

Well-posedness in critical spaces for the system of compressible Navier-Stokes in larger spaces

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Abstract

This paper is dedicated to the study of viscous compressible barotropic fluids in dimension $N \geq 2$. We address the question of well-posedness for *large* data having critical Besov regularity. Our result improves the analysis of R. Danchin in [13] and of B. Haspot in [15], by the fact that we may take initial density in $B_{p,1}^{\frac{N}{p}}$ with $1 \leq p < +\infty$. Our result relies on a new a priori estimate for the velocity, where we introduce a new structure to weaken one the coupling between the density and the velocity. In particular our result is the first where we obtain uniqueness without imposing hypothesis on the gradient of the density.

1 Introduction

The motion of a general barotropic compressible fluid is described by the following system:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)D(u)) - \nabla(\lambda(\rho)\operatorname{div}u) + \nabla P(\rho) = \rho f, \\ (\rho, u)_{/t=0} = (\rho_0, u_0). \end{cases} \quad (1.1)$$

Here $u = u(t, x) \in \mathbb{R}^N$ stands for the velocity field and $\rho = \rho(t, x) \in \mathbb{R}^+$ is the density. The pressure P is a suitable smooth function of ρ . We denote by λ and μ the two viscosity coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $\lambda + 2\mu > 0$ (in the sequel to simplify the calculus we will assume the viscosity coefficients are constant functions). Such a conditions ensures ellipticity for the momentum equation and is satisfied in the physical cases where $\lambda + \frac{2\mu}{N} > 0$. We supplement the problem with initial condition (ρ_0, u_0) and an outer force f . Throughout the paper, we assume that the space variable x is in \mathbb{R}^N or to the periodic box \mathcal{T}_a^N with period a_i , in the i -th direction. We restrict ourselves to the case $N \geq 2$.

The problem of existence of global solution in time for Navier-Stokes equations was addressed in one dimension for smooth enough data by Kazhikov and Shelukin in [29], and for discontinuous ones, but still with densities away from zero, by Serre in [36] and Hoff in [20]. Those results have been generalized to higher dimension by Matsumura

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and Nishida in [32] for smooth data close to equilibrium and by Hoff in the case of discontinuous data in [23, 26]. The existence and uniqueness of local classical solutions for (1.1) with smooth initial data such that the density ρ_0 is bounded and bounded away from zero (i.e., $0 < \underline{\rho} \leq \rho_0 \leq M$) has been stated by Nash in [34]. Let us emphasize that no stability condition was required there. On the other hand, for small smooth perturbations of a stable equilibrium with constant positive density, global well-posedness has been proved in [32]. Many works on the case of the one dimension have been devoted to the qualitative behavior of solutions for large time (see for example [20, 29]). Refined functional analysis has been used for the last decades, ranging from Sobolev, Besov, Lorentz and Triebel spaces to describe the regularity and long time behavior of solutions to the compressible model [37], [38], [22], [28]. For results of weak-strong uniqueness, we refer to the work of P. Germain [14].

Guided in our approach by numerous works dedicated to the incompressible Navier-Stokes equation (see e.g [33]):

$$(NS) \quad \begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \operatorname{div} v = 0, \end{cases}$$

we aim at solving (1.1) in the case where the data (ρ_0, u_0, f) have *critical* regularity. By critical, we mean that we want to solve the system in functional spaces with norm independent of the changes of scales which leave (1.1) invariant. In the case of barotropic fluids, it is easy to see that the transformations:

$$(\rho(t, x), u(t, x)) \longrightarrow (\rho(l^2 t, lx), lu(l^2 t, lx)), \quad l \in \mathbb{R}, \quad (1.2)$$

have that property, provided that the pressure term has been changed accordingly.

The use of critical functional frameworks led to several new well-posedness results for compressible fluids (see [10, 11, 13]). In addition to have a norm invariant by (1.2), appropriate functional space for solving (1.1) must provide a control on the L^∞ norm of the density (in order to avoid vacuum and loss of ellipticity). For that reason, we restricted our study to the case where the initial data (ρ_0, u_0) and external force f are such that, for some positive constant $\bar{\rho}$:

$$(\rho_0 - \bar{\rho}) \in B_{p,1}^{\frac{N}{p}-1}, \quad u_0 \in B_{p_1,1}^{\frac{N}{p_1}-1} \quad \text{and} \quad f \in L_{loc}^1(\mathbb{R}^+, \in B_{p_1,1}^{\frac{N}{p_1}-1})$$

for suitable choice of $(p, p_1) \in [1, +\infty[$. In [13], however, we had to have $p = p_1$ with the limitation $p < 2N$ for the existence of solutions and $p \leq N$ for the uniqueness, indeed in this article there exists a very strong coupling between the pressure and the velocity. To be more precise, the pressure term is considered as a remainder for the elliptic operator in the momentum equation of (1.1). This present paper improves the results of R. Danchin in [10, 13], in the sense that the initial density belongs to larger spaces $B_{p,1}^{\frac{N}{p}}$ with larger value $p \in [1, +\infty[$. The main idea of this paper is to introduce a new variable than the velocity in the goal to *kill* the relation of coupling between the velocity and the density. In the present paper, we address the question of local well-posedness in the critical functional framework under the assumption that the initial density belongs to critical Besov space with a index of integrability different of that the

velocity. We adapt the spirit of the results of [1] and [17] concerning dependent density incompressible Navier-Stokes system (at the difference that in these works the velocity and the density are naturally decoupled). To simplify the notation, we assume from now on that $\bar{\rho} = 1$. Hence as long as ρ does not vanish, the equations for $(a = \rho^{-1} - 1, u)$ read:

$$\begin{cases} \partial_t a + u \cdot \nabla a = (1 + a) \operatorname{div} u, \\ \partial_t u + u \cdot \nabla u - (1 + a) \mathcal{A}u + \nabla(g(a)) = f, \end{cases} \quad (1.3)$$

In the sequel we will note $\mathcal{A} = \mu \Delta + (\lambda + \mu) \nabla \operatorname{div}$ and g a smooth function which may be computed from the pressure function P .

One can now state our main result.

Theorem 1.1 *Assume that P is a suitably smooth function of the density and that $1 \leq p_1 \leq p < +\infty$ such that $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p}$. Let $u_0 \in B_{p_1,1}^{\frac{N}{p_1}-1}$, $f \in L_{loc}^1(\mathbb{R}^+, B_{p_1,1}^{\frac{N}{p_1}-1})$ and $a_0 \in B_{p,1}^{\frac{N}{p}}$ with $1 + a_0$ bounded away from zero. If $\frac{1}{p} + \frac{1}{p_1} > \frac{1}{N}$ there exists a positive time T such that system (1.1) has a solution (a, u) with $1 + a$ bounded away from zero,*

$$a \in \widetilde{C}([0, T], B_{p,1}^{\frac{N}{p}}), \quad u \in \widetilde{C}([0, T]; B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap L^1([0, T], B_{p,1}^{\frac{N}{p}+1}).$$

Moreover this solution is unique if $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{p_1}$.

Remark 1 *We refer the reader for the notation of $\widetilde{L}^p(B_{p,r}^s)$ (with $s \in \mathbb{R}$, $(p, r, \rho) \in [1, +\infty]^3$) to the definition 2.2.*

Remark 2 *It means that this theorem allow us to reach very critical spaces, in fact we are very close to get solution for initial data (a_0, u_0) in $B_{\infty,1}^0 \times B_{N,1}^1$. This space is absolutely critical for compressible Navier-Stokes system in the sense that $B_{N,1}^0$ is close from L^N which is critical for incompressible Navier-Stokes. Moreover in this case we do not ask any information on the derivatives of the initial density when a_0 is in $B_{\infty,1}^0$ (this is really new compared with the different previous results existing in the literature of the topic). In passing we can remark that $B_{\infty,1}^0$ is not far of L^∞ (L^∞ being in some sense the more general space to control the non linearities appearing on the density, for example the pressure). In this sense, we can consider that our result is quite optimal.*

Remark 3 *It seems possible to improve the theorem 1.1 in choosing initial data a_0 in $B_{p,\infty}^{\frac{N}{p}} \cap B_{\infty,1}^0$, for this we could use some arguments of density to deal with the variable coefficients of the heat equation, however some supplementary conditions appear on p_1 in this case.*

The key to theorem 1.1 is to introduce a new auxiliary velocity v_1 to control the velocity u . By this way we avoid the coupling between the density and the velocity, as the pressure term is included in the velocity v_1 . More precisely we write the gradient of the pressure as a Laplacian of a variable v , and we want to treat the variable $v_1 = u - v$. We can check easily that v_1 verifies a heat equation with some remainders where the pressure

has disappeared. We are able then to get a control on v_1 which can write roughly as $v_1 = u - \mathcal{G}P(\rho)$ where \mathcal{G} is a pseudodifferential operator of order -1 . By this way, we have canceled out the coupling between v_1 and the density, we next verify easily that we have a control Lipschitz of the gradient of u (it is crucial to estimate the density by the transport equation).

Remark 4 *In the present paper we did not strive for unnecessary generality which may hide the new ideas of our analysis. Hence we focused on the somewhat academic model of barotropic fluids. In physical contexts however, a coupling with the energy equation has to be introduced. Besides, the viscosity coefficients may depend on the density. We believe that our analysis may be carried out to these more general models. (See [15, 16]). The main point is only to define an effective velocity adapted to the problem and we refer for more details to [19].*

In [21], D. Hoff shows a very strong theorem of uniqueness for weak solutions when the pressure is of the specific form $P(\rho) = K\rho$ with $K > 0$. Similarly in [23], [26], [22], D. Hoff gets global weak solutions and point out regularizing effects on the velocity when the initial data are small. In particular when the pressure has this form, he does not need estimate on the gradient of the initial density. In the following corollary, we will observe that this type of pressure ensures a specific structure and avoid to impose some extra conditions for the uniqueness. In the following corollary, we will assume that:

Corollary 1 *Assume that $P(\rho) = K\rho$ with $K > 0$. Let $1 \leq p_1 \leq p \leq +\infty$ such that $\frac{1}{p_1} \leq \frac{1}{N} + \frac{1}{p}$. Assume that $u_0 \in B_{p_1,1}^{\frac{N}{p_1}-1}$, $f \in L_{loc}^1(\mathbb{R}^+, B_{p_1,1}^{\frac{N}{p_1}-1})$ and $a_0 \in B_{p,1}^{\frac{N}{p}}$ with $1 + a_0$ bounded away from zero.*

- *If $\frac{1}{p} + \frac{1}{p_1} > \frac{1}{N}$ there exists a positive time T such that system (1.1) has a solution (a, u) with $1 + a$ bounded away from zero,*

$$a \in \tilde{C}([0, T], B_{p,1}^{\frac{N}{p}}), \quad u \in \tilde{C}([0, T]; B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap L^1([0, T], B_{p,1}^{\frac{N}{p}+1}).$$

- *If moreover we assume that $\sqrt{\rho_0}u_0 \in L^2$, $\rho_0 - \bar{\rho} \in L_2^1$, $u_0 \in H^s$ with $s > 0$ if $N = 2$ and $s > \frac{1}{2}$ if $N = 3$. Finally we need to assume that u_0 belongs to $L^{2+\varepsilon}$ if $N = 2$ and to $L^{6+\varepsilon}$ if $N = 3$ with $\varepsilon > 0$. Moreover we assume that $0 < \lambda < \frac{5}{4}\mu$ if $N = 3$. Then the solution (a, u) is unique.*

Remark 5 *In the previous corollary we did not want strive with generalities which may hide the main functional spaces used on the initial data. But in fact we need of additional regularity on the source term f when $N = 3$ to obtain the previous corollary, we refer to the conditions (1.13) and (1.14) of [24].*

Remark 6 *Here L_2^1 defines the corresponding Orlicz space (see definition in [30]).*

Remark 7 *This corollary improves theorem 1.1 in the sense we do not need of the condition $\frac{2}{N} \leq \frac{1}{p} + \frac{1}{p_1}$ to get uniqueness.*

Remark 8 *Up to my knowledge, it seems that it is the first time that we get strong solution without any control on the gradient of the initial density $\nabla \rho_0$. Indeed in [12], we have $\nabla \rho_0 \in B_{1,N}^0$. In our case $\nabla \rho_0$ has a negative index of regularity, more precisely $\nabla \rho_0 \in B_{p,1}^{\frac{N}{p}-1}$ with $\frac{N}{p} - 1 < 0$.*

Remark 9 *Moreover we can observe that with this type of pressure we are very close to have existence of strong solution in finite time for initial data (a_0, u_0) in $B_{\infty,1}^0 \times B_{2,1}^{\frac{N}{2}-1}$. It means that this theorem bridges the gap between the result of D. Hoff (see [21]) where the initial density is assumed L^∞ but where we have no uniqueness in dimension $N = 3$ and the results of R. Danchin in [13] where the initial density is far from L^∞ . In fact we are slightly surcritical on the initial velocity and with an additional condition of type $u_0 \in L^{6+\varepsilon}$ with $\varepsilon > 0$ in dimension $N = 3$. However it is the first result of type strong solution where it is possible to reach the critical case $a_0 \in B_{\infty,1}^0$.*

Our paper is structured as follows. In section 2, we give a few notation and briefly introduce the basic Fourier analysis techniques needed to prove our result. Sections 3 and 4 are devoted to the proof of key estimates for the linearized system (1.1). In section 5, we prove the theorem 1.1 and corollary 1. Two inescapable technical commutator estimates and some theorems of ellipticity are postponed in an appendix.

2 Littlewood-Paley theory and Besov spaces

Throughout the paper, C stands for a constant whose exact meaning depends on the context. The notation $A \lesssim B$ means that $A \leq CB$. For all Banach space X , we denote by $C([0, T], X)$ the set of continuous functions on $[0, T]$ with values in X . For $p \in [1, +\infty]$, the notation $L^p(0, T, X)$ or $L_T^p(X)$ stands for the set of measurable functions on $(0, T)$ with values in X such that $t \rightarrow \|f(t)\|_X$ belongs to $L^p(0, T)$. Littlewood-Paley decomposition corresponds to a dyadic decomposition of the space in Fourier variables. Let $\alpha > 1$ and (φ, χ) be a couple of smooth functions valued in $[0, 1]$, such that φ is supported in the shell supported in $\{\xi \in \mathbb{R}^N / \alpha^{-1} \leq |\xi| \leq 2\alpha\}$, χ is supported in the ball $\{\xi \in \mathbb{R}^N / |\xi| \leq \alpha\}$ such that:

$$\forall \xi \in \mathbb{R}^N, \quad \chi(\xi) + \sum_{l \in \mathbb{N}} \varphi(2^{-l}\xi) = 1.$$

Denoting $h = \mathcal{F}^{-1}\varphi$, we then define the dyadic blocks by:

$$\begin{aligned} \Delta_{-1}u &= \chi(D)u = \tilde{h} * u \quad \text{with } \tilde{h} = \mathcal{F}^{-1}\chi, \\ \Delta_l u &= \varphi(2^{-l}D)u = 2^{lN} \int_{\mathbb{R}^N} h(2^l y) u(x - y) dy \quad \text{with } h = \mathcal{F}^{-1}\chi, \text{ if } l \geq 0, \\ S_l u &= \sum_{k \leq l-1} \Delta_k u. \end{aligned}$$

Formally, one can write that:

$$u = \sum_{k \geq -1} \Delta_k u.$$

This decomposition is called nonhomogeneous Littlewood-Paley decomposition.

2.1 Nonhomogeneous Besov spaces and first properties

Definition 2.1 For $s \in \mathbb{R}$, $p \in [1, +\infty]$, $q \in [1, +\infty]$, and $u \in \mathcal{S}'(\mathbb{R}^N)$ we set:

$$\|u\|_{B_{p,q}^s} = \left(\sum_{l \geq -1} (2^{ls} \|\Delta_l u\|_{L^p})^q \right)^{\frac{1}{q}}.$$

The Besov space $B_{p,q}^s$ is the set of temperate distribution u such that $\|u\|_{B_{p,q}^s} < +\infty$.

Remark 10 The above definition is a natural generalization of the nonhomogeneous Sobolev and Hölder spaces: one can show that $B_{\infty,\infty}^s$ is the nonhomogeneous Hölder space C^s and that $B_{2,2}^s$ is the nonhomogeneous space H^s .

Proposition 2.1 The following properties hold:

1. there exists a constant universal C such that:

$$C^{-1} \|u\|_{B_{p,r}^s} \leq \|\nabla u\|_{B_{p,r}^{s-1}} \leq C \|u\|_{B_{p,r}^s}.$$
2. If $p_1 < p_2$ and $r_1 \leq r_2$ then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-N(1/p_1-1/p_2)}$.
3. $B_{p,r_1}^{s'} \hookrightarrow B_{p,r}^s$ if $s' > s$ or if $s = s'$ and $r_1 \leq r$.

Before going further into the paraproduct for Besov spaces, let us state an important proposition.

Proposition 2.2 Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. Let $(u_q)_{q \geq -1}$ be a sequence of functions such that

$$\left(\sum_{q \geq -1} 2^{qs} \|u_q\|_{L^p}^r \right)^{\frac{1}{r}} < +\infty.$$

If $\text{supp } \hat{u}_1 \subset \mathcal{C}(0, 2^q R_1, 2^q R_2)$ for some $0 < R_1 < R_2$ then $u = \sum_{q \geq -1} u_q$ belongs to $B_{p,r}^s$ and there exists a universal constant C such that:

$$\|u\|_{B_{p,r}^s} \leq C^{1+|s|} \left(\sum_{q \geq -1} (2^{qs} \|u_q\|_{L^p})^r \right)^{\frac{1}{r}}.$$

Let now recall a few product laws in Besov spaces coming directly from the paradifferential calculus of J-M. Bony (see [4]) and rewrite on a generalized form in [1] by H. Abidi and M. Paicu (in this article the results are written in the case of homogeneous spaces but it can easily generalize for the nonhomogeneous Besov spaces).

Proposition 2.3 We have the following laws of product:

- For all $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$ we have:

$$\|uv\|_{B_{p,r}^s} \leq C(\|u\|_{L^\infty} \|v\|_{B_{p,r}^s} + \|v\|_{L^\infty} \|u\|_{B_{p,r}^s}). \quad (2.4)$$

- Let $(p, p_1, p_2, r, \lambda_1, \lambda_2) \in [1, +\infty]^2$ such that: $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{p_2}$, $p_1 \leq \lambda_2$, $p_2 \leq \lambda_1$, $\frac{1}{p} \leq \frac{1}{p_1} + \frac{1}{\lambda_1}$ and $\frac{1}{p} \leq \frac{1}{p_2} + \frac{1}{\lambda_2}$. We have then the following inequalities:
if $s_1 + s_2 + N \inf(0, 1 - \frac{1}{p_1} - \frac{1}{p_2}) > 0$, $s_1 + \frac{N}{\lambda_2} < \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} < \frac{N}{p_2}$ then:

$$\|uv\|_{B_{p,r}^{s_1+s_2-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B_{p_1,r}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}, \quad (2.5)$$

when $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ (resp $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$) we replace $\|u\|_{B_{p_1,r}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}$ (resp $\|v\|_{B_{p_2,\infty}^{s_2}}$) by $\|u\|_{B_{p_1,1}^{s_1}} \|v\|_{B_{p_2,r}^{s_2}}$ (resp $\|v\|_{B_{p_2,\infty}^{s_2} \cap L^\infty}$), if $s_1 + \frac{N}{\lambda_2} = \frac{N}{p_1}$ and $s_2 + \frac{N}{\lambda_1} = \frac{N}{p_2}$ we take $r = 1$.

If $s_1 + s_2 = 0$, $s_1 \in (\frac{N}{\lambda_1} - \frac{N}{p_2}, \frac{N}{p_1} - \frac{N}{\lambda_2}]$ and $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$ then:

$$\|uv\|_{B_{p,\infty}^{-N(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p})}} \lesssim \|u\|_{B_{p_1,1}^{s_1}} \|v\|_{B_{p_2,\infty}^{s_2}}. \quad (2.6)$$

If $|s| < \frac{N}{p}$ for $p \geq 2$ and $-\frac{N}{p'} < s < \frac{N}{p}$ else, we have:

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p,\infty}^{\frac{N}{p}} \cap L^\infty}. \quad (2.7)$$

Remark 11 In the sequel p will be either p_1 or p_2 and in this case $\frac{1}{\lambda} = \frac{1}{p_1} - \frac{1}{p_2}$ if $p_1 \leq p_2$, resp $\frac{1}{\lambda} = \frac{1}{p_2} - \frac{1}{p_1}$ if $p_2 \leq p_1$.

Corollary 2 Let $r \in [1, +\infty]$, $1 \leq p \leq p_1 \leq +\infty$ and s such that:

- $s \in (-\frac{N}{p_1}, \frac{N}{p_1})$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$,
- $s \in (-\frac{N}{p_1} + N(\frac{1}{p} + \frac{1}{p_1} - 1), \frac{N}{p_1})$ if $\frac{1}{p} + \frac{1}{p_1} > 1$,

then we have if $u \in B_{p,r}^s$ and $v \in B_{p_1,\infty}^{\frac{N}{p}} \cap L^\infty$:

$$\|uv\|_{B_{p,r}^s} \leq C \|u\|_{B_{p,r}^s} \|v\|_{B_{p_1,\infty}^{\frac{N}{p}} \cap L^\infty}.$$

We recall now a result concerning the composition for Besov spaces:

Proposition 2.4 Let I be an open interval of \mathbb{R} . Let $s > 0$ and σ be the smallest integer such that $\sigma \geq s$. Let $F : I \rightarrow \mathbb{R}$ satisfy $F(0) = 0$ and $F' \in W^{\sigma,\infty}(I; \mathbb{R})$. Assume that $v \in B_{p,r}^s$ has values in $J \subset I$. Then $F(v) \in B_{p,r}^s$ and there exists a constant C depending only on s , I , J , and N , and such that

$$\|F(v)\|_{B_{p,r}^s} \leq C(1 + \|v\|_{L^\infty})^\sigma \|F'\|_{W^{\sigma,\infty}} \|v\|_{B_{p,r}^s}.$$

The study of non stationary PDE's requires spaces of type $L^\rho(0, T, X)$ for appropriate Banach spaces X . In our case, we expect X to be a Besov space, so that it is natural to localize the equation through Littlewood-Paley decomposition. But, in doing so, we obtain bounds in spaces which are not type $L^\rho(0, T, X)$ (except if $r = p$). We are now going to define the spaces of Chemin-Lerner (see [8]) in which we will work, which are a refinement of the spaces $L_T^\rho(B_{p,r}^s)$.

Definition 2.2 Let $\rho \in [1, +\infty]$, $T \in [1, +\infty]$ and $s_1 \in \mathbb{R}$. We set:

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} = \left(\sum_{l \geq -1} 2^{lrs_1} \|\Delta_l u(t)\|_{L^\rho(L^p)}^r \right)^{\frac{1}{r}}.$$

We then define the space $\tilde{L}_T^\rho(B_{p,r}^{s_1})$ as the set of temperate distribution u over $(0, T) \times \mathbb{R}^N$ such that $\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} < +\infty$.

We set $\tilde{C}_T(\tilde{B}_{p,r}^{s_1}) = \tilde{L}_T^\infty(\tilde{B}_{p,r}^{s_1}) \cap \mathcal{C}([0, T], B_{p,r}^{s_1})$. Let us emphasize that, according to Minkowski inequality, we have:

$$\|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} \leq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \geq \rho, \quad \|u\|_{\tilde{L}_T^\rho(B_{p,r}^{s_1})} \geq \|u\|_{L_T^\rho(B_{p,r}^{s_1})} \text{ if } r \leq \rho.$$

Remark 12 It is easy to generalize proposition 2.3, to $\tilde{L}_T^\rho(B_{p,r}^{s_1})$ spaces. The indices s_1 , p , r behave just as in the stationary case whereas the time exponent ρ behaves according to Hölder inequality.

Here we recall a result of interpolation which explains the link of the space $B_{p,1}^s$ with the space $B_{p,\infty}^s$, see [9].

Proposition 2.5 There exists a constant C such that for all $s \in \mathbb{R}$, $\varepsilon > 0$ and $1 \leq p < +\infty$,

$$\|u\|_{\tilde{L}_T^\rho(B_{p,1}^s)} \leq C \frac{1+\varepsilon}{\varepsilon} \|u\|_{\tilde{L}_T^\rho(B_{p,\infty}^s)} \left(1 + \log \frac{\|u\|_{\tilde{L}_T^\rho(B_{p,\infty}^{s+\varepsilon})}}{\|u\|_{\tilde{L}_T^\rho(B_{p,\infty}^s)}} \right).$$

Now we give some result on the behavior of the Besov spaces via some pseudodifferential operator (see [9]).

Definition 2.3 Let $m \in \mathbb{R}$. A smooth function $f : \mathbb{R}^N \rightarrow \mathbb{R}$ is said to be a \mathcal{S}^m multiplier if for all multi-index α , there exists a constant C_α such that:

$$\forall \xi \in \mathbb{R}^N, \quad |\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}.$$

Proposition 2.6 Let $m \in \mathbb{R}$ and f be a \mathcal{S}^m multiplier. Then for all $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$ the operator $f(D)$ is continuous from $B_{p,r}^s$ to $B_{p,r}^{s-m}$.

3 Estimates for a parabolic system with variable coefficients

Let us first state estimates for the following constant coefficient parabolic system:

$$\begin{cases} \partial_t u - \mu \Delta u - (\lambda + \mu) \nabla \operatorname{div} u = f, \\ u|_{t=0} = u_0. \end{cases} \quad (3.8)$$

Proposition 3.7 *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$. Assume that $\mu \geq 0$ and that $\lambda + 2\mu \geq 0$. Then there exists a universal constant κ such that for all $s \in \mathbb{Z}$ and $T \in \mathbb{R}^+$,*

$$\begin{aligned} \|u\|_{\tilde{L}_T^\infty(B_{p_1,1}^s)} &\leq C(\|u_0\|_{B_{p_1,1}^s} + \|f\|_{L_T^1(B_{p_1,1}^s)}), \\ \kappa\nu\|u\|_{L_T^1(B_{p,r}^{s+2})} &\leq \sum_{l \geq 0} 2^{ls}(1 - e^{-\kappa\nu 2^{2l}T})(\|\Delta_l u_0\|_{L^{p_1}} + \|\Delta_l f\|_{L_T^1(L^{p_1})}) \\ &\quad + T(\|u_0\|_{B_{p,r}^s} + \|f\|_{L_T^1(B_{p,r}^s)}), \end{aligned}$$

with $\nu = \min(\mu, \lambda + 2\mu)$.

We now consider the following parabolic system which is obtained by linearizing the momentum equation:

$$\begin{cases} \partial_t u - b(\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u) = f + g, \\ u|_{t=0} = u_0. \end{cases} \quad (3.9)$$

Above u is the unknown function. We assume that $u_0 \in B_{p_1,1}^s$, $f \in L^1(0, T; B_{p_1,1}^s)$, $g \in L^1(0, T; B_{p_2,1}^s)$, that b is bounded by below by a positive constant \underline{b} and b belongs to $L^\infty(0, T; B_{p,1}^{\frac{N}{p}})$ with $p \in [1, +\infty]$.

Proposition 3.8 *Let $g = 0$. Let $\underline{\nu} = \underline{b} \min(\mu, \lambda + 2\mu)$ and $\bar{\nu} = \mu + |\lambda + \mu|$. Assume that $s \in (-\frac{N}{p}, \inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$ and $s \in (-\frac{N}{p_1}, \inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \geq 1$. Let $m \in \mathbb{Z}$ be such that $b_m = 1 + S_m a$ satisfies:*

$$\inf_{(t,x) \in [0,T) \times \mathbb{R}^N} b_m(t, x) \geq \frac{\underline{\nu}}{2}. \quad (3.10)$$

There exist three constants c, C and κ (with c, C , depending only on N and on s , and κ universal) such that if in addition we have:

$$\|1 - S_m a\|_{L^\infty(0,T; B_{p_1,1}^{\frac{N}{p_1}})} \leq c \frac{\underline{\nu}}{\bar{\nu}} \quad (3.11)$$

then setting:

$$Z_m(t) = 2^{2m} \bar{\nu}^2 \underline{\nu}^{-1} \int_0^t \|a\|_{B_{p_1,1}^{\frac{N}{p_1}}}^2 d\tau,$$

we have for all $t \in [0, T]$,

$$\begin{aligned} \|u\|_{\tilde{L}^\infty((0,T) \times B_{p_1,1}^s)} + \kappa \underline{\nu} \|u\|_{\tilde{L}^1((0,T) \times B_{p_1,1}^{s+2})} &\leq e^{C(1+T)Z_m(T)}((1+T)\|u_0\|_{B_{p_1,1}^s} \\ &\quad + \int_0^t e^{-C(1+T)Z_m(\tau)} \|f(\tau)\|_{B_{p_1,1}^s} d\tau). \end{aligned}$$

Remark 13 *Let us stress the fact that if $a \in \tilde{L}^\infty((0, T) \times B_{p_1,1}^{\frac{N}{p_1}})$ then assumption (3.10) and (3.11) are satisfied for m large enough. This will be used in the proof of theorem 1.1. Indeed, according to Bernstein inequality, we have:*

$$\|a - S_m a\|_{L^\infty((0,T) \times \mathbb{R}^N)} \leq \sum_{q \geq m} \|\Delta_q a\|_{L^\infty((0,T) \times \mathbb{R}^N)} \lesssim \sum_{q \geq m} 2^{q \frac{N}{p}} \|\Delta_q a\|_{L^\infty(L^p)}.$$

Because $a \in \tilde{L}^\infty((0, T) \times B_{p,1}^{\frac{N}{p}})$, the right-hand side is the remainder of a convergent series hence tends to zero when m goes to infinity. For a similar reason, (3.11) is satisfied for m large enough.

Proof: In the sequel, we will treat only the case $p_1 \leq p$, the other case is similar. Let us first rewrite (3) as follows:

$$\partial_t u - b_m(\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u) = f + E_m, \quad (3.12)$$

with $E_m = (\mu \Delta u + (\lambda + \mu) \nabla \operatorname{div} u)(\operatorname{Id} - S_m)a$. Note that, because $s \in (-\frac{N}{p}, \inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$ and $s \in (-\frac{N}{p_1}, \inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \geq 1$, the error term E_m may be estimated by:

$$\|E_m\|_{B_{p_1,1}^s} \lesssim \|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|D^2 u\|_{B_{p_1,1}^s}, \quad (3.13)$$

Now applying Δ_q to equation (3.12) yields for $q \geq 0$:

$$\frac{d}{dt} u_q - \mu \operatorname{div}(b_m \nabla u_q) - (\lambda + \mu) \nabla(b_m \operatorname{div} u_q) = f_q + E_{m,q} + \tilde{R}_q. \quad (3.14)$$

where we denote by $u_q = \Delta_q u$ and with:

$$\tilde{R}_q = \mu(\Delta_q(b_m \Delta u) - \operatorname{div}(b_m \nabla u_q)) + (\lambda + \mu)(\Delta_q(b_m \nabla \operatorname{div} u) - \nabla(b_m \operatorname{div} u_q)).$$

Next multiplying both sides by $|u_q|^{p_1-2} u_q$, and integrating by parts in the second, third and last term in the left-hand side, we get:

$$\begin{aligned} \frac{1}{p_1} \frac{d}{dt} \|u_q\|_{L^{p_1}}^{p_1} - \frac{1}{p_1} \int (|u_q|^{p_1} \operatorname{div} v + \mu \operatorname{div}(b_m \nabla u_q) |u_q|^{p_1-2} u_q + \xi \nabla(b_m \operatorname{div} u_q) |u_q|^{p_1-2} u_q) dx \\ \leq \|u_q\|_{L^{p_1}}^{p_1-1} (\|f_q\|_{L^{p_1}} + \|\Delta_q E_m\|_{L^{p_1}} + \|\tilde{R}_q\|_{L^{p_1}}). \end{aligned}$$

Hence denoting $\xi = \mu + \lambda$, $\nu = \min(\mu, \lambda + 2\mu)$ and using (3.10), lemma [A5] of [10] and Young's inequalities we get:

$$\frac{1}{p_1} \frac{d}{dt} \|u_q\|_{L^{p_1}}^{p_1} + \frac{\nu \underline{b}(p_1 - 1)}{p_1^2} 2^{2q} \|u_q\|_{L^{p_1}}^{p_1} \leq \|u_q\|_{L^{p_1}}^{p_1-1} (\|f_q\|_{L^{p_1}} + \|E_{m,q}\|_{L^{p_1}} + \|\tilde{R}_q\|_{L^{p_1}}),$$

which leads, after time integration to:

$$\begin{aligned} \|u_q\|_{L^{p_1}} + \frac{\nu \underline{b}(p_1 - 1)}{p_1} 2^{2q} \int_0^t \|u_q\|_{L^{p_1}} d\tau \leq \|\Delta_q u_0\|_{L^{p_1}} + \int_0^t (\|f_q\|_{L^{p_1}} + \|E_{m,q}\|_{L^{p_1}} \\ + \|\tilde{R}_q\|_{L^{p_1}}) d\tau, \end{aligned} \quad (3.15)$$

where $\underline{\nu} = \underline{b}\nu$. For commutator \tilde{R}_q , we have the following estimate (see lemma 2 in the appendix)

$$\|\tilde{R}_q\|_{L^{p_1}} \lesssim c_q \bar{\nu} 2^{-qs} \|S_m a\|_{B_{p,1}^{\frac{N}{p}+1}} \|Du\|_{B_{p_1,1}^s}, \quad (3.16)$$

where $(c_q)_{q \in \mathbb{Z}}$ is a positive sequence such that $\sum_{q \in \mathbb{Z}} c_q = 1$, and $\bar{\nu} = \mu + |\lambda + \mu|$. Note that, owing to Bernstein inequality, we have:

$$\|S_m a\|_{B_{p,1}^{\frac{N}{p}+1}} \lesssim 2^m \|a\|_{B_{p_1,1}^{\frac{N}{p_1}}}$$

Hence, plugging these latter estimates and (3.13) in (3.15), then multiplying by 2^{qs} and summing up on $q \geq 0$, we discover that, for all $t \in [0, T]$:

$$\begin{aligned} \|u - \Delta_1 u\|_{L_t^\infty(B_{p_1,1}^s)} + \frac{\nu \underline{b}(p_1 - 1)}{p} \|u - \Delta_1 u\|_{L_t^1(B_{p_1,1}^{s+2})} &\leq \|u_0\|_{B_{p_1,1}^s} + \|f\|_{L_t^1(B_{p_1,1}^s)} \\ &+ C\bar{\nu} \int_0^t (\|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+2}} + 2^m \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+1}}) d\tau, \end{aligned} \quad (3.17)$$

We need now to control the block $\Delta_1 u$ corresponding to the low frequencies. To treat the term $\Delta_1 u$ similarly we apply the operator Δ_{-1} to the equation and by energy inequalities, we get:

$$\|\Delta_{-1} u(t)\|_{L^{p_1}} \leq \|\Delta_{-1} u_0\|_{L^{p_1}} + \|f\|_{L^1(L^{p_1})} + \int_0^t (\|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+2}} + 2^m \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+1}}) ds.$$

and:

$$\|\Delta_{-1} u(t)\|_{L_t^1(L^{p_1})} \leq t(\|\Delta_{-1} u_0\|_{L^{p_1}} + \|f\|_{L^1(L^{p_1})}) + \int_0^t (\|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+2}} + 2^m \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+1}}) ds.$$

So we have by the two previous inequalities and (3.17):

$$\begin{aligned} \|u\|_{L_t^\infty(B_{p_1,1}^s)} + \frac{\nu \underline{b}(p_1 - 1)}{p} \|u\|_{L_t^1(B_{p_1,1}^{s+2})} &\leq C(1+t)(\|u_0\|_{B_{p_1,1}^s} + \|f\|_{L_t^1(B_{p_1,1}^s)}) \\ &+ \bar{\nu} \int_0^t (\|a - S_m a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+2}} + 2^m \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+1}}) d\tau, \end{aligned}$$

for a constant C depending only on N and s . Let $X(t) = \|u\|_{L_t^\infty(B_{p_1,1}^s)} + \nu \underline{b} \|u\|_{L_t^1(B_{p_1,1}^{s+2})}$. Assuming that m has been chosen so large as to satisfy:

$$C\bar{\nu} \|a - S_m a\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq \underline{\nu},$$

and using that by interpolation, we have:

$$C\bar{\nu} \|a\|_{B_{p,1}^{\frac{N}{p}}} \|u\|_{B_{p_1,1}^{s+2}} \leq \kappa \underline{\nu} + \frac{C^2 \bar{\nu}^2 2^{2m}}{4\kappa \underline{\nu}} \|a\|_{B_{p,1}^{\frac{N}{p}}}^2 \|u\|_{B_{p_1,1}^s},$$

we end up with:

$$X(t) \leq C(1+t)(\|u_0\|_{B_{p_1,1}^s} + \|f\|_{L_t^1(B_{p_1,1}^s)}) + C \frac{\bar{\nu}^2}{\underline{\nu}} \int_0^t 2^{2m} \|a\|_{B_{p,1}^{\frac{N}{2}}}^2 X(\tau) d\tau.$$

Grönwall lemma then leads to the desired inequality \square

In the following corollary, we generalize proposition 3.9 when $g \neq 0$ and $g \in \tilde{L}^1(B_{p_2,1}^{'s})$. Moreover here $u_0 = u_1 + u_2$ with $u_1 \in B_{p_1,1}^s$ and $u_2 \in B_{p_2,1}^{'s}$.

Corollary 3 Let $\underline{\nu} = \underline{b} \min(\mu, \lambda + 2\mu)$ and $\bar{\nu} = \mu + |\lambda + \mu|$. Assume that $s \in (-\frac{N}{p}, \inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \leq 1$ and $s \in (-\frac{N}{p_1}, \inf(\frac{1}{p}, \frac{1}{p_1})]$ if $\frac{1}{p} + \frac{1}{p_1} \geq 1$. Moreover we assume that: $s \in (-\frac{N}{p}, \inf(\frac{1}{p}, \frac{1}{p_2})]$ if $\frac{1}{p} + \frac{1}{p_2} \leq 1$ and $s \in (-\frac{N}{p_2}, \inf(\frac{1}{p}, \frac{1}{p_2})]$ if $\frac{1}{p} + \frac{1}{p_2} \geq 1$. Let $m \in \mathbb{Z}$ be such that $b_m = 1 + S_m a$ satisfies:

$$\inf_{(t,x) \in [0,T) \times \mathbb{R}^N} b_m(t,x) \geq \frac{\underline{b}}{2}. \quad (3.18)$$

There exist three constants c, C and κ (with c, C , depending only on N and on s , and κ universal) such that if in addition we have:

$$\|1 - S_m a\|_{L^\infty(0,T;B_{p,1}^{\frac{N}{p}})} \leq c \frac{\underline{\nu}}{\bar{\nu}} \quad (3.19)$$

then setting:

$$Z_m(t) = 2^{2m} \bar{\nu}^2 \underline{\nu}^{-1} \int_0^t \|a\|_{B_{p,1}^{\frac{N}{p}}}^2 d\tau,$$

We have for all $t \in [0, T]$,

$$\begin{aligned} \|u\|_{\tilde{L}_T^\infty(B_{p_1,1}^s + B_{p_2,1}^{s'})} + \kappa \underline{\nu} \|u\|_{\tilde{L}_T^1(B_{p_1,1}^{s+2} + B_{p_2,1}^{s'+2})} &\leq e^{C(1+T)Z_m(t)} ((1+T)(\|u_1\|_{B_{p_1,1}^s} + \\ &\|u_2\|_{B_{p_2,1}^{s'}}) + \int_0^t e^{-C(1+T)Z_m(\tau)} (\|f(\tau)\|_{B_{p_1,1}^s} + \|g(\tau)\|_{B_{p_2,1}^{s'}}) d\tau). \end{aligned}$$

Proof: We split the solution u in two parts u_1 and u_2 which verify the following equations:

$$\begin{cases} \partial_t u_1 + v \cdot \nabla u_1 + u_1 \cdot \nabla w - b(\mu \Delta u_1 + (\lambda + \mu) \nabla \operatorname{div} u_1) = f, \\ u|_{t=0} = u_1^0, \end{cases}$$

and:

$$\begin{cases} \partial_t u_2 + v \cdot \nabla u_2 + u_2 \cdot \nabla w - b(\mu \Delta u_2 + (\lambda + \mu) \nabla \operatorname{div} u_2) = g, \\ u|_{t=0} = u_2^0. \end{cases}$$

We have then $u = u_1 + u_2$ and we conclude by applying proposition 3.8. \square

Proposition 3.8 fails in the limit case $s = -\frac{N}{p}$. The reason why is that proposition 2.3 cannot be applied any longer. One can however state the following result which will be the key to the proof of uniqueness in dimension two.

Proposition 3.9 Under condition (3.10), there exists three constants c, C and κ (with c, C , depending only on N , and κ universal) such that if:

$$\|a - S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \leq c \frac{\underline{\nu}}{\bar{\nu}}, \quad (3.20)$$

then we have:

$$\|u\|_{L_t^\infty(B_{p_1,\infty}^{-\frac{N}{p_1}})} + \kappa \underline{\nu} \|u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})} \leq C(1+t) (\|u_0\|_{B_{p_1,\infty}^{-\frac{N}{p_1}}} + \|f\|_{\tilde{L}_t^1(B_{p_1,\infty}^{\frac{N}{p_1}})}),$$

whenever $t \in [0, T]$ satisfies:

$$\bar{\nu}^2 t (1+t) \|a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})}^2 \leq c 2^{-2m} \underline{\nu}. \quad (3.21)$$

Proof: We just point out the changes that have to be done compare to the proof of proposition 3.8. The first one is that instead of (3.13), we have in accordance with proposition 2.3:

$$\|E_m\|_{\tilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})} \lesssim \|a - S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \|D^2 u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})}, \quad (3.22)$$

The second change concerns the estimate of commutator \tilde{R}_q . According remark 15, we now have for all $q \in \mathbb{Z}$:

$$\|\tilde{R}_q\| \lesssim \bar{\nu} 2^{q\frac{N}{p_1}} \|S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}+1})} \|Du\|_{\tilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})}. \quad (3.23)$$

Plugging all these estimates in (3.15) then taking the supremum over $q \in \mathbb{Z}$, we get:

$$\begin{aligned} \|u\|_{L_t^\infty(B_{p_1,\infty}^{-\frac{N}{p_1}})} + 2\underline{\nu} \|u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})} &\leq (1+t) \left(\|u_0\|_{B_{p_1,1}^{-\frac{N}{p_1}}} + C\bar{\nu} \left(\|a - S_m a\|_{\tilde{L}_t^\infty(B_{p,1}^{\frac{N}{p}})} \|u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})} \right. \right. \\ &\quad \left. \left. + 2^m \|a\|_{L_t^\infty(B_{p,1}^{\frac{N}{p}})} \|u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{1-\frac{N}{p_1}})} + \|f\|_{\tilde{L}_t^1(B_{p_1,\infty}^{-\frac{N}{p_1}})} \right) \right). \end{aligned}$$

Using that:

$$\|u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{1-\frac{N}{p_1}})} \leq \sqrt{t} \|u\|_{\tilde{L}_t^1(B_{p_1,\infty}^{2-\frac{N}{p_1}})}^{\frac{1}{2}} \|u\|_{L_t^\infty(B_{p_1,\infty}^{\frac{N}{p_1}})}^{\frac{1}{2}},$$

and taking advantage of assumption (3.20) and (3.21), it is now easy to complete the proof. \square

4 The mass conservation equation

Let us first recall standard estimates in Besov spaces for the following linear transport equation:

$$(\mathcal{H}) \quad \begin{cases} \partial_t a + u \cdot \nabla a = g, \\ a|_{t=0} = a_0. \end{cases}$$

Proposition 4.10 *Let $1 \leq p_1 \leq p \leq +\infty$, $r \in [1, +\infty]$ and $s \in \mathbb{R}$ be such that:*

$$-N \min\left(\frac{1}{p_1}, \frac{1}{p}\right) < s < 1 + \frac{N}{p_1}.$$

There exists a constant C depending only on N , p , p_1 , r and s such that for all $a \in L^\infty([0, T], B_{p,r}^\sigma)$ of (\mathcal{H}) with initial data a_0 in $B_{p,r}^s$ and $g \in L^1([0, T], B_{p,r}^s)$, we have for a.e $t \in [0, T]$:

$$\|f\|_{\tilde{L}_t^\infty(B_{p,r}^s)} \leq e^{CU(t)} \left(\|f_0\|_{B_{p,r}^s} + \int_0^t e^{-CV(\tau)} \|F(\tau)\|_{B_{p_1,r}^s} d\tau \right), \quad (4.24)$$

with: $U(t) = \int_0^t \|\nabla u(\tau)\|_{B_{p_1,\infty}^{\frac{N}{p_1}} \cap L^\infty} d\tau$.

For the proof of proposition 4.10, see [3]. We now focus on the mass equation associated to (1.3):

$$\begin{cases} \partial_t a + v \cdot \nabla a = (1 + a) \operatorname{div} v, \\ a|_{t=0} = a_0. \end{cases} \quad (4.25)$$

Here we generalize a proof of R. Danchin in [13].

Proposition 4.11 *Let $r \in 1, +\infty$, $1 \leq p_1 \leq p \leq +\infty$ and $s \in (-\min(\frac{N}{p_1}, \frac{N}{p}, \frac{N}{p})]$. Assume that $a_0 \in B_{p,r}^s \cap L^\infty$, $v \in L^1(0, T; B_{p_1,1}^{\frac{N}{p_1}+1})$ and that $a \in \tilde{L}_T^\infty(B_{p,r}^s) \cap L_T^\infty$ satisfies (4.25). Let $V(t) = \int_0^t \|\nabla v(\tau)\|_{B_{p_1,1}^{\frac{N}{p_1}}} \frac{N}{p} d\tau$. There exists a constant C depending only on N such that for all $t \in [0, T]$ and $m \in \mathbb{Z}$, we have:*

$$\|a\|_{\tilde{L}_t^\infty(B_{p,r}^s \cap L^\infty)} \leq e^{2CV(t)} \|a_0\|_{B_{p,r}^s \cap L^\infty} + e^{2CV(t)} - 1, \quad (4.26)$$

$$\|a - S_m a\|_{B_{p,r}^s} \leq \|a_0 - S_m a_0\|_{B_{p,r}^s} + \frac{1}{2}(1 + \|a_0\|_{B_{p,r}^s \cap L^\infty})(e^{2CV(t)} - 1) + C\|a\|_{L^\infty} V(t), \quad (4.27)$$

$$\begin{aligned} \left(\sum_{l \leq m} 2^{lrs} \|\Delta_l(a - a_0)\|_{L_t^r(L^p)}^r\right)^{\frac{1}{r}} &\leq (1 + \|a_0\|_{B_{p,r}^s})(e^{CV(t)} - 1) \\ &\quad + C2^m \|a_0\|_{B_{p,r}^s} \int_0^t \|v\|_{B_{p_1,1}^{\frac{N}{p_1}}} \frac{N}{p} d\tau. \end{aligned} \quad (4.28)$$

Proof: Applying Δ_l to (4.25) yields:

$$\partial_t \Delta_l a + v \cdot \nabla \Delta_l a = R_l + \Delta_l((1 + a) \operatorname{div} v) \quad \text{with } R_l = [v \cdot \nabla, \Delta_l]a.$$

Multiplying by $\Delta_l a |\Delta_l a|^{p-2}$ then performing a time integration, we easily get:

$$\|\Delta_l a(t)\|_{L^p} \lesssim \|\Delta_l a_0\|_{L^p} + \int_0^t (\|R_l\|_{L^p} + \|\operatorname{div} v\|_{L^\infty} \|\Delta_l a\|_{L^p} + \|\Delta_l((1 + a) \operatorname{div} v)\|_{L^p}) d\tau.$$

According to proposition 2.3 and interpolation, there exists a constant C and a positive sequence $(c_l)_{l \in \mathbb{N}}$ in l^r with norm 1 such that:

$$\|\Delta_l((1 + a) \operatorname{div} v)\|_{L^p} \leq C c_l 2^{-ls} (1 + \|a\|_{B_{p,r}^s \cap L^\infty}) \|\operatorname{div} v\|_{B_{p_1,1}^{\frac{N}{p_1}}}.$$

Next the term $\|R_l\|_{L^p}$ may be bounded according to lemma 1 in appendix. We end up with:

$$\forall t \in [0, T], \forall l \in \mathbb{Z}, \quad 2^{ls} \|\Delta_l a(t)\|_{L^p} \leq 2^{ls} \|\Delta_l a_0\|_{L^p} + C \int_0^t c_l (1 + \|a\|_{B_{p,r}^s \cap L^\infty}) V' d\tau, \quad (4.29)$$

hence, summing up on \mathbb{Z} in l^r ,

$$\forall t \in [0, T], \forall l \in \mathbb{Z}, \quad \|a(t)\|_{B_{p,r}^s} \leq \|a_0\|_{B_{p,r}^s} + \int_0^t C V' \|a(\tau)\|_{B_{p,r}^s} d\tau + \int_0^t C (1 + \|a\|_{L_T^\infty}) V' d\tau.$$

Next we have:

$$\|a\|_{L_t^\infty} \leq \int_0^t (1 + \|a(\tau)\|_{L^\infty}) V'(\tau) d\tau.$$

By summing the two previous inequalities, applying Gronwall lemma and proposition 2.2 yields inequality (4.26). Let us now prove inequality (4.27). Starting from (4.29) and summing up over $l \geq m$ in l^r , we get:

$$\begin{aligned} \left(\sum_{l \geq m} 2^{lsr} \|\Delta_l a\|_{L_t^\infty(L^p)}^r \right)^{\frac{1}{r}} &\leq \left(\sum_{l \geq m} 2^{lsr} \|\Delta_l a_0\|_{L^p}^r \right)^{\frac{1}{r}} + C \int_0^t V'(e^{2CV} \|a_0\|_{B_{p,r}^s \cap L^\infty} + e^{2CV} - 1) d\tau \\ &\quad + \int_0^t C(1 + \|a\|_{L^\infty}) V' d\tau. \end{aligned}$$

Straightforward calculations then leads to (4.27). In order to prove (4.28), we use the fact that $\tilde{a} = a - a_0$ satisfies:

$$\partial_t \tilde{a} + v \cdot \nabla \tilde{a} = (1 + \tilde{a}) \operatorname{div} v + a_0 \operatorname{div} v - v \cdot \nabla a_0, \quad \tilde{a}|_{t=0} = 0.$$

Therefore, arguing as for proving (4.29), we get for all $t \in [0, T]$ and $l \in \mathbb{Z}$,

$$\begin{aligned} 2^{l \frac{N}{p}} \|\Delta_l \tilde{a}\|_{L^p} &\leq \int_0^t 2^{l \frac{N}{p}} (\|\Delta_l(a_0 \operatorname{div} v)\|_{L^p} + \|\Delta_l(v \cdot \nabla a_0)\|_{L^p}) d\tau \\ &\quad + C \int_0^t c_l (1 + \|a\|_{B_{p,1}^{\frac{N}{p}}}) V' d\tau. \end{aligned}$$

Since $B_{p,1}^{\frac{N}{p}}$ is an algebra and the product maps $B_{p,1}^{\frac{N}{p}} \times B_{p,1}^{\frac{N}{p}-1}$ in $B_{p,1}^{\frac{N}{p}-1}$, we discover that:

$$2^{l \frac{N}{p}} \|\Delta_l \tilde{a}\|_{L^\infty(L^p)} \leq C \left(\int_0^t 2^l c_l \|a_0\|_{B_{p,1}^{\frac{N}{p}}} \|v\|_{B_{p,1}^{\frac{N}{p}}} d\tau + \int_0^t c_l (1 + \|a_0\|_{B_{p,1}^{\frac{N}{p}}} + \|a\|_{B_{p,1}^{\frac{N}{p}}}) V' d\tau \right),$$

hence, summing up on $l \leq m$,

$$\sum_{l \leq m} 2^{l \frac{N}{p}} \|\Delta_l \tilde{a}\|_{L^\infty(L^p)} \leq C \left(\int_0^t 2^m \|a_0\|_{B_{p,1}^{\frac{N}{p}}} \|v\|_{B_{p,1}^{\frac{N}{p}}} d\tau + \int_0^t (1 + \|a_0\|_{B_{p,1}^{\frac{N}{p}}} + \|a\|_{B_{p,1}^{\frac{N}{p}}}) V' d\tau \right),$$

Plugging (4.26) in the right-hand side yields (4.28).

5 The proof of theorem 1.1

5.1 Strategy of the proof

To improve the results of R. Danchin in [10], [13], it is crucial to kill the coupling between the velocity and the pressure which intervene in the works of R. Danchin. In this goal, we need to integrate the pressure term in the study of the linearized equation of the momentum equation. For making, we will try to express the gradient of the pressure as a Laplacian term, so we set for $\bar{\rho} > 0$ a constant state:

$$\operatorname{div} v = P(\rho) - P(\bar{\rho}).$$

Let \mathcal{E} the fundamental solution of the Laplace operator.

We will set in the sequel: $v = \nabla \mathcal{E} * (P(\rho) - P(\bar{\rho})) = \nabla (\mathcal{E} * [P(\rho) - P(\bar{\rho})])$ ($*$ here means the operator of convolution). We verify next that:

$$\nabla \operatorname{div} v = \nabla \Delta (\mathcal{E} * [P(\rho) - P(\bar{\rho})]) = \Delta \nabla (\mathcal{E} * [P(\rho) - P(\bar{\rho})]) = \Delta v = \nabla P(\rho).$$

By this way we can now rewrite the momentum equation of (1.3). We obtain the following equation where we have set $\nu = 2\mu + \lambda$:

$$\partial_t u + u \cdot \nabla u - \frac{\mu}{\rho} \Delta \left(u - \frac{1}{\nu} v\right) - \frac{\lambda + \mu}{\rho} \nabla \operatorname{div} \left(u - \frac{1}{\nu} v\right) = f.$$

We want now calculate $\partial_t v$, by the transport equation we get:

$$\partial_t v = \nabla \mathcal{E} * \partial_t P(\rho) = -\nabla \mathcal{E} * (P'(\rho) \operatorname{div}(\rho u)).$$

We have finally:

$$\Delta(\partial_t v) = -P'(\rho) \operatorname{div}(\rho u).$$

Notation 1 *To simplify the notation, we will note in the sequel*

$$\nabla \mathcal{E} * (P'(\rho) \operatorname{div}(\rho u)) = \nabla (\Delta)^{-1} (P'(\rho) \operatorname{div}(\rho u)).$$

Finally we can now rewrite the system (1.3) as follows:

$$\begin{cases} \partial_t a + (v_1 + \frac{1}{\nu} v) \cdot \nabla a = (1 + a) \operatorname{div}(v_1 + \frac{1}{\nu} v), \\ \partial_t v_1 - (1 + a) \mathcal{A} v_1 = f - u \cdot \nabla u + \frac{1}{\nu} \nabla (\Delta)^{-1} (P'(\rho) \operatorname{div}(\rho u)), \\ a|_{t=0} = a_0, (v_1)|_{t=0} = (v_1)_0. \end{cases} \quad (5.30)$$

where $v_1 = u - \frac{1}{\nu} v$. In the sequel we will study this system by extracting some uniform bounds in Besov spaces on (a, v_1) as the in the following works [1], [17]. The advantage of the system (5.30) is that we have *kill* the coupling between v_1 and a term of pressure. Indeed in the works of R. Danchin [10], [13], the pressure was considered as a term of rest in the momentum equation, so it implied a strong relationship between the density and the velocity. In particular it was impossible to distinguish the index of integration for the Besov spaces.

5.2 Proof of the existence

Construction of approximate solutions

We use a standard scheme:

1. We smooth out the data and get a sequence of smooth solutions $(a^n, u^n)_{n \in \mathbb{N}}$ to (1.3) on a bounded interval $[0, T^n]$ which may depend on n . We set $v_1^n = u^n - v^n$ where $\operatorname{div} v^n = P(\rho^n) - P(\bar{\rho})$ with $v^n = \nabla \mathcal{E} * P(\rho^n) - P(\bar{\rho})$.

2. We exhibit a positive lower bound T for T^n , and prove uniform estimates on (a^n, u^n) in the space

$$E_T = \tilde{C}_T(B_{p,1}^{\frac{N}{p}}) \times (\tilde{C}_T(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap \tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})).$$

More precisely to get this bounds we will need to study the behavior of (a^n, v_1^n) .

3. We use compactness to prove that the sequence (a^n, u^n) converges, up to extraction, to a solution of (5.30).

Throughtout the proof, we denote $\underline{\nu} = \underline{b} \min(\mu, \lambda + 2\mu)$ and $\bar{\nu} = \mu + |\mu + \lambda|$, and we assume (with no loss of generality) that f belongs to $\tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}})$.

First step

We smooth out the data as follows:

$$a_0^n = S_n a_0, \quad u_0^n = S_n u_0 \quad \text{and} \quad f^n = S_n f.$$

Note that we have:

$$\forall l \in \mathbb{Z}, \quad \|\Delta_l a_0^n\|_{L^p} \leq \|\Delta_l a_0\|_{L^p} \quad \text{and} \quad \|a_0^n\|_{B_{p,\infty}^{\frac{N}{p}}} \leq \|a_0\|_{B_{p,\infty}^{\frac{N}{p}}},$$

and similar properties for u_0^n and f^n , a fact which will be used repeatedly during the next steps. Now, according [13], one can solve (1.3) with the smooth data (a_0^n, u_0^n, f^n) . We get a solution (a^n, u^n) on a non trivial time interval $[0, T_n]$ such that:

$$a^n \in \tilde{C}([0, T_n], B_{2,1}^N) \quad \text{and} \quad u^n \in \tilde{C}([0, T_n], B_{2,1}^{\frac{N}{2}-1}) \cap \tilde{L}_{T_n}^1(B_{2,1}^{\frac{N}{2}+1}). \quad (5.31)$$

Uniform bounds

Let T_n be the lifespan of (a_n, u_n) , that is the supremum of all $T > 0$ such that (1.1) with initial data (a_0^n, u_0^n) has a solution which satisfies (5.31). Let T be in $(0, T_n)$. We aim at getting uniform estimates in E_T for T small enough. For that, we need to introduce the solution u_L^n to the linear system:

$$\partial_t u_L^n - \mathcal{A} u_L^n = f^n, \quad u_L^n(0) = u_0^n - \frac{1}{\nu} \tilde{v}_0.$$

Now, we set $\tilde{u}^n = u^n - u_L^n$ and the vectorfield $\tilde{v}_1^n = \tilde{u}^n - \frac{1}{\nu} \tilde{v}^n$ with $\text{div} \tilde{v}^n = P(\rho^n) - P(\bar{\rho})$. We can check that \tilde{v}_1^n satisfies the parabolic system:

$$\begin{cases} \partial_t \tilde{v}_1^n - (1 + a^n) \mathcal{A} \tilde{v}_1^n = -(u_L^n + \frac{1}{\nu} \tilde{v}^n) \cdot \nabla \tilde{v}_1^n - \tilde{v}_1^n \cdot \nabla u^n + a^n \mathcal{A} u_L^n - \frac{1}{\nu} (u_L^n \cdot \nabla \tilde{v}^n \\ \quad + \frac{1}{\nu} \tilde{v}^n \cdot \nabla \tilde{v}^n) - u_L^n \cdot \nabla u_L^n + \frac{1}{\nu} \nabla(\Delta)^{-1} (P'(\rho^n) \text{div}(\rho^n u^n)), \\ (\tilde{v}_1^n)_{t=0} = 0. \end{cases} \quad (5.32)$$

which has been studied in proposition 3.8. Define $m \in \mathbb{Z}$ by:

$$m = \inf \{ p \in \mathbb{N} / 2\bar{\nu} \sum_{l \geq p} 2^{l \frac{N}{p}} \|\Delta_l a_0\|_{L^p} \leq c\bar{\nu} \} \quad (5.33)$$

where c is small enough positive constant (depending only N) to be fixed hereafter. In the sequel we will need of a control on $a - S_m a$ small to apply proposition 3.8, so here m is enough big (we explain how in the sequel). Let:

$$\bar{b} = 1 + \sup_{x \in \mathbb{R}^N} a_0(x), \quad A_0 = 1 + 2\|a_0\|_{B_{p,1}^{\frac{N}{p}}}, \quad U_0 = \|u_0\|_{B_{p_1,1}^{\frac{N}{p_1}-1}} + \|a_0\|_{B_{p,1}^{\frac{N}{p}}} + \|f\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}-1})},$$

and $\tilde{U}_0 = 2CU_0 + 4C\bar{\nu}A_0$ (where C' is a constant embedding and C stands for a large enough constant depending only N which will be determined when applying proposition 2.3, 3.8 and 4.10 in the following computations.) We assume that the following inequalities are fulfilled for some $\eta > 0$:

$$\begin{aligned} (\mathcal{H}_1) \quad & \|a^n - S_m a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq c\nu\bar{\nu}^{-1}, \\ (\mathcal{H}_2) \quad & C\bar{\nu}^2 T \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}^2 \leq 2^{-2m}\underline{\nu}, \\ (\mathcal{H}_3) \quad & \frac{1}{2}\underline{b} \leq 1 + a^n(t, x) \leq 2\bar{b} \text{ for all } (t, x) \in [0, T] \times \mathbb{R}^N, \\ (\mathcal{H}_4) \quad & \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq A_0, \\ (\mathcal{H}_5) \quad & \|u_L^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1})} \leq U_0, \quad \|u_L^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3})} \leq \eta, \\ (\mathcal{H}_6) \quad & \|\tilde{v}_1^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}})} + \underline{\nu} \|\tilde{v}_1^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})} \leq \tilde{U}_0 \eta, \\ (\mathcal{H}_7) \quad & \|\tilde{v}^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}+1})} \leq C' A_0, \\ (\mathcal{H}_8) \quad & \|\nabla u^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})} \leq (\underline{\nu}^{-1} \tilde{U}_0 + 1) \eta \end{aligned}$$

Remark that since:

$$1 + S_m a^n = 1 + a^n + (S_m a^n - a^n),$$

assumptions (\mathcal{H}_1) and (\mathcal{H}_3) combined with the embedding $B_{p,1}^{\frac{N}{p}} \hookrightarrow L^\infty$ insure that:

$$\inf_{(t,x) \in [0,T] \times \mathbb{R}^N} (1 + S_m a^n)(t, x) \geq \frac{1}{4}\underline{b}, \quad (5.34)$$

provided c has been chosen small enough (note that $\frac{\underline{\nu}}{\bar{\nu}} \leq \bar{b}$).

We are going to prove that under suitable assumptions on T and η (to be specified below) if condition (\mathcal{H}_1) to (\mathcal{H}_7) are satisfied, then they are actually satisfied with strict inequalities. Since all those conditions depend continuously on the time variable and are strictly satisfied initially, a basic bootstrap argument insures that (\mathcal{H}_1) to (\mathcal{H}_8) are indeed satisfied for T enough small (with a T which could depend of n). In the sequel, we

will see that these conditions on T do not depend of n and by a criterion of continuation we will see that our T check $T \leq T_n$.

First we shall assume that η and T satisfies:

$$C(1 + \nu^{-1}\tilde{U}_0)\eta + \frac{C'}{\nu}A_0T < \log 2 \quad (5.35)$$

so that denoting $\tilde{V}_1^n(t) = \int_0^t \|\nabla \tilde{v}_1^n\|_{B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1}} d\tau$, $\tilde{V}^n(t) = \frac{1}{\nu} \int_0^t \|\nabla \tilde{v}^n\|_{B_{p,1}^{\frac{N}{p}}} d\tau$ and $U_L^n(t) = \int_0^t \|\nabla u_L^n\|_{B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3}} d\tau$, we have, according to (\mathcal{H}_5) and (\mathcal{H}_6) :

$$e^{C(U_L^n + \tilde{V}_1^n + \tilde{V}^n)(T)} < 2 \quad \text{and} \quad e^{C(U_L^n + \tilde{V}_1^n + \tilde{V}^n)(T)} - 1 \leq 1. \quad (5.36)$$

In order to bound a^n in $\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})$, we apply inequality (4.26) and get:

$$\|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} < 1 + 2\|a_0\|_{B_{p,1}^{\frac{N}{p}}} = A_0. \quad (5.37)$$

Hence (\mathcal{H}_4) is satisfied with a strict inequality. (\mathcal{H}_7) verifies a strict inequality, it follows from proposition 2.6 and (\mathcal{H}_4) . Next, applying proposition 3.7 and proposition 2.6 yields:

$$\|u_L^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1})} \leq U_0, \quad (5.38)$$

$$\begin{aligned} \kappa\nu\|u_L^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3})} &\leq \sum_{l \geq 0} 2^{l(\frac{N}{p_1}-1)} (1 - e^{-\kappa\nu 2^{2l}T}) (\|\Delta_l u_0\|_{L^{p_1}} + \|\Delta_l a_0\|_{L^p} \\ &\quad + \|\Delta_l f\|_{L^1(\mathbb{R}^+, L^{p_1})}) + TU_0. \end{aligned} \quad (5.39)$$

Hence taking T such that:

$$\begin{aligned} \kappa\nu\|u_L^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3})} &\leq \sum_{l \geq 0} 2^{l(\frac{N}{p_1}-1)} (1 - e^{-\kappa\nu 2^{2l}T}) (\|\Delta_l u_0\|_{L^{p_1}} + \|\Delta_l a_0\|_{L^p} \\ &\quad + \|\Delta_l f\|_{L^1(\mathbb{R}^+, L^{p_1})}) + TU_0. < \kappa\eta\nu, \end{aligned} \quad (5.40)$$

insures that (\mathcal{H}_5) is strictly verified. Since (\mathcal{H}_1) , (\mathcal{H}_2) , (\mathcal{H}_5) , (\mathcal{H}_6) , (\mathcal{H}_7) and (5.34) are satisfied, proposition 3.8 may be applied, we obtain:

$$\begin{aligned} &\|\tilde{v}_1^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}})} + \nu\|\tilde{v}_1^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})} \\ &\leq Ce^{C(1+T)} \int_0^T (\|a^n \mathcal{A} u_L^n\|_{B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}}} + \|u_L^n \cdot \nabla u_L^n\|_{B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}}}) \\ &\quad + \|u_L^n \cdot \nabla \tilde{v}^n\|_{B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}}} + \|\tilde{v}^n \cdot \nabla \tilde{v}^n\|_{B_{p,1}^{\frac{N}{p}}} + \|\nabla(\Delta)^{-1}(P'(\rho^n) \operatorname{div}(\rho^n u^n))\|_{B_{p,1}^{\frac{N}{p}}}) \\ &\quad + \|(u_L^n + \frac{1}{\nu} \tilde{v}^n) \cdot \nabla \tilde{v}_1^n\|_{B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}}} + \|\tilde{v}_1^n \cdot \nabla u^n\|_{B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}}}) dt. \end{aligned}$$

As $\frac{N}{p} + \frac{N}{p_1} - 1 \geq 0$ and $P'(\rho^n) \operatorname{div}(\rho^n u^n) = \nabla P(\rho^n) \cdot u^n + P'(\rho^n) \rho^n \operatorname{div} u^n$, we can take advantage of proposition 2.3, 2.1 2.6. (In passing, we want mention that here a crucial

point is that $\Delta \tilde{v}_1^n$ belongs to $\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}} + B_{p_1,1}^{\frac{N}{p_1}-1})$, it means that we are able to give sense to the product $a^n \mathcal{A} \tilde{v}_1^n$ with the condition $\frac{N}{p} + \frac{N}{p_1} - 1 \geq 0$. It is the main novelty compared with the works of R. Danchin in [10, 13], indeed we are able to "kill" in some sense the coupling between the pressure term and the velocity. It means that by this way the constraints are less concerning the law of paraproduct for the term $a \Delta u$. In other words, we are able to ask no more that the hypothesis on p and p_1 used in the case of Navier-Stokes with dependent density (see [1] and [17]).

We get then with h and h_1 regular function checking the conditions of proposition 2.4:

$$\begin{aligned} & \|\nabla(\Delta)^{-1}(P'(\rho^n) \operatorname{div}(\rho^n u^n))\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})} \\ & \leq C(\|\nabla(\Delta)^{-1}(h(a^n) \operatorname{div}(h_1(a^n) u^n))\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})} + \|\nabla(\Delta)^{-1}(\operatorname{div}(u^n))\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})}) \\ & \leq C_P(\|u^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})} (1 + \|a^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})}^2)) \\ & \leq C_P(\sqrt{T}(\|u_L^n\|_{\tilde{L}_T^2(B_{p,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+2})} + \|\tilde{v}_1^n\|_{\tilde{L}_T^2(B_{p,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} + T\|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})})(1 + \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}^2)) \end{aligned}$$

The next term $\tilde{v}^n \cdot \nabla \tilde{v}^n$ determines the choice of working in the space $B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1}$, indeed we recall here that at the difference of the works in [10, 13], we have no coupling between the density and the velocity. So in this sense this term is crucial in the sense of the regularity of \tilde{v}_1^n .

$$\|\tilde{v}^n \cdot \nabla \tilde{v}^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})} \leq C_1 T \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}^2.$$

We proceed similarly for the other terms and we end up with:

$$\begin{aligned} & \|\tilde{v}_1^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}})} + \nu \|\tilde{v}_1^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})} \leq C e^{C(1+T)} (C_1 T \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}^2 \\ & + C_P(\sqrt{T}(\|u_L^n\|_{\tilde{L}_T^2(B_{p,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+2})} + \|\tilde{v}_1^n\|_{\tilde{L}_T^2(B_{p,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} + T\|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})})(1 + \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}^2) \\ & + \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \|u_L^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3})} + \|u_L^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1})} \|u_L^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3})}) \\ & + \sqrt{T} \|u_L^n\|_{\tilde{L}_T^2(B_{p,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+2})} \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} + T \|\tilde{v}_1^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}})} \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \\ & + (\|u_L^n\|_{\tilde{L}_T^2(B_{p,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+2})} + \sqrt{T} \|a^n\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}) \|\tilde{v}_1^n\|_{\tilde{L}_T^2(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} \\ & + \|\tilde{v}_1^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}})} (\|\tilde{v}_1^n\|_{\tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})} + \|u_L^n\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3})})) \end{aligned} \quad (5.41)$$

with $C = C(N)$ and $C_P = (N, P, \underline{b}, \bar{b})$. Now, using assumptions (\mathcal{H}_4) , (\mathcal{H}_5) and (\mathcal{H}_6) , and inserting (5.36) in (5.41) gives:

$$\|\tilde{v}_1^n\|_{\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1})} + \|\tilde{v}_1^n\|_{L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1})} \leq 2C(\bar{\nu} A_0 + \tilde{U}_0^2 \eta + U_0) \eta + C_1 T A_0 (1 + A_0) + \sqrt{T} A_0 U_0,$$

hence (\mathcal{H}_6) is satisfied with a strict inequality provided when T and η verifies:

$$2C(\bar{\nu} A_0 + U_0 + \tilde{U}_0^2 \eta) \eta + C_1 T A_0 (1 + A_0) + \sqrt{T} A_0 U_0 < C \bar{\nu} \eta. \quad (5.42)$$

(\mathcal{H}_8) verifies a strict inequality, it follows from proposition (\mathcal{H}_5) , (\mathcal{H}_6) and (\mathcal{H}_7) . We now have to check whether (\mathcal{H}_1) is satisfied with strict inequality. For that we apply proposition (4.11) which yields for all $m \in \mathbb{Z}$,

$$\sum_{l \geq m} 2^{l \frac{N}{2}} \|\Delta_l a^n\|_{L_T^\infty(L^p)} \leq \sum_{l \geq m} 2^{l \frac{N}{p}} \|\Delta_l a_0\|_{L^p} + (1 + \|a_0\|_{B_{p,1}^{\frac{N}{2}}})(e^{C(U_L^n + \tilde{U}^n)(T)} - 1). \quad (5.43)$$

Using (5.35) and (\mathcal{H}_5) , (\mathcal{H}_6) , we thus get:

$$\|a^n - S_m a^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq \sum_{l \geq m} 2^{l \frac{N}{p}} \|\Delta_l a_0\|_{L^p} + \frac{C}{\log 2} (1 + \|a_0\|_{B_{p,1}^{\frac{N}{p}}})(1 + \nu^{-1} \tilde{L}_0) \eta.$$

Hence (\mathcal{H}_1) is strictly satisfied provided that η further satisfies:

$$\frac{C}{\log 2} (1 + \|a_0\|_{B_{p,1}^{\frac{N}{p}}})(1 + \nu^{-1} \tilde{U}_0) \eta < \frac{c\nu}{2\nu}. \quad (5.44)$$

In order to check whether (\mathcal{H}_3) is satisfied, we use the fact that:

$$a^n - a_0 = S_m(a^n - a_0) + (Id - S_m)(a^n - a_0) + \sum_{l > n} \Delta_l a_0,$$

whence, using $B_{p,1}^{\frac{N}{p}} \hookrightarrow L^\infty$ and assuming (with no loss of generality) that $n \geq m$,

$$\begin{aligned} \|a^n - a_0\|_{L^\infty((0,T) \times \mathbb{R}^N)} &\leq C(\|S_m(a^n - a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} + \|a^n - S_m a^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \\ &\quad + 2 \sum_{l \geq m} 2^{l \frac{N}{p}} \|\Delta_l a_0\|_{L^p}). \end{aligned}$$

Changing the constant c in the definition of m and in (5.44) if necessary, one can, in view of the previous computations, assume that:

$$C(\|a^n - S_m a^n\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} + 2 \sum_{l \geq m} 2^{l \frac{N}{p}} \|\Delta_l a_0\|_{L^p}) \leq \frac{b}{4}.$$

As for the term $\|S_m(a^n - a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})}$, it may be bounded according proposition 4.11:

$$\begin{aligned} \|S_m(a^n - a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} &\leq (1 + \|a_0\|_{B_{p,1}^{\frac{N}{p}}})(e^{C(\tilde{V}_1^n + \tilde{V}^n + U_L^n)(T)} - 1) + C2^{2m} \sqrt{T} \|a_0\|_{B_{p,1}^{\frac{N}{p}}} \\ &\quad \times \|u^n\|_{L_T^2(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}})}. \end{aligned}$$

Note that under assumptions (\mathcal{H}_5) , (\mathcal{H}_6) , (5.35) and (5.44) (and changing c if necessary), the first term in the right-hand side may be bounded by $\frac{b}{8}$. Hence using interpolation, (5.38) and the assumptions (5.35) and (5.44), we end up with:

$$\|S_m(a^n - a_0)\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq \frac{b}{8} + C2^m \sqrt{T} \|a_0\|_{B_{2,1}^{\frac{N}{2}}} \sqrt{\eta(U_0 + \tilde{U}_0 \eta)(1 + \nu^{-1} \tilde{U}_0)}.$$

Assuming in addition that T satisfies:

$$C2^m\sqrt{T}\|a_0\|_{B_{p,1}^{\frac{N}{p}}}\sqrt{\eta(U_0 + \tilde{U}_0\eta)(1 + \nu^{-1}\tilde{U}_0)} < \frac{b}{8}, \quad (5.45)$$

and using the assumption $\underline{b} \leq 1 + a_0 \leq \bar{b}$ yields (\mathcal{H}_3) with a strict inequality.

One can now conclude that if $T < T^n$ has been chosen so that conditions (5.40), (5.42) and (5.45) are satisfied (with η verifying (5.35) and (5.44), and m defined in (5.33) and $n \geq m$ then (a^n, u^n) satisfies (\mathcal{H}_1) to (\mathcal{H}_8) , thus is bounded independently of n on $[0, T]$. We still have to state that T^n may be bounded by below by the supremum \bar{T} of all times T such that (5.40), (5.42) and (5.45) are satisfied. This is actually a consequence of the uniform bounds we have just obtained, and of remark ?? and proposition 4.10. Indeed, by combining all these informations, one can prove that if $T^n < \bar{T}$ then (a^n, u^n) is actually in:

$$\tilde{L}_{T^n}^\infty(B_{2,1}^{\frac{N}{2}} \cap B_{p,1}^{\frac{N}{p}}) \times \left(\tilde{L}_{T^n}^\infty(B_{2,1}^{\frac{N}{2}} \cap (B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1})) \cap L_{T^n}^1(B_{2,1}^{\frac{N}{2}+1} \cap (B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+2})) \right)^N$$

hence may be continued beyond \bar{T} (see the remark on the lifespan following the statement in [10]). We thus have $T^n \geq \bar{T}$.

Compactness arguments

We now have to prove that $(a^n, u^n)_{n \in \mathbb{N}}$ tends (up to a subsequence) to some function (a, u) which belongs to E_T . Here we recall that:

$$E_T = \tilde{C}([0, T], B_{p,1}^{\frac{N}{p}}) \times (\tilde{L}^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap \tilde{L}^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})).$$

The proof is based on Ascoli's theorem and compact embedding for Besov spaces. As similar arguments have been employed in [10] or [13], we only give the outlines of the proof.

- Convergence of $(a^n)_{n \in \mathbb{N}}$:

We use the fact that $\tilde{a}^n = a^n - a_0^n$ satisfies:

$$\partial_t \tilde{a}^n = -u^n \cdot \nabla a^n + (1 + a^n) \operatorname{div} u^n.$$

Since $(u^n)_{n \in \mathbb{N}}$ is uniformly bounded in $\tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1}) \cap L_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1})$, it is by interpolation and the fact that $p_1 \leq p$, also bounded in $L_T^r(B_{p,1}^{\frac{N}{p}-1+\frac{2}{r}})$ for any $r \in [1, +\infty]$. By using the standard product laws in Besov spaces, we thus easily gather that $(\partial_t \tilde{a}^n)$ is uniformly bounded in $\tilde{L}_T^2(B_{p,1}^{\frac{N}{p}-1})$. Hence $(\tilde{a}^n)_{n \in \mathbb{N}}$ is bounded in $\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1} \cap B_{p,1}^{\frac{N}{p}})$ and equicontinuous on $[0, T]$ with values in $B_{p,1}^{\frac{N}{p}-1}$. Since the embedding $B_{p,1}^{\frac{N}{p}-1} \cap B_{p,1}^{\frac{N}{p}}$ is (locally) compact, and $(a_0^n)_{n \in \mathbb{N}}$ tends to a_0 in $B_{p,1}^{\frac{N}{p}}$, we conclude that $(a^n)_{n \in \mathbb{N}}$ tends (up to extraction) to some distribution a . Given that $(a^n)_{n \in \mathbb{N}}$ is bounded in $\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})$, we actually have $a \in \tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})$.

- Convergence of $(u_L^n)_{n \in \mathbb{N}}$:

From the definition of u_L^n and proposition 3.7, it is clear that $(u_L^n)_{n \in \mathbb{N}}$ tends to solution u_L to:

$$\partial_t u_L - \mathcal{A}u_L = f, \quad u_L(0) = u_0 - \frac{1}{\nu}.$$

$$\text{in } \tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap \tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+3}).$$

- Convergence of $(\tilde{v}_1^n)_{n \in \mathbb{N}}$:

We use the fact that:

$$\begin{aligned} \partial_t \tilde{v}_1^n = & -(u_L^n + \frac{1}{\nu} \tilde{v}^n) \cdot \nabla \tilde{v}_1^n - \tilde{v}_1^n \cdot \nabla u^n - \frac{1}{\nu} (u_L^n \cdot \nabla \tilde{v}^n - \frac{1}{\nu} \tilde{v}^n \cdot \nabla \tilde{v}^n) + (1 + a^n) \mathcal{A} \tilde{v}_1^n \\ & + a^n \mathcal{A} u_L^n - u_L^n \cdot \nabla u_L^n + \frac{1}{\nu} \nabla(\Delta)^{-1} (P'(\rho^n) \operatorname{div}(\rho^n u^n)), \end{aligned}$$

As $(a^n)_{n \in \mathbb{N}}$ is uniformly bounded in $L_T^\infty(B_{p,1}^{\frac{N}{p}})$ and $(u^n)_{n \in \mathbb{N}}$ is uniformly bounded in $L_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap L^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1})$, it is easy to see that the right-hand side is uniformly bounded in $\tilde{L}_T^{\frac{4}{3}}(B_{p_1,1}^{\frac{N}{p_1}-\frac{3}{2}}) + \tilde{L}^\infty(B_{p,1}^{\frac{N}{p}-1})$. Hence $(\tilde{v}_1^n)_{n \in \mathbb{N}}$ is bounded in $\tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1})$ and equicontinuous on $[0, T]$ with values in $B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}-\frac{3}{2}}$. This enables to conclude that $(\tilde{v}_1^n)_{n \in \mathbb{N}}$ converges (up to extraction) to some function $\tilde{v}_1 \in \tilde{L}_T^\infty(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})$.

By interpolating with the bounds provided by the previous step, one obtains better results of convergence so that one can pass to the limit in the mass equation and in the momentum equation. Finally by setting $u = \tilde{v}_1 + \tilde{v} + u_L$, we conclude that (a, u) satisfies (1.3).

In order to prove continuity in time for a it suffices to make use of proposition 4.10. Indeed, a_0 is in $B_{p,1}^{\frac{N}{p}}$, and having $a \in \tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})$ and $u \in \tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1})$ insure that $\partial_t a + u \cdot \nabla a$ belongs to $\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}})$. Similarly, continuity for u may be proved by using that $(\tilde{v}_1)_0 \in B_{p_1,1}^{\frac{N}{p_1}-1}$ and that $(\partial_t v_1 - \mu \Delta v_1) \in \tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}})$. We conclude by using the fact that $u = v_1 + \frac{1}{\nu} v$.

5.3 The proof of the uniqueness

In this section, we are interested in proving the results of uniqueness of theorem 1.1, we will use similar arguments as in [10, 13, 16].

Uniqueness when $1 \leq p_1 < N$, $\frac{2}{N} < \frac{1}{p} + \frac{1}{p_1}$ and $N \geq 3$

In this section, we focus on the cases $1 \leq p_1 < N$, $\frac{2}{N} < \frac{1}{p} + \frac{1}{p_1}$, $N \geq 3$ and postpone the analysis of the other cases (which turns out to be critical) to the next section. Throughout the proof, we assume that we are given two solutions (a^1, u^1) and (a^2, u^2) of (1.3). In the sequel we will show that $a^1 = a^2$ and $v_1^1 = v_1^2$ where $u^i = v_1^i + \tilde{v}^i$ (for the notation,

we adapt the same as in the previous section). It will imply in particular that $u^1 = u^2$. We know that (a^1, v_1^1) and (a^2, v_1^2) belongs to:

$$\tilde{C}([0, T]; B_{p,1}^{\frac{N}{p}}) \times (\tilde{C}([0, T]; B_{p,1}^{\frac{N}{p_1}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap \tilde{L}^1(0, T; B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2}))^N.$$

Let $\delta a = a^2 - a^1$, $\delta v = \tilde{v}^2 - \tilde{v}^1$ and $\delta v_1 = v_1^2 - v_1^1$. The system for $(\delta a, \delta v_1)$ reads:

$$\begin{cases} \partial_t \delta a + u^2 \cdot \nabla \delta a = \delta a \operatorname{div} u^2 + (\delta v_1 + \frac{1}{\nu} \delta v) \cdot \nabla a^1 + (1 + a^1) \operatorname{div} (\delta v_1 + \frac{1}{\nu} \delta v), \\ \partial_t \delta v_1 + u^2 \cdot \delta \nabla v_1 + \delta v_1 \cdot \nabla u^1 - (1 + a^1) \mathcal{A} \delta v_1 = \delta a \mathcal{A} v_1^2 - \frac{1}{\nu} (u^2 \cdot \nabla \delta \tilde{v} \\ - \delta \tilde{v} \cdot \nabla u^1) + \nabla (\Delta)^{-1} \left((P'(\rho^2) - P'(\rho^1)) \operatorname{div} (\rho^2 u^2) + P'(\rho^1) \operatorname{div} (\rho^1 \delta u) \right. \\ \left. + P'(\rho^1) \operatorname{div} ((\rho^2 - \rho^1) u^2) \right). \end{cases} \quad (5.46)$$

The function δa may be estimated by taking advantage of proposition 4.10 with $s = \frac{N}{p} - 1$. Denoting $U^i(t) = \|\nabla u^i\|_{\tilde{L}^1(B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1})}$ for $i = 1, 2$, we get for all $t \in [0, T]$,

$$\begin{aligned} \|\delta a(t)\|_{B_{p,1}^{\frac{N}{p}-1}} &\leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} \|\delta a \operatorname{div} u^2 + (\delta v_1 + \frac{1}{\nu} \delta v) \cdot \nabla a^1 \\ &\quad + (1 + a^1) \operatorname{div} (\delta v_1 + \frac{1}{\nu} \delta v)\|_{B_{p,1}^{\frac{N}{p}-1}} d\tau, \end{aligned}$$

Next using proposition 2.3 and 2.6 we obtain:

$$\begin{aligned} \|\delta a(t)\|_{B_{p,1}^{\frac{N}{p}-1}} &\leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} \|\delta a\|_{B_{p,1}^{\frac{N}{p}-1}} (\|u^2\|_{B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1}} + (1 + 2\|a_1\|_{B_{p,1}^{\frac{N}{p}}})) \\ &\quad + (1 + 2\|a_1\|_{B_{p,1}^{\frac{N}{p}}}) \|\delta v_1\|_{B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1}} d\tau, \end{aligned}$$

Hence applying Grönwall lemma, we get:

$$\|\delta a(t)\|_{B_{p,1}^{\frac{N}{p}-1}} \leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} (1 + \|a^1\|_{B_{p,1}^{\frac{N}{p}}}) \|\delta v_1\|_{B_{p,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+1}} d\tau. \quad (5.47)$$

For bounding δv_1 , we aim at applying proposition 4.8 of [16] to the second equation of (5.46). So let us fix an integer m such that:

$$1 + \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} S_m a^1 \geq \frac{b}{2} \quad \text{and} \quad \|a^1 - S_m a^1\|_{L_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq c \frac{\nu}{\nu}. \quad (5.48)$$

Note since a^1 satisfies a transport equation with right-hand side in $\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}-1})$, proposition 4.10 guarantees that a^1 is in $\tilde{C}_T(B_{p,1}^{\frac{N}{p}})$. Hence such an integer does exist (see remark

13). Now applying proposition 4.8 of [16] with $s = \frac{N}{p_1} - 2$ and $s' = \frac{N}{p} - 1$ insures that for all time $t \in [0, T]$, we have:

$$\begin{aligned} \|\delta v_1\|_{L_t^1(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} &\leq C e^{C(1+t)U(t)} \int_0^t e^{-C(1+\tau)U(\tau)} \left(\|\delta a \mathcal{A} v_1^2 - \frac{1}{\nu} (\delta v \cdot \nabla v_1^1 + v_1^1 \cdot \nabla \delta v) \right. \\ &\quad \left. - \frac{1}{\nu^2} (v_1^1 \cdot \nabla \delta v + \delta v \cdot \nabla v^2) \right\|_{B_{p_1,1}^{\frac{N}{p_1}-2} + B_{p,1}^{\frac{N}{p}-1}} d\tau, \end{aligned}$$

with $U(t) = U^1(t) + U^2(t) + 2^{2m} \underline{\nu}^{-1} \bar{\nu}^2 \int_0^t \|a^1\|_{B_{p,1}^{\frac{N}{p}}}^2 d\tau$.

Hence, applying proposition 2.3 we get:

$$\begin{aligned} \|\delta v_1\|_{\tilde{L}_t^1(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} &\leq C e^{C(1+t)U(t)} \int_0^t e^{-C(1+\tau)U(\tau)} \left(1 + \|a^1\|_{B_{p,1}^{\frac{N}{p}}} + \|a^2\|_{B_{p,1}^{\frac{N}{p}}} \right. \\ &\quad \left. + \|v_1^2\|_{B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2}} \right) \|\delta a\|_{B_{p,1}^{\frac{N}{p}-1}} d\tau. \end{aligned} \quad (5.49)$$

Finally plugging (5.47) in (5.49), we get for all $t \in [0, T_1]$,

$$\begin{aligned} \|\delta v_1\|_{\tilde{L}_t^1(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} &\leq C \int_0^t \left(1 + \|a^1\|_{B_{p,1}^{\frac{N}{p}}} + \|a^2\|_{B_{p,1}^{\frac{N}{p}}} + \|v_1^2\|_{B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2}} \right) \\ &\quad \times \|\delta v_1\|_{\tilde{L}_\tau^1(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} d\tau. \end{aligned}$$

Since a^1 and a^2 are in $L^\infty(B_{p,1}^{\frac{N}{p}})$ and v_1^2 belongs to $L_T^1(B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2})$, applying Grönwall lemma yields $\delta v_1 = 0$, an $[0, T]$.

Uniqueness when: $\frac{2}{N} = \frac{1}{p_1} + \frac{1}{p}$ **or** $p_1 = N$ **or** $N = 2$.

The above proof fails in dimension two. One of the reasons why is that the product of functions does not map $B_{p,1}^{\frac{N}{p}} \times B_{p_1,1}^{\frac{N}{p_1}-2}$ in $B_{p_1,1}^{\frac{N}{p_1}-2}$ but only in the larger space $B_{p_1,\infty}^{\frac{N}{p_1}-2}$. This induces us to bound δa in $L_T^\infty(B_{p,\infty}^{\frac{N}{p}-1})$ and δv_1 in $L_T^\infty(B_{p_1,\infty}^{\frac{N}{p_1}} + B_{p,\infty}^{\frac{N}{p}}) \cap L_T^1(B_{p_1,\infty}^{\frac{N}{p_1}} + B_{p,\infty}^{\frac{N}{p}+1})$ (or rather, in the widetilde version of those spaces, see below). Yet, we are in trouble because due to $B_{p_1,\infty}^{\frac{N}{p_1}}$ is not embedded in L^∞ , the term $\delta v_1 \cdot \nabla a^1$ in the right hand-side of the first equation of (5.46) cannot be estimated properly. As noticed in [12], this second difficulty may be overcome by making use of logarithmic interpolation and Osgood lemma (a substitute for Gronwall inequality). Let us now tackle the proof. Fix an integer m such that:

$$1 + \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} S_m a^1 \geq \frac{b}{2} \quad \text{and} \quad \|a^1 - S_m a^1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq c \frac{\underline{\nu}}{\bar{\nu}}, \quad (5.50)$$

and define T_1 as the supremum of all positive times t such that:

$$t \leq T \quad \text{and} \quad t \bar{\nu}^2 \|a^1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})} \leq c 2^{-2m} \underline{\nu}. \quad (5.51)$$

Remark that the proposition 4.10 ensures that a^1 belongs to $\tilde{C}_T(B_{p,1}^{\frac{N}{p}})$ so that the above two assumptions are satisfied if m has been chosen large enough. For bounding δa in $L_T^\infty(B_{p,\infty}^{\frac{N}{p}-1})$, we apply proposition 4.10 with $r = +\infty$ and $s = 0$. We get (with the notation of the previous section):

$$\begin{aligned} \forall t \in [0, T], \quad \|\delta a(t)\|_{B_{p,\infty}^{\frac{N}{p}-1}} &\leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} \|\delta a \operatorname{div} u^2 + (\delta v_1 + \frac{1}{\nu} \delta v) \cdot \nabla a^1 \\ &\quad + (1 + a^1) \operatorname{div}(\delta v_1 + \frac{1}{\nu} \delta v)\|_{B_{p,\infty}^{\frac{N}{p}-1}} d\tau, \end{aligned}$$

hence using that the product of two functions maps $B_{p,\infty}^{\frac{N}{p}-1} \times B_{p,1}^{\frac{N}{p}}$ in $B_{p,\infty}^{\frac{N}{p}-1}$, and applying Gronwall lemma,

$$\|\delta a(t)\|_{B_{p,\infty}^{\frac{N}{p}-1}} \leq C e^{CU^2(t)} \int_0^t e^{-CU^2(\tau)} (1 + \|a^1\|_{B_{p,1}^{\frac{N}{p}}}) \|\delta v_1\|_{B_{p,1}^{\frac{N}{p}} + B_{p,1}^{\frac{N}{p}+1}} d\tau. \quad (5.52)$$

Next, using proposition 4.8 of [16] combined with proposition 2.3 and corollary 2 in order to bound the nonlinear terms, we get for all $t \in [0, T_1]$:

$$\begin{aligned} \|\delta v_1\|_{\tilde{L}_T^1(B_{p,1}^{\frac{N}{p}-2} + B_{p,\infty}^{\frac{N}{p}+1})} &\leq C e^{C(1+t)(U^1+U^2)(t)} \int_0^t (1 + \|a^1\|_{B_{p,1}^{\frac{N}{p}}} + \|a^2\|_{B_{p,1}^{\frac{N}{p}}}) \\ &\quad + \|v_1^2\|_{B_{p,1}^{\frac{N}{p}+1} + B_{p,1}^{\frac{N}{p}+2}} \|\delta a\|_{B_{p,\infty}^{\frac{N}{p}-1}} d\tau. \end{aligned} \quad (5.53)$$

In order to control the term $\|\delta v_1\|_{B_{p,1}^{\frac{N}{p}} + B_{p,1}^{\frac{N}{p}+1}}$ which appears in the right-hand side of (5.52), we make use of the following logarithmic interpolation inequality whose proof may be found in [12], page 120:

$$\begin{aligned} \|\delta v_1\|_{L_t^1(B_{p,1}^{\frac{N}{p}} + B_{p,1}^{\frac{N}{p}+1})} &\lesssim \\ \|\delta v_1\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}})} \log \left(e + \frac{\|\delta v_1\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}-1})} + \|\delta v_1\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}+1})}}{\|\delta v_1\|_{\tilde{L}_t^1(B_{p,1}^{\frac{N}{p}})}} \right) \\ &\quad + \|\delta v_1\|_{\tilde{L}_t^1(B_{p,\infty}^{\frac{N}{p}+1})} \log \left(e + \frac{\|\delta v_1\|_{\tilde{L}_t^1(B_{p,\infty}^{\frac{N}{p}})} + \|\delta v_1\|_{\tilde{L}_t^1(B_{p,\infty}^{\frac{N}{p}+2})}}{\|\delta v_1\|_{\tilde{L}_t^1(B_{p,\infty}^{\frac{N}{p}})}} \right). \end{aligned} \quad (5.54)$$

Because v_1^1 and v_2^2 belong to $\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}-1} + B_{p,1}^{\frac{N}{p}+1}) \cap L_T^1(B_{p,1}^{\frac{N}{p}+1} + B_{p,1}^{\frac{N}{p}+2})$, the numerator in the right-hand side may be bounded by some constant C_T depending only on T and on the norms of v_1^1 and v_2^2 . Therefore inserting (5.52) in (5.53) and taking advantage of

(5.54), we end up for all $t \in [0, T_1]$ with:

$$\begin{aligned} \|\delta v_1\|_{\tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} &\leq C(1 + \|a^1\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}})}) \\ &\times \int_0^t (1 + \|a^1\|_{B_{p,1}^{\frac{N}{p}}} + \|a^2\|_{B_{p,1}^{\frac{N}{p}}} + \|v_1^2\|_{B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2}}) \|\delta v_1\|_{\tilde{L}_t^1(B_{p_1,\infty}^{\frac{N}{p_1}})} \\ &\times \log(e + C_T \|\delta v_1\|_{\tilde{L}_\tau^1(B_{p_1,\infty}^{\frac{N}{p_1}} + B_{p,\infty}^{\frac{N}{p}+1})}^{-1}) d\tau. \end{aligned}$$

Since the function $t \rightarrow \|a^1(t)\|_{B_{p,1}^{\frac{N}{p}}} + \|a^2(t)\|_{B_{p,1}^{\frac{N}{p}}} + \|v_1^2(t)\|_{B_{p_1,1}^{\frac{N}{p_1}+1} + B_{p,1}^{\frac{N}{p}+2}}$ is integrable on $[0, T]$, and:

$$\int_0^1 \frac{dr}{r \log(e + C_T r^{-1})} = +\infty$$

Osgood lemma yields $\|\delta v_1\|_{\tilde{L}_T^1(B_{p_1,1}^{\frac{N}{p_1}} + B_{p,1}^{\frac{N}{p}+1})} = 0$. Note that the definition of m depends only on T and that (5.48) is satisfied on $[0, T]$. Hence, the above arguments may be repeated on $[T_1, 2T_1]$, $[2T_1, 3T_1]$, etc. until the whole interval $[0, T]$ is exhausted. This yields uniqueness on $[0, T]$ for a and v_1 which implies uniqueness for u .

5.4 Proof of corollary 1

The proof follows the same line as theorem 1.1 except concerning the uniqueness. In the sequel we will concentrate us only the result of uniqueness. For that we use the main theorem of D. Hoff in [21] which is a result of weak-strong uniqueness. More precisely, we recall the result of D. Hoff.

Let (ρ, u) a weak solution (see the definition of D. Hoff in [21]) with the following properties:

$$u \in C([0, T] \times \mathbb{R}^N) \cap L^r((0, T) \times \mathbb{R}^N) \cap L^1(0, T, W^{1,\infty}(\mathbb{R}^N)) \cap L_{loc}^\infty(L^\infty(\mathbb{R}^N)), \quad (5.55)$$

$$\rho - \bar{\rho}, u, f \in L^2((0, T) \times \mathbb{R}^N), \quad (5.56)$$

$$\frac{1}{\rho} \in L^\infty, \quad (5.57)$$

and

$$u \in L^r((0, T) \times \mathbb{R}^N), \quad (5.58)$$

with $r > N$. Let (ρ_1, u_1) a strong solution such that (5.55), (5.56) and (5.57) are verified and:

$$\int_0^T [\|u_1(\cdot, t)\|_{L^\infty} + t \|\nabla u_1(\cdot, t)\|_{L^\infty} + (t \|\nabla F_1(\cdot, t), \nabla \omega_1\|_{L^4}^2)^a] dt < +\infty, \quad (5.59)$$

with $F_1 = \operatorname{div} u_1 - P(\rho_1) + P(\bar{\rho})$ the effective pressure, ω_1 the curl of u_1 and with $a = \frac{2}{3}$ if $N = 2$ and $a = \frac{4}{5}$ for $N = 3$. We assume in the sequel that:

$$f \in L^1((0, T), L^{2q}(\mathbb{R}^N)), \quad (5.60)$$

for some $q \in [1, +\infty]$. And finally D. Hoff need to assume that:

$$\rho_0 - \bar{\rho} \in L^2 \cap L^{2p}. \quad (5.61)$$

We can now state the result that D. Hoff obtains in [21]:

Theorem 5.2 *Assuming that (ρ, u) and (ρ_1, u_1) are weak solution (for the precise definition see [21]), moreover (ρ, u) verify (5.55), (5.56), (5.57), (5.58) and (ρ_1, u_1) verify (5.55), (5.59) and (5.60). The initial data are chosen as in the corollary 1 with the additional condition (5.61). Let $P(\rho) = K\rho$ with $K > 0$. Then under the previous hypothesis:*

$$u = u_1 \text{ on } (0, T).$$

Remark 14 *Here (ρ_1, u_1) have to consider as the strong solution and (ρ, u) as the weak solution.*

Moreover in [24, 24], D. Hoff prove the existence of a solution (ρ, u) satisfying all the conditions (5.55), (5.59) and (5.60) except that $u \in L^1((0, T), W^{1,\infty}(\mathbb{R}^N))$ if we have the following conditions on the initial data (ρ_0, u_0) :

$$\begin{aligned} \rho_0 &\in L^\infty, \rho_0 - \bar{\rho} \in L_2^1, \\ \inf \rho_0 &> 0, \\ u_0 &\in H^s \text{ with } s > 0 \text{ if } N = 2 \text{ or } s > \frac{1}{2} \text{ if } N = 3. \\ u_0 &\in L^{q+\varepsilon}, \text{ with } q = 2 \text{ if } N = 2 \text{ or } q = 6 \text{ if } N = 3 \text{ (here } \varepsilon > 0). \end{aligned} \quad (5.62)$$

For proving these results D. Hoff uses essentially inequalities of energy (in his case the initial data are assumed small and he obtains existence of global weak solutions, for a similar case with large initial data see [18]). The main difficulty is so to control the norm of u_1 in $L^1((0, T), W^{1,\infty}(\mathbb{R}^N))$.

In our context, we want to verify that a solution $(\bar{\rho}, \bar{u})$ constructed in 1.1 with the additional conditions on the initial data of 1 verify all the hypothesis of theorem 5.2 (it means that $(\bar{\rho}, \bar{u})$ have to check the hypothesis of the strong solution and of the weak solution). In this case, we will be able to conclude that if we choose two solutions $(\bar{\rho}, \bar{u})$ and $(\bar{\rho}_1, \bar{u}_1)$ in the class of the solutions of theorem 1.1 then $(\bar{\rho}, \bar{u}) = (\bar{\rho}_1, \bar{u}_1)$.

As we explain previously, the regularizing effects on the velocity in [24, 24] result from energy inequalities combined with an argument of smallness to apply a bootstrap (see [18] for more details in the case of large initial data). It means that we can obtain the same regularizing effects on the velocity \bar{u} (where $(\bar{\rho}, \bar{u})$ is a solution coming from theorem 1.1) with our choice of initial data. Indeed we have combined for the initial data conditions, these of theorem 1.1 and these of [24, 24].

Then to achieve the proof of uniqueness we only have to prove that \bar{u} is in $L_T^1(W^{1,\infty}(\mathbb{R}^N))$ and that $(\bar{\rho}_1, \bar{u}_1)$ verify the condition of the weak solution of theorem 5.2. The fact that $\bar{u} \in L^1((0, T), W^{1,\infty}(\mathbb{R}^N))$ is just a consequence that \bar{u} belongs to $L_T^1(B_{p,1}^{\frac{N}{p}+1})$. For the same reason that previously, we can easily show that $(\bar{\rho}_1, \bar{u}_1)$ verify (5.55), (5.56), (5.57), (5.58).

6 Appendix

This section is devoted to the proof of commutator estimates which have been used in section 2 and 3. They are based on paradifferential calculus, a tool introduced by J.-M. Bony in [4]. The basic idea of paradifferential calculus is that any product of two distributions u and v can be formally decomposed into:

$$uv = T_u v + T_v u + R(u, v) = T_u v + T'_v u$$

where the paraproduct operator is defined by $T_u v = \sum_q S_{q-1} u \Delta_q v$, the remainder operator R , by $R(u, v) = \sum_q \Delta_q u (\Delta_{q-1} v + \Delta_q v + \Delta_{q+1} v)$ and $T'_v u = T_v u + R(u, v)$. The following lemma is useful to get estimate for the transport equation.

Lemma 1 *Let $1 \leq p_1 \leq p \leq +\infty$ and $\sigma \in (-\min(\frac{N}{p}, \frac{N}{p_1}), \frac{N}{p} + 1]$. There exists a sequence $c_q \in l^1(\mathbb{Z})$ such that $\|c_q\|_{l^1} = 1$ and a constant C depending only on N and σ such that:*

$$\forall q \in \mathbb{Z}, \quad \|[v \cdot \nabla, \Delta_q] a\|_{L^{p_1}} \leq C c_q 2^{-q\sigma} \|\nabla v\|_{B_{p,1}^{\frac{N}{p}}} \|a\|_{B_{p_1,1}^{\sigma}}. \quad (6.63)$$

In the limit case $\sigma = -\min(\frac{N}{p}, \frac{N}{p_1})$, we have:

$$\forall q \in \mathbb{Z}, \quad \|[v \cdot \nabla, \Delta_q] a\|_{L^{p_1}} \leq C c_q 2^{q\frac{N}{p}} \|\nabla v\|_{B_{p,1}^{\frac{N}{p}}} \|a\|_{B_{p,\infty}^{-\frac{N}{p_1}}}. \quad (6.64)$$

Finally, for all $\sigma > 0$ and $\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{p}$, there exists a constant C depending only on N and on σ and a sequence $c_q \in l^1(\mathbb{Z})$ with norm 1 such that:

$$\forall q \in \mathbb{Z}, \quad \|[v \cdot \nabla, \Delta_q] v\|_{L^p} \leq C c_q 2^{-q\sigma} (\|\nabla v\|_{L^\infty} \|v\|_{B_{p_1,1}^{\sigma}} + \|\nabla v\|_{L^{p_2}} \|\nabla v\|_{B_{p,1}^{\sigma-1}}). \quad (6.65)$$

Inequality (3.16) is a consequence of the following lemma:

Lemma 2 *Let $1 \leq p_1 \leq p \leq +\infty$ and $\alpha \in (1 - \frac{N}{p}, 1]$, $k \in \{1, \dots, N\}$ and $R_q = \Delta_q (a \partial_k w) - \partial_k (a \Delta_q w)$. There exists $c = c(\alpha, N, \sigma)$ such that:*

$$\sum_q 2^{q\sigma} \|R_q\|_{L^{p_1}} \leq C \|a\|_{B_{p,1}^{\frac{N}{p} + \alpha}} \|w\|_{B_{p_1,1}^{\sigma+1-\alpha}} \quad (6.66)$$

whenever $-\frac{N}{p} < \sigma \leq \alpha + \frac{N}{p}$.

In the limit case $\sigma = -\frac{N}{p}$, we have for some constant $C = C(\alpha, N)$:

$$\sup_q 2^{-q\frac{N}{p}} \|R_q\|_{L^{p_1}} \leq C \|a\|_{B_{p,1}^{\frac{N}{p} + \alpha}} \|w\|_{B_{p_1,\infty}^{-\frac{N}{p_1} + 1 - \alpha}}. \quad (6.67)$$

Proof The proof is almost the same as the one of lemma A3 in [10]. It is based on Bony's decomposition which enables us to split R_q into:

$$R_q = \underbrace{\partial_k [\Delta_q, T_a] w}_{R_q^1} - \underbrace{\Delta_q T_{\partial_k a} w}_{R_q^2} + \underbrace{\Delta_q T_{\partial_k w} w}_{R_q^3} + \underbrace{\Delta_q R(\partial_k w, a)}_{R_q^4} - \underbrace{\partial_k T'_{\Delta_q w} a}_{R_q^5}.$$

Using the fact that:

$$R_q^1 = \sum_{q'=q-4}^{q+4} \partial_k[\Delta_q, S_{q'-1}a]\Delta_{q'}w,$$

and the mean value theorem, we readily get under the hypothesis that $\alpha \leq 1$,

$$\sum_q 2^{q\sigma} \|R_q^1\|_{L^{p_1}} \lesssim \|\nabla a\|_{B_{\infty,1}^{\alpha-1}} \|w\|_{B_{p_1,1}^{\sigma+1-\alpha}}. \quad (6.68)$$

Standard continuity results for the paraproduct insure that R_q^2 satisfies (6.68) and that:

$$\sum_q 2^{q\sigma} \|R_q^1\|_{L^{p_1}} \lesssim \|\nabla w\|_{B_{\infty,1}^{\sigma-\alpha-\frac{N}{p_1}}} \|a\|_{B_{p,1}^{\frac{N}{p}+\alpha}}. \quad (6.69)$$

provided $\sigma - \alpha - \frac{N}{p} \leq 0$. Next, standard continuity result for the remainder insure that under the hypothesis $\sigma > -\frac{N}{p}$, we have:

$$\sum_q 2^{q\sigma} \|R_q^1\|_{L^{p_1}} \lesssim \|\nabla w\|_{B_{p_1,1}^{\sigma-\alpha}} \|a\|_{B_{p,1}^{\frac{N}{p}+\alpha}}. \quad (6.70)$$

For bounding R_q^5 we use the decomposition: $R_q^5 = \sum_{q' \geq q-3} \partial_k(S_{q'+2}\Delta_q w \Delta_{q'}a)$, which leads (after a suitable use of Bernstein and Hölder inequalities) to:

$$2^{q\sigma} \|R_q^5\|_{L^{p_1}} \lesssim \sum_{q' \geq q-2} 2^{(q-q')(\alpha+\frac{N}{p_1}-1)} 2^{q(\sigma+1-\alpha)} \|\Delta_q w\|_{L^{p_1}} 2^{q'(\frac{N}{p}+\alpha)} \|\Delta_{q'}a\|_{L^p}.$$

Hence, since $\alpha + \frac{N}{p} - 1 > 0$, we have:

$$\sum_q 2^{q\sigma} \|R_q^5\|_{L^p} \lesssim \|\nabla w\|_{B_{p_1,1}^{\sigma+1-\alpha}} \|a\|_{B_{p,1}^{\frac{N}{p}+\alpha}}.$$

Combining this latter inequality with (6.68), (6.69) and (6.70), and using the embedding $B_{p,1}^{\frac{N}{p}} \hookrightarrow B_{\infty,1}^{r-\frac{N}{p}}$ for $r = \frac{N}{p} + \alpha - 1$, σ_α completes the proof of (6.66).

The proof of (6.67) is almost the same: for bounding R_q^1 , R_q^2 , R_q^3 and R_q^5 , it is just a matter of changing \sum_q into \sup_q . \square

Remark 15 For proving proposition 3.9, we shall actually use the following non-stationary version of inequality (6.67):

$$\sup_q 2^{-q\frac{N}{p}} \|R_q\|_{L_T^1(L^{p_1})} \leq C \|a\|_{\tilde{L}_T^\infty(B_{p,1}^{\frac{N}{p}+\alpha})} \|w\|_{\tilde{L}_T^1(B_{p_1,\infty}^{-\frac{N}{p_1}+1-\alpha})},$$

which may be easily proved by following the computations of the previous proof, dealing with the time dependence according to Hölder inequality.

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