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ON THE REGULARIZATION OF THE COLLISION SOLUTIONS OF THE ONE-CENTER PROBLEM WITH WEAK FORCES

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Abstract. We study the possible regularization of collision solutions for one centre problems with a weak singularity. In the case of logarithmic singularities, we consider the method of regularization via smoothing of the potential. With this technique, we prove that the extended flow, where collision solutions are replaced with transmission trajectories, is continuous, though not differentiable, with respect to the initial data.

1. Introduction.

In this paper we deal with dynamical systems associated with conservative central forces which are singular at the origin. A *classical solution* does not interact with the singularity of the force, i.e., it is a path $u \in \mathcal{C}^2(T, \mathbb{R}^2 \setminus \{0\})$ which fulfils the initial value problem

$$P \ : \ \begin{cases} \ddot{u} = \nabla V(|u|) \\ \\ (u(0), \dot{u}(0)) = (q_0, p_0) \in (\mathbb{R}^2 \setminus \{0\} \times \mathbb{R}^2) \end{cases} \tag{1}$$

where $V(x) \in \mathcal{C}^2(\mathbb{R}^+, \mathbb{R})$ is the potential of the force and T denotes the maximal interval of existence. As well-known, the two-body problem with an interaction potential V can be reduced to a system of this form where $u(t)$ denotes the position of one of the particle with respect to the centre of mass. Accordingly, we shall term *collision* the configuration $u(t) = 0$. Since the force field diverges at $u = 0$, collisions are among the main sources of non-completeness of the associated flow. This work studies the possible extensions of the flow through the collision that make it continuous with respect to the initial conditions. We are concerned with weak singularities of the potential, namely logarithms.

The regularization of total and partial collisions in the N-body problem is a very classical subject and, in the years, different strategies have been developed in order to extend motions beyond the singularity [11, 10, 13, 12, 6, 15, 14, 16]. Very roughly, these classical methods rely upon suitable changes of space-time variables aimed at obtaining a smooth flow, possibly on an extended phase space; to this aim,

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the first step is to determine the asymptotic behavior of the collision solutions and then the phase space is extended either by means of a double covering, or with the attachment of a collision manifold.

In this paper we consider a further, non classical way of extending the flow, related to the technique of regularization via smoothing of the potential. Although this method is amply adopted by the astrophysical community in dealing with the singularities of the vector field associated to the N–body problem in the context of numerical simulations [1, 5, 9], first De Giorgi [4] proposed to consider the smoothing of the potential as a regularization technique and an exhaustive analysis, in case of homogeneous potentials, was performed by Bellettini, Fusco, Gronchi in [3]. This method consists in smoothing the singular potential and passing to the limit as the smoothing parameter ε and the angular momentum tend to zero simultaneously but in an independent manner (indeed we know that the only collision motions have vanishing angular momentum). This involves an in-depth analysis about the ways the smoothing of the potential coupled with the perturbation of initial conditions lead to define a global solution of the singular problem. This technique, when successful, has the advantage of being extremely robust with respect to the application of existence techniques such as the direct method of the calculus of variation. Let us mention that variational methods have been widely exploited in the recent literature in order to obtain selected symmetric trajectories for N-body problems with Kepler potentials [7].

To begin with, we remove the singularity at $x = 0$ and we denote with $V_\varepsilon(x)$ the smoothed function defined as

$$V_\varepsilon(x) = V(\sqrt{x^2+\varepsilon^2}) \ , \qquad \varepsilon > 0 \ .$$

Then we look at the regularized problem

$$P(\varepsilon) \ : \ \begin{cases} \ddot{u} = \nabla V_\varepsilon(|u|) \\ \\ (u(0), \dot{u}(0)) = (q_0, p_0) \quad \in \mathbb{R}^2 \times \mathbb{R}^2 \ . \end{cases} \tag{2}$$

Unlike (1), the differential equation (2) is no longer singular, so the initial value problem admits a global solution in $\mathbb{C}^\infty((-\infty,+\infty);\mathbb{R}^2)$ for every choice of the datum (q_0, p_0), provided $\nabla V(x)$ is sublinear at infinity[1]. Since we focus on the singularities due to collisions, we fix a ball $B_0(\bar{R})$ of radius $\bar{R}$ centered at the origin, where the collision is the only singularity that system (1) can develop and we denote with $\mathcal{S}(V) \subset \mathbb{R}^2 \times \mathbb{R}^2$ the set of initial conditions (q_0, p_0) leading to collision for the system P with $|q_0| \leq \bar{R}$. For every $\bar{\nu} \in \mathcal{S}(V)$ let $u_{\bar{\nu}}(t) \in \mathcal{C}^2(T, \mathbb{R}^2)$ be the collision solution where T denotes the maximal interval of existence such that $|u_{\bar{\nu}}(t)| \leq \bar{R}$. Denoting with $u_{\varepsilon,\nu}(t)$ the solution of (2) with initial data ν, we investigate the existence of the asymptotic limit of the paths $u_{\varepsilon,\nu}(t)$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$, its relationship with the collision solution $u_{\bar{\nu}}(t)$ of the singular system P and the continuity of the limit trajectory with respect to initial data. The definition of regularization considered in [3] is the following.

Definition 1.1. Let $V(x)$ be a singular potential. We say that the problem (1) is weakly regularizable via smoothing of the potential in $B_0(\bar{R})$ if, for every $\bar{\nu} \in \mathcal{S}(V)$,

[1]Without any additional assumption on the behaviour of the potential $V(x)$ far away from the origin, a solution of system (1) might have singularities other than collisions: for instance solutions could blow up in finite time.

there exist two sequences $(\varepsilon_k)_k$, $(\nu_k)_k$ tending to 0 and $\bar{\nu}$ respectively, such that there exists

$$\lim_{k\to\infty} u_{\varepsilon_k,\nu_k} = u_0$$

and the flow

$$\tilde{u}_\nu(t) = \begin{cases} u_\nu(t) & \nu \notin \mathcal{S}(V) \\ u_0(t) & \nu \in \mathcal{S}(V) \end{cases}$$

is continuous with respect to ν.

In addition we say that

Definition 1.2. The singular one centre problem (1) is strongly regularizable via smoothing of the potential if there exists $\bar{R}$ such that for every $\bar{\nu} \in \mathcal{S}(V)$ there exists

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} u_{\varepsilon,\nu} = u_0 \tag{3}$$

and the flow

$$\tilde{u}_\nu(t) = \begin{cases} u_\nu(t) & \nu \notin \mathcal{S}(V) \\ u_0(t) & \nu \in \mathcal{S}(V) \end{cases}$$

is continuous with respect to ν.

In both the definitions we mean that the limit of the regularizing paths $u_{\varepsilon,\nu}(t)$ and the continuity of the extended flow are held in the ball $B_0(\bar{R})$.

In [3] the authors prove that in the case of homogeneous potential of degree α, $V(x) = \frac{1}{|x|^\alpha}$, $\alpha > 0$, the one-centre problem is weakly regularizable via smoothing of the potential if and only if $\alpha > 2$ or α is in the form

$$\alpha = 2\left(1 - \frac{1}{n}\right)$$

where n is a positive integer. On the other hand it is shown that the homogeneous problem is never strongly regularizable via smoothing of the potential. Indeed, a necessary condition in order to achieve the uniform limit (3) is that the apsidal angle $\Delta\theta_l(u)$ of a solution of the system (1) has to converge to $\frac{\pi}{2}$ as the angular momentum l tends to zero (see the definition of apsidal angle in the next section). This condition is never satisfied by α-homogeneous potentials $\alpha > 0$, since $\Delta\theta_l(u) \to \frac{\pi}{2-\alpha} > \frac{\pi}{2}$ as $l \to 0$ [3, 17]. Conversely, when the logarithmic potential is considered, it can be proved [17] that the apsidal angle do indeed converge to $\pi/2$ as the angular momentum vanishes. Then there is no obstruction and we could expect that the limit (3) is attained. This fact suggests to extend the motion after a collision by reflecting it about the origin. We will show that, in this way, not only for the logarithmic potential, but for a larger class $\mathcal{V}^*$ of potentials, the problem is regularizable according to Definition 1.2.

The sets of potential functions we will consider in this paper are the following.

Definition 1.3 (The functions set $\mathcal{V}$). We define $\mathcal{V}$ the set of functions $V(x) \in \mathbb{C}^\infty(\mathbb{R}^+, \mathbb{R})$ with the properties:

i. $\lim_{x \to 0^+} V(x) = +\infty$

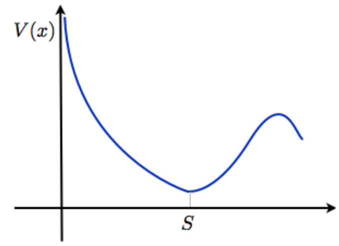
there exists $S > 0$ such that for every $x \in (0, S)$

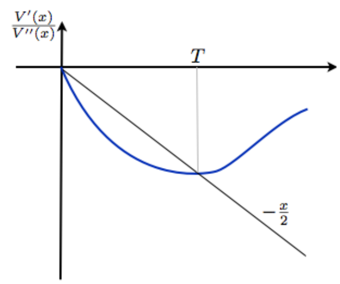
ii. $V'(x) < 0$, $V''(x) > 0$

iii. the function $\dfrac{V'(x)}{V''(x)}$ is decreasing with respect to x

and

iv. $\dfrac{d}{dx}\left(\dfrac{V'(x)}{V''(x)}\right)(0) < -\dfrac{1}{2}$.





The properties ***iii,iv*** guarantee the existence of a value $T > 0$ such that

$$\frac{V'(x)}{V''(x)} \leq -\frac{x}{2} \quad \text{in} \quad (0, \mathrm{T}) \ . \tag{4}$$

Let us define

$$\bar{R} := \min\{T, S\}. \tag{5}$$

Definition 1.4 (The functions set $\mathcal{V}^*$)**.** We denote with $\mathcal{V}^*$ the set of functions $V(x) \in \mathcal{V}$ with the further property

v. $\lim_{\lambda \to 0} \dfrac{V(\lambda x)}{V(\lambda)} = 1$ for every $x \geq 1$ uniformly in every compact $K = [1, M]$.

The set $\mathcal{V}$ includes potentials having homogeneous singularities and weaker. For instance the logarithmic potential, $V(x) = -\log(x)$, as well as the homogeneous potentials, $V(x) = |x|^{-\alpha}$, provided $\alpha \in (0, 1)$, belong to $\mathcal{V}$. On the other hand condition **v.** can be considered as a logarithmic type property or a zero-homogeneity property: indeed it is never satisfied by homogeneous potentials, while the logarithmic potential is a prototype of all the functions satisfying condition **v.**.

Our main goal is the following:

Main Theorem. *For every $V(x) \in \mathcal{V}^*$ the one centre problem is regularizable according to Definition 1.2 where $\bar{R}$ is given in* (5).

In the particular case of logarithmic potential, $V(x) = -\log(x)$, one has $\bar{R} = +\infty$, therefore

Corollary 1. *The logarithmic one central problem is globally regularizable via smoothing of the potential according to Definition 1.2.*

The paper is organized as follows. In Section 2 we follow the classical method for dealing with central problem based on first integrals and we derive the set $\mathcal{S}(V)$ of initial conditions leading to the collisions for the unperturbed system. Next, in Section 3, given any collision solution $u(t)$, we set the initial data $\bar{\nu} \in \mathcal{S}(V)$ and we define the family of paths $u_{\varepsilon,\nu}(t)$. Section 4 contains the proof of the Main Theorem and the analysis of the regularity of the extended flow. The main part of the proof consists in proving the existence of the limit of the path $u_{\varepsilon,\nu}(t)$ as $(\varepsilon,\nu) \to (0,\bar{\nu})$, especially for what that concerns the angular part, Theorem 4.2. This is the most delicate step, for the it involves the uniformity of the limit as $(\varepsilon,\nu) \to (0,\bar{\nu})$, and it allows to conclude the strong regularizability of the problem.

It results that the natural extension of the collision solution is the *transmission solution*, see Definition 4.4, obtained by reflecting the motion through the collision. The regularity of the extended flow is carried on in section 4.2: in Theorem 4.5 and Theorem 4.7 the continuity of the Poincaré map and the continuity of the Poincaré section with respect to initial data are achieved.

2. **Preliminaries.** For any choice of the potential function $V(x) \in \mathcal{C}^2(\mathbb{R}^+,\mathbb{R})$ the one centre problem (1) is a Hamiltonian system and it admits the two classical first integrals of motion: the energy E and the angular momentum l:

$$E = \frac{1}{2}|\dot{u}|^2 - V(|u|), \qquad l = \dot{u} \wedge u \ .$$

The conservation of the angular momentum implies the motion is planar, therefore we choose the horizontal plane as the orbital plane and, in the following, l is used to denote the third component of the angular momentum, rather than the vector. In polar coordinates (r,θ) the quantities E and l are expressed in the form

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}\frac{l^2}{r^2} - V(r) \qquad l = r^2\dot{\theta} \ . \tag{6}$$

By means of the function

$$f(r) = 2r^2(E + V(r)) \tag{7}$$

the relation (6) reads as $l^2 = f(r) - (r\dot{r})^2$, hence a solution of system (1) with energy E and angular momentum l exists only for those values of the radial coordinate $r \geq 0$ satisfying $f(r) \geq l^2$.

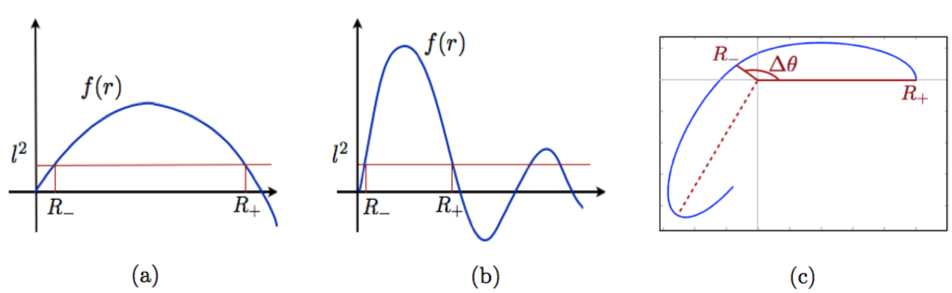


FIGURE 1. (a-b) Apsidal values $R_\pm$ for two different potential functions. (c) Representation of the apsidal angle $\Delta\theta$.

Depending on the behavior of the potential $V(x)$, more than one intervals of positive values of radial coordinate could fulfill the last relation. On the other hand, since the purpose of this work is to study the collision solutions, we will consider only the interval closest to the origin. Therefore, for any fixed value of the energy E and angular momentum l, let us define the *apsidal values* $R_\pm$ of the orbit as follows, see Figure 1(a-b)

- R_+, if it exists, the minimum positive value of r such that $f(r) = l^2$ and $f'(r) < 0$, $R_+ = +\infty$ otherwise
- R_-, if it exists, the minimum positive value of $r < R_+$ such that $f(r) = l^2$ and $f'(r) > 0$, $R_- = 0$ otherwise.

By definition, it descends that $R_- = 0$ for collision solutions and $R_+ = +\infty$ for unbounded orbits and, following the terminology adopted in celestial mechanics, we sometimes refer to R_+ and R_- respectively as the *apocentre* and the *pericentre* of the orbit. Moreover, when they are positive and finite, the apsidal values correspond to the stationary points for the radial motion and, from (6), they are solutions of equation $r^2(E + V(r)) = l^2$.

As it is well known [18], in case of non collision and bounded trajectories, the radial coordinate $r(t)$ oscillates periodically between its extremal values R_+ and R_- while the angular coordinate $\theta(t)$ covers an angle equal to

$$\Delta\theta_l(u) = \int_{R_-}^{R_+} \frac{1}{r\sqrt{\frac{2r^2}{l^2}(E + V(r)) - 1}} dr \tag{8}$$

between each singular oscillation of $r(t)$. We term the angle $\Delta\theta_l(u)$ the *apsidal angle*, see figure 1(c).

The knowledge of the apsidal values R_+ and R_- and the value of the apsidal angle is sufficient to determine the behavior of the solution since the whole trajectory is obtained repeating symmetrically and periodically the part of path between a point where $r(t)$ is maximum and the following point where $r(t)$ is minimal.

The definition of the apsidal angle extends in a natural way for unbounded and collision solutions: in the first case the orbits is composed by a single oscillation from infinity to the pericentral point and back to infinity and the apsidal angle represents half of the angle covered by the particle during this journey, while, if the orbit ends into a collision, the apsidal angle denotes the increment of the angular coordinate between the apocentral point and the collision itself. The value of the apsidal angle is obtained replacing in (8) $R_+ = +\infty$ in the former case and $R_- = 0$ in the latter.

In order to characterize the set $\mathcal{S}(V)$ of initial data leading to a collision we give the following definition.

Definition 2.1. We say that a potential function $V(x)$, singular in the origin, is of weak type if

$$\lim_{x\to 0^+} x^2 V(x) = 0 \ .$$

Otherwise we say that $V(x)$ is a strong type potential.

A similar classification of singular potentials can be found in a work of Gordon [8] where a potential $V(x)$ is said to satisfy a *strong force condition* at a point x_0 if $V(x)$ tends to infinity as x tends to x_0 and also there exists a function $U(x)$ with infinitely deep wells at x_0, such that

$$V(x) \geq |\nabla U(x)|^2$$

in a neighborhood of x_0. We notice that, among the homogeneous potentials, the set of potentials with the property to be of strong type and the ones satisfying the Gordon's strong force condition coincide. In particular, an α-homogeneous potential is of weak type if and only if $\alpha \in (0, 2)$ and, in these cases, a collision occurs only in zero angular momentum orbits [12], while if $\alpha \geq 2$ a collision solution exists also for non-zero values of the angular momentum [3, 12]. The next proposition extends this result.

Proposition 1. *If $V(x) \in \mathcal{C}^2(\mathbb{R}^+, \mathbb{R})$ is a weak type potential, a solution $u(t)$ of the dynamical system $\ddot{u} = \nabla V(|u|)$ ends into a collision if and only if the angular momentum is zero.*

Proof. Denote with E and l the energy and the angular momentum of the solution $u(t)$ and let $f(r)$ as in (7). As mentioned before, a solution exists only for the values of radial coordinate r satisfying $l^2 \leq f(r)$. Suppose $l = 0$: since $V(x)$ tends to infinity as x goes to zero, for every value of E there exists a neighborhood of the origin where $E + V(r) > 0$ then the solution presents a collision.

Conversely, since $V(x)$ is a weak type potential, it follows that $f(r) \to 0$ as $x \to 0^+$ thus for every value of $l \neq 0$ there exist a neighborhood of the origin where $l^2 > f(r)$. Hence the collision can not be attained on solutions with non zero angular momentum. □

Proposition 2. *Every $V(x) \in \mathcal{V}$ is a weak type potential.*

Proof. From relation (4), by integration, it follows the estimate

$$-V'(\xi) \leq \frac{C_1}{\xi^2}\ , \qquad C_1 > 0$$

for every $\xi \in (0, T)$. Therefore, again by integration, we infer

$$V(x) \leq \frac{C_1}{x} + C_2 \tag{9}$$

and we conclude

$$\lim_{x \to 0} x^2 V(x) = 0\ .$$

□

From Proposition 1 and Proposition 2 we deduce that, for any choice of $V(x) \in \mathcal{V}$, a solution of the system (1) ends into a collision if and only if the angular momentum is zero. Therefore the set $\mathcal{S}(V)$ of initial conditions $\bar{\nu} = (\bar{q}_0, \bar{p}_0)$ that lead the solutions into a singularity consists in those $(\bar{q}_0, \bar{p}_0)$ satisfying $l = |\bar{q}_0 \wedge \bar{p}_0| = 0$.

3. **Setting.** For every fixed $V(x) \in \mathcal{V}$ let $\bar{R}$ be the quantity defined in (5) and let $B_0(\bar{R})$ be used to denote the ball of radius $\bar{R}$ around the origin. It results that the collision is the only source of singularity inside $B_0(\bar{R})$ for the dynamical system (1). We also remind that the angular momentum l is zero for collision solution, hence the collision orbit is a straight line joining some point in the plane with the origin. As discussed in the introduction, given a collision solution $\bar{u}(t)$ for the one central problem, our intent is to define an extension beyond the collision.

To this aim, the first step consists in setting the initial conditions $\bar{\nu} = (\bar{q}_0, \bar{p}_0)$ for the singular path $\bar{u}(t)$: denoting with P the first positive solution of equation $f(r) = 0$, $P := +\infty$ if such value does not exists, we fix the initial data according to the following alternative :

Case 1 $P < \bar{R}$.

The collision solution $\bar{u}(t)$ is bounded in a ball centered in the origin of radius $P < \bar{R}$ and P represents the maximal value of the radial coordinate, see Figure 2. Without loss of generality, we can set the initial condition $\bar{\nu}$ of the collision solution $\bar{u}$ as

$$\bar{\nu} = (\bar{q}_0, 0), \qquad |\bar{q}_0| = P\ . \tag{10}$$

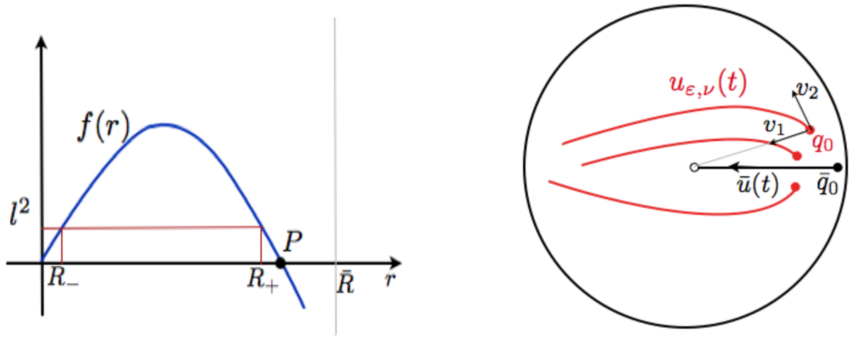


FIGURE 2. Case 1: the collision solution $\bar{u}(t)$ is within $B_0(\bar{R})$.

Case 2 $P \geq \bar{R}$

In this case the collision solution is not bounded in $B_0(\bar{R})$ and it could also be unbounded. We focus our analysis only on the portion of path bounded by $\bar{R}$ hence we select as initial condition for $\bar{u}(t)$ the couple

$$\bar{\nu} = (\bar{q}_0, \bar{p}_0), \qquad |\bar{q}_0| = \bar{R}, \qquad |\bar{p}_0|^2 = 2(E + V(\bar{R})) \tag{11}$$

where the initial velocity $\bar{p}_0$ is directed toward the center of attraction, see Figure 3.

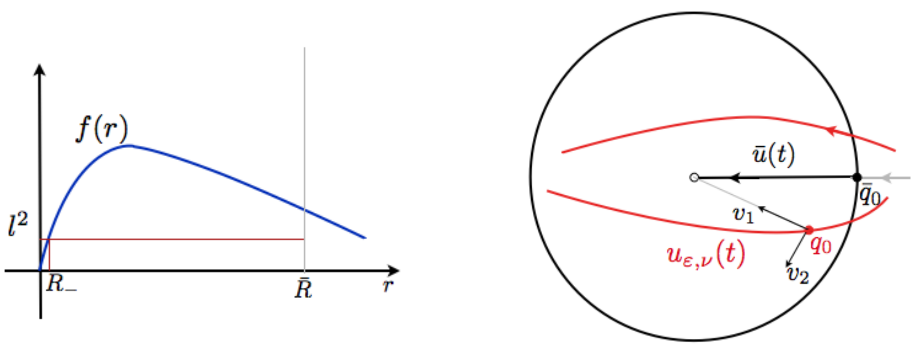


FIGURE 3. Case 2: the collision solution $\bar{u}(t)$ is not bounded in $B_0(\bar{R})$.

In both the cases, for $\varepsilon > 0$ let $u_{\varepsilon,\nu}(t)$ be the solution of system (2) leading from an initial data $\nu = (q_0, p_0)$. We refer to $u_{\varepsilon,\nu}(t)$ as *regularizing paths* in order to underline the purpose to define the extension for the singular solution $\bar{u}(t)$ as the limit of $u_{\varepsilon,\nu}(t)$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$. The smoothing of the potential does not affect the hamiltonian structure of the system, therefore the angular momentum l and the energy $E_{\varepsilon,\nu} = \frac{1}{2}|\dot{u}_{\varepsilon,\nu}|^2 - V_\varepsilon(|u_{\varepsilon,\nu}|)$ are conserved along the solution $u_{\varepsilon,\nu}(t)$. Decomposing the initial velocity p_0 in terms of the parallel and orthogonal component with respect to q_0,

$$p_0 = v_1 + v_2 \ , \qquad v_1 \parallel q_0, \quad v_2 \perp q_0$$

one has $|v_2|^2 = \frac{l^2}{|q_0|^2}$ and $E_{\varepsilon,\nu} = \frac{1}{2}|v_1|^2 + \frac{1}{2}\frac{l^2}{|q_0|^2} - V_\varepsilon(|q_0|)$. The condition $\nu \to \bar{\nu}$ is equivalent to

$$q_0 \to \bar{q}_0, \quad l \to 0, \quad v_1 \to \bar{p}_0$$

that, coupled with the condition $\varepsilon \to 0$, implies $E_{\varepsilon,\nu} \to E$. It turns out that also the apsidal values of the regularizing paths have to converge to the corresponding ones of $\bar{u}(t)$, indeed as $(\varepsilon, \nu) \to (0, \bar{\nu})$, the pericentre R_- of the solution $u_{\varepsilon,\nu}(t)$ tends to zero while the apocentre R_+ is bounded by $\bar{R}$ and tends to P in **case 1**, while $R_+ > \bar{R}$ and possibly $R_+ = +\infty$ in **case 2**.

In the following sections we will deal with the existence and the property of the limit for the paths $u_{\varepsilon,\nu}(t)$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$. As discussed in [3], the behavior of the angular coordinate of the regularizing paths plays a fundamental role for the existence of the uniform limit of $u_{\varepsilon,\nu}(t)$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$ rather then for subsequences $(\varepsilon_k, \nu_k) \to (0, \bar{\nu})$. For this reason and since we focus our analysis only inside the ball $B_0(\bar{R})$, we extend the definition of the apsidal angle for the solution $u_{\varepsilon,\nu}(t)$ as follows: in **case 1** we denote with $\Delta\theta(u_{\varepsilon,\nu})$ the apsidal angle of the path $u_{\varepsilon,\nu}(t)$ as it is defined in (8),

$$\Delta\theta(u_{\varepsilon,\nu}) = \int_{R_-}^{R_+} \frac{l}{r^2 \dot{r}} dr \tag{12}$$

otherwise, in **case 2**, we denote with $\Delta\theta(u_{\varepsilon,\nu})$ the angle covered by the path $u_{\varepsilon,\nu}(t)$ between the point where $u_{\varepsilon,\nu}(t)$ enter in the ball $B_0(\bar{R})$ and the point of minimal distance from the origin

$$\Delta\theta(u_{\varepsilon,\nu}) = \int_{R_-}^{\bar{R}} \frac{l}{r^2 \dot{r}} dr \ . \tag{13}$$

4. **Proof of Main Theorem and property of the extended flow.** The proof of the Main Theorem is composed by two parts: first, in Section 4.1 we prove the existence of the limit of the trajectories $u_{\varepsilon,\nu}(t)$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$ and we define the extension of the singular solution, then in Section 4.2 we study the regularity of the extended flow.

A necessary condition for the existence of the limit (3) is the existence of the limit of the apsidal angle of the regularized solutions. Theorem 4.2 concerns the asymptotic of $\Delta\theta(u_{\varepsilon,\nu})$ as $(\varepsilon, \nu) \to (0, \bar{\nu})$: to this aim we first prove in Lemma 4.3 the $\mathbb{L}^1$ boundedness of the integrand in (12) and (13) then we apply the dominated convergence theorem and pass to the limit under the integral sign. The boundedness of the integrand is a consequence of a technical estimate stated in the Proposition 3 and it is attained for every potential $V(x) \in \mathcal{V}$, while the existence of the limit is a consequence of Proposition 4 based on the assumption **v.**. The result we obtain suggest to define the extension $u_0(t)$ of the collision solution $\bar{u}(t)$ beyond the singularity as a *transmission solution*, Definition 4.4.

In order to gain the regularity of the extension, we analyse, in Theorems 4.5 and Theorem 4.7, the continuity of the Poincaré map and the continuity of the Poincaré section of the extended flow in the phase space.

4.1. **The existence of the uniform limit of $u_{\varepsilon,\nu}(t)$.**

Proposition 3. *Let $V(x) \in \mathcal{V}$ and $\bar{R}$ as in (5). Then for every $\bar{r} < \bar{R}$ there exists an $\bar{\varepsilon}$ such that $\forall \varepsilon < \bar{\varepsilon}$ and for every $0 < y < \bar{r}$ it holds*

$$F_{(y,\bar{r})}(\varepsilon, x) := \frac{(\bar{r}+x)}{\left(\frac{x}{y}-1\right)}\left[\left(\frac{x}{y}\right)^2\left(\frac{V_\varepsilon(x)-V_\varepsilon(\bar{r})}{V_\varepsilon(y)-V_\varepsilon(\bar{r})}\right)\frac{(\bar{r}^2-y^2)}{(\bar{r}^2-x^2)}-1\right] \geq \bar{r}$$

for every $y \leq x \leq \bar{r}$.

To prove the Proposition 3 we first show in the next Lemma that there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon < \bar{\varepsilon}$ the function $F_{(y,\bar{r})}(\varepsilon, x) > F_{(y,\bar{r})}(0, x)$ for every $0 < y \leq x \leq \bar{r} < S$, then we prove that $F_{(y,\bar{r})}(0, x) \geq \bar{r}$ for every $0 < y \leq x \leq \bar{r} < \bar{R}$.

Lemma 4.1. *Let $V(x)$ be a function satisfying the properties **i.-iii.** in Definition 1.3. Then for every choice of $0 < y < \bar{r} < S$ there exists $\bar{\varepsilon} > 0$ such that $\forall \varepsilon < \bar{\varepsilon}$ and $\forall x \in (y, \bar{r})$ it holds*

$$F_{(y,\bar{r})}(\varepsilon, x) > F_{(y,\bar{r})}(0, x) .$$

Proof. The statement follows from the inequality

$$Q_{(y,\bar{r})}(\varepsilon, x) := \frac{V_\varepsilon(x)-V_\varepsilon(\bar{r})}{V_\varepsilon(y)-V_\varepsilon(\bar{r})} \geq \frac{V(x)-V(\bar{r})}{V(y)-V(\bar{r})}$$

that, by means of straightforward calculations and reminding the definition of smoothed potential $V_\varepsilon(\cdot) = V(\sqrt{(\cdot)^2+\varepsilon^2})$, is equivalent to

$$\frac{V(\sqrt{y^2+\varepsilon^2})-V(\sqrt{x^2+\varepsilon^2})}{V(\sqrt{x^2+\varepsilon^2})-V(\sqrt{\bar{r}^2+\varepsilon^2})} \leq \frac{V(y)-V(x)}{V(x)-V(\bar{r})} .$$

For every $s > 0$ we define the function $U(s) = V(\sqrt{s})$. Obviously the function $U(s)$ inherits property **i.** and property **ii.** for every $s \in (0, S^2)$, while the relation

$$2\sqrt{s}\frac{U''(s)}{U'(s)} = \frac{V''(\sqrt{s})}{V'(\sqrt{s})} - \frac{1}{2\sqrt{s}}$$

together with property **iii.** implies that the function $\sqrt{s}\dfrac{U''(s)}{U'(s)}$ is increasing for every $s \in (0, S^2)$. In terms of the function $U(s)$, it's enough to prove that, for every choice of $0 < y < \bar{r} < S^2$, there exists $\bar{\varepsilon} > 0$ such that $\forall \varepsilon < \bar{\varepsilon}$ the relation

$$g(\varepsilon, x) := \frac{U(y+\varepsilon)-U(x+\varepsilon)}{U(x+\varepsilon)-U(\bar{r}+\varepsilon)} \leq \frac{U(y)-U(x)}{U(x)-U(\bar{r})} = g(0, x)$$

holds for all $x \in (y, \bar{r})$. We infer this result showing that

$$\frac{dg}{d\varepsilon}(0, x) < 0 \quad \forall\ 0 < y < x < \bar{r} < S^2 .$$

The sign of the derivative is given by the sign of the function

$$G_{(y,\bar{r})}(x) := U'(y)\Big(U(x)-U(\bar{r})\Big)+U(y)\Big(U'(\bar{r})-U'(x)\Big)+U'(x)U(\bar{r})-U'(\bar{r})U(x) .$$

Since $G_{(y,\bar{r})}(y) = G_{(y,\bar{r})}(\bar{r}) = 0$ and $G_{(y,\bar{r})}(x)$ is continuous, there exists at least one point $\bar{x} \in (y, \bar{r})$ where $G'_{(y,\bar{r})}(\bar{x}) = 0$; the proof of the Lemma follows once we prove the inequalities

$$G'_{(y,\bar{r})}(x) < 0\ \forall x \in (y, \bar{x}) \quad \text{and} \quad G'_{(y,\bar{r})}(x) > 0\ \forall x \in (\bar{x}, \bar{r}) . \tag{14}$$

Let be $N(x)$ the function

$$N(x) := \frac{\sqrt{x}G'_{y,\bar{r}}(x)}{U'(x)(U(\bar{r})-U(y))} = \sqrt{x}\frac{U''(x)}{U'(x)} + \sqrt{x}\frac{(U'(y)-U'(\bar{r}))}{(U(\bar{r})-U(y))} \ .$$

$N(\bar{x}) = 0$ and since $\sqrt{s}\dfrac{U''(s)}{U'(s)}$ is increasing for every $s \in (0, S^2)$ and the factor $\dfrac{(U'(y)-U'(\bar{r}))}{(U(\bar{r})-U(y))}$ is positive, we infer that $N(x)$ is increasing in x. Thus $N(x) < 0$ for $x < \bar{x}$ and $N(x) > 0$ otherwise and, since $U'(x)(U(\bar{r})-U(y)) > 0$, inequalities (14) hold. □

Proof of Proposition 3.
We fix $0 < y < \bar{r} < \bar{R}$. For Lemma 4.1 there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in [0, \bar{\varepsilon}]$ and for every $x \in [y, \bar{r}]$ it holds $F_{(y,\bar{r})}(\varepsilon, x) \geq F_{(y,\bar{r})}(0, x)$. Hence it's sufficient to prove

$$F_{(\bar{r},y)}(x) := \frac{(\bar{r}+x)}{(\frac{x}{y}-1)}\left[\left(\frac{x}{y}\right)^2\left(\frac{V(x)-V(\bar{r})}{V(y)-V(\bar{r})}\right)\frac{(\bar{r}^2-y^2)}{(\bar{r}^2-x^2)} - 1\right] \geq \bar{r} \ .$$

Suppose for a moment that relation

$$\frac{V(x)-V(\bar{r})}{V(y)-V(\bar{r})} \geq \left(\frac{\bar{r}-x}{\bar{r}-y}\right)\frac{y}{x} \tag{15}$$

holds for every $x \in (y, \bar{r})$. Replacing into $F_{(\bar{r},y)}(x)$ we obtain

$$F_{(\bar{r},y)}(x) \geq \frac{(\bar{r}+x)}{(\frac{x}{y}-1)}\left[\left(\frac{x}{y}\right)\frac{(\bar{r}+y)}{(\bar{r}+x)} - 1\right] \geq \bar{r} \ .$$

In order to prove relation (15) we rewrite it as

$$\frac{(V(x)-V(\bar{r}))}{(\bar{r}-x)}x \geq \frac{(V(y)-V(\bar{r}))}{(\bar{r}-y)}y \quad \forall x \in (y, \bar{r}) \ . \tag{16}$$

For $x = y$ the inequality is verified; moreover, denoting with $N(x)$ the numerator of the derivative

$$\frac{d}{dx}\left(\frac{(V(x)-V(\bar{r}))}{(\bar{r}-x)}x\right) = \frac{x(\bar{r}-x)V'(x) + \bar{r}(V(x)-V(\bar{r}))}{(\bar{r}-x)^2} \tag{17}$$

one has $N(\bar{r}) = 0$ and $\frac{dN}{dx}(x) = (\bar{r}-x)\Big(2V'(x) + xV''(x)\Big)$. For (4), for every $x \in (0, \bar{R})$, $\frac{dN}{dx}(x) \leq 0$, hence the derivative in (17) is positive and relation (16) holds for every $x \in (y, \bar{r})$. □

Proposition 4. *Let $V(x) \in \mathcal{V}^*$. Then for every $\rho > 1$*

$$\lim_{(\delta,\varepsilon)\to(0,0)} \frac{V_\varepsilon(\rho\delta)}{V_\varepsilon(\delta)} = 1 \ , \qquad \delta, \varepsilon > 0 \ .$$

Proof. We rewrite the above limit in the form

$$\lim_{(\delta,\varepsilon)\to(0,0)} \frac{V\left(\sqrt{\delta^2+\varepsilon^2}\sqrt{\frac{\rho^2\delta^2+\varepsilon^2}{\delta^2+\varepsilon^2}}\right)}{V(\sqrt{\delta^2+\varepsilon^2})}.$$

Since $\rho > 1$, for every choice of positive values of ε and δ, it holds

$$1 \leq \frac{\rho^2\delta^2+\varepsilon^2}{\delta^2+\varepsilon^2} \leq \rho^2$$

then, from Definition (1.4) of $\mathcal{V}^*$, replacing λ and M with $\delta^2+\varepsilon^2$ and ρ respectively, we infer the statement of the proposition. □

According with the previous setting, as in Section 3, let $V(x) \in \mathcal{V}^*$ and let $\bar{u}(t)$ be any collision solution of the system (1) with energy E, leading from the initial condition $\bar{\nu} = (\bar{q}_0, \bar{p}_0)$ in the form (10) or (11) and, for every sufficiently small $\varepsilon > 0$, let $u_{\varepsilon,\nu}(t)$ be the solution of the regularized system (2) with initial condition ν and $\Delta\theta(u_{\varepsilon,\nu})$ as in (12), (13).

Theorem 4.2. *There exists*

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} \Delta\theta(u_{\varepsilon,\nu})$$

and such limit is $\frac{\pi}{2}$.

Proof. Reminding the definition of $\bar{R}$ and R_+, we define

$$\beta = \min\{\bar{R}, R_+\}$$

therefore, for every $u_{\varepsilon,\nu}(t)$, regardless they are bounded or not by $\bar{R}$, we write

$$\Delta\theta(u_{\varepsilon,\nu}) = \int_{R_-}^{\beta} \frac{l}{r^2\dot{r}} dr \ .$$

In order to deal with the convergence of $\Delta\theta(u_{\varepsilon,\nu})$ we first rewrite the integrand in a different way. From the conservation of energy it follows

$$\dot{r}^2 = 2(E_{\varepsilon,\nu} + V_\varepsilon(r)) - \frac{l^2}{r^2}$$

thus, replacing the radial velocity and extracting the roots R_- and β from the denominator, we infer

$$\begin{aligned}\Delta\theta(u_{\varepsilon,\nu}) &= \int_{R_-}^{\beta} \frac{1}{r\sqrt{\frac{2r^2}{l^2}(E_{\varepsilon,\nu} + V_\varepsilon(r)) - 1}} dr \\ &= \int_{R_-}^{\beta} \frac{1}{r\sqrt{(r-R_-)(\beta-r)}} \sqrt{\frac{(r-R_-)(\beta-r)}{\frac{2r^2}{l^2}(E_{\varepsilon,\nu} + V_\varepsilon(r)) - 1}}\end{aligned}$$

By means of the change of variables $\rho = \frac{r}{R_-}$

$$\begin{aligned}\Delta\theta(u_{\varepsilon,\nu}) &= \int_1^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}} \sqrt{\frac{(\beta-\rho R_-)(\rho-1)}{\frac{2R_-^2\rho^2}{l^2}\Big(E_{\varepsilon,\nu} + V_\varepsilon(\rho R_-)\Big) - 1}} d\rho \\ &= \int_1^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(\bar{R}-\rho R_-)(\rho-1)}} \sqrt{K} d\rho\end{aligned} \tag{18}$$

We proceed as follows: first, in the next Lemma, we exhibit an uniform bound for the function K provided ε small enough is taken, then we apply the Lebesgue's theorem in order to obtain the limit of $\Delta\theta(u_{\varepsilon,\nu})$ as $(\varepsilon,\nu) \to (0,\bar{\nu})$ and we'll prove that such limit exists if $V(x) \in \mathcal{V}^*$.

Lemma 4.3. *Let $V(x) \in \mathcal{V}$, then for ε small enough the function K is bounded by a constant in its domain.*

Proof. Replacing in K the relation $E_{\varepsilon,\nu} = \frac{1}{2}v^2 + \frac{1}{2}\frac{l^2}{\beta^2} - V_\varepsilon(\beta)$ we obtain

$$K = \frac{(\beta - \rho R_-)(\rho - 1)}{\frac{2R_-^2\rho^2}{l^2}\left(\frac{1}{2}v^2 + \frac{1}{2}\frac{l^2}{\beta^2} - V_\varepsilon(\beta) + V_\varepsilon(\rho R_-)\right) - 1} .$$

We observe that v is the zero in case $\beta = R_+$.

Subtracting the energy formula $E_{\varepsilon,\nu} = \frac{1}{2}|\dot{u}_{\varepsilon,l}|^2 - V_\varepsilon(u_{\varepsilon,l})$ evaluated in R_- from the same evaluated in β, we infer

$$\frac{2R_-^2}{l^2} = \frac{1}{(V_\varepsilon(R_-) - V_\varepsilon(\beta) + \frac{1}{2}v^2)}\frac{\beta^2 - R_-^2}{\beta^2}$$

that, replaced into K, implies

$$K = \frac{(\beta - \rho R_-)(\rho - 1)}{\left(\frac{R_-^2\rho^2}{\beta^2} - 1\right) + \rho^2\left(\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\beta) + \frac{v_1^2}{2}}{V_\varepsilon(R_-) - V_\varepsilon(\beta) + \frac{v_1^2}{2}}\right)\frac{\beta^2 - R_-^2}{\beta^2}} .$$

We note that for every $0 < a < b$, the function $f(z) = \frac{a+z}{b+z}$ is increasing for positive z. The condition **ii.** in Definition 1.3 implies that, for every small enough ε, the function $V_\varepsilon(x)$ is decreasing with respect to x for every $x < \beta$. This yields the relations $V_\varepsilon(\beta) < V_\varepsilon(\rho R_-) < V_\varepsilon(R_-)$, thus replacing $a = V_\varepsilon(\rho R_-) - V_\varepsilon(\beta)$, $b = V_\varepsilon(R_-) - V_\varepsilon(\beta)$ and $z = \frac{v_1^2}{2}$ in $f(z)$, it follows

$$\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\beta) + \frac{v^2}{2}}{V_\varepsilon(R_-) - V_\varepsilon(\beta) + \frac{v^2}{2}} > \frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\beta)}{V_\varepsilon(R_-) - V_\varepsilon(\beta)} \qquad \forall \rho \in \left(1, \frac{\beta}{R_-}\right),\ \forall v^2 .$$

Therefore

$$K < \frac{\beta^2}{\frac{(\beta + \rho R_-)}{\rho - 1}\left(\rho^2\left(\frac{V_\varepsilon(\rho R_-) - V_\varepsilon(\beta)}{V_\varepsilon(R_-) - V_\varepsilon(\beta)}\right)\frac{\beta^2 - R_-^2}{\beta^2 - R_-^2\rho^2} - 1\right)}$$

and, by means of the substitutions $x = \rho R_-$ and $y = R_-$,

$$K < \frac{\beta^2}{\frac{(\bar{R} + x)}{\frac{x}{y} - 1}\left(\left(\frac{x}{y}\right)^2\left(\frac{V_\varepsilon(x) - V_\varepsilon(\beta)}{V_\varepsilon(y) - V_\varepsilon(\beta)}\right)\frac{\beta^2 - y^2}{\beta^2 - x^2} - 1\right)} \qquad x \in (y, \beta) .$$

Finally, applying Proposition 3, we obtain the uniform bound

$$K < \beta \quad \forall \rho \in \left(1, \frac{\beta}{R_-}\right) . \tag{19}$$

□

Continue the proof of Theorem 4.2.
The boundedness of the functions K_1 and K_2 and the formula

$$\int_1^\xi \frac{1}{x\sqrt{(x-1)(1-\frac{x}{\xi})}}dx = \pi , \quad \xi > 1 \tag{20}$$

implies that the integral (18) is bounded by $\pi\beta$.

In order to apply the Lebesgue dominated convergence theorem we need an uniform bound of all the integrands, independently on the values of (ε, ν) in the neighborhood of $(0, \bar{\nu})$.

The difficulties arise since the integrand functions are singular at the end-points and, moreover, one of these is not fixed but it changes as (ε, ν) varies. To overcome these problems, we proceed splitting the integral into the sum of two integrals whose integrand is singular in only one end-point. Precisely

$$\begin{aligned}\Delta\theta(u_{\varepsilon,\nu}) &= \int_1^{\frac{\beta}{R_-}} \frac{\sqrt{K}}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}}d\rho \\ &= \int_1^{\sqrt{\frac{\beta}{R_-}}} \frac{\sqrt{K}}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}}d\rho + \int_{\sqrt{\frac{\beta}{R_-}}}^{\frac{\beta}{R_-}} \frac{\sqrt{K}}{\rho\sqrt{(\beta-\rho R_-)(\rho-1)}}d\rho \\ &= I_1 + I_2\end{aligned}$$

and

$$\lim_{(\varepsilon,\nu)\to(0,\bar\nu)} \Delta\theta(u_{\varepsilon,\nu}) = \lim_{(\varepsilon,\nu)\to(0,\bar\nu)} I_1 + \lim_{(\varepsilon,\nu)\to(0,\bar\nu)} I_2 \ .$$

First we calculate I_2 and we check that it is infinitesimal as $(\varepsilon,\nu)\to(0,\bar\nu)$: since K is bounded, $K\leq\beta$,

$$I_2 = \frac{1}{\sqrt{\beta}}\int_{\sqrt{\frac{\beta}{R_-}}}^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(1-\rho\frac{R_-}{\beta})(\rho-1)}}\sqrt{K}d\rho < \int_{\sqrt{\frac{\beta}{R_-}}}^{\frac{\beta}{R_-}} \frac{1}{\rho\sqrt{(1-\rho\frac{R_-}{\beta})(\rho-1)}}d\rho$$

hence, for (20),

$$\begin{aligned}I_2 &< \pi - \int_1^{\sqrt{\frac{\beta}{R_-}}} \frac{1}{\rho\sqrt{(1-\rho\frac{R_-}{\beta})(\rho-1)}}d\rho = \pi - 2\arctan\left[\sqrt{\frac{\rho-1}{1-\frac{\rho R_-}{\beta}}}\right]\Bigg|_1^{\sqrt{\frac{\beta}{R_-}}} \\ &= 2\left(\frac{\pi}{2} - \arctan\sqrt{\frac{\sqrt{\beta}}{\sqrt{R_-}}}\right) = 2\arctan\sqrt{\frac{\sqrt{R_-}}{\sqrt{\beta}}} \approx 2\sqrt[4]{\frac{R_-}{\beta}}\end{aligned}$$

where, in the last passage, the relation $\arctan(x)+\arctan(\frac{1}{x}) = \frac{\pi}{2}$ is used. Passing to the limit, reminding that $R_-\to 0$ as $(\varepsilon,\nu)\to(0,\bar\nu)$, we infer

$$\lim_{(\varepsilon,\nu)\to(0,\bar\nu)} I_2 = 0 \ .$$

It remains to prove the existence of the limit for I_1. We define $F_{\varepsilon,\nu}(\rho):\mathbb{R}^+\to\mathbb{R}^+$ the function

$$F_{\varepsilon,\nu}(\rho) := \frac{1}{\rho\sqrt{(\rho-1)(\beta-\rho R_-)}}\sqrt{K}\ \chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} \ .$$

Again, for (19), it follows that $\forall\rho\in(1,+\infty)$

$$\begin{aligned}F_{\varepsilon,\nu}(\rho) &< \frac{1}{\rho\sqrt{(\rho-1)\left(1-\frac{\rho R_-}{\beta}\right)}}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} \\ &< \frac{1}{\rho\sqrt{(\rho-1)\left(1-\sqrt{\frac{R_-}{\beta}}\right)}}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} < \frac{C}{\rho\sqrt{\rho-1}}\end{aligned}$$

where the constant $C>0$ is independent on (ε,ν). Since all the functions $F_{\varepsilon,\nu}(\rho)$ are dominated by a function $\tilde F\in\mathbb{L}^1([1,\infty])$, by the Lebesgue theorem, we are allowed to pass the pointwise limit of $F_{\varepsilon,\nu}(\rho)$ as $(\varepsilon,\nu)\to(0,\bar\nu)$ under the integral sign and obtain the limit of I_1.

Using K in the form (4.1) we write

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} \frac{1}{\rho\sqrt{(\rho-1)(\beta-\rho R_-)}}\sqrt{K}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]} =$$

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} \frac{1}{\rho}\sqrt{\frac{\beta^2}{(\beta^2-R_-^2\rho^2)\left(\rho^2\frac{\left(V_\varepsilon(\rho R_-)-V_\varepsilon(\beta)+\frac{v^2}{2}\right)}{\left(V_\varepsilon(R_-)-V_\varepsilon(\beta)+\frac{v^2}{2}\right)}\frac{(\beta^2-R_-^2)}{(\beta^2-R_-^2\rho^2)}-1\right)}}\chi_{\left[1,\sqrt{\frac{\beta}{R_-}}\right]}$$

hence

$$F_{\varepsilon,\nu}(\rho) \sim \frac{1}{\rho}\sqrt{\frac{1}{\rho^2\frac{V_\varepsilon(\rho R_-)}{V_\varepsilon(R_-)}-1}} \quad \text{as} \quad (\varepsilon,\nu)\to(0,\bar{\nu}).$$

Therefore for every $V(x)\in\mathcal{V}^*$, thank to Proposition (4), we infer

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} F_{\varepsilon,\nu}(\rho) = \frac{1}{\rho\sqrt{\rho^2-1}}$$

uniformly in ε and l. This yields

$$\lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} \Delta\theta(u_{\varepsilon,\nu}) = \lim_{(\varepsilon,\nu)\to(0,\bar{\nu})} I_1 = \int_1^\infty \frac{1}{\rho\sqrt{\rho^2-1}}d\rho = \frac{\pi}{2}\ .$$

□

Remark 1. The contribution to the apsidal angle due to the portion of the orbit far from the center of attraction is negligible as the angular momentum approaches to zero.

More precisely, for any value $C\in(R_-,R_+)$, the angle $\Delta_C\theta(u)$ covered by the orbit $u(t)$ between the apocentre R_+ and the point where $|u(t)|=C$, tends to zero as the angular momentum vanishes. Indeed, following the same argument as before,

$$\Delta_C\theta(u) = \int_C^{R_+} \frac{1}{r\sqrt{\frac{2r^2}{l^2}(E_{\varepsilon,\nu}+V_\varepsilon(r))-1}}dr < 2\arctan\sqrt{\frac{(R_+-C)R_-}{R_+(C-R_-)}} \to 0$$

as l goes to zero. It follows that the pointwise limit of the sequence of trajectories $u_{\varepsilon,\nu}(t)$ as $(\varepsilon,\nu)\to(0,\bar{\nu})$ is a straight line crossing the origin: this fact, together with the limit value of the apsidal angle obtained in the previous theorem, suggests to extend the collision solution $\bar{u}(t)$ beyond the singularity replacing symmetrically the solution itself forward the collision point in the same direction.

Definition 4.4. Let $\bar{u}(t)$, $t\in[0,T_0)$, be a collision path, T_0 the collision instant. Define the **transmission solution** $u_0(t)$, $t\in[0,2T_0]$ as

$$\begin{cases} u_0(t) = \bar{u}(t) & t\in[0,T_0] \\ u_0(t) = -\bar{u}(2T_0-t) & t\in[T_0,2T_0] \end{cases}$$

To complete the proof of the Main Theorem it remains to show that, for every $t\in[0,2T_0]$, the sequence $\{u_{\varepsilon,\nu}(t)\}$ pointwise converges to $u_0(t)$ as $(\varepsilon,\nu)\to(0,\bar{\nu})$ and that the flow obtained replacing the collision solution $\bar{u}$ with the transmission solution u_0 is continuous with respect to initial data.

4.2. **The regularity of the extended flow.** We denote with $\mathcal{F} = \{y = (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2\}$ the phase space of planar motion and we consider the initial value problems defined on $\mathcal{F}$ equivalent to systems (1) and (2)

$$P(0) = \begin{cases} y' = f(y) \\ x(0) = p \in \mathbb{R}^2 \backslash \{0\} \\ v(0) = q \in \mathbb{R}^2 \end{cases}, \quad P(\varepsilon) = \begin{cases} y' = f_\varepsilon(y) \\ x(0) = p \in \mathbb{R}^2 \\ v(0) = q \in \mathbb{R}^2 \end{cases}$$

where $f(x, v) = (v, \nabla V(|x|))$ and $f_\varepsilon(x, v) = (v, \nabla V_\varepsilon(|x|))$.

For every initial data $\bar{y} = (\bar{q}_0, \bar{p}_0)$, $|\bar{q}_0| \leq \bar{R}$, leading to collision for the system $P(0)$, let $\bar{y}(t) = (\bar{x}(t), \bar{v}(t)) : [0, T_0) \to \mathcal{F}$ be the corresponding singular solution where T_0 denote the collision time and $|\bar{x}(t)| \leq \bar{R}$ for every $t \in [0, T_0)$. We extend $\bar{y}(t)$ according to the Definition 4.4 defining $y_0(t) = (x_0(t), v_0(t))$ as

$$y_0(t) = \bar{y}(t) \qquad t \in [0, T_0)$$

$$y_0(t) = \begin{cases} x_0(t) = -\bar{x}(2T_0 - t) \\ v_0(t) = \bar{v}(2T_0 - t) \end{cases} \qquad t \in (T_0, 2T_0)$$

Let $\Phi_0(y, t) : \mathcal{F} \times \mathbb{R}^+ \to \mathcal{F}$ be used to indicate the extended flow related to system $P(0)$ and $\Phi_\varepsilon(y, t)$ the flow associated to the system $P(\varepsilon)$.

Moreover we denote with $\Phi_T(y, \varepsilon)$ the Poincaré map defined as the solution at time T of the system $P(\varepsilon)$ with initial value y. The first result concerning the regularity of the extension and that conclude the proof of the Main Theorem is the continuity of the Poincaré map for $T \neq T_0$, i.e.

$$\lim_{\substack{y \to \bar{y} \\ \varepsilon \to 0}} \Phi_T(y, \varepsilon) = \Phi_T(\bar{y}, 0) \ .$$

Remark 2. We can not expect the continuity of the Poincaré map in T_0 because, even if the configurations $x_{T_0}(y, \varepsilon)$ would converge to $x_{T_0}(\bar{y}, 0)$, the limit can not be attained by the sequence $v_{T_0}(y, \varepsilon)$, since $v_{T_0}(\bar{y}, 0)$ is unbounded.

For those $T < T_0$ no problem arises, indeed the above limit comes from the classical theorem of continuity with respect to initial data of ordinary differential equations. On the other hand, for $T > T_0$ the continuity of the Poincaré map is stated in the following theorem.

Theorem 4.5. *Let $V(x) \in \mathcal{V}^*$ and suppose $\bar{y} = (\bar{q}_0, \bar{p}_0)$, $|\bar{q}_0| \leq \bar{R}$ be an initial condition leading to collision for the system $P(0)$ at time T_0. Then*

$$\lim_{\substack{y \to \bar{y} \\ \varepsilon \to 0}} \Phi_T(y, \varepsilon) = \Phi_T(\bar{y}, 0)$$

for every $T > T_0$ such that $v(T) \neq 0$.

In the proof of Theorem 4.5 we will need the following classical Lemma.

Lemma 4.6. *Let $H(r, p)$ and $H_0(p)$ be real continuous functions with respect to a set of parameters p, and suppose H be strictly increasing as function of r. Then for every T the function $r = r(T, p)$, implicit solution of equation*

$$T = H_0(p) + H\Big(r(T, p), p\Big)$$

is continuous as function of p.

Proof of Theorem 4.5
As usual we set (r,θ) the polar coordinates of the plane then a point $y=(x,v)$ in the phase space is replaced by

$$x = (r\cos\theta, r\sin\theta)$$
$$v = (\dot r\cos\theta - r\dot\theta\sin\theta, \dot r\sin\theta + r\dot\theta\cos\theta)$$

From the definition of $\Phi_T(y,\varepsilon)$, it remains well defined the set of functions $r_T(y,\varepsilon)$, $\theta_T(y,\varepsilon)$, $\dot r_T(y,\varepsilon)$, $\dot\theta_T(y,\varepsilon)$ denoting, respectively, the values of the radial and angular coordinate and their velocity at time $T\neq T_0$ of a solution of the systems $P(\varepsilon)$ and $P(0)$ with initial data y. The continuity of $\Phi_T(y,\varepsilon)$ is equivalent to the continuity of each one of the previous functions.

Let $\bar y = (\bar r, \bar\theta, \dot{\bar r}, \dot{\bar\theta})$ be an initial data in the phase space leading to collision with nonzero initial velocity, $\dot{\bar r}<0$, and denote with $\bar E = \frac{1}{2}\dot{\bar r}^2 - V(\bar r)$ the energy of the collision solution.

A point $y\in\mathcal{F}$, $y=(r_0,\theta_0,\dot r_0,\dot\theta_0)$, tends to $\bar y$ if it holds

$$|r_0-\bar r|\to 0\ , \qquad |\theta_0-\bar\theta|\to 0 \mod 2\pi$$
$$(E,l)\to(\bar E,0)\ .$$

To begin with, we show the continuity of the function $r_T(y,\varepsilon)$ as $y\to\bar y$ and $\varepsilon\to 0$. The value of $r_T(y,\varepsilon)$ is governed by the equation $\dot r = \sqrt{2(E+V_\varepsilon(r)) - \frac{l^2}{r^2}}$, thus $r_T(y,\varepsilon)$ is a function of the initial position r_0, the couple E, l and the parameter ε. Define $\mathcal{T}_0(E,l,r_0,\varepsilon)$ as the time necessary to the solution $r(y,t)$ to reach the minimal value $R_-=R_-(E,l)$, then

$$\begin{aligned} T &= \int_{R_-}^{r_0} \frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}d\rho + \int_{R_-}^{r_T(y,\varepsilon)} \frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}d\rho \\ &= \mathcal{T}_0(E,l,r_0,\varepsilon) + \mathcal{T}(r_T(y,\varepsilon),E,l) \end{aligned}$$

Claim *The function $\mathcal{T}_0$ and $\mathcal{T}$ are continuous with respect to r_0,E,l,ε.*

Suppose for the moment that the claim is true, since the function $\mathcal{T}$ is strictly increasing with respect to $r_T(y,\varepsilon)$, for Lemma (4.6), the function $r_T(y,\varepsilon)$ is continuous with respect to the set of parameters E,l,r_0,ε. Therefore

$$\lim_{\substack{y\to\bar y\\ \varepsilon\to 0}} r_T(y,\varepsilon) = r_T(\bar y,0)\ .$$

The continuity of $\theta_T(y,\varepsilon)$ is equivalent to the continuity of $\Delta\theta_T(y,\varepsilon) := \theta_T(y,\varepsilon)-\theta_0$. By definition of transmission solution, it holds $\Delta\theta_T(\bar y)=\pi$, hence, for Theorem 4.2 and Remark 1, we gain

$$\lim_{\substack{y\to\bar y\\ \varepsilon\to 0}} \theta_T(y,\varepsilon) = \theta_T(\bar y,0)\ .$$

The continuity of $\dot r_T(y,\varepsilon)$ and $\dot\theta_T(y,\varepsilon)$ follows immediately by the continuity of $r_T(y,\varepsilon)$ and relations

$$\dot r_T(y,\varepsilon) = 2\sqrt{E+V_\varepsilon(r_T(y,\varepsilon)) - \frac{1}{2}\frac{l^2}{r_T(y,\varepsilon)^2}}\ , \qquad \dot\theta_T(y,\varepsilon) = \frac{l}{r_T(y,\varepsilon)^2}\ .$$

□

Proof of the claim

Denoting with p and $\bar p$ the sets of parameters $p = (E, l, r_0, \varepsilon)$ and $\bar p = (\bar E, 0, \bar r, 0)$, we have to show that

$$\lim_{p\to\bar p} \mathcal{T}_0(p) = \mathcal{T}_0(\bar p)\ .$$

We want to apply the dominated convergence theorem and pass the limit under the integral sign in

$$\lim_{p\to\bar p}\int_{R_-}^{r_0} \frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}d\rho\ .$$

To this aim we first exhibit an $\mathbb{L}^1$ bound for the integrand function. We observe that the only singularity for the integrand is in R_- since $|\ddot r|$ and , by continuity, $|\dot r|$ are supposed to be positive. Given $\gamma \in (R_-, r_0)$ we write

$$\begin{aligned}&\int_{R_-}^{r_0} \frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}d\rho\\ &= \int_{R_-}^{\gamma} \underbrace{\frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}}_{I}d\rho + \int_{\gamma}^{r_0} \underbrace{\frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}}_{II}d\rho\ .\end{aligned}$$

The second part II is easily bounded by a constant, while, for what that concerns the first part, using the definition of energy, we rewrite I in the form

$$I = \frac{1}{\sqrt{\dot r_0^2 + \frac{l^2}{r_0^2} - 2V_\varepsilon(r_0) + 2V_\varepsilon(\rho) - \frac{l^2}{\rho^2}}} = \frac{1}{\sqrt{\dot r_0^2 + l^2\left(\frac{\rho^2 - r_0^2}{\rho^2 r_0^2}\right) + 2\big(V_\varepsilon(\rho) - V_\varepsilon(r_0)\big)}}\ .$$

Again from the definition of energy, evaluated in r_0 and in the pericentre R_-, it descends

$$l^2 = \left(\frac{r_0^2 R_-^2}{R_-^2 - r_0^2}\right)\Big(2\big(V_\varepsilon(r_0) - V_\varepsilon(R_-)\big) - \dot r_0^2\Big)$$

that, replaced in the integrand, produces

$$\begin{aligned}I &= \frac{1}{\sqrt{\dot r_0^2\left[1 - \frac{R_-^2(r_0^2-\rho^2)}{\rho^2(r_0^2 - R_-^2)}\right] + \frac{R_-^2}{\rho^2}\left(\frac{r_0^2-\rho^2}{r_0^2-R_-^2}\right)2\big(V_\varepsilon(r_0) - V_\varepsilon(R_-)\big) + 2\big(V_\varepsilon(\rho) - V_\varepsilon(r_0)\big)}}\\ &\le \frac{1}{\sqrt{2\big(V_\varepsilon(\rho) - V_\varepsilon(r_0)\big)}}\frac{1}{\sqrt{1 - \frac{R_-^2}{\rho^2}\left(\frac{r_0^2-\rho^2}{r_0^2-R_-^2}\right)\frac{V_\varepsilon(R_-)-V_\varepsilon(r_0)}{V_\varepsilon(\rho)-V_\varepsilon(r_0)}}}\ .\end{aligned}$$

Proposition 3 and straightforward calculation yield the inequality

$$1 - \frac{R_-^2}{\rho^2}\left(\frac{r_0^2-\rho^2}{r_0^2-R_-^2}\right)\frac{V_\varepsilon(R_-)-V_\varepsilon(r_0)}{V_\varepsilon(\rho)-V_\varepsilon(r_0)} > \frac{r_0(\rho - R_-)}{\rho(r_0+R_-)}$$

provided ε small enough is chosen. It follows

$$I < \frac{C}{\sqrt{\rho - R_-}} \in \mathbb{L}^1(R_-, \gamma)\ .$$

We now apply the Lebesgue theorem and we obtain

$$\lim_{p\to\bar p}\int_{R_-}^{r_0} \frac{1}{\sqrt{2(E+V_\varepsilon(\rho)) - \frac{l^2}{\rho^2}}}d\rho = \int_0^{\bar r}\frac{1}{\sqrt{2(\bar E + V(\rho))}}d\rho$$

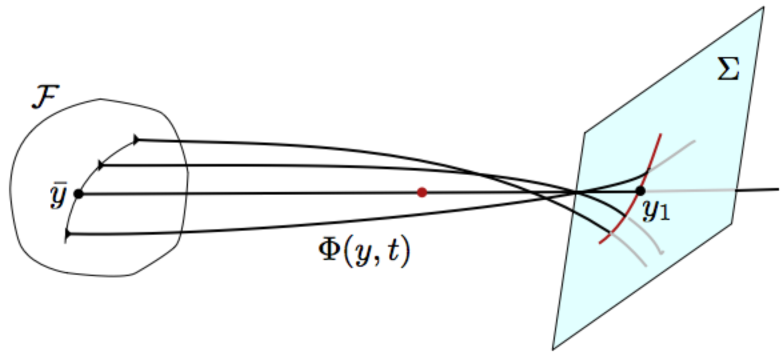


FIGURE 4. Poincaré section

thus the function $\mathcal{T}_0$ and, for the same reason, the function $\mathcal{T}$ are continuous with respect to the set of parameters p.

□

Let again $\bar{y}$ be used to indicate an initial condition leading to collision for the system $P(0)$, T_0 the collision instant and, for $T > T_0$, denote $y_1 = \Phi(\bar{y}, T)$ according to Definition 4.4 of extension of the collision solution. Given Σ an hyperplane in the phase space passing through y_1 and transversal to the flow, we show that, for every initial data y near $\bar{y}$, there exists a time $t = \tau(y)$ when the trajectory $\Phi(y,t)$ intersects the hyperplane Σ. As the data y changes, the trace $S(y) = \Phi\big(y, \tau(y)\big)$ drawn on Σ is called the *Poincaré section* of the flow on Σ, see Figure 4. In the next theorem we prove the continuity of the map $\tau(y)$ and the continuity of the Poincaré section in a neighborhood of $\bar{y}$.

Theorem 4.7. *Let $V(x) \in \mathcal{V}^*$. Then there exists a $\delta > 0$ and a continuous function $\tau(y)$ defined in a δ-neighborhood of $\bar{y}$, $N_\delta(\bar{y})$, such that $\tau(\bar{y}) = T$ and*

$$\Phi(y, \tau(y)) \in \Sigma \ .$$

Moreover the Poincaré section is continuous in $N_\delta(\bar{y})$.

Proof. Given a vector F in the phase space such that $F \cdot f(y_1) > 0$, we consider the hyperplane

$$\Sigma = \{y : (y - y_1) \cdot F = 0\} \ .$$

By definition of y_1 it descends $\big(\Phi(\bar{y}, T) - y_1\big) \cdot F = 0$ and, since

$$\frac{d}{dt}\big(\Phi(\bar{y}, t) - y_1\big) \cdot F\Big|_{t=T} = f(y_1) \cdot F > 0 \ ,$$

there exists an $\xi > 0$ such that

$$\big(\Phi(\bar{y}, T - \xi) - y_1\big) \cdot F < 0 \quad \text{and} \quad \big(\Phi(\bar{y}, T + \xi) - y_1\big) \cdot F > 0 \ .$$

For Theorem (4.5) and for the sign permanence theorem it follows that there exists a $\delta > 0$ such that for every $|\bar{y} - y| < \delta$

$$\big(\Phi(y, T - \xi) - y_1\big) \cdot F < 0 \quad \text{and} \quad \big(\Phi(y, T + \xi) - y_1\big) \cdot F > 0 \ .$$

For every fixed y the function $(\Phi(y,t) - y_1) \cdot F$ is increasing in t: indeed

$$\frac{d}{dt}\big(\Phi(y, t) - y_1\big) \cdot F = f\big(\Phi(y, t)\big) \cdot F \ .$$

Hence the continuity of the vector space $f(y)$ out of collision points together with the continuity of the orbit $\Phi(y,t)$ with respect to both the variables, implies that $f\big(\Phi(y,t)\big)\cdot F>0$. Therefore, for every $y\in N_\delta(\bar y)$, there exists a time $\tau(y)$ satisfying $\Phi\big(y,\tau(y)\big)\in\Sigma$ and $\tau(\bar y)=T$. In order to prove the continuity of $\tau(y)$ we define

$$H(y,\tau)=(\Phi(y,\tau)-y_1)\cdot F$$

then $\tau(y)$ is the implicit solution of $H(y,\tau(y))=0$. Since H is continuous in y and it is continuous and increasing with respect to τ, for Lemma (4.6) the function $\tau(y)$ is continuous. Moreover, for composition of continuous functions we infer the continuity of the Poincaré section.

□

5. **Variational property of the collision solutions.** In this section we join a variational approach that consists in seeking solutions of the system (1) as critical points of the action functional

$$\mathcal{A}(u)=\int_T \mathcal{L}(u,\dot u)dt$$

where

$$\mathcal{L}(u,\dot u)=\frac{1}{2}|\dot u(t)|^2+V(|u(t)|)$$

is the lagrangian function associated to the equation of motion. This method is well known in the literature and it has been extensively exploited in order to find periodic solutions for the N-body problem, see for instance [2, 7, 8] and the references therein. Besides the discussion about the existence of minimal paths of the action functional, an interesting question is whether such minimal paths could have a collision in the interior of their domain. In presence of Coulombic potential and in general of homogeneous potential of weak type, it is known that the collision solutions have finite action but are not minimal paths. If a potential with a weaker singularity is considered, as like the logarithmic potential, one could expect that the contribution of the potential is negligible by a variational point of view. The following theorem shows that this is not the case, indeed, it is proved that, despite of the weakness of the singularity, any collision path is not a minimum for the action functional whenever a potential $V(x)\in\mathcal{V}^*$ is chosen.

Theorem 5.1. *For every $V(x)\in\mathcal{V}^*$, let $u_0:[-T,T]\to\mathbb{R}^2$ be the extension of a collision solution of system (1) according with the Definition 4.4 of transmission solution. Then $u_0(t)$ is not a minimal path for the action functional $\mathcal{A}_{|\Lambda}$, where*

$$\Lambda=\{u(t)\in C([-T,T],\mathbb{R}^2):\dot u\in\mathbb{L}^2([-T,T];\mathbb{R}^2),u(-T)=u_0(-T),u(T)=u_0(T)\}$$

denotes the set of paths joining the end points of u_0.

Proof. Without loss of generality we suppose $t=0$ be the collision instant. We first prove that u_0 has finite action. Since a collision solution has zero angular momentum, the path $u_0(t)$ satisfies the energy integral $E=\frac{1}{2}|\dot u_0|^2-V(|u_0|)$ from which we infer the asymptotic estimate

$$|\dot u_0|^2\sim V(|u_0|) \tag{21}$$

as $|u_0|$ tends to zero. We write the action of $u_0(t)$ as

$$\mathcal{A}(u_0)=2\int_0^T\big(E+2V(|u_0(t)|)\big)dt\le C\int_0^T V(|u_0(t)|)\,dt=C\int_0^R\frac{V(r)}{\dot r}\,dr$$

where $r(t) = |u_0(t)|$ and C is a positive constant. From (21) and (9) it follows $\mathcal{A}(u_0) < +\infty$.

In order to prove that $u_0(t)$ is not the minima for the action $\mathcal{A}$ among all the paths in Λ, we perform a variation on the trajectory u_0 that removes the collision and makes the action decrease.

Let $u_1(t) = u_0(t) + v^\delta(t)$ where $v^\delta(t)$ is the standard variation

$$v^\delta(t) = \begin{cases} \delta & |t| < T_1 \\ \frac{(T-t)}{(T_1-T)}\delta & T_1 < |t| < T \end{cases}$$

directed orthogonally to $u_0(t)$.

Let us compute the difference $\Delta\mathcal{A} = \mathcal{A}(u_0) - \mathcal{A}(u_1)$.

$$\begin{aligned}\Delta\mathcal{A} &= \int_{-T}^{T} \Big(\mathcal{L}(u_0,\dot u_0) - \mathcal{L}(u_1,\dot u_1)\Big)dt \\ &= \int_{-T}^{T} \left(\frac12|\dot u_0|^2 + V(|u_0|)\right) - \left(\frac12|\dot u_1|^2 + V(|u_1|)\right) dt \\ &= \int_{-T}^{T} \frac12\left(|\dot u_0|^2 - |\dot u_1|^2\right) dt + \int_{-T}^{T} V(|u_0|) - V(|u_1|)dt = \Delta\mathcal{K} + \Delta V \ .\end{aligned}$$

We study separately the kinetic and the potential contribute. Since the variation v^δ is directed orthogonally to u_0

$$|\dot u_1(t)|^2 = \begin{cases} |\dot u_0(t)|^2 & |t| \le T_1 \\ |\dot u_0(t)|^2 + \frac{\delta^2}{(T-T_1)^2} & T_1 < |t| \le T \end{cases}$$

then

$$\Delta\mathcal{K} = -2\int_{T_1}^{T} \frac12 \frac{\delta^2}{(T-T_1)^2}dt = -\frac{\delta^2}{(T-T_1)} \ . \tag{22}$$

We show that, for every δ small enough, the contribution of the potential part is larger than the penalising contribution, due to the kinetic part.

$$\begin{aligned}\Delta V &= 2\int_0^{T} V(|u_0|) - V(|u_1|)\,dt > 2\int_0^{T_1} V(|u_0|) - V(|u_1|)\,dt \\ &\ge \int_0^{T_1} V(|u_0|) - V\big(\sqrt{u_0^2+\delta^2}\big)\,dt \ .\end{aligned}$$

For every t fixed

$$V(|u_0|) - V\Big(\sqrt{u_0^2+\delta^2}\Big) = -\int_0^1 \frac{d}{d\xi} V\left(\sqrt{u_0^2+\xi\delta^2}\right) d\xi$$

hence

$$\begin{aligned}\Delta V &\ge 2\int_0^{T_1}\int_0^1 -\frac{V'\big(\sqrt{u_0^2+\xi\delta^2}\big)}{2\sqrt{u_0^2+\xi\delta^2}}\delta^2\,d\xi\,dt = \delta^2\int_0^{T_1}\int_0^1 -\frac{V'\big(\sqrt{u_0^2+\xi\delta^2}\big)}{\sqrt{u_0^2+\xi\delta^2}}\,d\xi\,dt \\ &= \delta^2\int_0^{T_1} f_\delta(t)\,dt = \delta^2 R_\delta \ .\end{aligned}$$

For δ sufficiently small the functions $f_\delta(t)$ are positive and by Fatou's Lemma

$$\lim_{\delta\to0}\int_0^{T_1}\int_0^1 -\frac{V'\big(\sqrt{u_0^2+\xi\delta^2}\big)}{\sqrt{u_0^2+\xi\delta^2}}\,d\xi\,dt \ge \int_0^{T_1}\frac{-V'(|u_0(t)|)}{|u_0(t)|}\,dt$$

Since $|u_0(t)|$ is bounded, for a suitable positive constant C we write

$$\int_0^{T_1} \frac{-V'(|u_0(t)|)}{|u_0(t)|} > C \int_0^{T_1} -V'(|u_0(t)|)\, dt = C\big(\dot r(T_1) - \lim_{t\to 0} \dot r(t)\big) = +\infty \ . \quad (23)$$

The last relation holds because the function $r(t) = |u_0(t)|$ solves the differential equation $\ddot r(t) = -V'(r(t))$, being the collision solution u_0 a radial solution. Combining (22), (23) and (23), for every small enough δ,

$$\Delta\mathcal{A} = \Delta\mathcal{K} + \Delta\mathcal{V} = \delta^2\Big(-\frac{1}{T-T_1} + R_\delta\Big) > 0$$

then we conclude that u_0 is not a minimum for $\mathcal{A}$. □

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