arguments may be repeated on $[T_1,2T_1]$, $[2T_1,3T_1]$,etc. until the whole interval $[0,T]$ is exhausted. This yields uniqueness on $[0,T]$ for a and v_1 which implies uniqueness for u. □

5.4 Proof of theorem 1.2

The proof follows the same line as theorem 1.1 except concerning the uniqueness. In the sequel we will concentrate us only on the result of uniqueness which improves significantly in the specific case $P(\rho) = K\rho$ with $K > 0$ the theorem 1.1. Indeed we will be able to reach the critical case for the initial density . For that we use the main theorem of D. Hoff in [21] which is a result of weak-strong uniqueness. For the completeness of the proof we would like to recall the result of D. Hoff (see [21] for more details).
Let (ρ, u) a weak solution of the system (1.1) (see the definition of D. Hoff in [21]) with the following properties:

$$u \in C((0,T] \times \mathbb{R}^N) \cap L^r((0,T) \times \mathbb{R}^N) \cap L^1(0,T,W^{1,\infty}(\mathbb{R}^N)) \cap L^\infty_{loc}(L^\infty(\mathbb{R}^N)), \quad (5.63)$$

$$\rho - \bar{\rho}, u, f \in L^2((0,T) \times \mathbb{R}^N), \quad (5.64)$$

$$\frac{1}{\rho} \in L^\infty, \quad (5.65)$$

and

$$u \in L^r((0,T) \times \mathbb{R}^N), \quad (5.66)$$

with $r > N$. Let (ρ_1, u_1) a strong solution such that (5.63), (5.64) and (5.65) are verified and:

$$\begin{aligned} \int_0^T [\|u_1(t,\cdot)\|^2_{L^\infty} + t\|\nabla u_1(t,\cdot))\|^2_{L^\infty} + t\|\nabla F_1(\cdot,t), \nabla\omega_1\|^2_{L^2} \\ + (t\|\nabla F_1(\cdot,t), \nabla\omega_1\|^2_{L^4})^a]dt < +\infty, \end{aligned} \quad (5.67)$$

with $F_1 = \text{div}u_1 - P(\rho_1) + P(\bar{\rho})$ the effective pressure, ω_1 the curl of u_1 and with $a = \frac{2}{3}$ if $N = 2$ and $a = \frac{4}{5}$ for $N = 3$. We assume in the sequel that:

$$f \in L^1((0,T), L^{2q}(\mathbb{R}^N), \quad (5.68)$$

for some $q \in [1, +\infty]$. And finally D. Hoff needs to assume that:

$$\rho_0 - \bar{\rho} \in L^2 \cap L^{2p}, \quad (5.69)$$

where p is the Hölder conjugate of q.
We can now state the result that D. Hoff obtains in [21]:

Theorem 5.5 *Assuming that (ρ, u) and (ρ_1, u_1) are weak solutions (for the precise definition see [21]), moreover (ρ, u) verify (5.63), (5.64), (5.65), (5.66) and (ρ_1, u_1) verify (5.63), (5.64), (5.65), (5.67) and (5.68) . The initial data are chosen as in the theorem 1.2 with the additional condition (5.69). Let $P(\rho) = K\rho$ with $K > 0$. Then under the previous hypothesis:*

$$u = u_1 \;\; \textit{and} \;\; u = u_1 \;\; \textit{on} \; (0,T).$$

Remark 20 *Here (ρ_1, u_1) have to consider as the strong solution and (ρ, u) as the weak solution.*

Furthermore in [24], D. Hoff exhibits a class of solutions (ρ_1, u_1) satisfying all the conditions (5.63), (5.64), (5.65), (5.67) and (5.68) except that $u_1 \in L^1((0,T), W^{1,\infty}(\mathbb{R}^N))$. For this D. Hoff assume the following conditions on the initial data:

$$\begin{aligned}
&\rho_0 \in L^\infty,\ \rho_0 - \bar{\rho} \in L^1_2,\\
&\inf \rho_0 > 0,\\
&u_0 \in H^s \text{ with } s > 0 \text{ if } N = 2 \text{ or } s > \frac{1}{2} \text{ if } N = 3.\\
&u_0 \in L^q, \text{ with } q > 2 \text{ if } N = 2 \text{ or } q = 6 \text{ if } N = 3.
\end{aligned} \tag{5.70}$$

For proving these results D. Hoff uses essentially inequalities of energy (in his case the initial data are assumed small and he obtains existence of global weak solutions, for a similar case with large initial data we refer to [18]). The main difficulty for using the theorem 5.5 is then to prove the Lipchitz condition on u_1, i.e u_1 in $L^1((0,T), W^{1,\infty}(\mathbb{R}^N))$.

In our context, we want to verify that a solution $(\bar{\rho}, \bar{u})$ constructed in theorem 1.1 with the additional conditions on the initial data of theorem 1.2 verify all the hypothesis of theorem 5.5 . It means that $(\bar{\rho}, \bar{u})$ have to check the hypothesis of the class of the strong solution and of the class of weak solution. In this case, we will be able to conclude that if we choose two solutions $(\bar{\rho}, \bar{u})$ and $(\bar{\rho}_1, \bar{u}_1)$ in the class of the solutions of theorem 1.1 then $(\bar{\rho}, \bar{u}) = (\bar{\rho}_1, \bar{u}_1)$.
As we explain previously, the regularizing effects obtained on the velocity in [24] result from energy inequalities combined with an argument of smallness to apply a bootstrap. This idea has been developed in [18] in the case of large initial data. In particular it is shown in [24, 18] that with our choice on the initial data in theorem 1.2 the solution of theorem 1.1 satisfy all the conditions (5.63), (5.64), (5.65), (5.67) and (5.68). Indeed for proving (5.63), (5.64), (5.65), (5.67) and (5.68), it suffices to use the same argument than in [24, 18], it means tricky energy inequalities.
The only new point compared with [24, 18] to achieve the proof of uniqueness corresponds to prove that $\bar{u}$ is in $L^1_T(W^{1,\infty}(\mathbb{R}^N))$ and that $(\bar{\rho}, \bar{u})$ verify the condition of the weak solution of theorem 5.5. The last point is evident. We only want to point out that $\bar{u}$ is in $L^1((0,T), W^{1,\infty}(\mathbb{R}^N))$ because $\bar{u}$ belongs to $L^1_T(B^{\frac{N}{p}+1}_{p,1})$. This conclude the proof of the uniqueness. □

5.5 Proof of theorem 1.3

We follow here exactly the same lines than the proof of theorem 1.1 except that we introduce a new effective velocity. Indeed in our case the viscosity coefficients are variable, so we set v which verifies the following elliptic equation:

$$(2\mu + \lambda)\Delta v + \mathrm{div}\big(f_1(q)Dv\big) + \nabla\big(f_2(q)\mathrm{div}v\big) = \nabla P(\rho). \tag{5.71}$$

with $f_1(q) = \mu(\rho) - \mu(1)$ and $f_2(q) = \lambda(\rho) - \lambda(1)$. We can resolve this elliptic equation as $\mu \geq c > 0$ and $2\mu + \lambda \geq c > 0$, indeed in our case we work away from the vacuum. To do this we have to use the estimates on the Lamé operator of the appendix in [17]. The idea is in fact to treat the system (5.71) as a Lamé operator with regular variable viscosity coefficient that we perturb by a remainder with small variable viscosity coefficient. We use only the fact that the functions in C^0_∞ are dense in $B^{\frac{N}{p}}_{p,1}$. The idea is exactly the same as in the proof of proposition 3.8. More precisely we have as $q \in \widetilde{L}^\infty(B^{\frac{N}{p}}_{p,1})$ for $r \geq 1$, $p \geq 1$ and $|s| \leq \frac{N}{p}$ (for more details we refer to [17]):

$$\|v\|_{\widetilde{L}^r(B^s_{p,1})} \leq C\|q\|_{\widetilde{L}^r(B^{s-1}_{p,1})}.$$

It means that as in the proof of theorem 1.1, v is one derivative more regular than q and that we can estimate v in function of q. Moreover we have $\partial_t v$ which verifies the following elliptic equation:

$$\mathrm{div}(\mu(\rho)D\partial_t v) + \nabla(\lambda(\rho)\mathrm{div}\partial_t v) = \nabla\partial_t P(\rho) - div(\partial_t\mu(\rho)Dv) - \nabla(\partial_t\lambda(\rho)\mathrm{div}v).$$

We can in a similar way getting estimates on $\partial_t v$ in function of q, u and v, so in function of q and u. To do this, we have one time more to apply elliptic estimates in Chemin-Lerner spaces. In the sequel as in the proof in theorem 1.1, we will have to consider an effective velocity defined by $v_1 = u - v$. The rest of the proof is exactly similar to the proof of the theorem 1.1 and is nothing than tedious verifications. It is left to the reader.
□

6 Proof of theorem 1.4

We now want to prove theorem 1.4. We have assumed here that $\rho_0^{\frac{1}{p_1}} u_0 \in L^{p_1}$ with $p_1 = N + \varepsilon$ with ε arbitrary small. We would like to show that with our hypothesis in particular that a is in L^∞_T and then we are able to prove that $\rho^{\frac{1}{p_1}} u \in L^\infty_T(L^{p_1})$. First as in [18] we can show that if we control $\|\rho$ in norm L^∞_T then we control the vacuum or more precisely $\frac{1}{\rho}$ in L^∞. We refer to [18] for more details. Let now showing that we can control $\rho^{\frac{1}{p_1}} u$ in $L^\infty_T(L^{p_1})$.
We multiply the momentum equation by $u|u|^{p_1-2}$ and we get after integration by part:

$$\begin{aligned}
&\frac{1}{p_1}\int_{\mathbb{R}^N} \rho|u|^{p_1}(t,x)dx + \mu\int_0^t |u|^{p_1-2}|\nabla u|^2(t,x)dtdx + \frac{p_1-2}{4}\mu\int_0^t |u|^{p_1-4}|\nabla|u|^2|^2(t,x)dxdt \\
&+\lambda\int_0^t\int_{\mathbb{R}^N}(\mathrm{div}u)^2|u|^{p_1-2}(t,x)dtdx + \lambda\frac{p_1-2}{2}\int_0^t\int_{\mathbb{R}^N}\mathrm{div}u\sum_i u_i\partial_i|u|^2|u|^{p_1-4}(t,x)dtdx- \\
&\int_0^t\int_{\mathbb{R}^N}\big(P(\rho)-P(\bar{\rho})\big)\big(\mathrm{div}u|u|^{p_1-2} + (p_1-2)\sum_{i,k}u_iu_k\partial_iu_k|u|^{p_1-4}\big)(t,x)dtdx \\
&\leq \int_{\mathbb{R}^N}\rho_0|u_0|^{p_1}dx.
\end{aligned}$$

By Young's inequalities, inequality (1.5) and the fact that $P(\rho)-P(\bar{\rho})$ belongs in $L^\infty(L^1\cap L^\infty)$ we conclude that $\rho^{\frac{1}{p_1}} u$ is in $L^\infty_T(L^{p_1})$ and that u is in $L^\infty_T(L^{p_1})$ as $\frac{1}{\rho}$ is in L^∞.

Now as u, $\frac{1}{\rho}$ and ρ are bounded respectively in $L^\infty_T(L^{p_1}) \hookrightarrow \widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1+\frac{\varepsilon}{2}}_{p_1,1})$, $L^\infty_T(L^\infty)$ and in $\widetilde{L}^\infty_T(B^{\frac{N}{p}+\frac{\varepsilon}{2}}_{p,1})$. We now can use the remark 12 (for a proof we refer to [10]). It

means that there exists a time $T \geq c > 0$ where c depends only on the dimension N, the viscosity coefficients and on $\|q\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p}+\frac{\epsilon}{2}}_{p,1})}$ and $\|u\|_{\widetilde{L}^\infty_T(B^{\frac{N}{p_1}-1+\frac{\epsilon}{2}}_{p_1,1})}$. In fact it suffices only to verify how the conditions (5.47), (5.49) and (5.52) are verifies.
It means that we can construct by theorem 1.1 a solution (a_1, u_1) on $(T-\alpha, T-\alpha+T^{'})$ with initial data $(a(T-\alpha), u(T-\alpha))$ (here $\alpha < T^{'}$). The only difficulty is to prove that on $(T-\alpha, T)$ we have:

$$(\rho_1, u_1) = (\rho, u).$$

Now we can use our supplementary condition on the initial data, i.e $(a_0, u_0) \in B^1_{N,1} \times B^0_{N,1}$. Indeed by persistency results as in [19], we can show that:

$$a \in \widetilde{L}^\infty_T(B^1_{N,1}),\ u \in \widetilde{L}^\infty_T(B^0_{N,1}) \text{ and } u \in \widetilde{L}^1_T(B^2_{N,1}).$$

It means that $(a(T-\alpha), u(T-\alpha))$ and $(a_1(T-\alpha), u_1(T-\alpha))$ are in $B^1_{N,1} \times B^0_{N,1}$. We can show then by theorem 1.1 that $(\rho_1, u_1) = (\rho, u)$ on $(T-\alpha, T)$, because (a_1, u_1) and (a, u) are in the class on $(T-\alpha, T)$ of the solutions of theorem 1.1 which are unique.

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