Boundary Unique Continuation for Stochastic Parabolic Equations

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Abstract

This paper is devoted to the study the boundary unique continuation property for forward stochastic parabolic equations, i.e., determine the value of the solution by means of the observation on arbitrary open subset of the boundary. We establish a quantitative version of the boundary unique continuation property by a global Carleman estimate for forward stochastic parabolic equations.

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1 Introduction

Let $T > 0$, $G \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a given bounded domain with the C^2 boundary ∂G, and Γ be a given nonempty open subset of ∂G.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{B(t)\}_{t\geq 0}$ is defined. Let H be a Banach space. We denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(|X(\cdot)|^2_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega; C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geq 0}$-adapted continuous processes X such that $\mathbb{E}(|X|^2_{C([0,T];H)}) < \infty$, with the canonical norm.

Throughout this paper, we make the following assumptions on the coefficients $a^{ij} : \Omega \times [0,T] \times \overline{G} \to \mathbb{R}^{n\times n}$ $(i,j = 1, 2, \cdots, n)$:

(H1) $a^{ij} \in L^2_{\mathcal{F}}(\Omega; C^1([0,T]; W^{1,\infty}(G)))$ *and* $a^{ij} = a^{ji}$;

(H2) *There is some constant* $\beta_0 > 0$ *such that*

$$\sum_{i,j} a^{ij}(\omega,t,x)\xi^i\xi^j \geq \beta_0|\xi|^2, \qquad (\omega,t,x,\xi) \equiv (\omega,t,x,\xi^1,\cdots,\xi^n) \in \Omega \times (0,T) \times G \times \mathbb{R}^n. \tag{1.1}$$

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Here and henceforth, for simplicity, we denote $\sum_{i,j=1}^{n}$ by $\sum_{i,j}$. For the same reason, we also use the notation $y_i \equiv y_i(x) = \partial y(x)/\partial x_i$, where x_i is the i-th coordinate of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. In a similar manner, we use the notation z_i, v_i, etc. for the partial derivatives of z and v with respect to x_i. Also, we denote the scalar product in $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle$, and use C to denote a generic positive constant depending only on T, G, Γ and $(a^{ij})_{n\times n}$, which may change from line to line.

The main purpose of this paper is to study the quantitative boundary unique continuation of the following forward stochastic parabolic equation:

$$\begin{cases} dy - \sum_{i,j}(a^{ij}y_i)_j dt = [\langle a_1, \nabla y \rangle + a_2 y]dt + a_3 y \, dB(t) & \text{in } (0,T)\times G, \\ y = 0 & \text{on } (0,T)\times \Gamma. \end{cases} \tag{1.2}$$

Here $a_1 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G;\mathbb{R}^n))$, $a_2 \in L^\infty_{\mathcal{F}}(0,T;L^\infty(G))$ and $a_3 \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G))$.

To begin with, we first give the following definition of the solution for equation (1.2).

Definition 1.1 *We call $y \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G))) \cap L^2_{\mathcal{F}}(0,T;H^2_\Gamma(G))$ a solution of equation (1.2) if for any $t \in [0,T]$ and any $\eta \in H^1_0(G)$, it holds*

$$\begin{aligned} &\int_G z(t,x)\eta(x)dx - \int_G z(0,x)\eta(x)dx \\ = &\int_0^t \int_G \Big\{ -\sum_{i,j} a^{ij}(s,x) z_i(s,x)\eta_j(x) + [\langle a_1(s,x), \nabla z(s,x) \rangle + a_2(s,x)z(s,x)]\eta(x)\Big\} dxds \\ &+ \int_0^t \int_G a_3(s,x)z(s,x)\eta(x)dxdB(s), \qquad P-\text{a.s.} \end{aligned} \tag{1.3}$$

Here $H^2_\Gamma(G) \overset{\triangle}{=} \big\{u \in H^1(G) : u = 0 \text{ on } \Gamma\big\}$.

The main result of this paper is stated as follows:

Theorem 1.1 *For any given $G' \subset\subset G$ and $0 < \kappa < \frac{1}{2}$, there exists a constant $C > 0$ such that for any $y \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G))) \bigcap L^2_{\mathcal{F}}(0,T;H^2_\Gamma(G))$ solving equation (1.2) and any $\varepsilon > 0$, it holds that*

$$|y|^2_{L^2_{\mathcal{F}}((\frac{T}{2}-\kappa T, \frac{T}{2}+\kappa T)\times G')} \le C\varepsilon |y|^2_{L^2_{\mathcal{F}}((0,T);H^1(G))} + e^{\frac{C}{\varepsilon}} \int_0^T \int_\Gamma \Big|\frac{\partial y}{\partial \nu}\Big|^2 d\Gamma dt. \tag{1.4}$$

Particularly, any solution $y \in L^2_{\mathcal{F}}(\Omega;C([0,T];L^2(G))) \bigcap L^2_{\mathcal{F}}(0,T;H^1_\Gamma(G))$ of equation (1.2) vanishes identically in Q, P-a.s. provided that $\frac{\partial y}{\partial \nu} = 0$ in $(0,T)\times\Gamma$, P-a.s.

The above result is a boundary unique continuation theorem for stochastic parabolic equations. Roughly speaking, a unique continuation result is any statement as one of the following types:

1. *Given a partial differential operator P and two regions $\mathcal{O}_1 \subset \mathcal{O}_2$, if u satisfies that $Pu = 0$, then u is uniquely determined in $\mathcal{O}_2$ by its values in $\mathcal{O}_1$.*

2. *Given a partial differential operator P and a domain $\mathcal{O}$. Let $x_0 \in \mathcal{O}$. If u satisfies that $Pu = 0$, then u is uniquely determined in $\mathcal{O}$ by the values of all the derivatives of u at x_0.*

Obviously, statement of type 2 is stronger than statement of type 1. Statement of type 2 is also called *strong unique continuation property*. In this paper, we do not consider this kind of unique continuation.

If P is a linear operator, then the unique continuation property is equivalent to the following statement:

If u satisfies that $Pu = 0$, then $u = 0$ in $\mathcal{O}_2$ if $u = 0$ in $\mathcal{O}_1$.

Unique continuation is an important problem in partial differential equations. The study of it may date back to the classical results of Holmgren and Carleman at the very beginning of the 20th century. In 1950-70's, there is a climax of the study of it. At that time, most of the existing works are devoting to the local unique continuation property. [5] and [14] are very good references for those works. In the recent 20 years, due to the need from Control/Inverse Problems of partial differential equations(see [7, 13] for example), the study of the global unique continuation for partial differential equations is very active. There are a great many references for the unique continuation property of deterministic parabolic equations (see [2, 3, 8, 9, 11] and the references cited therein). It would be quite interesting to extend the results for deterministic parabolic equations to the stochastic ones. However, as far as we know, [12] is the only published paper for the stochastic counterpart. There are many things which remain to be done.

Generally speaking, there are two classical tools for studying the unique continuation property for deterministic parabolic equations. One is Carleman estimate (see [2, 3] for eample) and the other is utilizing doubling property for the solution (see [8, 9] for example). In this paper, we employ Carleman estimate to prove Theorem 1.1. The usual approach to utilize Carleman estimate for the unique continuation needs to localize the problem. Unfortunately, one cannot simply localize the problem as usual because the classical localization technique may change the adaptedness of solutions, which is a key feature in the stochastic setting. We use a global Carleman estimate and get help from the property of the weight function to overcome this difficulty. Obtaining the unique continuation property of stochastic parabolic equations by establishing suitable doubling property for the solution will be studied in some later papers. In this paper, in order to present the key idea in the simplest way, we do not pursue the full technical generality.

The rest of this paper is organized as follows. In Section 2, we give some preliminaries. Section 3 is addressed to prove our main result, that is, Theorem 1.1.

2 Some preliminaries

Let $M \subset \mathbb{R}^n (n \in \mathbb{N})$ be a given bounded domain with the C^2 boundary ∂M. For any nonnegative and nonzero function $\phi \in C^2(M)$ and any (large) parameters $\lambda > 1$ and $\mu > 1$, put

$$\ell = \lambda\alpha, \quad \alpha(t,x) = \frac{e^{\mu\phi(x)} - e^{2\mu|\phi|_{C(\overline{M})}}}{t^2(T-t)^2}, \quad \varphi(t,x) = \frac{e^{\mu\phi(x)}}{t^2(T-t)^2}. \tag{2.1}$$

In the sequel, for a positive integer r, we denote by $O(\mu^r)$ a function of order μ^r for large μ (which is independent of λ and T); by $O_\mu(\lambda^r)$ a function of order λ^r for fixed μ and for large λ, which is independent of T, either.

Lemma 2.1 *[10, Theorem 4.1] Let $g^{ij} \in L^2_{\mathcal{F}}(\Omega; C^1([0,T]; W^{1,\infty}(M)))$ satisfy $g^{ij} = g^{ji}$ $(i,j = 1,2,\cdots,n)$. Assume that either $(g^{ij})_{n\times n}$ or $-(g^{ij})_{n\times n}$ is a uniformly positive definite matrix, and its smallest eigenvalue is $\gamma_0 > 0$. Let u be an $H^2(M)$-valued semi-martingale. Set*

$$\theta = e^{\ell}, \quad v = \theta u, \quad \Psi = 2\sum_{i,j} g^{ij}\ell_{ij}. \tag{2.2}$$

Then for a.e. $x \in M$ *and* P-a.s. $\omega \in \Omega$,

$$\begin{aligned}
&2\int_0^T \theta\Big[-\sum_{i,j}(g^{ij}v_i)_j + Av\Big]\Big[du - \sum_{i,j}(g^{ij}u_i)_j dt\Big] + 2\int_0^T \sum_{i,j}(g^{ij}v_i dv)_j \\
&\quad +2\int_0^T \sum_{i,j}\Big[\sum_{i',j'}\Big(2g^{ij}g^{i'j'}\ell_{i'}v_iv_{j'} - g^{ij}g^{i'j'}\ell_i v_{i'}v_{j'}\Big) + \Psi g^{ij}v_i v \\
&\qquad\qquad -g^{ij}\Big(A\ell_i + \frac{\Psi_i}{2}\Big)v^2\Big]_j dt \qquad (2.3)\\
&\geq 2\sum_{i,j}\int_0^T c^{ij}v_iv_j dt + \int_0^T Bv^2 dt + \int_0^T \Big| -\sum_{i,j}(g^{ij}v_i)_j + Av\Big|^2 dt \\
&\quad -\int_0^T \theta^2 \sum_{i,j} g^{ij}(du_i + \ell_i du)(du_j + \ell_j du) - \int_0^T \theta^2 A(du)^2,
\end{aligned}$$

where

$$\begin{cases}
A \stackrel{\triangle}{=} -\displaystyle\sum_{i,j}(g^{ij}\ell_i\ell_j - g^{ij}_j\ell_i - g^{ij}\ell_{ij}) - \Psi,\\
B \stackrel{\triangle}{=} 2\Big[A\Psi - \displaystyle\sum_{i,j}(Ag^{ij}\ell_i)_j\Big] - A_t - \displaystyle\sum_{i,j}(g^{ij}\Psi_j)_i - \ell_t^2,\\
c^{ij} \stackrel{\triangle}{=} \displaystyle\sum_{i',j'}\Big[2g^{ij'}(g^{i'j}\ell_{i'})_{j'} - (g^{ij}g^{i'j'}\ell_{i'})_{j'}\Big] - \frac{g^{ij}_t}{2} + \Psi g^{ij}.
\end{cases} \qquad (2.4)$$

Moreover, for λ *and* μ *large enough, it holds*

$$\begin{aligned}
&A = -\lambda^2\mu^2\varphi^2\sum_{i,j}g^{ij}\phi_i\phi_j + \lambda\varphi O(\mu^2),\\
&B \geq 2s_0^2\lambda^3\mu^4\varphi^3|\nabla\phi|^4 + \lambda^3\varphi^3 O(\mu^3) + \lambda^2\varphi^2 O(\mu^4) + \lambda\varphi O(\mu^4)\\
&\qquad +\lambda^2T^2\varphi^3 O(e^{4\mu|\phi|_{C(\overline{G})}}) + \lambda^2 T\varphi^3 O(\mu^2) + \lambda T\varphi^2 O(\mu^2), \qquad (2.5)\\
&\sum_{i,j}c^{ij}v_iv_j \geq \big[\gamma_0^2\lambda\mu^2\varphi|\nabla\phi|^2 + \lambda\varphi O(\mu)\big]|\nabla v|^2.
\end{aligned}$$

3 Proof of Theorem 1.1

Proof of Theorem 1.1 : We choose a bounded domain $\widetilde{G}$ with C^2 boundary $\partial\widetilde{G}$ such that

$$G \subset \widetilde{G}, \quad \overline{\partial G \cap \widetilde{G}} = \Gamma, \quad \partial G \setminus \Gamma \subset \partial\widetilde{G} \quad \text{and } \widetilde{G}\setminus G \text{ contains some nonempty open subset.} \qquad (3.1)$$

In fact, $\widetilde{G}$ can be constructed as follows: Take a bounded domain $J \subset \mathbb{R}^n$ with C^2 boundary ∂J such that $\partial J \cap \overline{G} = \Gamma$ and let $\widetilde{G} = J \cup G \cup \Gamma$. Let $G_0 \subset\subset \widetilde{G}\setminus G$ be an open subdomain. We know that there is a $\psi \in C^2(\widetilde{G})$ satisfying (see [4] for example)

$$\begin{cases}
\psi > 0 & \text{in } \widetilde{G},\\
\psi = 0 & \text{on } \partial\widetilde{G},\\
|\nabla\psi| > 0 & \text{in } G \subset \widetilde{G}\setminus G_0.
\end{cases} \qquad (3.2)$$

Let $N > 4$ be a positive number, and set

$$\begin{aligned}&\widetilde{G}_1 \triangleq \Big\{x:\ \psi(x) \geq \frac{4|\psi|_{L^\infty(\widetilde{G})}}{N}\Big\}, \quad \widetilde{G}_2 \triangleq \Big\{x:\ \psi(x) \geq \frac{2|\psi|_{L^\infty(\widetilde{G})}}{N}\Big\},\\ &\widetilde{G}_3 \triangleq \Big\{x:\ \psi(x) \geq \frac{|\psi|_{L^\infty(\widetilde{G})}}{N}\Big\}.\end{aligned} \tag{3.3}$$

Put $G_i = G \cap \widetilde{G}_i (i = 1, 2, 3)$. Since $\psi \in C^2(\widetilde{G})$, we know that $\partial\widetilde{G}_i (i = 1, 2, 3)$ is C^2. Thus, ∂G_i is C^2 and $\partial G_i \cap \partial G \subset \Gamma (i = 1, 2, 3)$. Let $\eta \in C_0^\infty(G_3)$ such that $\eta = 1$ in G_2 and $0 \leq \eta \leq 1$. For any y solving equation (1.2), let $z = \eta y$, then z solves

$$\begin{cases} dz - \displaystyle\sum_{i,j}(a^{ij}z_i)_j dt = \big(\langle\, a_1, \nabla z\,\rangle + a_2 z + f\big)dt + a_3 z dB(t) & \text{in } (0,T) \times G_3,\\ z = \dfrac{\partial z}{\partial \nu} = 0 & \text{on } (0,T) \times \partial G_3. \end{cases} \tag{3.4}$$

Here $f = -\sum_{i,j}(a^{ij}_j \eta_i y + a^{ij}\eta_{ij} y + 2a^{ij}\eta_i y_j) - \langle\, a_1, \nabla\eta\,\rangle\, y \in L^2_{\mathcal{F}}(0,T; L^2(G_3))$.

Applying Lemma 2.1 to equation (3.4) with $u = z$, $g^{ij} = a^{ij}$, $M = G_3$ and $\phi = \chi_{G_3}\psi$, integrating it in G_3 and taking mathematical expectation, we see

$$\begin{aligned}&2\mathbb{E}\int_0^T\int_{G_3}\theta\Big[-\sum_{i,j}(g^{ij}v_i)_j + Av\Big]\Big[dz - \sum_{i,j}(a^{ij}z_i)_j dt\Big]dx + 2\mathbb{E}\int_0^T\int_{G_3}\sum_{i,j}(a^{ij}v_i dv)_j dx\\ &+2\mathbb{E}\int_0^T\int_{G_3}\sum_{i,j}\Big[\sum_{i',j'}\Big(2a^{ij}a^{i'j'}\ell_{i'}v_i v_{j'} - a^{ij}a^{i'j'}\ell_i v_{i'}v_{j'}\Big) + \Psi a^{ij}v_i v - a^{ij}\Big(A\ell_i + \frac{\Psi_i}{2}\Big)v^2\Big]_j dxdt\\ &\geq 2\mathbb{E}\sum_{i,j}\int_0^T\int_{G_3}c^{ij}v_i v_j dxdt + \mathbb{E}\int_0^T\int_{G_3}Bv^2 dxdt + \mathbb{E}\int_0^T\Big|-\sum_{i,j}(a^{ij}v_i)_j + Av\Big|^2 dxdt\\ &-\mathbb{E}\int_0^T\int_{G_3}\theta^2\sum_{i,j}a^{ij}(dz_i + \ell_i dz)(dz_j + \ell_j dz)dx - \mathbb{E}\int_0^T\int_{G_3}\theta^2 A(dz)^2 dx,\end{aligned} \tag{3.5}$$

Owing to that z is a solution to equation (3.4), we find

$$\begin{aligned}&2\mathbb{E}\int_0^T\int_{G_3}\theta\Big[-\sum_{i,j}(g^{ij}v_i)_j + Av\Big]\Big[dz - \sum_{i,j}(a^{ij}z_i)_j dt\Big]dx\\ &\leq \mathbb{E}\int_0^T\int_{G_3}\Big[-\sum_{i,j}(g^{ij}v_i)_j + Av\Big]^2 dxdt + \mathbb{E}\int_0^T\int_{G_3}\theta^2\Big(\langle\, a_1, \nabla z\,\rangle + a_2 z + f\Big)^2 dxdt\\ &\leq \mathbb{E}\int_0^T\int_{G_3}\Big[-\sum_{i,j}(g^{ij}v_i)_j + Av\Big]^2 dxdt + 3|a_1|^2_{L^\infty(0,T;L^\infty(G;\mathbb{R}^n))}\mathbb{E}\int_0^T\int_{G_3}\theta^2|\nabla z|^2 dxdt\\ &\quad+3|a_2|^2_{L^\infty(0,T;L^\infty(G))}\mathbb{E}\int_0^T\int_{G_3}\theta^2 z^2 dxdt + 3\mathbb{E}\int_0^T\int_{G_3}\theta^2 f^2 dxdt.\end{aligned} \tag{3.6}$$

Since $z = 0$ on ∂G_3, we have

$$\mathbb{E}\int_0^T\int_{G_3}\sum_{i,j}(a^{ij}v_i dv)_j dx = 0. \tag{3.7}$$

By means of $z=0$ on ∂G_3 and $\frac{\partial z}{\partial \nu}=0$ on $\partial G_3 \subset \Gamma$, we find

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_3}\sum_{i,j}\Big[\sum_{i',j'}\Big(2a^{ij}a^{i'j'}\ell_{i'}v_iv_{j'}-a^{ij}a^{i'j'}\ell_iv_{i'}v_{j'}\Big)+\Psi a^{ij}v_iv-a^{ij}\Big(A\ell_i+\frac{\Psi_i}{2}\Big)v^2\Big]_j dxdt\\
&=\lambda\mu\mathbb{E}\int_0^T\int_\Gamma \theta^2\varphi\sum_{i,j}a^{ij}\psi_i\nu^j\sum_{i',j'}a^{i'j'}\nu^{i'}\nu^{j'}\Big|\frac{\partial z}{\partial\nu}\Big|^2 d\Gamma dt \qquad (3.8)\\
&\le C\lambda\mu\mathbb{E}\int_0^T\int_\Gamma\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt.
\end{aligned}
$$

From (2.4) and (2.5), we know that there is a $\mu_0>0$ such that for every $\mu\ge\mu_0$, one can find a $\lambda_0(\mu_0)>0$ so that for all $\lambda\ge\lambda_0(\mu_0)$, it holds that

$$
\mathbb{E}\sum_{i,j}\int_0^T\int_{G_3}c^{ij}v_iv_jdxdt\ge C\lambda\mu^2\mathbb{E}\int_0^T\int_{G_3}|\nabla v|^2dxdt. \qquad (3.9)
$$

and that

$$
\mathbb{E}\int_0^T\int_{G_3}Bv^2dxdt\ge C\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_3}v^2dxdt. \qquad (3.10)
$$

Utilizing that z solves equation (3.4) again, we get

$$
\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_3}\theta^2\sum_{i,j}a^{ij}(dz_i+\ell_idz)(dz_j+\ell_jdz)dx\\
&=\mathbb{E}\int_0^T\int_{G_3}\theta^2\sum_{i,j}a^{ij}\Big[(a_3z)_i(a_3z)_j+2\lambda\mu\varphi\psi_i(a_3z)_ja_3z+\lambda^2\mu^2\varphi^2\psi_i\psi_ja_3^2z^2\Big]dxdt \qquad (3.11)\\
&\le C|a_3|^2_{L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G))}\Big(\mathbb{E}\int_0^T\int_{G_3}\theta^2\big(z^2+|\nabla z|^2\big)dxdt+\lambda^2\mu^2\mathbb{E}\int_0^T\int_{G_3}\theta^2\varphi^2|z|^2dxdt\Big).
\end{aligned}
$$

By $z_i=\theta^{-1}(v_i-\ell_iv)=\theta^{-1}(v_i-\lambda\mu\varphi\psi_iv)$ and $v_i=\theta(z_i+\ell_iz)=\theta(z_i+\lambda\mu\varphi\psi_iz)$, we get

$$
\frac{1}{C}\theta^2(|\nabla z|^2+\lambda^2\varphi^2\mu^2z^2)\le|\nabla v|^2+\lambda^2\mu^2\varphi^2v^2\le C\theta^2(|\nabla z|^2+\lambda^2\varphi^2\mu^2z^2). \qquad (3.12)
$$

From (3.5)–(3.12), we know that there is a $\lambda_1\ge\lambda_0$ such that for all $\lambda\ge\lambda_1$, we can find a $\mu_1(\lambda_1)\ge\mu_0(\lambda_0)$ so that for every $\mu\ge\mu_1(\lambda_1)$, it holds that

$$
\begin{aligned}
&\alpha\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_3}\theta^2z^2dxdt+\lambda\mu^2\mathbb{E}\int_0^T\int_{G_3}\theta^2|\nabla z|^2dxdt\\
&\le C\mathbb{E}\int_0^T\int_{G_3}\theta^2f^2dxdt+C\lambda\mu\mathbb{E}\int_0^T\int_\Gamma\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2d\Gamma dt.
\end{aligned} \qquad (3.13)
$$

Recalling that $f=-\sum_{i,j}(a^{ij}_j\eta_iy+a^{ij}\eta_{ij}y+2a^{ij}\eta_iy_j)-\langle\, a_1,\nabla\eta\,\rangle\, y$, from inequality (3.13), we find that

$$
\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_3}\theta^2z^2dxdt+\lambda\mu^2\mathbb{E}\int_0^T\int_{G_3}\theta^2|\nabla z|^2dxdt\\
&\le C\mathbb{E}\int_0^T\int_{G_3\setminus G_2}\theta^2\big(|y|^2+|\nabla y|^2\big)dxdt+C\lambda\mu\mathbb{E}\int_0^T\int_\Gamma\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2d\Gamma dt,
\end{aligned} \qquad (3.14)
$$

which, together with $z = \eta y$, implies that

$$\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_1}\theta^2 y^2 dxdt + \lambda\mu^2\mathbb{E}\int_0^T\int_{G_1}\theta^2|\nabla y|^2 dxdt\\
&\leq C\mathbb{E}\int_0^T\int_{G_3\setminus G_2}\theta^2\big(|y|^2+|\nabla y|^2\big)dxdt + C\lambda\mu\mathbb{E}\int_0^T\int_\Gamma\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt.
\end{aligned} \tag{3.15}$$

Fix any $\kappa\in(0,\frac{1}{2})$. Noting that $\theta=\theta(t,\cdot)$ is increasing (*resp.* decreasing) with respect to t in $[0,T/2]$ (*resp.* $(T/2,T]$), we deduce that

$$\begin{aligned}
&\lambda^3\mu^4\mathbb{E}\int_0^T\int_{G_1}\theta^2 y^2 dxdt + \lambda\mu^2\mathbb{E}\int_0^T\int_{G_1}\theta^2|\nabla y|^2 dxdt\\
&\geq \lambda^3\mu^4\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G_1}\theta^2 y^2 dxdt + \lambda\mu^2\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G_1}\theta^2|\nabla y|^2 dxdt\\
&\geq \inf_{x\in G_1}\theta^2\Big(\frac{T}{2}-\kappa T\Big)\lambda\mu^2\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G_1}(y^2+|\nabla y|^2)dxdt,
\end{aligned} \tag{3.16}$$

that

$$\begin{aligned}
&\mathbb{E}\int_0^T\int_{G_3\setminus G_2}\theta^2\big(|y|^2+|\nabla y|^2\big)dxdt\\
&\leq \sup_{x\in G\setminus G_2}\theta^2\Big(\frac{T}{2}\Big)\mathbb{E}\int_0^T\int_{G_3\setminus G_2}\big(|y|^2+|\nabla y|^2\big)dxdt.
\end{aligned} \tag{3.17}$$

and that

$$\begin{aligned}
&\lambda\mu\mathbb{E}\int_0^T\int_\Gamma\theta^2\varphi\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt\\
&\leq \lambda\mu\sup_{x\in G}\Big[\theta^2\Big(\frac{T}{2}\Big)\varphi\Big(\frac{T}{2}\Big)\Big]\mathbb{E}\int_0^T\int_\Gamma\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt.
\end{aligned} \tag{3.18}$$

From (3.16) and (3.17), we find

$$\begin{aligned}
&\inf_{x\in G_1}\theta^2\Big(\frac{T}{2}-\kappa T\Big)\lambda\mu^2\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G_1}(y^2+|\nabla y|^2)dxdt\\
&\leq C\sup_{x\in G_3\setminus G_2}\theta^2\Big(\frac{T}{2}\Big)\mathbb{E}\int_0^T\int_{G_3\setminus G_2}\big(|y|^2+|\nabla y|^2\big)dxdt + C\lambda\mu\sup_{x\in G}\Big[\theta^2\Big(\frac{T}{2}\Big)\varphi\Big(\frac{T}{2}\Big)\Big]\mathbb{E}\int_0^T\int_\Gamma\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt.
\end{aligned} \tag{3.19}$$

Let us choose N in (3.3) large enough such that $G'\subset G_1=\tilde{G}_1\cap G$. Since $\inf_{x\in G_1}\psi>\sup_{x\in G_3\setminus G_2}\psi$, we know that for any given $\kappa\in(0,\frac{1}{2})$, there is a $\mu_2>0$ and $\lambda_2(\mu_2)>0$ such that for all $\mu\geq\mu_2$ and $\lambda\geq\lambda_2(\mu_2)$, it holds that

$$\inf_{x\in G_1}\theta^2\Big(\frac{T}{2}-\kappa T\Big)\geq\sup_{x\in G\setminus G_2}\theta^2\Big(\frac{T}{2}\Big).$$

Hence, for any $\lambda\geq\max\{\lambda_1,\lambda_2\}$ and $\mu\geq\max\{\mu_1,\mu_2\}$, we have

$$\begin{aligned}
&\lambda\mu^2\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G'}(y^2+|\nabla y|^2)dxdt\\
&\leq C\mathbb{E}\int_0^T\int_{G_3\setminus G_2}\big(|y|^2+|\nabla y|^2\big)dxdt + C\lambda\mu\sup_{x\in G}\Big[\theta^2\Big(\frac{T}{2}\Big)\varphi\Big(\frac{T}{2}\Big)\Big]\mathbb{E}\int_0^T\int_\Gamma\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt.
\end{aligned} \tag{3.20}$$

Let $\varepsilon_0 = \frac{1}{\max\{\lambda_1,\lambda_2\}}$, from inequality (3.20), we have that for arbitrary $\varepsilon \in (0, \varepsilon_0]$, it holds that

$$\begin{aligned} &\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G'}(y^2+|\nabla y|^2)dxdt \\ &\leq C\varepsilon\mathbb{E}\int_0^T\int_G\big(|y|^2+|\nabla y|^2\big)dxdt + e^{\frac{C}{\varepsilon}}\mathbb{E}\int_0^T\int_\Gamma\Big|\frac{\partial y}{\partial\nu}\Big|^2 d\Gamma dt, \end{aligned} \tag{3.21}$$

which in turn implies that the above inequality holds for any $\varepsilon > 0$.

If $\frac{\partial y}{\partial \nu} = 0$ on $(0,T)\times\Gamma$, then we know that foe arbitrary $\varepsilon > 0$, it holds that

$$\mathbb{E}\int_{\frac{T}{2}-\kappa T}^{\frac{T}{2}+\kappa T}\int_{G'}(y^2+|\nabla y|^2)dxdt \leq C\varepsilon\mathbb{E}\int_0^T\int_G\big(|y|^2+|\nabla y|^2\big)dxdt. \tag{3.22}$$

Therefore, we find $y = 0$ in $(\frac{T}{2}-\kappa T, \frac{T}{2}+\kappa T)\times G'$, P-a.s. Because $\kappa \in (0, \frac{1}{2})$ is arbitrary, we have $y = 0$ in $(0,T)\times G'$, P-a.s. At last, noting that G' is arbitrary subset satisfying $G' \subset\subset G$, we know $y = 0$ in $(0,T)\times G$, P-a.s.

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