Incompressible limit of the non-isentropic Navier-Stokes equations with well-prepared initial data in three-dimensional bounded domains

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Abstract

This paper studies the incompressible limit of the non-isentropic Navier-Stokes equations for viscous polytropic flows with zero thermal coefficient in three-dimensional bounded C^4-domains. The uniform estimates in the Mach number , which exclude the estimate of high-order derivatives of the velocity in the normal directions to the boundary, are established within a short time interval independent of Mach number $\epsilon \in (0,1]$, provided that the initial data are well-prepared.

Résumé

Cet article étudie les limites incompressibles des équations non isentropiques de Navier-Stokes pour des écoulements polytropiques avec coefficient thermique nul en dimension trois d'espace dans des domaines bornés C^4. Des estimations uniformes en fonction du nombre de Mach $\epsilon \in (0,1]$ sont établies (à l'exception de certaines dérivées sur la vitesse dans la direction normale à la frontière) sur un petit intervalle de temps lorsque les données initiales sont bien posées.

Keywords: Incompressible limit; Navier-Stokes equations; Mach number
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1. Introduction

This paper is concerned with the low Mach number limit (or incompressible limit) of compressible flows which are described by the following non-dimensional Navier-Stokes equations in a three-dimensional bounded domain Ω:

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u I) + \frac{1}{\epsilon^2}\nabla P = 0, \tag{1.2}$$

$$(\rho e)_t + \mathrm{div}(\rho u e) + P\mathrm{div} u - \mathrm{div}(\kappa \nabla\theta) = \epsilon^2\big(2\mu|D(u)|^2 + \lambda(\mathrm{div} u)^2\big), \tag{1.3}$$

where $\rho,\ u = (u^1, u^2, u^3),\ P,\ e,\ \theta$ stand for the density, velocity, pressure, internal energy and temperature, respectively. The constants μ and λ are viscous coefficients with $\mu > 0$, $\mu + 2\lambda/3 \geq 0$, ϵ is the Mach number, κ is the heat conductivity coefficient, and $D(u) = (\nabla u + \nabla u^t)/2$. Moreover, we assume that the fluid is a polytropic ideal gas, that is,

$$e = c_v\theta, \quad P = R\rho\theta \tag{1.4}$$

with $c_v > 0$ and R being the specific heat at constant volume and the generic gas constant respectively. The ratio of specific heats is denoted by $\gamma = 1 + R/c_v$.

Formally, as ϵ tends to zero, the solutions to (1.1)-(1.4) will converge to the solution (ρ, u, π) of the following problem:

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{1.5}$$

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(2\mu D(u) + \lambda \mathrm{div} u I) + \nabla\pi = 0, \tag{1.6}$$

$$\gamma\, \mathrm{div} u = (\gamma - 1)\mathrm{div}\big[\kappa\nabla\big(\frac{1}{\rho}\big)\big]. \tag{1.7}$$

Especially, when the flows are isentropic or $\kappa = 0$, the limit velocity is divergence-free. This procedure is a singular limit, namely, the low Mach number limit (or incompressible limit), which is a physically interesting problem. Due to the large parameter $1/\epsilon^2$ in the momentum equation (1.2), mathematically, it is difficult to obtain uniform estimates in Mach number, which are necessary for the convergence to the background incompressible flows.

Many results have been achieved during the last three decades on the rigorous verification of the incompressible limit, which is a special case of the low Mach number approximation, with the limit velocity being divergence-free. The incompressible limit was studied by Beirao da Veiga [3], Isozaki [23], Klainerman & Majda [25], Schochet [35], Secchi [37, 38], Ukai [39] and others, for the inviscid ($\mu = \lambda = \kappa = 0$) isentropic ($P = P(\rho)$) fluids; and by Danchin [7, 8], Desjardins & Grenier [9], Desjardins, Grenier, Lions & Masmoudi [10], Donatelli, Feireisl & Novotný [12], Grenier [20], Hoff [22], Lions & Masmoudi [27, 28], Masmoudi [29], and others, for viscous (μ and λ are constants) isentropic fluids. It is interesting to note that the group method, which was developed by Grenier [20] and Schochet [35], is widely used in the study on various singular limits, both in the isentropic and non-isentropic regime, concerning the analysis of the fast acoustic waves in the periodic domains or bounded domains.

In the non-isentropic case, the large pressure gradient of $O(1/\epsilon^2)$ in the momentum equations is related to the behavior of both the density and the temperature. Thus, the low Mach number analysis is much more complicated in view of the trouble on estimating the vorticity. For the non-isentropic Euler equations and general initial data, Métivier & Schochet have proved some results [35, 36] which solve the cases without the solid boundary quite satisfactory. Later on, these results are extended in [1] to the boundary case by Alazard. In particular they have proved the existence of classical solutions on a time interval independent of ϵ.

The analysis for the non-isentropic Navier-Stokes equations is more difficult, due to the complex structure of the hyperbolic-parabolic coupled system and the fact that the entropy is not purely transported. The diffusion terms do not contribute more regularities, but produce more singular terms of $O(1/\epsilon)$. In the case that both the density and the temperature vary in a small range of $O(\epsilon^2)$ near positive constant states, Hagstrom and Lorenz [21] proved a uniform existence result for all time in the whole space provided that the background incompressible flows are sufficiently smooth. In the case that the temperature variation is not small and the viscous heating and thermal diffusion are negligible, Bresch et al [5] analyzed the acoustic waves by a method of characteristic expansions and gave a formal asymptotics as $\epsilon \to 0$ in $\mathbb{T}^n$. It is interesting to note that the group method in [20, 35] has been extended to the non-isentropic case in their result, and it provides new insight to the study of the low Mach number limit in the non-isentropic regime.

Another simplified situation, which allows the viscous heating, is that the thermal conductivity coefficient κ vanishes. In this case, Kim and Lee [24] verified the incompressible limit of local strong solutions in $\mathbb{R}^3$ with "well-prepared" initial data in the sense that,

$$\|(\rho_0^\epsilon - 1, q_0^\epsilon, u_0^\epsilon)\|_{H^2} + \frac{1}{\epsilon}\|(\nabla q_0^\epsilon, \mathrm{div} u_0^\epsilon)\|_{L^2} + \Big\|\frac{1}{\rho_0^\epsilon}\Big\|_{L^\infty} \leq C,$$

where ρ_0^ϵ, q_0^ϵ, u_0^ϵ are the initial data of ρ^ϵ, $(P^\epsilon - 1)/\epsilon$, and u^ϵ, respectively.

Concerning the full compressible Navier-Stokes equations, Alazard [2] recently studied this singular limit for local H^s solutions, $s > 2+n/2$, in $\mathbb{R}^n$ for "ill-prepared" initial data by employing the technique of pseudo-differential operators. The localized energy estimate is composed of high frequency estimate and low frequency estimate. Generalizing the method used in [5], Feireisl and Novotny [15] considered the low Mach number limit for the periodic "variational solutions" to the full Navier-Stokes-Fourier equations of certain radiative gases for "ill-prepared" initial data in the periodic domains, which dealt with the fast acoustic waves successfully. Related interesting results on various boundary conditions, including the cases with solid boundary, can be found in [13, 16] and the references therein. Unfortunately, the case of polytropic ideal gases is not included in their results.

The low Mach number limit for the non-isentropic Navier-Stokes equations governing the polytropic gases in bounded domains is not yet proved completely so far. In a recent work, Ou [34] studied the incompressible limit of the 1D non-isentropic Navier-Stokes equations with zero thermal conductivity in a finite interval. If the initial data are "well-prepared" in the sense that time derivatives up to order two are bounded initially:

$$\|(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon)\|_{H^2} + \|(\rho_t^\epsilon, u_t^\epsilon, q_t^\epsilon)|_{t=0}\|_{H^1} + \|(\rho_{tt}^\epsilon, u_{tt}^\epsilon, q_{tt}^\epsilon)|_{t=0}\|_{L^2} + \Big\|\frac{1}{\rho_0^\epsilon}\Big\|_{L^\infty} \leq C,$$

then the solutions are bounded uniformly in the Mach number in the same class as the initial data in a local time interval independent of Mach number. This implies that the limiting solution is exactly an incompressible profile.

The study of the low Mach number limit is a vast subject that we could only recommend one part of the references here. Within our knowledge, there are also interesting results [6, 14, 30] on the singular limits which vanish the Froude number too. Moreover, the reader may refer to Desjardins & Lin [11], Gallagher [19] and Schochet [36] for well-written survey papers.

The purpose of this paper is to verify the incompressible limit for the system (1.1)-(1.4) with zero heat-conductivity and "well-prepared" initial data in bounded domains in $\mathbb{R}^3$, thus generalizing the work of Kim and Lee [24] in the whole space. The main difficulty comparing to the periodic case and the whole space case ([2, 5, 21, 24]) is the uniform high-norm estimates with respect to the Mach number and a time interval independent of the Mach number. In a bounded domain, the geometry of the domain leads to the boundary effects of the acoustic waves, which will reflect against the boundary, thus increasing the complexity of the low Mach number analysis. From the viewpoint of mathematical analysis, the difficulty is that, the techniques of Fourier transform (used in [2, 21]) or pseudo-differential operators (used in [2]), especially the commutator estimates, cannot be employed in the case of solid boundary. Moreover, the integrating by parts for the high-order derivatives is invalid in the normal direction to the boundary, in contrast to the analysis in [24]. Thus these difficulties prevent us from using the usual way to balance the singular differential operators in the whole space or periodic space. We also remark further differences between our case and the abovementioned cases without solid boundary. In contrast to [21], we allow large variations of $O(1)$ for the density and the temperature, instead of variations of $O(1/\epsilon^2)$. Comparing to [5], the energy (or temperature) equation here loses the structure that the dependent variable is purely transported, since the viscous heating is allowed in this paper.

Moreover, a priori uniform estimates are more difficult to obtain than the 1D boundary case. In the analysis of [34], estimates of high order spatial derivatives can nearly be controlled by the temporal estimates which also satisfy the Dirichlet boundary condition. However, in three spatial dimensions, this is no longer true. Note that the temporal derivatives are not the only tangential derivatives to the boundary, and moreover, the estimates of tangential derivatives are strongly coupled with the estimates of normal derivatives. Thus the high order normal derivatives cannot be controlled by applying the anti-symmetry property of the penalty differential operators. To circumvent this difficulty, we take a new way other than the one in [24] to obtain the uniform estimates. Regarding the Navier-Stokes equations as a Stokes problem, we can see that the estimate of $L^2(0,t;H^3(\Omega))$-norm of the velocity u could eventually be reduced to the boundedness of $\|\text{div}u\|_{L^2(0,t;H^2(\Omega))}$, which is a quantity measuring incompressibility and easier to control. Moreover, $\|\nabla q/\epsilon^2\|_{L^2(0,t;H^1(\Omega))}$ is also estimated at the same time, which is very useful in deriving the uniform estimates. The most difficult part, namely $\|\nabla^2\text{div}u\|_{L^2(0,t;L^2(\Omega))}$, is estimated by a localized strategy following the idea in the works of Valli and Zajaczkowski [40, 41]. That is, we introduce the isothermal coordinates in local regions to estimate div$u$ and other higher order spatial derivatives of the velocity u and the pressure variation $(P-1)/\epsilon$ near the boundary. The difficulty comparing to [40, 41] is that, some of the estimates could contain the terms of order $1/\epsilon$ or even the quantities of higher order. Therefore, we should treat all these estimates with respect to the Mach number very carefully, so that the estimate for the full norm is closed.

We will prove the local existence of strong solutions to the non-isentropic Navier-Stokes equations with the uniform-in-ϵ estimates in a small time interval. Here we only require that the initial data are "well-prepared" in the following sense:

$$\big\|(\rho_0^\epsilon, u_0^\epsilon, \frac{q_0^\epsilon}{\epsilon})\big\|_{H^2} + \big\|\big(u_t^\epsilon, \frac{q_t^\epsilon}{\epsilon}\big)\big|_{t=0}\big\|_{L^2} + \big\|\frac{1}{\rho_0^\epsilon}\big\|_{L^\infty} \le C,$$

which relaxes the assumptions on the initial data in [34]. We should remark here that without the restriction of initial data, the uniform estimates and the convergence to the solution of the incompressible Navier-Stokes equations do not necessarily hold. The uniform-in-Mach number estimates, which exclude the estimates of high-order derivatives of the velocity in the normal directions to the boundary, are established within a short time interval. The reason that we cannot obtain the uniform estimates in full norm is the interaction between the solid boundary and the acoustic waves in bounded domains which are even though in a "slow" scale.

To state the main result of this paper, we consider the case $\kappa = 0$ in (1.1)-(1.3). In this case, the third limit equation reduces to

$$\text{div} u = 0.$$

Introducing the *pressure variation* q by

$$P = 1 + q,$$

and letting $\nu = \mu + \lambda$, we can rewrite the non-dimensional system (1.1)-(1.3) as follows

$$\rho_t + \text{div}(\rho u) = 0, \tag{1.8}$$

$$\rho(u_t + u\cdot\nabla u) + \frac{1}{\epsilon^2}\nabla q = \mu\Delta u + \nu\nabla \text{div} u, \tag{1.9}$$

$$q_t + u\cdot\nabla q + \gamma(q+1)\text{div} u = (\gamma-1)\epsilon^2\big(2\mu|D(u)|^2 + \lambda(\text{div} u)^2\big). \tag{1.10}$$

We impose the following initial and boundary conditions:

$$(\rho, u, q)|_{t=0} = (\rho_0, u_0, q_0) \quad \text{in } \Omega, \tag{1.11}$$

$$u|_{\partial\Omega} = 0, \tag{1.12}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with C^4-boundary $\partial\Omega$. Note that this requirement is clear when we deal with the localized estimates near the boundary. Similarly, the non-negativeness of the second viscosity coefficient is required in the localized estimates.

Our main result is the following, which gives the local existence and a uniform-in-ϵ estimate of strong solutions to (1.8)-(1.12), and the corresponding incompressible limit.

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with C^4-boundary $\partial\Omega$ and $\nu = \mu + \lambda \geq 0$. Assume that the initial data $\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon$ in (1.11) satisfy $\rho_0^\epsilon \geq C_0^{-1} > 0$,*

$$\big\|\big(\rho_0^\epsilon, u_0^\epsilon, \frac{q_0^\epsilon}{\epsilon}\big)\big\|_{H^2} + \big\|\big(u_t^\epsilon(0), \frac{q_t^\epsilon}{\epsilon}(0)\big)\big\|_{L^2} \leq C_0$$

and the compatibility conditions

$$u_0^\epsilon = u_t^\epsilon(0) = 0 \quad \text{on} \quad \partial\Omega,$$

where C_0 is a positive constant independent of $\epsilon \in (0,1]$. Then, the initial-boundary value problem (1.8)-(1.12) admits a unique solution (ρ^ϵ,u^ϵ, q^ϵ) in $\Omega \times [0, T_0]$, for some positive constant $T_0 = T_0(C_0)$ independent of $\epsilon \in (0,1]$. Moreover, $(\rho^\epsilon, u^\epsilon, q^\epsilon)$ satisfies the uniform estimate:

$$\max_{0\leq t\leq T_0} \Big(\big\|(\rho^\epsilon, \frac{q^\epsilon}{\epsilon})\big\|_{H^2} + \|u^\epsilon\|_{H^1} + \big\|(\rho_t^\epsilon, u_t^\epsilon, \frac{q_t^\epsilon}{\epsilon})\big\|_{L^2} + \big\|\frac{1}{\rho^\epsilon}\big\|_{L^\infty}\Big)(t) + \left[\int_0^{T_0} \Big(\|u^\epsilon\|_{H^3}^2 + \|u_t^\epsilon\|_{H^1}^2 + \big\|\frac{\nabla q^\epsilon}{\epsilon^2}\big\|_{H^1}^2\Big) dt\right]^{1/2} \leq C,$$

where $C = C(C_0)$ is a positive constant independent of $\epsilon \in (0,1]$. Furthermore, $(\rho^\epsilon, u^\epsilon)$ converges to (ρ, u) in certain Sobolev spaces as $\epsilon \to 0$, and there exists a function $\pi(x,t)$, such that $(\rho, u, \pi) \in C(0, T_0; H^2 \times H^2 \times H^1)$ solves *the following initial-boundary value problem of inhomogeneous incompressible Navier-Stokes equations:*

$$\begin{aligned} &\rho_t + u\cdot\nabla\rho = 0, \quad \operatorname{div} u = 0 \quad in \quad \Omega\times(0,T_0),\\ &\rho(u_t + u\cdot\nabla u) + \nabla\pi = \mu\Delta u \quad in \quad \Omega\times(0,T_0),\\ &u|_{\partial\Omega} = 0 \qquad in \ \ (0,T_0),\\ &(\rho,u)|_{t=0} = (\rho_0,u_0) \qquad in \ \ \Omega, \end{aligned} \tag{1.13}$$

where (ρ_0, u_0) *is supposed to be the weak limit of* $(\rho_0^\epsilon, u_0^\epsilon)$ *in* $H^2(\Omega)$ *with* $\mathrm{div} u_0 = 0$ *a.e. in* Ω.

Remark 1. *To simplify the statement, we have used "*$u_t^\epsilon(0)$*" to signifies the quantity* $u_t^\epsilon|_{t=0} := -u_0^\epsilon \cdot \nabla u_0^\epsilon + (-\nabla q_0^e/\epsilon^2 + \mu\Delta u_0^e + \nu\nabla \operatorname{div} u_0^e)/\rho_0^\epsilon$ *obtained through the equation* (1.9). *"*$q_t^\epsilon(0)$*" is given through the equation* (1.10) *in the same manner.*

Roughly speaking, the key ingredients in the proof of Theorem 1 are the global existence and the uniform-in-ϵ estimates, in particular the boundary estimates, in a small time interval, for the linearized system. To derive the uniform estimates on both temporal and spatial derivatives of the solution to the linearized equations near boundary, we first show the uniform estimates for ρ by utilizing the fact that the density equation is of order $O(1)$ and decouple from the momentum and energy equations. Then, we show that the quantities $\|(q/\epsilon, u)\|_{C([0,t],H^1)}$, $\|(q_t/\epsilon, u)\|_{C([0,t],L^2)}$, $\|u\|_{L^2(0,t;H^3)}$ and $\|\nabla q\|_{L^2(0,t;H^1)}$ are controlled by $\|\mathrm{div} u\|_{L^2(0,t;H^2)}$. The boundedness of $\|\mathrm{div} u\|_{L^2(0,t;H^2)}$ and higher order derivatives of q is obtained by estimating the tangential and normal derivatives on boundary respectively, and using the isothermal coordinates in local regions when deriving boundedness of the normal derivatives. Since the large parameter $1/\epsilon$ is involved in the momentum equations, the estimates near the boundary are very subtle and delicate, especially after the change of coordinates. Note that some of these estimates contain large terms with respect to Mach number, thus we should treat them carefully in order to close the estimate for the full norm. Finally, we apply the Schauder fixed point theorem and the derived uniform estimates for the linearized equations to obtain Theorem 1.

This article is organized as follows. In Section 2, we state some elementary lemmas and calculus inequalities for the convenience of the reader. In Section 3, uniform estimates for the linearized equations are shown by a elaborate analysis on both temporal and spatial derivatives near boundary. In Section 4, we give the proof of Theorem 1.

We end this section by introducing the notation used throughout this paper. By $W^{k,p}(\Omega)$ and $H^k(\Omega)$ we denote the usual Sobolev spaces with the norms $\|\cdot\|_{W^{k,p}}$ and $\|\cdot\|_{H^k}$, respectively. By C, and $F_0(\cdot)$, $F(\cdot)$, $F_i(\cdot)$, $i=1,2,\cdots$, we denote a generic positive constant, and positive continuous functions of their argument respectively, which are independent of ϵ. Note that $F_0(\cdot)$ may usually depend on small positive constants δ and η; however, $F_i(\cdot)$ is always independent of δ and η. We shall use the following abbreviations:

$$L_t^p(H^k)\equiv L^p(0,t;H^k(\Omega)),\quad C_t(H^k)\equiv C([0,t],H^k(\Omega)),$$
$$\|\cdot\|_{L_t^p(H^k)}\equiv\|\cdot\|_{L^p(0,t;H^k)},\quad \|\cdot\|_{C_t(H^k)}\equiv\|\cdot\|_{C([0,t],H^k)}.$$

Furthermore, we denote partial derivatives by subscripts and the components of a vector by superscripts. For example, u^j means the j-th component of a vector u, and u_y stands for the partial derivative of u with respect to y.

2. Preliminaries

In this section, we list some lemmas which will be frequently used throughout this paper.

Lemma 1. (Brezis and Bourguignon [4]) *Assume $F\in C([0,T];W^{k,p}(\Omega;\mathbb{R}^N))$ with $1\le p<\infty$, and $k>N/p+1$. Then the problem*

$$\frac{d\chi}{dt}(x,t)=F(\chi(x,t),t),\quad \chi(x,0)=x,$$

has a solution $\chi\in C^1([0,T];D^{k,p}(\Omega))$, where

$$D^{k,p}(\Omega)=\left\{\eta\in W^{k,p}(\Omega)|\ \eta:\bar\Omega\to\bar\Omega\ \text{ is bijective, }\eta^{-1}\in W^{k,p}(\Omega)\right\}.$$

Lemma 2. (Brezis and Bourguignon [4]) *Let $k\ge 2$ be an integer, and let $1\le p\le q\le\infty$ be such that $p<\infty$ and $k>\frac{N}{q}+1$. If $F\in W^{k,p}(\Omega)$, then the mapping $G\mapsto F\circ G$ is continuous from $D^{k,q}(\Omega)$ into $W^{k,p}(\Omega)$.*

Lemma 3. (Interpolation inequality, cf. Friedman [17], Part 1, Theorem 10.1) *Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with C^k-boundary, and let u be any* *function in* $W^{k,r}(\Omega)\cap L^q(\Omega)$ *with* $1\le r,q\le\infty$. *For any integer* j *with* $0\le j<k$, *and for any number* a *in the interval* $[j/k,1]$, *set*

$$\frac{1}{p}=\frac{j}{N}+a(\frac{1}{r}-\frac{k}{N})+(1-a)\frac{1}{q}.$$

If $k-j-N/r$ *is not a nonnegative integer, then*

$$\|D^j u\|_{L^p(\Omega)}\le C\|u\|^a_{W^{k,r}(\Omega)}\|u\|^{1-a}_{L^q(\Omega)}. \tag{2.1}$$

If $k-j-N/r$ *is a nonnegative integer, then* (2.1) *only holds for* $a=j/k$. *The constant* C *depends only on* Ω, r, q, k, j, a. □

For the reader's convenience, we give some special cases of (2.1) in $\mathbb{R}^3$ which will be frequently applied, but without mentioning explicitly, throughout this paper:

$$\|\nabla^2 u\|_{L^2}\le C\|u\|^{\frac{2}{3}}_{H^3}\|u\|^{\frac{1}{3}}_{L^2},\quad \|\nabla^2 u\|_{L^2}\le C\|u\|^{\frac{1}{2}}_{H^3}\|\nabla u\|^{\frac{1}{2}}_{L^2},$$

$$\|u\|_{L^3}\le C\|u\|^{\frac{1}{2}}_{H^1}\|u\|^{\frac{1}{2}}_{L^2},\quad \|u\|_{L^4}\le C\|u\|^{\frac{3}{4}}_{H^1}\|u\|^{\frac{1}{4}}_{L^2}.$$

Moreover, by the Sobolev embedding theorem, we have

$$\|u\|_{L^\infty}\le C\|u\|_{W^{1,4}}\le C\|u\|^{\frac{3}{4}}_{H^2}\|u\|^{\frac{1}{4}}_{H^1}\le C\|u\|^{\frac{7}{8}}_{H^2}\|u\|^{\frac{1}{8}}_{L^2}.$$

3. Uniform estimates for the linearized equations

Motivated by the idea of Kim and Lee [24], we shall use the method of linearization to obtain the existence and uniform estimates for the Navier-Stokes equations. Consider the following linearized equations in $\Omega\times(0,T)$:

$$\rho_t+\mathrm{div}(\rho v)=0, \tag{3.1}$$

$$\rho(u_t+v\cdot\nabla u)+\frac{1}{\epsilon^2}\nabla q=\mu\Delta u+\nu\nabla\mathrm{div}u, \tag{3.2}$$

$$\frac{1}{\gamma}(q_t+v\cdot\nabla q)+q\mathrm{div}v+\mathrm{div}u=\frac{\gamma-1}{\gamma}\epsilon^2\big(2\mu|D(v)|^2+\lambda(\mathrm{div}v)^2\big), \tag{3.3}$$

together with the initial data

$$(\rho,u,q)|_{t=0}=(\rho_0,u_0,q_0)\quad \text{in } \Omega \tag{3.4}$$

and the boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega \times (0,T), \tag{3.5}$$

where v is a given function satisfying $v|_{t=0} = u_0$, $v \in C([0,T], H_0^1) \cap L^2(0,T;H^3)$ and $v_t \in C([0,T];L^2) \cap L^2(0,T;H^1)$.

Before deriving the uniform estimates, we state a global existence theorem for the problem (3.1)-(3.5) in $\Omega \times [0,T]$ for any given $T > 0$.

Theorem 2. *Suppose that the initial data $(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon)$ in ((3.4)) satisfy*

$$(\rho_0^\epsilon, u_0^\epsilon, q_0^\epsilon) \in H^2(\Omega), \quad (u_t^\epsilon(0), q_t(0)) \in L^2(\Omega), \quad \rho_0^\epsilon \geq \delta_0 \text{ for some } \delta_0 > 0,$$

and the compatibility conditions: $u_0^\epsilon|_{\partial\Omega} = u_t^\epsilon(0)|_{\partial\Omega} = 0$. Then, the problem (3.1)-(3.5) admits a unique solution $(\rho^\epsilon, u^\epsilon, q^\epsilon)$ in $\Omega \times (0,T)$, satisfying $\inf\limits_{t,x} \rho^\epsilon(t,x) > 0$ in $\Omega \times (0,T)$ and

$$\begin{aligned}
&(\rho^\epsilon, q^\epsilon) \in C([0,T], H^2),\ u^\epsilon \in C([0,T], H_0^1 \cap H^2) \cap L^2(0,T;H^3),\\
&(\rho_t^\epsilon, q_t^\epsilon) \in C([0,T], L^2),\ u_t \in C([0,T], L^2) \cap L^2(0,T;H_0^1).
\end{aligned}$$

For the sake of simplicity, we will drop the superscript ϵ from now on. We define

$$M_0 := \big\|(\rho_0, \frac{q_0}{\epsilon})\big\|_{H^2} + \|u_0\|_{H^1} + \big\|\big(u_t(0), \frac{q_t}{\epsilon}(0)\big)\big\|_{L^2} + \|\rho_0^{-1}\|_{L^\infty}$$

$$M := \|v\|_{C_T(H^1)} + \|v_t\|_{C_T(L^2)} + \|v\|_{L^2_T(H^3)} + \|v_t\|_{L^2_T(H^1)},$$

and will show the following uniform estimate for (3.1)-(3.5) in this section, which play a key role in the proof of Theorem 1.

Proposition 1. *Assume that the initial data ρ_0, u_0, q_0 satisfy*

$$M_0 \leq C_0 \quad \text{for some constant } C_0 > 0 \text{ independent of } \epsilon \in (0,1].$$

Suppose that (ρ, u, q) is a solution obtained in Theorem 2. Then there exist the positive constants $T_0 = T_0(C_0) \leq T$ and $C = C(C_0)$, independent of $\epsilon \in (0,1]$, such that

$$\begin{aligned}
&\max_{0\leq t\leq T_0} \big(\|\frac{1}{\rho}\|_{L^\infty} + \big\|(\rho, \frac{q}{\epsilon})\big\|_{H^2} + \|u\|_{H^1} + \big\|(\rho_t, u_t, \frac{q_t}{\epsilon})\big\|_{L^2}\big)(t)\\
&\quad + \Big[\int_0^{T_0} \big(\|u\|_{H^3}^2 + \|u_t\|_{H^1}^2 + \|\frac{\nabla q}{\epsilon^2}\|_{H^1}^2\big)dt\Big]^{1/2} \leq C.
\end{aligned} \tag{3.6}$$

In the sequel, we estimate ρ and (q, u) separately, since the equation (3.1) doesn't contain any quantity of order $O(1/\epsilon^2)$ and thus can be decoupled from (1.9) and (1.10).

3.1. Estimates for ρ

Define the particle path $\chi(x, s; t)$ through (x, s), to be the solution of

$$\begin{cases} \dfrac{d}{dt}\chi(x,s;t) = v(\chi(x,s;t),t), \quad t,s \in [0,T], x \in \bar{\Omega}, \\ \hfill \chi(x,s;s) = x. \end{cases}$$

Then ρ can be expressed explicitly as

$$\rho(x,t) = \rho_0(\chi(x,t;0)) \exp\left(-\int_0^t \mathrm{div} v(\chi(x,t;s),s)ds\right). \tag{3.7}$$

It is easy to see that on the particle path, $\dot{\chi} = v(\chi, t)$. Since $v \in L^2(0,T;W^{2,6})$, by Lemma 1 and the fact that C^1 is dense in the class C of continuous functions, we have $\chi \in C([0,T]^2; D^{2,6})$. From (3.7) and Lemma 2, we get $\rho \in C(0,T;H^2)$ since $\rho_0 \in H^2$. This together with the equation (3.1) immediately yields $\rho_t \in C(0,T;L^2)$. Moreover, (3.7) implies $\inf_{t,x} \rho(t,x) > 0$ provided $\inf_{x\in\Omega} \rho_0(x) > 0$.

To derive bounds of ρ, we use the explicit formula (3.7) to deduce that

$$\begin{aligned} \|\rho^{-1}\|_{L^\infty_{x,t}} &\le \|\rho_0^{-1}\|_{L^\infty} \exp\left(C\sqrt{t}\|v\|_{L^2_T(H^3)}\right) \\ &\le M_0 \exp\left(C\sqrt{t}M\right) \le CM_0, \quad 0 \le t \le T_1, \end{aligned} \tag{3.8}$$

where $T_1 := \min\{T, (1+M^2)^{-1}\}$. A H^2-bound of ρ can be derived in a routine manner. Let α be a multi-index with $0 \le |\alpha| \le 2$. Taking ∂^α to (3.1), then multiplying the resulting equation by $\partial^\alpha\rho$, integrating over Ω, and summing over α, we obtain

$$\frac{d}{dt}\|\rho\|_{H^2}^2 \le C\|v\|_{H^3}\|\rho\|_{H^2}^2,$$

which implies that, for any $0 < \epsilon \le 1$ and $0 < t \le T_1$,

$$\|\rho(t)\|_{H^2} \le \|\rho_0\|_{H^2} \exp\left(C\int_0^t \|v(s)\|_{H^3}ds\right) \le CM_0. \tag{3.9}$$

Similarly, it follows from applying ∂_t to (3.1) that, for any $0 < \epsilon \le 1$ and $0 < t \le T_1$,

$$\|\rho_t(t)\|_{L^2} \le C(M_0^2 + 1). \tag{3.10}$$

We summarize the estimates for ρ as follows.

Proposition 2. *There exist positive constants C and $T_1(M)$, and a positive continuous function $F_1(\cdot)$, such that*

$$\|\rho^{-1}(t)\|_{L^\infty} + \|\rho(t)\|_{H^2} + \|\rho_t(t)\|_{L^2} \le F_1(M_0),$$

for any $0 \le t \le T_1$ and $0 < \epsilon \le 1$.

Note that all the functions $F_i(\cdot)$, $i = 1, 2, \cdots$, are independent of ϵ and any small positive constants δ and η throughout this paper.

3.2. L^2-estimates for (u, q)

The L^2-boundedness of (u, q) follows easily from the energy method. Namely, we multiply (3.2) by u and (3.3) by q/ϵ^2 in $L^2(\Omega)$, and integrate by parts to get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (\rho|u|^2 + \frac{q^2}{2\gamma\epsilon^2})dx + \int_\Omega \big(\mu|\nabla u|^2 + \nu(\mathrm{div}u)^2\big)dx \\
&\qquad \le \|\mathrm{div}v\|_{L^\infty}\|\frac{q}{\epsilon}\|_{L^2}^2 + C\epsilon\|\nabla v\|_{L^2}^{\frac{5}{4}}\|\nabla v\|_{H^2}^{\frac{3}{4}}\|\frac{q}{\epsilon}\|_{L^2},
\end{aligned}$$

where we have used the interpolation inequality in Lemma 3. Thus, applying the Gronwall inequality to the above inequality and using Sobolev's imbedding theorem, we deduce that, for any $0 \le t \le T_2 := \min(T, (1 + M^{16/5})^{-1})$ and $0 < \epsilon \le 1$,

$$\begin{aligned}
&\int_\Omega \big(\rho|u|^2 + \frac{q^2}{2\gamma\epsilon^2}\big)(t)dx + \int_0^t\int_\Omega (\mu|\nabla u|^2 + \nu(\mathrm{div}u)^2)dxds \\
&\quad \le \exp\big(C\int_0^t \|\mathrm{div}v\|_{H^2}ds\big)\big(F_0(M_0) + C\int_0^t \|v\|_{H^1}^{\frac{5}{4}}\|v\|_{H^3}^{\frac{3}{4}}ds\big) \\
&\quad \le \exp\big(C\sqrt{t}M\big)(F_0(M_0) + Ct^{\frac{5}{8}}M^2) \le F_0(M_0).
\end{aligned} \tag{3.11}$$

3.3. Estimates for first-order derivatives of (u, q)

Step 1. First, we estimate the first-order spatial derivatives of u. Taking the time derivative ∂_t to (3.2) and integrating the resulting equation by u in $L^2(\Omega)$, one finds

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega(\mu|\nabla u|^2+\nu(\text{div}u)^2)dx+\frac{1}{\epsilon^2}\int_\Omega\nabla q_t\cdot udx \\ &=-\int_\Omega\big(\rho_t(u_t+v\cdot\nabla u)+\rho(v\cdot\nabla u)_t\big)\cdot udx-\int_\Omega\rho u\cdot u_{tt}dx \\ &\equiv I_1(t)+I_2(t), \end{aligned} \tag{3.12}$$

where it is easy to see by the Poincaré and interpolation inequalities as well as a straightforward calculation that $|\int_0^t I_1(s)ds|$ is bounded from above by

$$\begin{aligned} &C\int_0^t\|\rho_t\|_{L^2}\|u\|_{H^1}(\|u_t\|_{H^1}+\|v\|_{H^1}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla u\|_{H^2}^{\frac{1}{2}})ds \\ &+C\int_0^t\|\rho\|_{H^2}\|u\|_{H^1}(\|v_t\|_{H^1}\|u\|_{H^1}+\|v\|_{H^1}\|\nabla u_t\|_{L^2})ds \\ &\quad\le F_0(M_0)\int_0^t(1+M^2+\|v_t\|_{H^1})\|\nabla u\|_{L^2}^2ds \\ &\quad+\delta\big(\|u_t\|_{L^2_t(H^1)}^2+\|u\|_{L^2_t(H^3)}^2\big),\ \forall\, 0\le t\le T_2,\ 0<\epsilon\le 1. \end{aligned}$$

To bound $I_2(t)$, we integrate by parts to see that $|\int_0^t I_2(s)ds|$ equals to

$$\begin{aligned} &\Big|-\int_\Omega\rho u\cdot u_tdx\Big|_0^t+\int_0^t\int_\Omega(\rho|u_t|^2+\rho_tu\cdot u_t)dxds\Big| \\ &\le C\|\rho_0\|_{H^2}\|u_0\|_{L^2}\|u_t(0)\|_{L^2}+C\|\rho\|_{C_T(H^2)}\|u(t)\|_{L^2}\|u_t(t)\|_{L^2} \\ &+\int_0^t(\|\rho\|_{H^2}\|u_t\|_{L^2}^2+\|\rho_t\|_{L^2}\|u\|_{H^1}\|u_t\|_{H^1})ds \\ &\le F_0(M_0)(1+\int_0^t\|u_t\|_{L^2}^2ds)+\delta\big(\|u_t(t)\|_{L^2}^2+\|u_t\|_{L^2_t(H^1)}^2\big). \end{aligned}$$

Substituting the above two inequalities into the integration of (3.12) in $(0,t)$ and using (3.11), we conclude that

$$\begin{aligned}
&\int_\Omega(\mu|\nabla u|^2+\nu(\mathrm{div}u)^2)(t)dx+\frac{1}{\epsilon^2}\int_0^t\int_\Omega\nabla q_t\cdot udxds\\
&\le F_0(M_0)\big(1+\int_0^t(1+M^2+\|v_t\|_{H^1})\|(\nabla u,u_t)\|_{L^2}^2ds\big)\\
&+\delta\big(\|u_t(t)\|_{L^2}+\|u_t\|_{L^2_t(H^1)}\big),\quad\forall\,0\le t\le T_2,\ \ 0<\epsilon\le 1.
\end{aligned}\tag{3.13}$$

Note that the function F_0 may depend on both M_0 and small constants δ and η from now on, but we still write it as $F_0(M_0)$ for the sake of simplicity.

On the other hand, we take the inner product of (3.3) and q_t^2/ϵ^2 in $L^2((0,t)\times\Omega)$, and make use of the interpolation inequality and (3.11) to get

$$\begin{aligned}
&\frac{1}{\gamma\epsilon^2}\int_0^t\int_\Omega q_t^2dxds+\frac{1}{\epsilon^2}\int_0^t\int_\Omega\mathrm{div}u\,q_tdxds\\
&\le C\int_0^t(\|v\|_{H^2}\big\|\frac{\nabla q}{\epsilon}\big\|_{L^2}+\big\|\frac{q}{\epsilon}\big\|_{H^1}\|\mathrm{div}v\|_{H^1}+\epsilon\|\nabla v\|_{L^4}^2)\big\|\frac{q_t}{\epsilon}\big\|_{L^2}ds\\
&\le C\int_0^t\|v\|_{H^3}\big\|(\frac{\nabla q}{\epsilon},\frac{q_t}{\epsilon})\big\|_{L^2}^2ds+\sqrt{t}\|v\|_{L^2_T(H^3)}\big\|\frac{q}{\epsilon}\big\|_{C_{T_2}(L^2)}^2\\
&+t^{\frac34}\|v\|_{C_T(H^1)}^{\frac52}\|v\|_{L^2_T(H^3)}^{\frac12},
\end{aligned}$$

which combined with (3.13) implies that for any $0\le t\le T_3:=\min(T,(1+M^4)^{-1})$ and $0<\epsilon\le 1$,

$$\begin{aligned}
&\int_\Omega(\mu|\nabla u|^2+\nu(\mathrm{div}u)^2)dx(t)+\frac{1}{\gamma\epsilon^2}\int_0^t\int_\Omega q_t^2dxds\\
&\le F_0(M_0)\int_0^t(1+M^2+\|v\|_{H^3}+\|v_t\|_{H^1})\big\|(\nabla u,u_t,\frac{\nabla q}{\epsilon},\frac{q_t}{\epsilon})\big\|_{L^2}^2ds\\
&+\delta(\|u_t(t)\|_{L^2}+\|u_t\|_{L^2_t(H^1)})+F_0(M_0).
\end{aligned}\tag{3.14}$$

Step 2. Next, we estimate the term $\|\nabla q/\epsilon\|_{C_t(L^2)}$. To this end, we differentiate (3.3) with respect to the spatial variables to get

$$\frac{1}{\gamma}(\nabla q_t+v\cdot\nabla^2q+\nabla v\nabla q)+\nabla(q\mathrm{div}v)+\nabla\mathrm{div}u=\epsilon^2\nabla\big(2\mu|D(v)|^2+\lambda(\mathrm{div}v)^2\big).$$

Multiplying this equation by $\nabla q/\epsilon^2$ in $L^2(\Omega)$ and integrating by parts, one obtains

$$\begin{aligned}\frac{d}{dt}\Big\|\frac{\nabla q}{\sqrt{2\gamma}\epsilon}\Big\|_{L^2}^2 \le& C\big(\|\nabla v\|_{L^\infty}\Big\|\frac{\nabla q}{\epsilon}\Big\|_{L^2}^2+\|\nabla \mathrm{div} v\|_{H^1}\Big\|\frac{q}{\epsilon}\Big\|_{H^1}\Big\|\frac{\nabla q}{\epsilon}\Big\|_{L^2}\big)\\ &+C\big(\|\nabla \mathrm{div} u\|_{L^2}\Big\|\frac{\nabla q}{\epsilon^2}\Big\|_{L^2}+\epsilon\|\nabla v\|_{H^1}\|\nabla^2 v\|_{L^3}\Big\|\frac{\nabla q}{\epsilon}\Big\|_{L^2}\big)\\ \le&\delta\Big\|\frac{\nabla q}{\epsilon^2}\Big\|_{L^2}^2+C_\delta\|\nabla \mathrm{div} u\|_{L^2}^2+C\big(\|v\|_{H^3}\Big\|\frac{q}{\epsilon}\Big\|_{H^1}^2+\|v\|_{H^1}^{\frac{3}{2}}\|v\|_{H^3}^{\frac{1}{2}}\big),\end{aligned}$$

from which, (3.11) and the inequality

$$\|\mathrm{div} u\|_{H^1}^2 \le \delta\|u\|_{H^3}^2+C_\delta\|u\|_{H^1}^2, \tag{3.15}$$

it follows that, for any $0\le t\le T_3,\ 0<\epsilon\le 1$,

$$\begin{aligned}\Big\|\frac{\nabla q}{\epsilon}(t)\Big\|_{L^2}^2 \le& F_0(M_0)+C\int_0^t\|v\|_{H^3}\Big\|\frac{\nabla q}{\epsilon}\Big\|_{L^2}^2 ds\\ &+\delta\big(\Big\|\frac{\nabla q}{\epsilon^2}\Big\|_{L_t^2(L^2)}^2+\|u\|_{L_t^2(H^3)}^2\big).\end{aligned} \tag{3.16}$$

Step 3. It remains to bound u_t and q_t/ϵ to finish the estimate of the first-order derivatives. For this purpose, we differentiate (3.2) with respect to t and take the inner product of the resulting equation and u_t in $L^2(\Omega)$ to deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega\rho|u_t|^2dx+&\int_\Omega(\mu|\nabla u_t|^2+\nu(\mathrm{div} u_t)^2)dx+\frac{1}{\epsilon^2}\int_\Omega\nabla q_t\cdot u_t dx\\ =&-\int_\Omega\rho_t(u_t+v\cdot\nabla u)\cdot u_t dx-\int_\Omega\rho(v_t\cdot\nabla)u\cdot u_t dx\\ \equiv& A_1(t)+A_2(t),\end{aligned} \tag{3.17}$$

where the terms on the right-hand side can be estimated as follows:

$$\begin{aligned}|A_1(t)|\le& C\|\rho_t\|_{L^2}(\|u_t\|_{L^4}^2+\|v\|_{H^2}\|\nabla u\|_{H^2}\|u_t\|_{L^2})\\ \le&\frac{\delta}{2}(\|u\|_{H^3}^2+\|u_t\|_{H^1}^2)+C_\delta\|\rho_t\|_{L^2}^2(\|\rho_t\|_{L^2}^2+\|v\|_{H^1}\|v\|_{H^3})\|u_t\|_{L^2}^2,\end{aligned}$$

and similarly

$$|A_2(t)|\le\frac{\delta}{2}\|u\|_{H^3}^2+C_\delta\|\rho\|_{H^2}^2\|v_t\|_{L^2}^2\|u_t\|_{L^2}^2.$$

Thus, it follows from (3.17), (3.9), (3.10), and the estimates for $A_1(t)$ and $A_2(t)$ that

$$\begin{aligned}\frac{1}{2}\int_\Omega \rho|u_t|^2dx(t)&+\underline{\mu}\int_0^t\|u_t\|_{H^1}^2ds+\frac{1}{\epsilon^2}\int_0^t\int_\Omega \nabla q_t\cdot u_tdxds\\ &\le F_0(M_0)\big(1+\int_0^t\big(1+M^2+M\|v\|_{H^3})\|u_t\|_{L^2}^2ds\big)\\ &+\delta\big(\|u\|_{L_t^2(H^3)}^2+\|u_t\|_{L_t^2(H^1)}^2\big),\end{aligned} \tag{3.18}$$

for some positive constant $\underline{\mu}$ and for any $0\le t\le T_3$, $0<\epsilon\le 1$.

Similarly, we differentiate (3.3) with respect to t and multiply the resulting equation by q_t/ϵ^2 in $L^2(\Omega)$ to infer that

$$\begin{aligned}\frac{1}{2\gamma}\frac{d}{dt}\|\frac{q_t}{\epsilon}\|_{L^2}^2&+\frac{1}{\epsilon^2}\int_\Omega q_t\mathrm{div}u_tdx\\ &=\frac{1-2\gamma}{2\gamma}\int_\Omega \mathrm{div}v\frac{q_t^2}{\epsilon^2}dx-\int_\Omega(\frac{v_t}{\gamma}\cdot\frac{\nabla q}{\epsilon}+\mathrm{div}v_t\frac{q}{\epsilon})\frac{q_t}{\epsilon}dx\\ &+\epsilon\int_\Omega(2\mu|D(v)|^2+\lambda(\mathrm{div}v)^2)_t\frac{q_t}{\epsilon}dx\\ &\equiv B_1(t)+B_2(t)+B_3(t),\end{aligned} \tag{3.19}$$

where $B_1(t)$, $B_2(t)$ and $B_3(t)$ can be bounded as follows:

$$|B_1(t)|\le C\|\mathrm{div}\,v\|_{L^\infty}\|\frac{q_t}{\epsilon}\|_{L^2}^2\le C\|v\|_{H^3}\|\frac{q_t}{\epsilon}\|_{L^2}^2,$$

$$\begin{aligned}|B_2(t)|&\le C\|v_t\|_{H^1}\big(\|\frac{\nabla q}{\epsilon}\|_{L^3}+\|\frac{q}{\epsilon}\|_{L^\infty}\big)\|\frac{q_t}{\epsilon}\|_{L^2}\\ &\le C\|v_t\|_{H^1}\big(\|\frac{\nabla q}{\epsilon}\|_{L^2}^{\frac12}\|\frac{\nabla q}{\epsilon}\|_{H^1}^{\frac12}+\|\frac{q}{\epsilon}\|_{H^2}^{\frac78}\|\frac{q}{\epsilon}\|_{L^2}^{\frac18}\big)\|\frac{q_t}{\epsilon}\|_{L^2}\\ &\le C\|v_t\|_{H^1}^{\frac{15}{8}}\|\frac{q_t}{\epsilon}\|_{L^2}^2+\delta\|\frac{\nabla q}{\epsilon}\|_{H^1}^2\\ &+C_\delta\big(\|v_t\|_{H^1}^{\frac14}\|\frac{\nabla q}{\epsilon}\|_{L^2}^2+(1+\|v_t\|_{H^1})\|\frac{q}{\epsilon}\|_{L^2}^2\big),\end{aligned}$$

and analogously

$$|B_3(t)|\le\|v_t\|_{H^1}^{\frac{15}{8}}\|\frac{q_t}{\epsilon}\|_{L^2}^2+C\|v\|_{H^1}^{\frac14}(\|v\|_{H^3}^{\frac{15}{8}}+\|v_t\|_{H^1}^{\frac{15}{8}}).$$

Hence, inserting the above estimates of $B_1(t)$, $B_2(t)$ and $B_3(t)$ into ((3.19)), we get

$$\frac{1}{2\gamma}\big\|\frac{q_t}{\epsilon}(t)\big\|_{L^2}^2+\frac{1}{\epsilon^2}\int_0^t\int_\Omega q_t\mathrm{div}u_t dxds$$
$$\le F_0(M_0)+\delta\big\|\frac{\nabla q}{\epsilon}\big\|_{L_t^2(H^1)}^2+C_\delta\int_0^t(1+\|v\|_{H^3}+\|v_t\|_{H^1}^{\frac{15}{8}})\big\|(\frac{\nabla q}{\epsilon},\frac{q_t}{\epsilon})\big\|_{L^2}^2ds, \tag{3.20}$$

for any $0\le t\le T_4:=\min(T,(1+M^{34})^{-1})$ and $0<\epsilon\le 1$. Combining (3.18) with (3.20), one concludes

$$\int_\Omega\big(\rho|u_t|^2+\frac{q_t^2}{\gamma\epsilon^2}\big)(t)dx+\int_0^t\int_\Omega(\mu|\nabla u_t|^2+\nu(\mathrm{div}u_t)^2)dxds$$
$$\le F_0(M_0)+\delta\big(\|u\|_{L_t^2(H^3)}^2+\big\|\frac{\nabla q}{\epsilon}\big\|_{L_t^2(H^1)}^2\big)$$
$$+F_0(M_0)\int_0^t\big(1+(1+M)\|v\|_{H^3}+\|v_t\|_{H^1}^{\frac{15}{8}}+M^2\big)\big\|(\nabla u,u_t,\frac{\nabla q}{\epsilon},\frac{q_t}{\epsilon})\big\|_{L^2}^2ds \tag{3.21}$$

for any $0\le t\le T_4$ and $0<\epsilon\le 1$.

3.4. Stokes problem

We rewrite (3.2) and (3.4) as an inhomogeneous Stokes problem in order to derive the desired bounds for $\|u\|_{L_t^2(H^3)}$ and $\|\nabla q/\epsilon^2\|_{L_t^2(H^1)}$:

$$\begin{cases}-\mu\Delta u+\frac{1}{\epsilon^2}\nabla q=\nu\nabla\mathrm{div}u-\rho(u_t+v\cdot\nabla u)\equiv f\quad in\ \Omega,\\ \mathrm{div}u=g\quad in\ \Omega,\\ u|_{\partial\Omega}=0.\end{cases} \tag{3.22}$$

We begin with the estimates of low order. By the usual estimates for the steady Stokes problem (cf. Chapter IV of Galdi's book [18]), Sobolev's embedding $H^2\hookrightarrow L^\infty$ and (3.15), we have

$$\|u\|_{L_t^2(H^2)}^2+\frac{1}{\epsilon^4}\|\nabla q\|_{L_t^2(L^2)}^2\le C\big(\|g\|_{L_t^2(H^1)}^2+\|f\|_{L_t^2(L^2)}^2\big)$$
$$\le C\|\mathrm{div}u\|_{L_t^2(H^1)}^2+C\|\rho\|_{L_{x,t}^\infty}^2\big(\|u_t\|_{L_t^2(L^2)}^2+\int_0^t\|v\|_{L^\infty}^2\|\nabla u\|_{L^2}^2ds\big)$$
$$\le\delta\|u\|_{L_t^2(H^3)}^2+F_2(M_0)\|u_t\|_{L_t^2(L^2)}^2+F_0(M_0)\int_0^t(1+M\|v\|_{H^3})\|\nabla u\|_{L^2}^2ds \tag{3.23}$$

for certain $F_2(M_0) > 1$ and any $0 < t \le T_4$.

Similarly, derivatives of higher order of (u, q) can be bounded as follows.

$$
\begin{aligned}
\|u\|^2_{L^2_t(H^3)} &+ \frac{1}{\epsilon^4}\|\nabla q\|^2_{L^2_t(H^1)} \\
&\le C\|\mathrm{div} u\|^2_{L^2_t(H^2)} + \delta\|u\|^2_{L^2_t(H^3)} + F_0(M_0)\|u_t\|^2_{L^2_t(H^1)} \\
&+ C_\delta F_0(M_0)\int_0^t \big(1 + (1+M^3)\|v\|_{H^3} + M^8\big)\|\nabla u\|^2_{L^2} ds,
\end{aligned} \tag{3.24}
$$

which together with (3.23) yields

$$
\begin{aligned}
\|u\|^2_{L^2_t(H^3)} + \frac{1}{\epsilon^4}\|\nabla q\|^2_{L^2_t(H^1)} &\le C\|\nabla^2 \mathrm{div} u\|^2_{L^2_t(L^2)} + F_3(M_0)\|u_t\|^2_{L^2_t(H^1)} \\
&+ F_0(M_0)\int_0^t \big(1 + (1+M^3)\|v\|_{H^3} + M^8\big)\|\nabla u\|^2_{L^2} ds,
\end{aligned} \tag{3.25}
$$

for some continuous function $F_3(\cdot) > 1$. Finally, combining (3.14), (3.16), (3.21) and (3.25), we have proved the following proposition.

Proposition 3. *There exist the positive constants C_1 and $T_4(M)$, such that for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$,*

$$
\begin{aligned}
\Big\|(\nabla u, & u_t, \frac{\nabla q}{\epsilon}, \frac{q_t}{\epsilon})(t)\Big\|^2_{L^2} + \|u\|^2_{L^2_t(H^3)} + \Big\|(u_t, \frac{\nabla q}{\epsilon^2})\Big\|^2_{L^2_t(H^1)} \\
\le & F_0(M_0)\int_0^t \big(1 + M^8 + (1+M)\|v\|_{H^3} + \|v_t\|^{\frac{15}{8}}_{H^1}\big)\Big\|(\nabla u, u_t, \frac{\nabla q}{\epsilon}, \frac{q_t}{\epsilon})\Big\|^2_{L^2} ds \\
& + F_0(M_0) + C_1\|\nabla^2 \mathrm{div} u\|^2_{L^2_t(L^2)}.
\end{aligned} \tag{3.26}
$$

3.5. Estimate of $\|\nabla^2 \mathrm{div} u\|_{L^2_t(L^2)}$

We will adapt the idea due to Valli et al. [40, 41] to estimate $\|\nabla^2 \mathrm{div} u\|_{L^2_t(L^2)}$ by dividing it into the interior part and the part near the boundary. Comparing to [40, 41], the major difference is that the large parameter $1/\epsilon^2$ is involved in (3.2). Thus it needs more careful analysis in order to close the estimate of the full norm.

3.5.1. Interior estimate

In fact, we will give an interior estimate of $\|\nabla^3 u\|_{L^2_t(L^2)}$. For the sake of convenience, we will use the Einstein summation convention in what follows.

Let χ_0 be a C_0^∞-function. Applying ∂_{jk} to (3.2), taking the inner product of the resulting equation and $\chi_0^2\partial_{jk}u$ in $L^2(\Omega)$, we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \chi_0^2\rho|\partial_{jk}u|^2dx+\frac{1}{\epsilon^2}\int_\Omega \chi_0^2\partial_{jk}\nabla q\cdot\partial_{jk}udx+\mu\int_\Omega \chi_0^2|\partial_{jk}\nabla u|^2dx\\
&\qquad\le \frac{1}{2}\int_\Omega \rho_t\chi_0^2|\partial_{jk}u|^2dx-\int_\Omega \rho\partial_{jk}(v\cdot\nabla u)\chi_0^2\partial_{jk}udx\\
&\qquad-2\int_\Omega \chi_0\big(\mu\partial_{jk}\nabla u:(\nabla x_0\partial_{jk}u)+\lambda\partial_{jk}\mathrm{div}u\nabla\chi_0\cdot\partial_{jk}u\big)dx\\
&\qquad-\int_\Omega \big(\partial_j\rho\partial_k(u_t+v\cdot\nabla u)+\partial_k\rho\partial_j(u_t+v\cdot\nabla u)\big)\chi_0^2\partial_{jk}udx\\
&\qquad-\int_\Omega \partial_{jk}\rho(u_t+v\cdot\nabla u)\chi_0^2\partial_{jk}udx\equiv\sum_{i=1}^5 \mathrm{II}_i.
\end{aligned}\tag{3.27}$$

We have to estimate II_i on the right-hand side of ((3.27)). From (3.1) we get

$$\begin{aligned}
\mathrm{II}_2=&-\int_\Omega \rho(\partial_j v^l\partial_{kl}u^i+\partial_k v^l\partial_{jl}u^i+\partial_{jk}v^l\partial_l u^i)\chi_0^2\partial_{jk}u^idx\\
&+\int_\Omega \chi_0\partial_l\chi_0\rho v^l|\partial_{jk}u|^2dx-\mathrm{II}_1,
\end{aligned}$$

which yields

$$\mathrm{II}_1+\mathrm{II}_2\le\delta\|u\|_{H^3}^2+C_\delta\|\rho\|_{H^2}\|v\|_{H^1}\|v\|_{H^3}\|\chi_0\sqrt{\rho}\partial_{jk}u\|_{L^2}^2.$$

Analogously, applying the interpolation inequalities, one finds that

$$\mathrm{II}_3\le\delta\|u\|_{H^3}^2+C_\delta\|\rho^{-1}\|_{L^\infty}^2\|\chi_0\sqrt{\rho}\partial_{jk}u\|_{L^2}^2,$$

$$\begin{aligned}
\mathrm{II}_4\le&\,\delta(\|u\|_{H^3}^2+\|u_t\|_{H^1}^2)\\
&+C_\delta\|\rho^{-1}\|_{L^\infty}(1+\|\rho\|_{H^2}^4)(1+\|v\|_{H^1}\|v\|_{H^3})\|\chi_0\sqrt{\rho}\partial_{jk}u\|_{L^2}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{II}_5\le&\,\delta(\|u\|_{H^3}^2+\|u_t\|_{H^1}^2)+C_\delta\|\rho^{-1}\|_{L^\infty}\|\rho\|_{H^2}^4\|\chi_0\sqrt{\rho}\partial_{jk}u)\|_{L^2}^2\\
&+C_\delta\|\rho^{-1}\|_{L^\infty}\|\rho\|_{H^2}^2\|v\|_{H^1}\|v\|_{H^3}\|\chi_0\sqrt{\rho}\partial_{jk}u\|_{L^2}^2.
\end{aligned}$$

Substituting the estimates of II_i $(i=1,\cdots,5)$ into (3.27) and integrating with respect to t, we arrive at

$$\|\chi_0\sqrt{\rho}\partial_{jk}u(t)\|_{L^2}^2+\mu\|\chi_0\partial_{jk}\nabla u\|_{L_t^2(L^2)}^2-\frac{1}{\epsilon^2}\int_0^t\int_\Omega\chi_0^2\partial_{jk}q\,\partial_{jk}\mathrm{div}u dxds$$
$$\le F_0(M_0)\big(1+\int_0^t(1+M\|v\|_{H^3})\|\chi_0\sqrt{\rho}\partial_{jk}u\|_{L^2}^2ds\big)$$
$$+\delta\big(\|u\|_{L_t^2(H^3)}^2+\|\frac{\nabla^2q}{\epsilon^2}\|_{L_t^2(L^2)}^2+\|u_t\|_{L_t^2(H^1)}^2\big). \tag{3.28}$$

Here the third term on the left-hand side of (3.28) is a large quantity without sign, which can be fortunately balanced by the following estimate. Similar to (3.27), one deduces

$$\frac{1}{2\gamma}\frac{d}{dt}\|\chi_0\frac{\partial_{jk}q}{\epsilon}\|_{L^2}^2+\frac{1}{\epsilon^2}\int_\Omega\chi_0^2\partial_{jk}q\partial_{jk}\mathrm{div}u dx$$
$$\le\int_\Omega\big|\frac{1}{2}\mathrm{div}v\chi_0^2+v\cdot\nabla\chi_0\chi_0\big|\big|\frac{\partial_{jk}q}{\epsilon}\big|^2dx+\int_\Omega\big(|\nabla v|\big|\frac{\nabla^2q}{\epsilon}\big|+|\nabla^2v|\big|\frac{\nabla q}{\epsilon}\big|$$
$$+|\partial_{jk}\mathrm{div}v|\big|\frac{q}{\epsilon}\big|+\epsilon(|\nabla^2v|^2+|\nabla v||\nabla^3v|)\big)\chi_0^2\big|\frac{\partial_{jk}q}{\epsilon}\big|dx.$$

By the interpolation inequality, the right hand side of the above inequality is less than

$$\delta\big\|\frac{\nabla q}{\epsilon}\big\|_{H^1}^2+C_\delta(\|v\|_{H^3}+\|v\|_{H^1}^{\frac14}\|v\|_{H^3}^{\frac74})\big\|\chi_0\frac{\partial_{jk}q}{\epsilon}\big\|_{L^2}^2$$
$$+C\big(\|v\|_{H^3}^{\frac18}\big\|\frac{q}{\epsilon}\big\|_{L^2}^{\frac14}\big\|\frac{\nabla q}{\epsilon}\big\|_{H^1}^{\frac74}+\|v\|_{H^3}^{\frac{15}{8}}\big\|\chi_0\frac{\partial_{jk}q}{\epsilon}\big\|_{L^2}^2\big)$$
$$+C\|v\|_{H^3}\big\|\frac{q}{\epsilon}\big\|_{L^2}\big\|\chi_0\frac{\partial_{jk}q}{\epsilon}\big\|_{L^2}+C\|v\|_{H^1}^{\frac18}\|v\|_{H^3}^{\frac{15}{8}}\|\chi_0\frac{\partial_{jk}q}{\epsilon}\|_{L^2},$$

which leads to

$$\frac{1}{2\gamma}\|\chi_0\frac{\partial_{jk}q}{\epsilon}(t)\|_{L^2}^2+\frac{1}{\epsilon^2}\int_0^t\int_\Omega\chi_0^2\partial_{jk}q\partial_{jk}\mathrm{div}u dxds$$
$$\le F_0(M_0)+\delta\big\|\frac{\nabla^2q}{\epsilon}\big\|_{L_t^2(L^2)}^2+C\int_0^tM^{\frac14}\|v\|_{H^3}^{\frac{15}{8}}ds \tag{3.29}$$
$$+C_\delta\int_0^t\big(1+\|v\|_{H^3}^{\frac{15}{8}}+M^{\frac14}\|v\|_{H^3}^{\frac74}\big)\big\|\chi_0\frac{\partial_{jk}q}{\epsilon}\big\|_{L^2}^2ds$$

for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$, where the third term on the right-hand side is also bounded by $F_0(M_0)$. Therefore, if we add (3.29) to (3.28) and recall that ρ is bounded from below by a constant depending only on M_0, we obtain that, for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$,

$$\begin{aligned}
&\big\|\chi_0(\nabla^2 u, \frac{\nabla^2 q}{\gamma\epsilon})(t)\big\|_{L^2}^2 + \|\chi_0\nabla^3 u\|_{L_t^2(L^2)}^2 \\
&\quad\le F_0(M_0) + \delta\big(\|u\|_{L_t^2(H^3)}^2 + \big\|\frac{\nabla^2 q}{\epsilon^2}\big\|_{L_t^2(L^2)}^2 + \|u_t\|_{L_t^2(H^1)}^2\big) \\
&\quad+ F_0(M_0)\int_0^t (1 + \|v\|_{H^3}^{\frac{15}{8}} + M^{\frac{15}{4}})\big\|(\chi_0\nabla^2 u, \frac{\chi_0\nabla^2 q}{\epsilon})\big\|_{L^2}^2 ds.
\end{aligned} \tag{3.30}$$

3.5.2. Boundary estimate

We construct the local coordinates by the isothermal coordinates $\lambda(\psi, \phi)$ to derive an estimate near the boundary (see also [40, 41]), where

$$\lambda_\psi \cdot \lambda_\psi > 0, \quad \lambda_\phi \cdot \lambda_\phi > 0, \quad \text{and } \lambda_\psi \cdot \lambda_\phi = 0.$$

We cover the boundary $\partial\Omega$ by a finite number of bounded open sets $W^k \subset \mathbb{R}^3$, $k = 1, 2, \cdots, L$, such that for any $x \in W^k \cap \Omega$,

$$x = \Lambda^k(\psi, \phi, r) \equiv \lambda^k(\psi, \phi) + rn(\lambda^k(\psi, \phi)), \tag{3.31}$$

where $\lambda^k(\psi, \phi)$ is the isothermal coordinates and n is the unit outer normal to $\partial\Omega$.

For simplicity, in what follows we will omit the superscript k in each W^k. We construct the orthonormal system corresponding to the local coordinates by

$$e_1 := \frac{\lambda_\psi}{|\lambda_\psi|}, \quad e_2 := \frac{\lambda_\phi}{|\lambda_\phi|}, \quad e_3 := e_1 \times e_2 \equiv n(\lambda). \tag{3.32}$$

By a straightforward calculation, we see that for sufficiently small r and $J \in C^2$,

$$\begin{aligned}
J :=& \det\frac{\partial x}{\partial(\psi, \phi, r)} = (\Lambda_\psi \times \Lambda_\phi) \cdot e_3 \\
=& |\lambda_\psi||\lambda_\phi| + r(|\lambda_\psi| n_\phi \cdot e_2 + |\lambda_\phi| n_\psi \cdot e_1) \\
&+ r^2\big[(n_\psi \cdot e_1)(n_\phi \cdot e_2) - (n_\psi \cdot e_2)(n_\phi \cdot e_1)\big] > 0.
\end{aligned}$$

Moreover, we can easily derive the following relations (see also [40]):

$$[\nabla(\Lambda)^1] \circ \Lambda = J^{-1}(\Lambda_\phi \times e_3), \tag{3.33}$$

$$[\nabla(\Lambda)^2] \circ \Lambda = J^{-1}(e_3 \times \Lambda_\psi), \tag{3.34}$$

$$[\nabla(\Lambda)^3] \circ \Lambda = J^{-1}(\Lambda_\psi \times \Lambda_\phi) = e_3, \tag{3.35}$$

where the notation "$\circ$" stands for the composite of operators. Set $y := (\psi, \phi, r)$ and denote by D_i the partial derivative with respect to y_i in local coordinates. Then, we rewrite the equations (3.2)-(3.3) in $[0,T] \times \tilde{\Omega}$, where $\tilde{\Omega} := \Lambda^{-1}(W \cap \Omega)$, as

$$\begin{aligned} &R(U_t^i + V^j a_{kj} D_k U^i) + \frac{1}{\epsilon^2} a_{ki} D_k Q \\ &\qquad = \mu a_{kj} D_k(a_{lj} D_l U^i) + \nu a_{ki} D_k(a_{lj} D_l U^j), \end{aligned} \tag{3.36}$$

$$\begin{aligned} &\frac{1}{\gamma}(Q_t + V^j a_{kj} D_k Q) + Q a_{kj} D_k V^j + a_{kj} D_k U^j \\ &\qquad = \epsilon^2 \big(\frac{\mu}{2}(a_{ki} D_k V^j + a_{kj} D_k V^i)^2 + \lambda(a_{kj} D_k V^j)^2\big), \end{aligned} \tag{3.37}$$

where the unknowns in local coordinates $U(t,y) := u(t,\Lambda(y))$, $Q(t,y) := q(t,\Lambda(y))$, $V(t,y) := v(t,\Lambda(y))$, $R(t,y) := \rho(t,\Lambda(y))$, and a_{ij} is the (i,j)-th entry of the matrix $Jac(\Lambda^{-1}) \equiv \{\frac{\partial y}{\partial x}\}$. Clearly, a_{ij} is a C^2-function, and it follows from (3.33)–(3.35) that

$$\sum_{j=1}^{3} a_{3j} a_{3j} = |n|^2 = 1, \quad \sum_{j=1}^{3} a_{1j} a_{3j} = \sum_{j=1}^{3} a_{2j} a_{3j} = 0. \tag{3.38}$$

The initial and boundary conditions for (R,U,Q) are

$$(R,U,Q)|_{t=0} = (R_0, U_0, Q_0)(y), \quad y \in \tilde{\Omega}, \tag{3.39}$$

$$U|_{\partial\tilde{\Omega}} = 0. \tag{3.40}$$

Moreover, this localized system has the following properties (see also [40]):

Proposition 4. *$D_i(Ja_{ij}) = 0$, for $j = 1,2,3$; $\chi D_\tau U = 0$, $\chi D_\tau D_\xi U = 0$ on $\partial\tilde{\Omega}$ in the tangential directions $\tau, \xi = 1,2$, where $\chi \in C_0^\infty(\Lambda^{-1}(W))$. Similarly, $\chi D_\tau V = 0$, $\chi D_\tau D_\xi V = 0$.* □

Recalling $D_j = \sum_{i=1}^3 a_{ji}\partial_i$, we will frequently make use of the following relations without pointing out explicitly in subsequent calculations:

$$\|D_y U\|_{L^p(\tilde{\Omega})} \le C\|\nabla_x u\|_{L^p(\Omega)}, \quad \|D_y^2 U\|_{L^p(\tilde{\Omega})} \le C\|\nabla_x u\|_{W^{1,p}(\Omega)}, \quad 1 \le p \le \infty.$$

The above inequalities apply to Q, R and V, too.

By virtue of the interpolation $\|\cdot\|_{H^2}^2 \le \delta\|\cdot\|_{H^3}^2 + C_\delta\|\cdot\|_{H^1}^2$, the estimate of $\|\nabla^2 \mathrm{div} u\|_{L_t^2(L^2)}$ reduces to the estimate of

$$\int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_y^2(a_{ji}D_jU^i)| dyds,$$

where χ is a $C_0^\infty(\tilde{\Omega})$-function.

Estimate of the derivatives in the tangential directions. We apply $D_{\xi\tau}$ to $(3.36)_i$ with ξ, τ being the tangential directions to $\partial\tilde{\Omega}$, then multiply the resulting equation by $J\chi^2 D_{\xi\tau}U^i$ and integrate to deduce that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\tilde{\Omega}} J\chi^2 R|D_{\xi\tau}U^i|^2 dy + \mu\int_{\tilde{\Omega}} J\chi^2 a_{kj}D_{k\xi\tau}U^i a_{lj}D_{l\xi\tau}U^i dy \\
&\quad + \nu\int_{\tilde{\Omega}} J\chi^2 a_{ki}D_{k\xi\tau}U^i a_{lj}D_{l\xi\tau}U^j dy + \frac{1}{\epsilon^2}\int_{\tilde{\Omega}} J\chi^2 D_{\xi\tau}(a_{ki}D_kQ)D_{\xi\tau}U^i dy \\
&\quad = \frac{1}{2}\int_{\tilde{\Omega}} D_k(J\chi^2 a_{kj})RV^j|D_{\xi\tau}U^i|^2 dy \\
&\quad + \int_{\tilde{\Omega}} J\chi^2 D_{\xi\tau}U^i(D_\xi R D_\tau U_t^i + D_\tau R D_\xi U_t^i + D_{\xi\tau}RU_t^i)dy \\
&\quad + \int_{\tilde{\Omega}} J\chi^2 D_{\xi\tau}U^i[D_\xi(RV^ja_{kj})D_{k\tau}U^i \\
&\quad + D_\tau(RV^ja_{kj})D_{k\xi}U^i + D_{\xi\tau}(RV^ja_{kj})D_kU^i]dy \\
&\quad - \int_{\tilde{\Omega}} a_{lj}D_{\xi\tau}U^i[\mu D_k(J\chi^2 a_{kj})D_{l\xi\tau}U^i + \nu D_k(J\chi^2 a_{ki})D_{l\xi\tau}U^j] \\
&\quad + D_{\xi\tau}a_{lj}[\mu D_lU^iD_k(J\chi^2 a_{kj}D_{\xi\tau}U^i) + \nu D_lU^jD_k(J\chi^2 a_{ki}D_{\xi\tau}U^i)]dy \\
&\quad + \int_{\tilde{\Omega}} J\chi^2 D_{\xi\tau}U^i\big\{\mu\big[a_{kj}D_k(D_\xi a_{lj}D_{l\tau}U^i + D_\tau a_{lj}D_{l\xi}U^i) \\
&\quad + D_\xi a_{kj}D_{k\tau}(a_{lj}D_lU^i) + D_\tau a_{kj}D_{k\xi}(a_{lj}D_lU^i) \\
&\quad + D_{\xi\tau}a_{kj}D_k(a_{lj}D_lU^i)\big] + \nu\big[a_{ki}D_k(D_\xi a_{lj}D_{l\tau}U^j \\
&\quad + D_\tau a_{lj}D_{l\xi}U^j) + D_\xi a_{ki}D_{k\tau}(a_{lj}D_lU^j) \\
&\quad + D_\tau a_{ki}D_{k\xi}(a_{lj}D_lU^j) + D_{\xi\tau}a_{ki}D_k(a_{lj}D_lU^j)\big]\big\}dy \equiv \sum_{i=1}^5 J_i,
\end{aligned} \tag{3.41}$$

where each term on the right-hand side can be bounded as follows.

$$
\begin{aligned}
|J_1| &\le C\|\rho\|_{H^2}\|v\|_{H^2}\|\nabla u\|_{H^2}\|\nabla u\|_{L^2}\\
&\le \delta\|u\|_{H^3}^2 + C_\delta\|\rho\|_{H^2}^2\|v\|_{H^1}\|v\|_{H^3}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
|J_2| &\le C\|\nabla u\|_{W^{1,3}}\|\rho\|_{H^2}\|u_t\|_{H^1}\\
&\le \delta(\|u_t\|_{H^1}^2 + \|u\|_{H^3}^2) + C_\delta\|\rho\|_{H^2}^8\|\nabla u\|_{L^2}^2,\\
|J_3| &\le C\|\nabla u\|_{W^{1,3}}\|\rho\|_{H^2}\|v\|_{H^2}\|\nabla u\|_{H^1}\\
&\le \delta\|u\|_{H^3}^2 + C_\delta\|\rho\|_{H^2}^{\frac{8}{3}}\|v\|_{H^1}^{\frac{4}{3}}\|v\|_{H^3}^{\frac{4}{3}}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

and finally

$$
|J_4| + |J_5| \le C\|u\|_{H^2}\|u\|_{H^3} \le \delta\|u\|_{H^3}^2 + C_\delta\|\nabla u\|_{L^2}^2.
$$

Observing that the matrix $\{\sum_k a_{ki}a_{kj}\}$ is strictly positive-definite, we have

$$
\mu\sum_{i,j} a_{kj}D_{k\xi\tau}U^i a_{lj}D_{l\xi\tau}U^i \ge \underline{\mu}|D_{y\xi\tau}U|^2.
$$

for some positive constant $\underline{\mu}$. Recalling $\nu \ge 0$, we insert the estimates for $J_1, \cdots, J_5$ into (3.41) to obtain that, for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$,

$$
\begin{aligned}
&\int_{\tilde\Omega} J\chi^2 R|D_{\xi\tau}U|^2dy + \underline{\mu}\int_0^t\int_{\tilde\Omega} J\chi^2|D_{y\xi\tau}U|^2dyds\\
&\quad + \frac{1}{\epsilon^2}\int_0^t\int_{\tilde\Omega} J\chi^2 D_{\xi\tau}(a_{ki}D_kQ)D_{\xi\tau}U^i dyds\\
&\quad \le F_0(M_0)\big(1 + \int_0^t (1 + M^{\frac{4}{3}}\|v\|_{H^3}^{\frac{4}{3}})\|\nabla u\|_{L^2}^2 ds\big)\\
&\quad + \delta\big(\|u_t\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^2(H^3)}^2\big).
\end{aligned}
\tag{3.42}
$$

To control the large term involving $1/\epsilon^2$ on the left-hand side of (3.42), which will be denoted by J_6 and is more involved to estimate, we argue, similarly to before, to find (cf. (3.29))

$$\begin{aligned}
&\frac{1}{2\gamma}\frac{d}{dt}\int_{\tilde\Omega} J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy+\frac{1}{\epsilon^2}\int_{\tilde\Omega} J\chi^2 D_{\xi\tau}(a_{kj}D_kU^j)D_{\xi\tau}Qdy\\
&\quad=\frac{1}{2\gamma}\int_{\tilde\Omega} D_k(J\chi^2V^ja_{kj})\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy-\int_{\tilde\Omega} J\chi^2\frac{D_{\xi\tau}Q}{\epsilon}D_{\xi\tau}\big(\frac{Q}{\epsilon}a_{kj}D_kV^j\big)dy\\
&\quad-\frac{1}{\gamma}\int_{\tilde\Omega} J\chi^2\frac{D_{\xi\tau}Q}{\epsilon}\Big(D_\xi(V^ja_{kj})\frac{D_{k\tau}Q}{\epsilon}+D_\xi\big(D_\tau(V^ja_{kj})\frac{D_kQ}{\epsilon}\big)\Big)dy\\
&\quad+\epsilon\int_{\tilde\Omega} J\chi^2\frac{D_{\xi\tau}Q}{\epsilon}D_{\xi\tau}\big(\frac{\mu}{2}(a_{ki}D_kV^j+a_{kj}D_kV^i)^2+\lambda(a_{kj}D_kV^j)^2\big)dy\\
&\quad\equiv\sum_{i=1}^4K_i,
\end{aligned}\tag{3.43}$$

where K_1 can be bounded as follows, using Sobolev's embedding theorem and the interpolation inequalities.

$$\begin{aligned}
|K_1|\le&\frac{1}{2\gamma}\big|\int_{\tilde\Omega} J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2\big(D_k(V^ja_{kj})+V^ja_{kj}D_kJJ^{-1}\big)dy\big|\\
&+\frac{1}{\gamma}\big|\int_{\tilde\Omega} V^ja_{kj}D_k\chi J\chi\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy\big|\\
\le&C_\delta(\|v\|_{W^{1,\infty}}+\|v\|_{L^\infty}^2)\int_{\tilde\Omega} J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy+\delta\big\|\frac{\nabla q}{\epsilon}\big\|_{H^1}^2\\
\le&C_\delta(\|v\|_{H^1}^{\frac14}\|v\|_{H^3}^{\frac34}+\|v\|_{H^1}\|v\|_{H^3})\int_{\tilde\Omega} J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2+\delta\big\|\frac{\nabla q}{\epsilon}\big\|_{H^1}^2.
\end{aligned}$$

Similarly, we can have

$$\begin{aligned}
|K_3|\le&\delta\big\|\frac{\nabla q}{\epsilon}\big\|_{H^1}^2+C_\delta\|v\|_{H^1}^{\frac14}\|v\|_{H^3}^{\frac74}\int_{\tilde\Omega} J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy,\\
|K_4|\le&C\|v\|_{H^3}^{\frac{15}{8}}+C_\delta\|v\|_{H^1}^{\frac14}\|v\|_{H^3}^{\frac{15}{8}}\int_{\tilde\Omega} J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy.
\end{aligned}$$

To control K_2, we observe that

$$\begin{aligned}
|K_2|\le&C\int_{\tilde\Omega}\big[(|D_y^3V|+|D_y^2V|+|D_yV|)\big|\frac{Q}{\epsilon}\big|\\
&+(|D_y^2V|+|D_yV|)\big|\frac{D_yQ}{\epsilon}\big|\big]J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|dy\\
&+C\int_{\tilde\Omega}|D_yV|J\chi^2\big|\frac{D_{\xi\tau}Q}{\epsilon}\big|^2dy\equiv K_{21}+K_{22},
\end{aligned}$$

where obviously,

$$K_{22} \leq C\|v\|_{H^3} \int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2 dy$$

and by the interpolation inequality,

$$\begin{aligned} K_{21} \leq & C\|v\|_{H^3} \big(\big\|\frac{q}{\epsilon}\big\|_{L^\infty} + \big\|\frac{\nabla q}{\epsilon}\big\|_{L^3}\big)\big(\int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2 dy\big)^{\frac{1}{2}} \\ \leq & \delta \big\|\frac{\nabla q}{\epsilon}\big\|_{H^1}^2 + C_\delta \|v\|_{H^3} \big\|\frac{q}{\epsilon}\big\|_{L^2}^2 + C(1 + \|v\|_{H^3}^{\frac{15}{8}}) \int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2 dy. \end{aligned}$$

Now, we integrate (3.43) in t and utilize the estimates for $K_1, \cdots, K_4$ to reach that

$$\begin{aligned} & \frac{1}{2\gamma} \int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2 (t) dy + \frac{1}{\epsilon^2} \int_0^t \int_{\tilde{\Omega}} J\chi^2 D_{\xi\tau}(a_{kj} D_k U^j) D_{\xi\tau} Q dy ds \\ & \leq F_0(M_0) \int_0^t (1 + \|v\|_{H^3}^{\frac{15}{8}})(1 + M^{\frac{1}{4}}) \int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2 dy ds \\ & + \delta \big\|\frac{\nabla q}{\epsilon}\big\|_{L_t^2(H^1)}^2 + F_0(M_0), \end{aligned} \tag{3.44}$$

for any $0 \leq t \leq T_4$ and $0 < \epsilon \leq 1$. Denoting by K_5 the second integral on the left-hand side of (3.44) and recalling the definition of J_6, we employ integration by parts and an interpolation inequality to deduce

$$|J_6 + K_5| \leq \delta\big(\big\|\frac{\nabla q}{\epsilon^2}\big\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^2(H^3)}^2\big) + C_\delta \|\nabla u\|_{L_t^2(L^2)}^2.$$

Consequently, the following estimate for the tangential derivatives of higher order follows from (3.42) and (3.44).

$$\begin{aligned} & \int_{\tilde{\Omega}} J\chi^2 \big(R|D_{\xi\tau}U|^2 + \frac{1}{2\gamma}\Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2\big)(t) dy + \int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_{y\xi\tau}U|^2 dy ds \\ & \leq F_0(M_0) \int_0^t \big(M^{\frac{4}{3}} \|v\|_{H^3}^{\frac{4}{3}} + (1 + M^{\frac{1}{4}})(1 + \|v\|_{H^3}^{\frac{15}{8}})\big) \\ & \times \big(\int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\xi\tau}Q}{\epsilon}\Big|^2 dy + \int_\Omega |\nabla u|^2 dx\big) ds \\ & + \delta\big(\|u_t\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^2(H^3)}^2 + \big\|\frac{\nabla q}{\epsilon^2}\big\|_{L_t^2(H^1)}^2\big) + F_0(M_0), \end{aligned} \tag{3.45}$$

for any $0 \leq t \leq T_4$ and $0 < \epsilon \leq 1$.

Estimate of the derivatives in the normal direction. We will adapt an idea in Valli's paper [40] to handle the components of higher order derivatives in the normal direction to $\partial\tilde{\Omega}$. Multiplying (3.36) by a_{3i}, we have

$$\begin{aligned}(\mu+\nu)D_3(a_{lj}D_lU^j) = & R(U_t^i + V_j a_{kj}D_kU^i)a_{3i} + \frac{1}{\epsilon^2}D_3Q \\ & + \mu\big(D_3(a_{lj}D_lU^j) - a_{kj}a_{3i}D_k(a_{lj}D_lU^i)\big).\end{aligned} \tag{3.46}$$

Recalling that $\sum_{j=1}^3 a_{3j}a_{3j} = 1$ and $\sum_{j=1}^3 a_{1j}a_{3j} = \sum_{j=1}^3 a_{2j}a_{3j} = 0$, the last term on the right-hand side of (3.46) can be written as

$$\begin{aligned}&\mu\big(D_3a_{3j}D_3U^j + D_3a_{\tau j}D_\tau U^j + a_{\tau j}D_{3\tau}U^j - a_{3j}D_3a_{3j}a_{3i}D_3U^i \\ &- a_{\tau j}a_{3i}D_\tau a_{lj}D_lU^i - a_{\tau j}a_{\xi j}a_{3i}D_{\tau\xi}U^i - a_{3j}a_{3i}D_3a_{\tau j}D_\tau U^i\big) \quad \text{for } \tau,\xi = 1,2.\end{aligned}$$

Thus, we can see that the 2nd-order normal derivative $D_{33}U$ is not involved on the right-hand side of (3.46). This is a critical observation in the estimates for derivatives.

Step 1. To continue our estimate, we differentiate (3.46) with respect to y_τ ($\tau = 1,2$), then multiply by $J\chi^2 D_{\tau 3}(a_{lj}D_lU^j)$ in $L^2((0,t)\times\Omega)$ to get, for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$,

$$\begin{aligned}&\frac{\mu+\nu}{2}\int_0^t\int_{\tilde{\Omega}} J\chi^2|D_{\tau 3}(a_{lj}D_lU^j)|^2dyds - \frac{1}{\epsilon^2}\int_0^t\int_{\tilde{\Omega}} J\chi^2 D_{\tau 3}QD_{\tau 3}(a_{lj}D_lU^j)dyds \\ &\quad\le C\int_0^t \big\{\|\rho\|_{H^2}^2[\|u_t\|_{H^1}^2 + (\|v\|_{L^\infty}^2 + \|v\|_{W^{1,3}}^2)\|\nabla u\|_{H^1}^2] \\ &\quad+\|\nabla u\|_{H^1}^2 + \int_{\tilde{\Omega}} J\chi^2|D_{\tau\xi y}U|^2dy\big\}ds \\ &\quad\le\delta\|u\|_{L_t^2(H^3)}^2 + C_\delta F_0(M_0)\int_0^t (1+M^{\frac{5}{2}}\|v\|_{H^3}^{\frac{3}{2}})\|(\nabla u, u_t)\|_{L^2}^2ds \\ &\quad+C\int_0^t\int_{\tilde{\Omega}} J\chi^2|D_{\tau\xi y}U|^2dyds + F_0(M_0)\|u_t\|_{L_t^2(H^1)}^2.\end{aligned} \tag{3.47}$$

In the mean while, we apply $D_{\tau 3}$ to (3.37) and take the product of the resulting equation with $J\chi^2 D_{\tau 3}Q/\epsilon^2$ to infer that

$$\begin{aligned}
&\frac{1}{2\gamma}\frac{d}{dt}\int_{\tilde{\Omega}} J\chi^2\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2 dy + \frac{1}{\epsilon^2}\int_{\tilde{\Omega}} J\chi^2 D_{\tau 3}Q D_{\tau 3}(a_{kj}D_k U^j)dy \\
&\qquad = -\int_{\tilde{\Omega}} J\chi^2\frac{D_{\tau 3}Q}{\epsilon}\Big\{\frac{1}{\gamma}\Big(D_\tau(V^j a_{kj})\frac{D_{k3}Q}{\epsilon} \\
&\qquad + D_\tau\Big[D_3(V^j a_{kj})\frac{D_k Q}{\epsilon}\Big]\Big) + D_{\tau 3}\Big(\frac{Q}{\epsilon}a_{kj}D_k V^j\Big)\Big\}dy \\
&\qquad + \epsilon\int_{\tilde{\Omega}} J\chi^2\frac{D_{\tau 3}Q}{\epsilon}D_{\tau 3}\Big(\frac{\mu}{2}(a_{ki}D_k V^j + a_{kj}D_k V^i)^2 + \lambda(a_{kj}D_k V^j)^2\Big)dy \\
&\qquad + \frac{1}{2\gamma}\int_{\tilde{\Omega}} D_k(V^j a_{kj}J\chi^2)\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2 dy \equiv \sum_{i=1}^{3} L_i,
\end{aligned} \tag{3.48}$$

where L_i $(i = 1, 2, 3)$ are estimated as follows.

$$\begin{aligned}
|L_1| \le& C_\delta(\|v\|_{H^3}^{\frac{15}{8}} + \|v\|_{H^1}^{\frac{1}{4}}\|v\|_{H^3}^{\frac{7}{4}})\int_{\tilde{\Omega}} J\chi^2\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2 dy \\
&+ C_\delta(1 + \|v\|_{H^3})\Big\|\frac{q}{\epsilon}\Big\|_{L^2}^2 + \delta\Big\|\frac{\nabla q}{\epsilon}\Big\|_{H^1}^2, \\
|L_2| \le& C\Big(\|v\|_{H^3}^{\frac{15}{8}} + \|v\|_{H^1}^{\frac{1}{4}}\|v\|_{H^3}^{\frac{15}{8}}\int_{\tilde{\Omega}} J\chi^2\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2 dy\Big),
\end{aligned}$$

and

$$|L_3| \le C\|v\|_{H^3}\int_{\tilde{\Omega}} J\chi^2\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2 dy.$$

Therefore, we integrate (3.48) over $(0, t)$ to conclude

$$\begin{aligned}
&\frac{1}{2\gamma}\int_{\tilde{\Omega}} J\chi^2\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2(t, y)dy + \frac{1}{\epsilon^2}\int_0^t\int_{\tilde{\Omega}} J\chi^2 D_{\tau 3}Q D_{\tau 3}(a_{kj}D_k U^j)dyds \\
&\qquad \le F_0(M_0)\int_0^t (1 + M^{\frac{1}{4}})(1 + \|v\|_{H^3}^{\frac{15}{8}})\int_{\tilde{\Omega}} J\chi^2\Big|\frac{D_{\tau 3}Q}{\epsilon}\Big|^2 dyds \\
&\qquad + \delta\Big\|\frac{\nabla q}{\epsilon}\Big\|_{L_t^2(H^1)}^2 + F_0(M_0),
\end{aligned} \tag{3.49}$$

for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$. Thus, an addition of (3.47) to (3.48) yields

$$
\begin{aligned}
&\int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\tau 3} Q}{\epsilon}\Big|^2(t) dy + \int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_{\tau 3}(a_{lj} D_l U^j)|^2 dyds \\
&\quad \le \delta\big(\|(u_t, \frac{\nabla q}{\epsilon})\|^2_{L^2_t(H^1)} + \|u\|^2_{L^2_t(H^3)}\big) + C_2 \int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_{\tau\xi y} U|^2 dyds \\
&\quad + F_4(M_0)\|u_t\|^2_{L^2_t(H^1)} + F_0(M_0)\big(1 + \int_0^t (1 + M^{\frac{5}{2}} \|v\|^{\frac{3}{2}}_{H^3}) \|(\nabla u, u_t)\|^2_{L^2} ds\big) \\
&\quad + F_0(M_0) \int_0^t (1 + M^{\frac{1}{4}})(1 + \|v\|^{\frac{15}{8}}_{H^3}) \int_{\tilde{\Omega}} J\chi^2 \Big|\frac{D_{\tau 3} Q}{\epsilon}\Big|^2 dyds,
\end{aligned}
\tag{3.50}
$$

for some constant $C_2 > 0$ and continuous function $F_4(M_0) > 1$, and for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$.

Step 2. Now, it suffices to estimate $\|D_{33}(a_{ij} D_i U^j)\|_{L^2((0,t)\times\tilde{\Omega})}$ in order to close the estimates for div$u$. Applying the normal derivative D_3 to (3.46), we see that

$$
\begin{aligned}
&(\mu + \nu) D_{33}(a_{lj} D_l U^j) - \frac{1}{\epsilon^2} D_{33} Q = D_3(R a_{3i})(U^i_t + V^j a_{kj} D_k U^i) \\
&\quad + R a_{3i}(D_3 U^i_t + D_3(V^j a_{kj} D_k U^i)) + O(1)(D_{33\tau} U^j + D_{3l} U^j + D_l U^j),
\end{aligned}
\tag{3.51}
$$

which, multiplied by $J\chi^2 D_{33}(a_{lj} D_l U^j)$ and integrated, gives

$$
\begin{aligned}
&\frac{\mu + \nu}{2} \int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_{33}(a_{lj} D_l U^j)|^2 dyds \\
&\quad - \frac{1}{\epsilon^2} \int_0^t \int_{\tilde{\Omega}} J\chi^2 D_{33} Q D_{33}(a_{lj} D_l U^j) dyds \\
&\quad \le C \int_0^t \big(\|\rho\|^2_{H^2}(\|v\|^2_{H^2} \|\nabla u\|^2_{L^3} + \|u_t\|^2_{H^1}) \\
&\quad + (1 + \|\rho\|^2_{H^2} \|v\|^2_{L^\infty}) \|\nabla u\|^2_{H^1} + \int_{\tilde{\Omega}} J\chi^2 |D_{33\tau} U|^2 dy\big) ds \\
&\quad \le F_0(M_0) \|u_t\|^2_{L^2_t(H^1)} + \delta \|u\|^2_{L^2_t(H^3)} + C \int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_{33\tau} U|^2 dyds \\
&\quad + F_0(M_0) \int_0^t (1 + M^{\frac{4}{3}} \|v\|^{\frac{4}{3}}_{H^3} + M^{\frac{5}{2}} \|v\|^{\frac{3}{2}}_{H^3}) \|(\nabla u, u_t)\|^2_{L^2} ds,
\end{aligned}
\tag{3.52}
$$

for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$.

Correspondingly, we differentiate (3.37) twice with respect to y_3 and multiply the resulting equation by $J\chi^2 D_{33}Q/\epsilon^2$, and then integrate to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\tilde\Omega} J\chi^2 \frac{|D_{33}Q|^2}{\epsilon^2}dy + \frac{1}{\epsilon^2}\int_{\tilde\Omega} J\chi^2 D_{33}Q D_{33}(a_{lj}D_l U^j)dyds\\
&\qquad \le \delta\|\frac{\nabla q}{\epsilon}\|_{H^1}^2 + \|v\|_{H^3}^{\frac{15}{8}} + C_\delta \|v\|_{H^3}\|\frac{q}{\epsilon}\|_{L^2}^2\\
&\qquad + C_\delta(1+\|v\|_{H^1}^{\frac{1}{4}})(1+\|v\|_{H^3}^{\frac{15}{8}})\int_{\tilde\Omega} J\chi^2\frac{|D_{33}Q|^2}{\epsilon^2}dy.
\end{aligned}
\tag{3.53}
$$

Combining (3.53) with (3.52), we see that

$$
\begin{aligned}
&\int_{\tilde\Omega} J\chi^2\frac{|D_{33}Q|^2}{\epsilon^2}dy(t) + \int_0^t\int_{\tilde\Omega} J\chi^2|D_{33}(a_{lj}D_lU^j)|^2dyds\\
&\quad \le \delta\big(\|u\|_{L_t^2(H^3)}^2 + \|\frac{\nabla q}{\epsilon}\|_{L_t^2(H^1)}^2\big) + F_0(M_0)\\
&\quad + F_5(M_0)\|u_t\|_{L^2(H^1)}^2 + C_3\int_0^t\int_{\tilde\Omega} J\chi^2|D_{33\tau}U|^2dyds\\
&\quad + F_0(M_0)\int_0^t \big(1+M^{\frac{4}{3}}\|v\|_{H^3}^{\frac{4}{3}} + M^{\frac{5}{2}}\|v\|_{H^3}^{\frac{3}{2}}\big)\|(\nabla u, u_t)\|_{L^2}^2ds\\
&\quad + C\int_0^t(1+M^{\frac{1}{4}})(1+\|v\|_{H^3}^{\frac{15}{8}})\int_{\tilde\Omega} J\chi^2\frac{|D_{33}Q|^2}{\epsilon^2}dy.
\end{aligned}
\tag{3.54}
$$

for some constant $C_3 > 0$ and continuous function $F_5(M_0) > 1$, and for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$.

Step 3. To estimate the rest terms in the third-order normal derivatives, we introduce an auxiliary Stokes problem in the original coordinates in the region near the boundary:

$$
\begin{cases}
-\mu\Delta_x\big[(\chi D_\tau U)\circ\Lambda^{-1}\big] + \frac{1}{\epsilon^2}\nabla_x\big[(\chi D_\tau Q)\circ\Lambda^{-1}\big] = G_1 & \text{in } W\cap\Omega,\\
\mathrm{div}_x\big[(\chi D_\tau U)\circ\Lambda^{-1}\big] = G_2 & \text{in } W\cap\Omega,\\
(\chi D_\tau U)\circ\Lambda^{-1} = 0 & \text{on } \partial(W\cap\Omega),
\end{cases}
\tag{3.55}
$$

where $G_1(x)$ and $G_2(x)$ will be given later.

By the regularity theory of the Stokes problem (see [18]), one has

$$
\int_\Omega |\Delta_x(\chi D_\tau U)\circ\Lambda^{-1}(x)|^2dx \le C(\|G_1\|_{L^2(W\cap\Omega)}^2 + \|G_2\|_{H^1(W\cap\Omega)}^2). \tag{3.56}
$$

Noticing that the left-hand side of (3.56) is equivalent to

$$\int_{\tilde\Omega} J\Big|\sum_{j=1}^3\sum_{k=1}^3 a_{kj}D_k\big(\sum_{l=1}^3 a_{lj}D_l(\chi D_\tau U)\big)\Big|^2 dy$$
$$\le \int_{\tilde\Omega} J\chi^2\Big|\sum_{j,k,l=1}^3 a_{kj}a_{lj}D_{kl\tau}U\Big|^2 dy + O(1)\int_{\tilde\Omega}(|D_\tau U|^2+|D_{y\tau}U|^2)dy,$$

and moreover, by (3.38),

$$D_{33\tau}U=\sum_{k,l=1}^3(\sum_{j=1}^3 a_{kj}a_{lj})D_{kl\tau}U-\sum_{1\le k,l\le 2}\sum_{j=1}^3 a_{kj}a_{lj}D_{kl\tau}U,$$

we obtain

$$\int_{\tilde\Omega} J\chi^2|D_{33\tau}U|^2dy \le C(\|G_1\|^2_{L^2(W\cap\Omega)}+\|G_2\|_{H^1(W\cap\Omega)})$$
$$+C\int_{\tilde\Omega} J\chi^2|D_{\zeta\xi\tau}U|^2dy + C_\delta\|\nabla u\|^2_{L^2}+\delta\|u\|^2_{H^3} \qquad (3.57)$$

for any tangential directions $\zeta,\xi,\tau=1,2$.

By virtue of (3.45), it suffices to control the L^2-norm of G_1, G_2 and $\nabla_x G_2$ in $W\cap\Omega$. To this end, we give an expression of G_1 and G_2 by applying χD_τ to (3.36). It is not difficult to see that

$$G_1^i=\chi D_\tau(R(U_t^i+V^ja_{kj}D_kU^i)-\nu a_{ki}D_k(a_{lj}D_lU^j))$$
$$+O(1)(D_lU^i+D_{kl}U^i+\frac{1}{\epsilon^2}D_kQ)$$

which gives immediately

$$\|G_1\|^2_{L^2}\le C\big(\|\frac{\nabla q}{\epsilon^2}\|^2_{L^2}+\int_{\tilde\Omega} J\chi^2|D_{\tau k}(a_{lj}D_lU^j)|^2dy\big)+\delta\|u\|^2_{H^3}$$
$$+C_\delta\big(1+\|\rho\|^4_{H^2}\|v\|^{\frac52}_{H^1}\|v\|^{\frac32}_{H^3}\big)\|\nabla u\|^2_{L^2}+C\|\rho\|^2_{H^2}\|u_t\|^2_{H^1}. \qquad (3.58)$$

In the same manner, we have

$$G_2^i=O(1)(D_\tau U^j+D_kU^j+D_{\tau k}U^j),$$

$$\nabla_x G_2^i(\Lambda^{-1}(x))=O(1)\big[D_lU^j+D_{kl}U^j+\chi D_{k\tau}(a_{lj}D_lU^j)\big].$$

Hence,

$$\|G_2\|_{H^1}^2 \le \delta \|u\|_{H^3}^2 + C_\delta \|\nabla u\|_{L^2}^2 + C \int_{\tilde{\Omega}} J\chi^2 |D_{k\tau}(a_{lj} D_l U^j)|^2 dy. \tag{3.59}$$

Substituting (3.58), (3.59) into (3.57) and integrating over $(0, t) \times \tilde{\Omega}$, we obtain

$$\begin{aligned}
(C_3 + 1) \int_0^t \int_{\tilde{\Omega}} J\chi^2 |D_{33\tau} U|^2 dyds \\
\le & \delta \|u\|_{L_t^2(H^3)}^2 + F_6(M_0) \|u_t\|_{L_t^2(H^1)}^2 + C_4 \Big\| \frac{\nabla q}{\epsilon^2} \Big\|_{L_t^2(L^2)}^2 \\
& + C_4 \int_0^t \int_{\tilde{\Omega}} J\chi^2 (|D_{\xi\tau y} U|^2 + |D_{3\tau}(a_{lj} D_l U^j)|^2) dyds \\
& + F_0(M_0) \int_0^t (1 + M^{\frac{5}{2}} \|v\|_{H^3}^{\frac{3}{2}}) \|\nabla u\|_{L^2}^2 ds,
\end{aligned} \tag{3.60}$$

for some constant $C_4 > 0$ and continuous function $F_6(M_0) > 1$ and for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$.

Step 4. Conclusions of estimates in normal direction. Now, denoting

$$\begin{aligned}
\Phi_\chi(t) := & \int_{\tilde{\Omega}} J\chi^2 \big(C_5 R |D_{\xi\tau} U|^2 + C_5 \Big| \frac{D_{\xi\tau} Q}{\epsilon} \Big|^2 \\
& + (C_4 + 1) \Big| \frac{D_{\tau 3} Q}{\epsilon} \Big|^2 + \Big| \frac{D_{33} Q}{\epsilon} \Big|^2 \big)(t) dy
\end{aligned}$$

with $C_5 := (C_2 + 1)(C_4 + 1)$, and

$$\begin{aligned}
\Psi_\chi(t) := & \int_0^t \int_{\tilde{\Omega}} J\chi^2 (|D_{y\xi\tau} U|^2 + |D_{\tau 3}(a_{lj} D_l U^j)|^2 \\
& + |D_{33}(a_{lj} D_l U^j)|^2 + |D_{33\tau} U|^2) dyds,
\end{aligned}$$

we deduce from (3.45), (3.50), (3.54) and (3.60) that

$$\begin{aligned}
\Phi_\chi(t) + \Psi_\chi(t) \le & \eta \big(\Big\| (u_t, \frac{\nabla q}{\epsilon^2}) \Big\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^2(H^3)}^2 \big) \\
& + C_4 \Big\| \frac{\nabla q}{\epsilon^2} \Big\|_{L_t^2(L^2)}^2 + F_7(M_0) \|u_t\|_{L_t^2(H^1)}^2 + F_0(M_0) \\
& + F_0(M_0) \int_0^t K(M, \|v\|_{H^3}) (\Phi_\chi(t) + \|(\nabla u, u_t)\|_{L^2}^2) ds,
\end{aligned}$$

where $F_7(M_0) := (C_4+1)F_4(M_0) + F_5(M_0) + F_6(M_0)$, and

$$K(M, \|v\|_{H^3}) := M^{\frac{5}{2}}\|v\|_{H^3}^{\frac{3}{2}} + M^{\frac{4}{3}}\|v\|_{H^3}^{\frac{4}{3}} + (1+M^{\frac{1}{4}})(1+\|v\|_{H^3}^{\frac{15}{8}}).$$

Inserting (3.30) into the above inequality and transforming the local coordinates into usual ones, we have

$$\begin{aligned}
&\big\|(\chi_0\nabla^2 u, \frac{\nabla^2 q}{\epsilon})(t)\big\|_{L^2}^2 + \int_{\tilde{\Omega}} J\chi^2 R|D_{\xi\tau}U(t,y)|^2 dy + \|\operatorname{div} u\|_{L_t^2(H^2)}^2 \\
&\le \eta\big(\|\frac{\nabla q}{\epsilon^2}\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^2(H^3)}^2\big) + C_5\big(\|\frac{\nabla q}{\epsilon^2}\|_{L_t^2(L^2)}^2 + \|\frac{\nabla q}{\epsilon}(t)\|_{L^2}^2\big) \\
&+ F_7(M_0)\|u_t\|_{L_t^2(H^1)}^2 + F_0(M_0)\big\{1 + \int_0^t (K(M, \|v\|_{H^3}) + M^4)\times \\
&\times\big(\int_{\tilde{\Omega}} J\chi^2|D_{\xi\tau}U|^2 dy + \big\|(\nabla u, u_t, \frac{\nabla q}{\epsilon}, \frac{\nabla^2 q}{\epsilon})\big\|_{L^2}^2\big)ds\big\},
\end{aligned} \tag{3.61}$$

for some constant $C_5 > 0$.

Finally, submitting (3.16), (3.21) and (3.23) into (3.61), we obtain the following proposition.

Proposition 5. *There exist a positive continuous function $F_8(\cdot)$ and the positive constants C_6 and $T_4(M)$, such that for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$,*

$$\begin{aligned}
&\big\|(\chi_0\nabla^2 u, \frac{\nabla^2 q}{\epsilon})(t)\big\|_{L^2}^2 + \int_{\tilde{\Omega}} J\chi^2 R|D_{\xi\tau}U(t,y)|^2 dy + \|\operatorname{div} u\|_{L_t^2(H^2)}^2 \\
&\le F_0(M_0)\int_0^t (K(M, \|v\|_{H^3}) + M^{\frac{15}{4}} + \|v_t\|_{H^1}^{\frac{15}{8}})\times \\
&\times\big(\int_{\tilde{\Omega}} J\chi^2|D_{\xi\tau}U|^2 dy + \big\|(\nabla u, u_t, \frac{\nabla q}{\epsilon}, \frac{\nabla^2 q}{\epsilon})\big\|_{L^2}^2\big)ds \\
&+ F_0(M_0) + \eta F_8(M_0)\big(\|\frac{\nabla q}{\epsilon^2}\|_{L_t^2(H^1)}^2 + \|u\|_{L_t^2(H^3)}^2\big),
\end{aligned} \tag{3.62}$$

where η is small constant which is to be chosen later. □

Now, by the L^2-estimates (3.11), Proposition 3, and Proposition 5 with η suitably small, we conclude

Proposition 6. *There exist a positive continuous function $F_9(\cdot)$ and a positive constant $T_4(M)$, such that*

$$\|u(t)\|_{H^1}^2 + \big\|\frac{q(t)}{\epsilon}\big\|_{H^2}^2 + \big|[\nabla^2 u(t)]\big|_{tan} + \big\|(u_t, \frac{\nabla q}{\epsilon^2})\big\|_{L_t^2(H^1)}^2 + \big\|(u_t, \frac{q_t}{\epsilon})(t)\big\|_{L^2}^2 + \|u\|_{L_t^2(H^3)}^2 \le F_9(M_0)$$

for any $0 \le t \le T_4$ and $0 < \epsilon \le 1$, where $|[\nabla^2 u(t)]|_{tan}$ is the L^2-norm of the second-order derivatives of u except the normal components to $\partial\Omega$.

We are now in a position to prove Proposition 1.

Proof of Proposition 1. Let $F_{10}(M_0) := F_1(M_0) + F_9(M_0)$, where $F_1(\cdot)$ and $F_9(\cdot)$ are defined in Propositions (2) and (6), respectively. Choosing M and then T_0, such that

$$M = F_{10}(M_0), \quad 0 < T_0 \le T_4 := T_4(M(M_0)), \tag{3.63}$$

we obtain that for any $0 < \epsilon \le 1$,

$$\max_{0\le t\le T_4}\big\{\big\|(\rho, \frac{q}{\epsilon})\big\|_{H^2} + \|u\|_{H^1} + \big\|(\rho_t, u_t, \frac{q_t}{\epsilon})\big\|_{L^2} + \big\|\frac{1}{\rho}\big\|_{L^\infty}\big\}(t) + \Big[\int_0^{T_0}\big(\|u\|_{H^3}^2 + \|u_t\|_{H^1}^2 + \big\|\frac{q}{\epsilon^2}\big\|_{H^2}^2\big)dt\Big]^{1/2} \le F_{10}(M_0), \tag{3.64}$$

which proves Proposition 1. □

4. Proof of the main theorem

In this section, we prove Theorem 1 by using the fixed-point techniques, the global existence for the linearized equations in Theorem 2 and the uniform-in-ϵ estimates in Proposition 1.

Proof of Theorem 1. Define the mapping L: $v \mapsto u$, where (ρ, q, u) is the unique solution to the linearized problem (3.1)–(3.5). Moreover, we define

$$R_{M,T^*} := \big\{u \mid u \in C([0,T^*], H_0^1),\ u_t \in C([0,T^*], L^2) \cap L^2(0,T^*; H_0^1), \max_{0\le t\le T^*}(\|u\|_{H^1} + \|u_t\|_{L^2})(t) + \|u\|_{L^2_{T^*}(H^3)} + \|u_t\|_{L^2_{T^*}(H^1)} \le M\big\},$$

where M depends on M_0, but not on $\epsilon \in (0,1]$.

It is obvious that R_{M,T^*} is a non-empty, convex, closed subset in $X := C([0,T^*], L^2)$, and is pre-compact in X by the Arzelà-Ascoli theorem. Given $M = F_{10}(M_0)$ large enough, and for sufficiently small $T^* \le T$, L maps R_{M,T^*} into itself by the uniform estimate ((3.6)) in Proposition 1. Note that $\epsilon \in (0,1]$ is arbitrary and T^* is independent of $\epsilon \in (0,1]$, the continuity of L in X can be shown in a routine manner. Assume $u^n = L(v^n)$, $u = L(v)$, and $v^n \to v$ in X, we can easily show that $u^n \to u$ in X by the Gronwall inequality.

Due to the Schauder fixed point theorem, L has a fixed point u in R_{M,T^*}. That is, there exists $(\rho^\epsilon, u^\epsilon, q^\epsilon)$ which solve the nonlinear problem (1.8)–(1.12) in $\Omega \times [0,T^*]$ and satisfy

$$\begin{aligned} &\max_{0\le t\le T^*}\Big\{\big\|(\rho^\epsilon, \frac{q^\epsilon}{\epsilon})\big\|_{H^2} + \|u\|_{H^1} + \big\|(\rho^\epsilon_t, u^\epsilon_t, \frac{q^\epsilon_t}{\epsilon})\big\|_{L^2} + \big\|\frac{1}{\rho^\epsilon}\big\|_{L^\infty}\Big\}(t) \\ &\quad + \Big[\int_0^{T^*} \big(\|u^\epsilon\|^2_{H^3} + \|u^\epsilon_t\|^2_{H^1} + \big\|\frac{q}{\epsilon^2}\big\|^2_{H^2}\big)dt\Big]^{1/2} \le C, \end{aligned} \tag{4.1}$$

where C, $T^* > 0$ are constants independent of $\epsilon \in (0,1]$. The uniqueness (in lower norms) can be easily shown by applying the Gronwall inequality. Thus, we have proved the uniform existence in Theorem 1.

From ((4.1)) and the Arzelà-Ascoli theorem, it follows that there exists a subsequence of $(\rho^\epsilon, u^\epsilon, q^\epsilon)$, still denoted by $(\rho^\epsilon, u^\epsilon, q^\epsilon)$, such that as $\epsilon \to 0$,

$$\rho^\epsilon \to \rho \quad \text{in} \quad L^\infty(0,T^*;H^s) \;\text{ for any } s < 2;$$

$$q^\epsilon \to 0 \;\text{ in } L^\infty(0,T^*;H^2); \quad u^\epsilon \to u \quad \text{in} \quad L^2(0,T^*;H^2);$$

$$\rho^\epsilon \rightharpoonup \rho \quad \text{in} \quad L^\infty(0,T^*;H^2); \quad u^\epsilon \rightharpoonup u \quad \text{in} \quad L^2(0,T^*;H^3);$$

$$(\rho^\epsilon_t, u^\epsilon_t, q^\epsilon_t) \rightharpoonup (\rho_t, u_t, 0) \quad \text{in} \quad L^\infty(0,T^*;L^2);$$

$$u^\epsilon_t \rightharpoonup u_t \quad \text{in} \quad L^2(0,T^*;H^1)$$

It follows from the general regularity theory that $\rho \in C(0,T^*;H^1)$ and $u \in C(0,T^*;H^2)$. By (1.10) and the uniform estimates in (4.1), we have

$$\text{div} u = 0 \quad a.e. \quad in \quad \Omega \times T^*.$$

Moreover, there exists a function $\pi(x,t)$ uniquely determined up to a constant, such that (ρ, u, π) solves the initial-boundary value problem (1.13) in $\Omega \times (0,T^*)$. Note that the density ρ satisfies a purely transported equation

$$\rho_t + u \cdot \nabla \rho = 0,$$

and $\rho_0 \in H^2$. Thus we obtain $\rho \in C(0, T^*; H^2)$ by repeating the arguments in Section 3.1.

To finish the proof, it remains to establish the global existence for the linearized problem (3.1)–(3.5) in $[0, T] \times \Omega$ for any $T > 0$. Since the global existence of ρ is available, it suffices to show the global existence of (u, q) to the problem (3.2)–(3.5). To this end, we first derive the local existence theorem by applying the Galerkin method and the fixed-point arguments (see [41]). The proof is standard and the key point is to derive the energy estimates which depend only on ϵ, the initial data and the given functions ρ and v. We take $\epsilon = 1$ without the loss of generality.

Lemma 4. (Local existence) *Let $(u_0, q_0) \in H^2$. Assume that u_0, ρ, v satisfy $u_0|_{\partial\Omega} = u_t(0)|_{\partial\Omega} = 0$, and*

$$\begin{aligned}
&v \in C([0, T], H_0^1) \cap L^2(0, T; H^3),\ \ \rho \in C([0, T], H^2),\\
&v_t \in C([0, T], L^2) \cap L^2(0, T; H_0^1),\ \ \rho_t \in C([0, T], L^2),\\
&\inf \rho > 0 \ \ in\ \Omega \times (0, T),\ \ (v, \rho)|_{t=0} = (u_0, \rho_0).
\end{aligned}$$

Then, there exists a $T' > 0$, such that the problem (3.2)–(3.5) admits a unique local solution (u, q) in $(0, T') \times \Omega$, satisfying

$$\begin{aligned}
&q \in C([0, T'], H^2),\ \ u \in C([0, T'], H_0^1 \cap H^2) \cap L^2(0, T'; H^3),\\
&q_t \in C([0, T'], L^2),\ \ u_t \in C([0, T'], L^2) \cap L^2(0, T'; H_0^1).
\end{aligned} \tag{4.2}$$

Proof. For given functions u, q satisfying (4.2) in $[0, T] \times \Omega$, we consider the following linearized problem of (3.2)–(3.5):

$$\rho U_t - \mu \Delta U - \nu \nabla \mathrm{div} U = G_1 \quad in \quad \Omega \times (0, T), \tag{4.3}$$

$$Q_t + v \cdot \nabla Q - \gamma Q \mathrm{div} v = G_2 \quad in \quad \Omega \times (0, T), \tag{4.4}$$

$$U = 0 \quad on \quad \partial\Omega \times (0, T) \tag{4.5}$$

$$(U, Q)|_{t=0} = (u_0, q_0)(x), \quad x \in \Omega, \tag{4.6}$$

where

$$G_1 := -\nabla q - \rho v \cdot \nabla u,\ \ G_2 := -\gamma \mathrm{div} u + (\gamma - 1)(2\mu |D(v)|^2 + \lambda (\mathrm{div} v)^2).$$

Thus $G_1 \in C([0, T], L^2) \cap L^2(0, T; H^1)$, $G_{1t} \in C([0, T], H^{-1})$, $G_2 \in L^1(0, T; H^2)$ and $G_{2t} \in L^1(0, T; L^2)$.

We outline the proof here. First, by applying the Galerkin method, we can show that there exists a solution U to (4.3), (4.5), (4.6) with $U \in L^\infty(0,T;H^1)\cap L^2(0,T;H^3)$ and $U_t \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)$. The general regularity theory implies that $U \in C([0,T],H^1)$ and thus $U_t \in C([0,T],L^2)$ by (4.3). Then, we prove the existence and regularity of Q by the method of characteristics. Finally, we define the mapping L_0: $(U,Q) := L_0(u,q)$, and the existence follows easily from the Schauder fixed-point theorem and the uniqueness from the Gronwall inequality. The proof is standard and we thus omit its details here. □

The global existence for the linearized equations (3.1)–(3.5) is a result of the continuation argument.

Lemma 5. (Global existence) *Let (u,q) be the local solution established in Lemma 4. Then (u,q) can be extended onto the time interval $[0,T]$. Moreover, (u,q) satisfies*

$$\begin{aligned}&u \in C([0,T],H_0^1\cap H^2)\cap L^2(0,T;H^3),\ u_t \in C([0,T],L^2)\cap L^2(0,T;H_0^1),\\ &q \in C([0,T],H^2),\ q_t \in C([0,T],L^2).\end{aligned} \tag{4.7}$$

Proof. Assume that (u,q) satisfies the regularities in (4.7) with T' being replaced by $T_1 := T_1(M_0,M,\epsilon)$. Clearly, $\|u(t)\|_{H^2}$ and $\|q(t)\|_{H^2}$ are finite a.e. $t \in [0,T_1]$. By redefining the value of $\|u(t)\|_{H^2}$ and $\|q(t)\|_{H^2}$ at $t = T_1$ if necessary, we have

$$\big(\|(u,q)\|_{H^2} + \|(u_t,q_t)\|_{L^2}\big)(T_1) \le \tilde{F}(M_0,M,\epsilon),$$

with $u(x,T_1) = u_t(x,T_1) = 0$ on $\partial\Omega$. Assume that $T_1 < T$ without loss of generality. Repeating all the procedures in Section 3, we have

$$\big(\|(u,q)\|_{H^2} + \|(u_t,q_t)\|_{L^2}\big)(T_1) \le F(M_0,M), \qquad \forall\, 0<\epsilon\le 1.$$

Again, we can apply Lemma 4 with the initial time $t = T_1$ to find that there exists a $T_2 := T_2(M_0,M,\epsilon)$, such that (u,q) solves (3.2)–(3.5) in $[0,T_1+T_2]\times\Omega$. Since $F(M_0,M)$ is bounded uniformly in ϵ on $[0,T]$, we can repeat this process until (u,q) is continued to the time interval $[0,T]$. Thus, the global existence is shown. Finally, the regularity $u \in C([0,T],H^2)$ easily follows from $u \in L^2(0,T;H^3)$ and $u_t \in L^2(0,T;H^1)$.

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