Chapter 6

A Geometric Toolbox for Tetrahedral Finite Elements Partitions

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Abstract: In this work we present a survey of some geometric results on tetrahedral partitions and their refinements in a unified manner. They can be used for mesh generation and adaptivity in practical calculations by the finite element method (FEM), and also in theoretical finite element (FE) analysis. Special emphasis is laid on the correspondence between relevant results and terminology used in FE computations, and those established in the area of discrete and computational geometry (DCG).

Keywords: *finite element method, tetrahedron, polyhedral domain, linear finite element, angle and ball conditions, convergence rate, mesh regularity, discrete maximum principle, mesh adaptivity, red, green and yellow refinements, bisection algorithm*

1 Introduction and Motivation

Many geometric facts about tetrahedra and partitions of polyhedra into tetrahedra are known, and some of them already for quite some time. Even so, with the appearance and permanent growth in speed and capacity of modern computers, together with the practical needs originating from various numerical methods such as the finite element method (FEM), new challenges still appear in this context.

Tetrahedra seem to be the most natural "basic shapes" for dissection or approximation of complicated 3D domains. As a result, constructing tetrahedral partitions and their refinements are among the most challenging problems in finite element discretization of three-dimensional partial differential equations that arise for instance in mathematical physics and engineering. In this survey, we discuss both mathematical and numerical issues related to this topic.

To start, we briefly present two motivating examples. First, it is commonly believed, not only among FEM practitioners but also in discrete and computational geometry (DCG), that the use of near degenerate tetrahedra in a partition should, if possible, be avoided. However, we will point out (see

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Section 4) that not all such tetrahedra are that bad, and, moreover, that some are even unavoidable in certain situations, e.g. for covering thin slots, gaps or strips of different materials (see [1, p. 76]).
Second, note that a single obtuse triangle or tetrahedron in a finite element triangulation can destroy the validity of the discrete maximum principle (DMP) for the Poisson equation $-\Delta u = f$ with homogeneous Dirichlet boundary conditions (see e.g. [2]). For instance, let the domain $(0,4)\times(0,2)$ be triangulated as in Figure **(1)** below. The space of continuous piecewise linear functions relative to this triangulation that satisfy the boundary conditions has dimension three. Their degrees of freedom are the values at the vertices $v_1=(1,1), v_2=(3,1)$, and $v_3=(2,1+p)$, which are indicated with dots. The triangle with vertices v_1,v_2,v_3 is obtuse for all $p\in(0,1)$. It can be easily verified that the discrete Laplacian does not have a non-negative inverse. For example, for $p=\frac{1}{2}$ this inverse equals

$$\begin{pmatrix} \frac{63}{248} & -\frac{1}{248} & \frac{1}{16} \\ -\frac{1}{248} & \frac{63}{248} & \frac{1}{16} \\ \frac{1}{16} & \frac{1}{16} & \frac{37}{160} \end{pmatrix}.$$

Therefore, each non-positive continuous function $f\neq 0$ whose support does not intersect the supports of the finite element functions that vanish at v_1, gives rise to an approximation u_h of u that is positive at v_2, hence violating the DMP (cf. Remark 6.11 below).

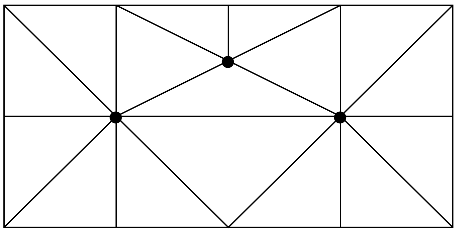


Figure **(1)**: Triangulation with a single obtuse triangle for $p=\frac{1}{2}$.

The above two examples show that the geometric properties of partitions used are important.
In what follows, we assume that $\Omega\subset\mathbf{R}^3$ is a given domain. If the boundary $\partial\overline{\Omega}$ of $\overline{\Omega}$ is contained in a finite number of planes, then $\overline{\Omega}$ is called a *polyhedral domain*. If $\overline{\Omega}$ is bounded, it is called a *polyhedron*. Further, let $L^2(\Omega)$ be the space of square integrable functions over Ω equipped with the standard norm. Sobolev spaces are denoted by $H^s(\Omega)$. The symbol c stands for a generic constant, and vol_d stands for the d-dimensional Euclidean volume.

2 Tetrahedra

2.1 Main geometric characteristics

Let $A=(A_1,A_2,A_3)$, $B=(B_1,B_2,B_3)$, $C=(C_1,C_2,C_3)$, and $D=(D_1,D_2,D_3)$ be points in $\mathbb{R}^3$ that are not contained in one plane. We denote by T the tetrahedron with vertices A, B, C, and D (see Figure **(2)**). It is the simplest closed convex polyhedron, which has 4 triangular faces and 6 edges.
The volume of T can, for example, be calculated by the following formula:

$$\mathrm{vol}_3 T = \frac{|\delta|}{6}, \tag{2.1}$$

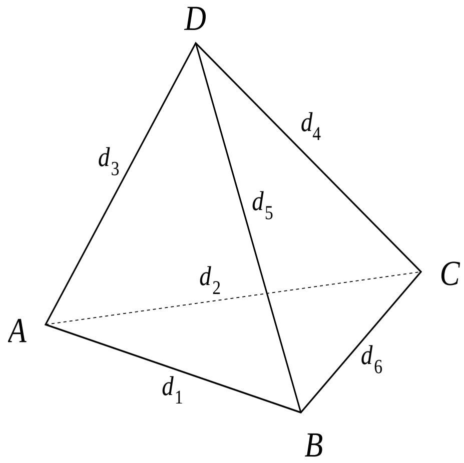


Figure **(2)**: Tetrahedron T with denotation.

where

$$\delta = \det \begin{bmatrix} B_1 - A_1 & B_2 - A_2 & B_3 - A_3 \\ C_1 - A_1 & C_2 - A_2 & C_3 - A_3 \\ D_1 - A_1 & D_2 - A_2 & D_3 - A_3 \end{bmatrix} = \det \begin{bmatrix} 1 & A_1 & A_2 & A_3 \\ 1 & B_1 & B_2 & B_3 \\ 1 & C_1 & C_2 & C_3 \\ 1 & D_1 & D_2 & D_3 \end{bmatrix}, \tag{2.2}$$

see [3, Sect. 6.2]. Further,

$$r_T = \frac{3\,\mathrm{vol}_3 T}{\mathrm{vol}_2 \partial T} \tag{2.3}$$

is the radius of the inscribed ball of T, where ∂T is the boundary of T.
By [4, p. 316], the radius of the circumscribed ball about T can be computed as

$$R_T = \frac{\sqrt{Z_T}}{24\,\mathrm{vol}_3 T}, \tag{2.4}$$

where

$$Z_T = 2d_1^2 d_2^2 d_4^2 d_5^2 + 2d_1^2 d_3^2 d_4^2 d_6^2 + 2d_2^2 d_3^2 d_5^2 d_6^2 - d_1^4 d_4^4 - d_2^4 d_5^4 - d_3^4 d_6^4. \tag{2.5}$$

In the above formula d_i and d_{i+3} are the Euclidean lengths of opposite edges of T for $i = 1, 2, 3$:

$$d_1 = \|A - B\|,\ d_2 = \|A - C\|,\ d_3 = \|A - D\|,\ d_4 = \|C - D\|,\ d_5 = \|B - D\|,\ d_6 = \|B - C\|.$$

See Figure **(2)**.
The dihedral angles of a tetrahedron are the six angles between each pair of faces of T. They are defined as the complementary angles of outward unit normals to those facets and can be calculated by means of the inner product (see [5, p. 385], [6]):

$$\cos\alpha = -n_1 \cdot n_2, \tag{2.6}$$

where n_1 and n_2 are outward unit normals of particular faces.

2.2 On the shapes of tetrahedra

It is common in both FE analysis and DCG to qualitatively distinguish between so-called "well-shaped" (i.e. close to regular) tetrahedra and "badly-shaped" ones (i.e. close to degenerate). Some classifications of badly-shaped tetrahedra are given in [7, p. 191], [8, p. 3], [9, p. 195], [10, p. 286], and [11, p. 256].

The classification in Figure **(3)** is taken from [8, 9], which distinguishes between so-called "skinny" and "flat" badly-shaped tetrahedra, based on closeness of vertices of such tetrahedra to one line (skinny) or to one face (flat). In practice (see [12, p. 794]), the degree of degeneration of a tetra-

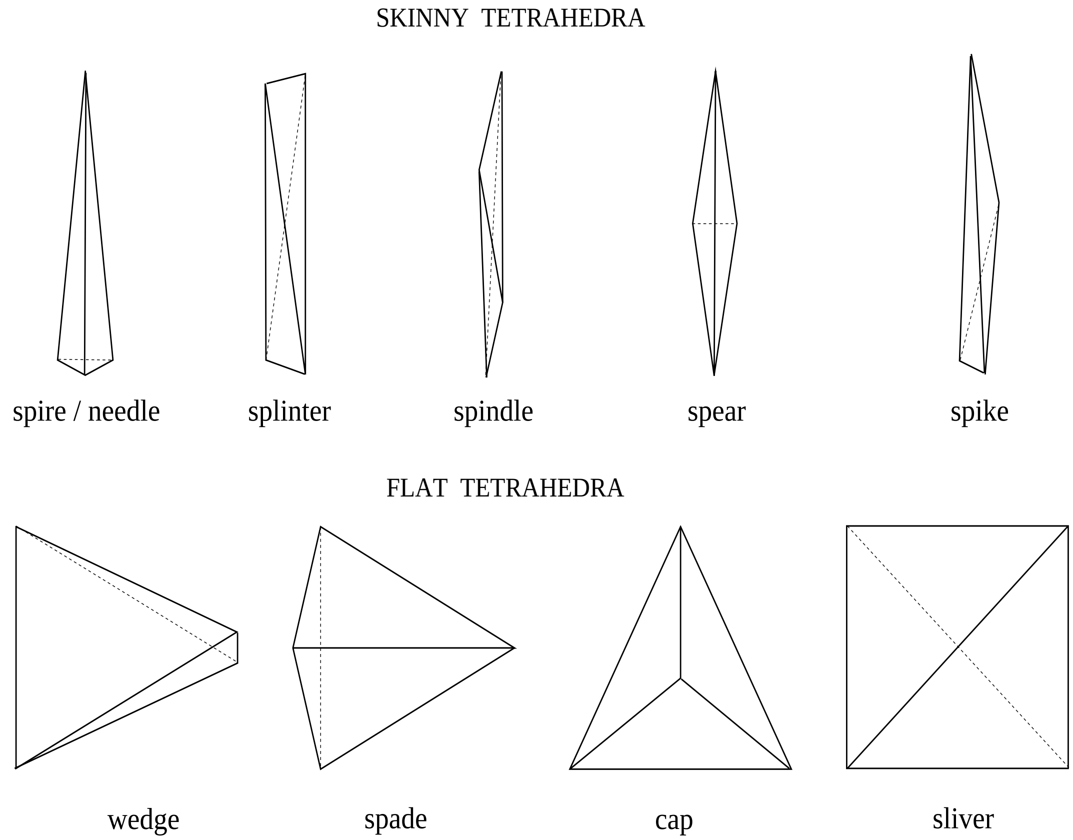


Figure **(3)**: Classification of "badly-shaped" tetrahedra according to [8, 9]. However, some tetrahedra (needles, splinters, wedges) satisfy the maximum angle condition (see (4.5)–(4.6) in below), which guarantees that their shape does not influence the nodal interpolation error (and, therefore, theoretical and also practical convergence of FE approximations) in a negative way.

hedron T is often measured in terms of the *quality indicator*

$$Q_T = 3\frac{r_T}{R_T} \in (0, 1], \tag{2.7}$$

with r_T and R_T defined in (2.3) and (2.4). Tetrahedra with quality indicator Q_T near 1 are almost regular, whereas those with Q_T near 0 are nearly degenerate. Other quality indicators used in DCG and FEMs (and their comparison) can be found e.g. in [9–13].

In Section 4 we shall introduce several regularity conditions in terms of angles and balls that are used in finite element convergence proofs.

3 Tetrahedral Partitions of Polyhedral Domains

3.1 On face-to-face partitions of polyhedra into tetrahedra

Definition 6.1. A finite set of tetrahedra is a (face-to-face) partition of a polyhedron $\overline{\Omega}$ if

i) the union of all the tetrahedra is $\overline{\Omega}$,

ii) the interiors of the tetrahedra are mutually disjoint,

iii) any face of any tetrahedron from the set is either a face of another tetrahedron in the set, or a subset of $\partial\Omega$.

Alternative terminology (commonly used in both FEM and DCG) is a simplicial complex, decomposition, dissection, division, grid, lattice, mesh, net, network, triangulation, space discretization, subdivision, tetrahedralization, etcetera.

Theorem 6.1. *For any polyhedron there exists a partition into tetrahedra.*

The main idea of the detailed constructive proof presented in [1,14] is the following. Denote the faces of a given polyhedron $\overline{\Omega}$ by $F_1, \dots, F_m$. Consider the planes $P_1, \dots, P_m \subset \mathbb{R}^3$ such that

$$F_i \subset P_i,\ i = 1, \dots, m.$$

It can be shown that all components of the set

$$\overline{\Omega} \setminus \cup_{i=1}^m P_i$$

are open convex polyhedra. Their closures can be decomposed into tetrahedra as follows. First, we triangulate each of its polygonal faces as sketched in Figure **(4)**. Second, we take the convex hull of the gravity center of the convex polyhedron with each of the triangles on its surface. If all common faces are triangulated in the same way, a partition of $\overline{\Omega}$ into tetrahedra satisfying the conditions of Definition 6.1 results.

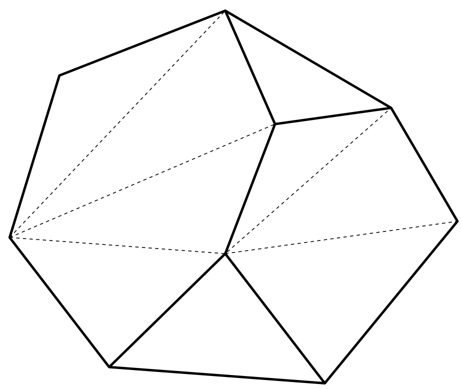


Figure **(4)**: Partition of a convex polyhedron into tetrahedra. Each polygonal face of its surface is divided into triangles.

For a given partition $\mathcal{T}_h$ the *discretization parameter h* stands for the maximum length of all edges in the partition, i.e.,

$$h = \max_{T \in \mathcal{T}_h} h_T,$$

where

$$h_T = \operatorname{diam} T.$$

3.2 Various refinement techniques for tetrahedral partitions

In FE analysis and computation, one needs sequences (infinite or finite) of partitions that have certain properties. They are usually constructed by face-to-face refinements of a given coarse partition [15, 16].

Definition 6.2. An infinite sequence $\mathcal{F} = \{\mathcal{T}_h\}_{h\to 0}$ of partitions of $\overline{\Omega}$ is called a *family of partitions* if for every $\varepsilon > 0$ there exists $\mathcal{T}_h \in \mathcal{F}$ with $h < \varepsilon$.

One can define various kinds of "well-shapedness", usually called regularity, in the sense that certain properties of the tetrahedral elements are supposed to hold uniformly over all partitions of the family.

Definition 6.3. A family $\mathcal{F}$ of partitions is *regular (strongly regular)* if there exists a constant $c > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$ we have

$$\mathrm{vol}_3 T \geq c h_T^3 \quad (\mathrm{vol}_3 T \geq c h^3). \tag{3.1}$$

Remark 6.1. It is easy to construct strongly regular families of triangulations of a polygonal domain into triangles in the sense that $\mathrm{vol}_2 T \geq ch^2$. This is because each triangle can be subdivided into four congruent triangles similar to the original one. Also techniques based on bisection can be used, see e.g. [17] for details. In three dimensions it is generally not possible to subdivide a tetrahedron into congruent tetrahedra similar to the original one (cf. [14]).

Nevertheless, the following theorem is valid.

Theorem 6.2. *For any tetrahedron there exists a strongly regular family of partitions into tetrahedra.*

For a detailed constructive proof see [14], or [1]. The main idea is that the reference tetrahedron $\tilde{T} = \tilde{A}\tilde{B}\tilde{C}\tilde{D}$, whose opposite edges $\tilde{A}\tilde{B}$ and $\tilde{C}\tilde{D}$ have length 2 and the length of the remaining edges is $\sqrt{3}$, can be divided into 8 congruent subtetrahedra which are similar to $\tilde{T}$ (cf. Figure (**5**)). This is the only tetrahedron (up to scaling) with such a property. An arbitrary tetrahedron T can now be decomposed into 8 tetrahedra (cf. Figure (**5**)) using an affine one-to-one mapping between $\tilde{T}$ and T. Such a refinement is called *red*.

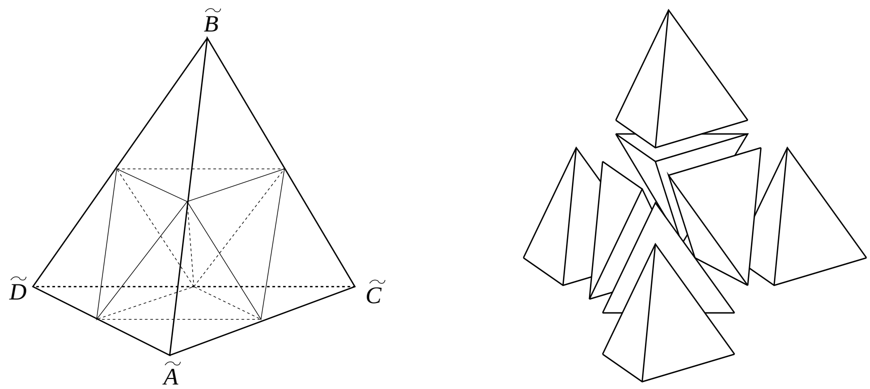


Figure (**5**): Red refinement in 3D.

Remark 6.2. An interesting observation on the performance of the 3D red refinement is presented in [14] and [18]. The convex hull of a vertex of a tetrahedron T with the midpoints of the outgoing edges is a tetrahedron similar to the original one. The octahedron that remains after cutting away the four tetrahedra corresponding to each of the four vertices of T has three spatial diagonals (see Figure (**5**)). Therefore, there are three possibilities for refining a given tetrahedron into 8 subtetrahedra so that its boundary triangles are divided by midlines. However, only choosing the shortest interior diagonal of the octahedron leads to a regular family of tetrahedral face-to-face partitions.

Theorem 6.3. *For any polyhedron there exists a strongly regular family of partitions into tetrahedra.*

Its proof follows immediately from Theorems 6.1 and 6.2.

Local refinements of tetrahedral partitions are needed at those regions in Ω, where singularities or large variations of the solution of PDEs and its derivatives occur. This usually happens near vertices and edges of the polyhedron Ω, or where jumps in coefficients occur, or where the type of boundary condition changes, or near the so-called interfaces (see, e.g., [19, 20]).
In two dimensions, such refinements are usually done with the help of midlines and medians of triangles. Triangles that are divided by midlines are called *red* and by medians *green*, see [21]. The corresponding refinements are also called red and green [22]. Other refinement techniques exist, such as *red** refinement [23], *blue* refinement [24], and *yellow* refinement [25] (see Figure (**6**)).

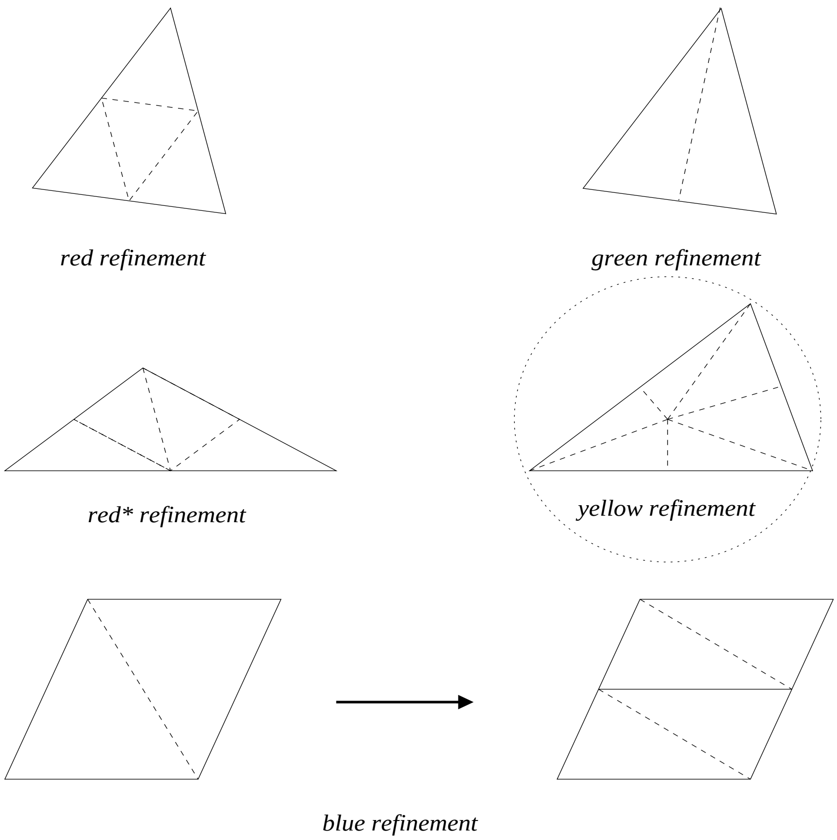


Figure (**6**): Refinement techniques in 2D.

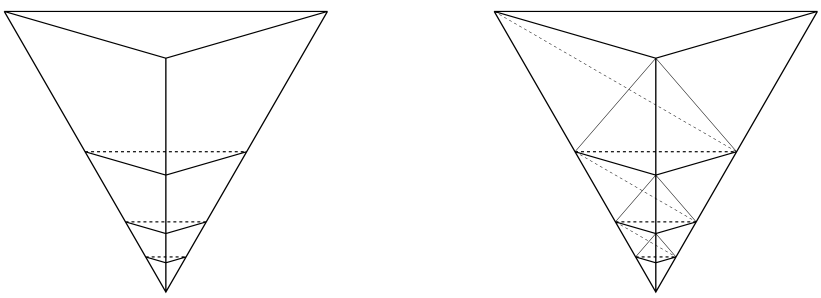
Three-dimensional analogues of green and red refinement are sketched in Figures **(7)** and **(5)**. In Figure **(7)**, we also depict a hybrid red-green refinement: one face of the tetrahedron is divided by midlines and the other faces by medians. A three-dimensional analogue of yellow refinement from Figure **(6)** will be introduced in Section 5.1.
It is worth noting that green refinements from Figures **(6)** and **(7)** are also known in the literature as *bisections*, see e.g. [17,22,26].

green refinement *red-green refinement*

Figure **(7)**: 3D analogues of green and red refinements.

Remark 6.3. In [26], a simple algorithm is presented that generates local refinements of tetrahedral partitions using green and red-green refinement of tetrahedra. They induce a regular family $\mathcal{F}$. Moreover, it can proved that there exists a constant $c > 0$ such that $Q_T \geq c$ for all tetrahedra $T \in \mathcal{T}_h$ and all $\mathcal{T}_h \in \mathcal{F}$, where Q_T is the quality indicator of T defined in (2.7).

Remark 6.4. In Figure **(8)**, we depict another local refinement procedure to treat vertex singularities proposed by B. Guo in [27]. A tetrahedron is first decomposed into one tetrahedron and several pentahedra as in Figure **(8)**a. Then, each pentahedron is decomposed into three tetrahedra as in Figure **(8)**b.



a) b)

Figure **(8)**: Local refinement technique from [27].

In [5] we describe an algorithm that generates local refinements of nonobtuse tetrahedra towards a vertex. The main idea is based on recursive use of Coxeter's trisection [28] on the left of Figure **(9)**.

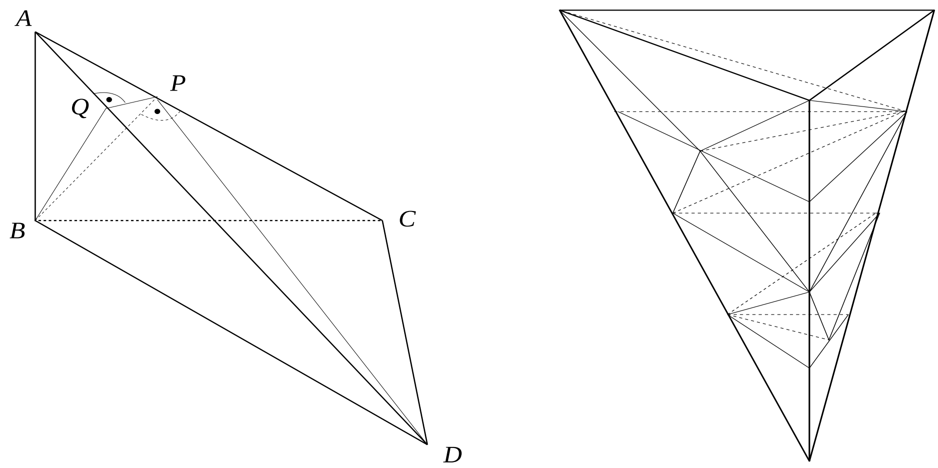


Figure **(9)**: Examples of local nonobtuse refinements towards a vertex.

Remark 6.5. Domains with curved boundaries are usually approximated by polyhedra. Doing this, a so-called variational crime is committed. The remainder of the domain can be handled, e.g. by special curved hat and slice elements. See [29].

Remark 6.6. Uniform or almost uniform partitions usually produce various superconvergence phenomena, see e.g. [30, 31] and references therein.

4 Mesh Regularity: Angle and Ball Conditions in FE Analysis

In [32], one can find results on interpolation estimates for piecewise polynomial functions relative to a family of partitions of the domain, and their relation to the approximation error in FEM. Some of them also follow from the theorems given in the section below.

4.1 The minimum angle condition

The following four regularity conditions for families of simplicial partitions are commonly used in the FE analysis (cf. Section 3.2). The constants c_i in those conditions may depend on the dimension $d \in \{2,3\}$.

Condition 1: There exists $c_1 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$

$$\mathrm{vol}_d\, T \geq c_1 h_T^d\,. \tag{4.1}$$

Condition 2: There exists $c_2 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$ there exists a ball $b \subset T$ with radius r_T such that

$$r_T \geq c_2 h_T\,. \tag{4.2}$$

Condition 3: There exists $c_3 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$

$$\mathrm{vol}_d\, T \geq c_3\, \mathrm{vol}_d\, B, \tag{4.3}$$

where $B \supset T$ is the circumscribed ball about T.

Condition 4: There exists $c_4 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$, any $T \in \mathcal{T}_h$, and any dihedral angle α and, for $d = 3$, also any angle α within a triangular face of T, we have

$$\alpha \geq c_4. \tag{4.4}$$

Theorem 6.4. *The above four regularity conditions are equivalent for $d = 2, 3$.*

The proof can be found in [33]. Condition 2 is sometimes called the *inscribed ball condition* [32]. Condition 4 is usually called the *minimum angle condition*. In the 2D case it was introduced by M. Zlámal in [34].

Remark 6.7. If the quality factor (2.7) is bounded from below by a constant $c > 0$ independently of h, then the family $\mathcal{F}$ of simplicial partitions is regular, since (4.2) is valid:

$$r_T \geq \frac{c}{3} R_T \geq \frac{c}{6} h_T$$

as $2R_T \geq h_T$.

4.2 Maximum angle condition

Definition 6.4. A family $\mathcal{F}$ of partitions of a polyhedron into tetrahedra is said to be *semiregular* if there exist a $c_5 < \pi$ such that for any $\mathcal{T}_h \in \mathcal{F}$, any $T \in \mathcal{T}_h$, any dihedral angle γ between faces of T and any angle φ within a triangular face of T, we have

$$\gamma \leq c_5, \tag{4.5}$$

$$\varphi \leq c_5. \tag{4.6}$$

The maximum angle condition (4.6) for triangles was first introduced by J. L. Synge [35] and for tetrahedra first by M. Křížek [36].

Theorem 6.5. *Any regular family of partitions of a polyhedron into tetrahedra is semiregular.*

For the proof see [36], where it is also shown that the converse implication does not hold. Semiregular families can contain needles, wedges, and splinters of arbitrary thinness. See Figure **(3)**.
For any tetrahedron T and function $v \in C(T)$, we write $\pi_T v$ for the nodal Lagrange linear interpolant of v on T, further, $\|\cdot\|_{k,\infty,T}$ is the norm and $|\cdot|_{k,\infty,T}$ is the seminorm in the Sobolev space $W^{k,\infty}(T)$.

Theorem 6.6. *Let $\mathcal{F}$ be a semiregular family of partitions of a polyhedron into tetrahedra. Then there exists $c_6 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $T \in \mathcal{T}_h$ we have*

$$\|v - \pi_T v\|_{1,\infty,T} \leq c_6 h_T |v|_{2,\infty,T} \quad \forall v \in C^2(T). \tag{4.7}$$

For the proof see [1, pp. 85–87].

Remark 6.8. With a sliver tetrahedron (cf. Figure **(3)**)

$$A = (-h,0,0), \quad B = (0,h^3,-h), \quad C = (h,0,0), \quad D = (0,h^3,h),$$

we see that (4.6) holds, since $\varphi < \frac{\pi}{2}$, but (4.5) is violated for $h \to 0$. Similarly we observe that (4.6) is not valid and that (4.5) holds for $h \to 0$ if we consider a spike tetrahedron:

$$A = (0,0,0), \quad B = (h,0,0), \quad C = (h,0,h^3), \quad D = (-h,h^3,0).$$

These two examples show that conditions (4.5) and (4.6) are independent. In both the examples $\pi_T v$ loses its optimal order error behavior (4.7). See [36].

Remark 6.9. Theorem 6.6 shows that some badly-shaped tetrahedra preserve the optimal interpolation properties. They can therefore be safely used to fill narrow gaps and slots, see e.g. [1, p. 76], and also [37–43].

Remark 6.10. The maximum angle condition represents only a sufficient condition for the convergence of linear finite elements due to Theorem 6.6 and the famous Céa's lemma. According to [41], this condition is not necessary for the convergence of the FEM.

5 Discrete Maximum Principles for Linear Tetrahedral Finite Elements

The FEM uses piecewise polynomials to approximate solutions of partial differential equations. If these solutions satisfy certain maximum principles, it is desirable that their finite element approximations satisfy their discrete analogues (called discrete maximum principles, or DMPs in short). Nonobtuse and acute tetrahedral partitions indeed yield finite element approximations that satisfy DMPs for several elliptic [44–49] and parabolic problems [50–52] by means of continuous piecewise linear functions.

A key observation in this context is that the gradient of a non-zero linear function on a simplex T that vanishes on a face F_j of T is a constant non-zero normal to F_j. Hence, the sign of inner products between pairs of gradients of two distinct functions on T with this property is in one-to-one correspondence with the type of dihedral angle. To be more explicit, for $d \geq 1$ we have the following expression, which was derived in [44] directly from [5] and [49],

$$(\nabla v_i)^\top \nabla v_j = -\frac{\mathrm{vol}_{d-1} F_i \, \mathrm{vol}_{d-1} F_j}{(d \, \mathrm{vol}_d T)^2} \cos \alpha_{ij}, \quad i,j = 1,\dots,4, \quad i \neq j, \tag{5.1}$$

where α_{ij} is the dihedral angle between F_i and F_j, and v_ℓ is the linear function that vanishes on F_ℓ and has value one at the vertex B_ℓ opposite F_ℓ (see Figure **(10)**). A similar expression was introduced in [53].

Basically, the discrete Laplacian that results from the standard finite element method has a nonnegative inverse if each of the above inner products in the partition is non-positive for distinct i and j, which is the case for nonobtuse partitions. If the partition is in fact acute, the discrete Laplacian has a positive inverse and then reaction terms of small enough size can be handled using perturbation arguments. See for instance the papers [44, 54] where the presence of a reaction term in a reaction-diffusion problem led to the condition that the partition should be acute and the diameters of the simplices small enough.

Remark 6.11. It is also of practical interest that the DMP holds in order to avoid negative numerical values of nonnegative physical quantities like concentration, temperature, density, and pressure, see e.g. [45] for some real-life examples. Also a discrete heat flux may have an opposite sign than the continuous flux when the DMP is violated.

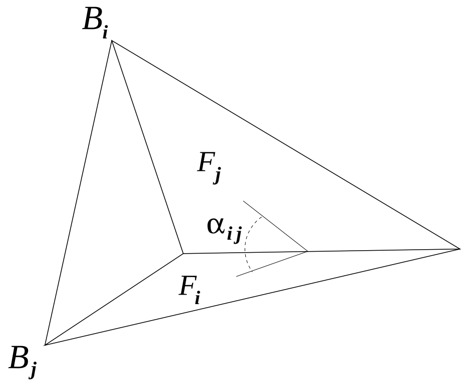


Figure **(10)**: Illustration for the above formula (5.1).

5.1 Nonobtuse tetrahedral partitions and their refinements

To increase the accuracy of FE calculations, we often need to perform various global or local refinements of the partitions. In this context, the techniques presented in Section 3 can be used. However, if we are interested in the preservation of the DMP on more refined partitions, then we should be able to guarantee the preservation of geometric properties of acuteness or nonobtuseness in the refining process.

For convenience, in Figure **(11)** we present several examples of nonobtuse tetrahedra, which are also mentioned in what follows. The left one, called a *path tetrahedron*, has three mutually orthogonal edges that form a path (in the sense of graph theory), the middle one, called a *cube corner tetraheron*, has three mutually orthogonal edges that share a common vertex.

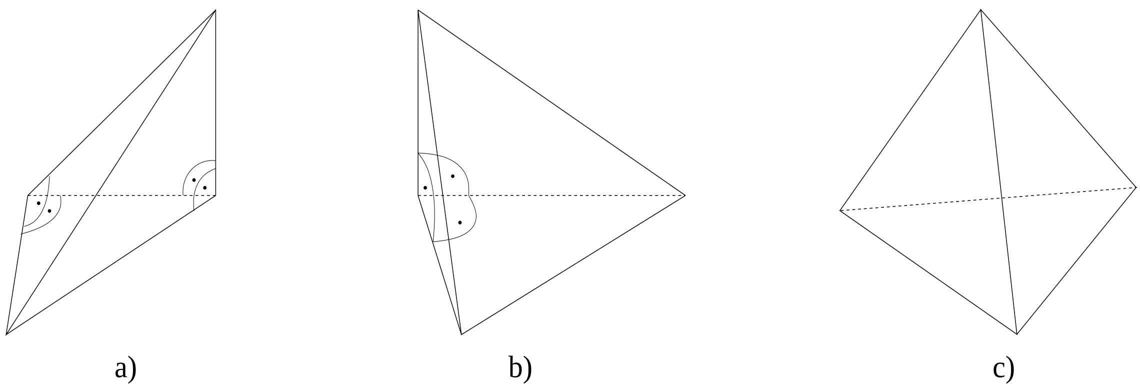


Figure **(11)**: Examples of nonobtuse tetrahedra: a) path, b) cube corner, and c) regular.

In [25], we presented sufficient conditions for the existence of partitions into path-tetrahedra with an arbitrarily small mesh size, as formulated in the following theorem.

Theorem 6.7. *Let each tetrahedron in a given nonobtuse partition of a polyhedron contain its circumcenter. Then there exists a family of partitions into path-tetrahedra.*

Its proof is constructive. Each face is first partitioned into four or six right triangles whose common vertex is the center of its circumscribed circle. Then each tetrahedron from the initial partition is divided into path-tetrahedra, by taking the convex hulls of the right triangles on its surface with its circumcenter (see Figure **(12)** (left)). Such a refinement technique is called *yellow* (cf. Figure **(6)**). In this case, common faces of adjacent tetrahedra from the initial partition are partitioned in the same manner. The proof then proceeds by induction.

Remark 6.12. In [55] the nonobtuseness assumption in Theorem 6.7 is replaced by a weaker condition that requires that only faces are nonobtuse. This enables us to apply the above technique also to degenerated tetrahedra (like needles, wedges, slivers, and splinters).

One technique for local nonobtuse tetrahedral refinements (towards a vertex) is presented in Figure **(12)**, see [56] for details.

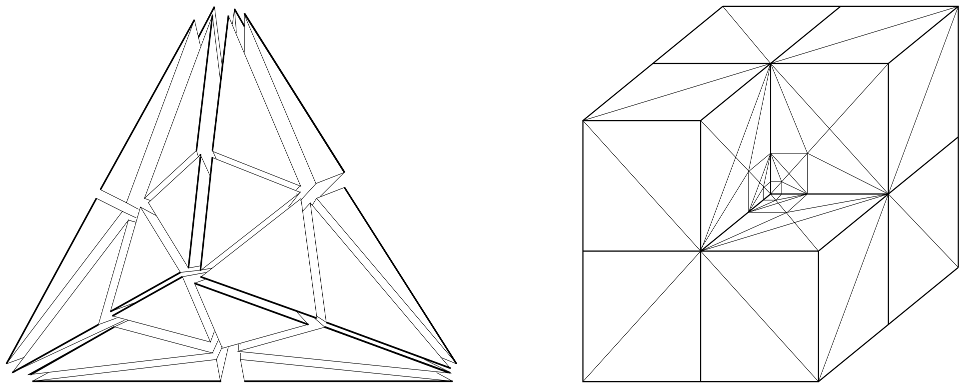


Figure **(12)**: Global and local nonobtuse tetrahedral refinements from [25, 56].

Further, we present the key idea and also an illustration from the recent work [57] (see Figures **(13)** and **(14)**) on nonobtuse tetrahedral refinements towards a flat face of (or interface inside) the solution domain. For this purpose we take a square prism (e.g. a cube) and its adjacent square prism. Denote their vertices and some other nodes as sketched in Figure **(13)**, where also partitions of some faces are given.

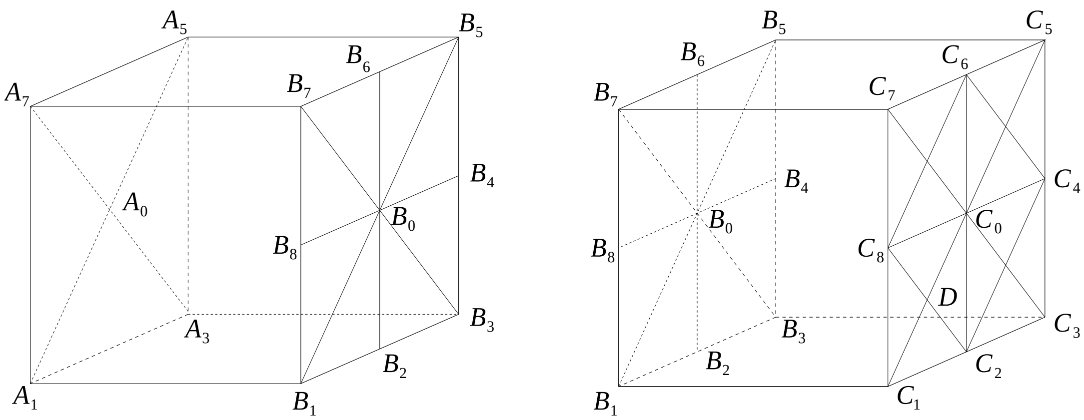


Figure **(13)**: A sketch of a decomposition of two adjacent square prisms into nonobtuse tetrahedra.

In what follows, let $s = |B_1B_3| = |B_3B_5|$ denote the lengths of the edges of the square faces of the prisms, and let $l_1 = |A_0B_0|$ and $l_2 = |B_0C_0|$ be their thicknesses.

First, we decompose the left square prism $A_1A_3A_5A_7B_1B_3B_5B_7$ of Figure **(13)** into four triangular prisms whose common edge is A_0B_0. Second, we decompose each triangular prism into four tetrahedra. For instance, the triangular prism $A_0A_1A_3B_0B_1B_3$ will be divided in the following way (see Figure **(14)**):

$A_0A_1A_3B_0$ (cube corner tetrahedron), $A_1B_1B_2B_0$ (path tetrahedron),
$A_3B_3B_2B_0$ (path tetrahedron), and $A_1A_3B_0B_2$.

The first three resulting tetrahedra are clearly nonobtuse. The last tetrahedron $A_1A_3B_0B_2$ is the union of two path tetrahedra whose common face is $A_2B_0B_2$, where A_2 is the midpoint of A_1A_3. We see that $A_1A_3B_0B_2$ is nonobtuse if and only if

$$|B_1B_3| \leq 2|A_0B_0|, \quad \text{i.e.} \quad l_1 \geq \frac{s}{2}. \tag{5.2}$$

The other three triangular prisms, $A_0A_3A_5B_0B_3B_5$, $A_0A_5A_7B_0B_5B_7$, and $A_0A_1A_7B_0B_1B_7$, can be subdivided similarly.

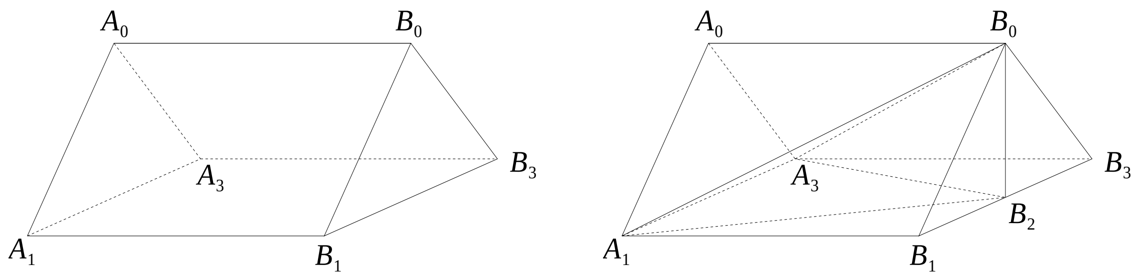


Figure **(14)**: Decomposition of a triangular prism $A_0A_1A_3B_0B_1B_3$ into four tetrahedra.

Next, we decompose the right adjacent square prisms $B_1B_3B_5B_7C_1C_3C_5C_7$ of Figure **(13)** into eight triangular prisms whose common edge is B_0C_0. Further, the triangular prism $B_0B_1B_2C_0C_1C_2$ will be divided into four tetrahedra like in the previous step:

$B_0B_1B_2C_2$ (cube corner tetrahedron), $B_0C_0DC_2$ (path tetrahedron),
$B_1C_1DC_2$ (path tetrahedron), and $B_0B_1DC_2$.

The last tetrahedron is nonobtuse provided

$$|B_0B_1| \leq 2|B_0C_0|, \quad \text{i.e.} \quad l_2 \geq \frac{\sqrt{2}s}{4}. \tag{5.3}$$

This condition is necessary and sufficient to guarantee a nonobtuse decomposition of the triangular prism $B_0B_1B_2C_0C_1C_2$ into four nonobtuse tetrahedra as described above.
The other seven triangular prisms can be divided into nonobtuse tetrahedra similarly. In this way (i.e., under conditions (5.2) and (5.3)) we get a face-to-face nonobtuse partition of two adjacent square prisms. The left square prism of Figure **(13)** is subdivided into 16 and the right prism into 32 nonobtuse tetrahedra. This enables us to form layers and use this process repeatedly (see examples in [57]).

5.2 On acute tetrahedral partitions and their refinements

The following theorem states a relationship between dihedral angles and angles in triangular faces.

Theorem 6.8. *Let ABCD be an acute tetrahedron. Let α be the dihedral angle at the edge AD and let φ be the angle $\angle BAC$ with vertex at A. Then*

$$\varphi < \alpha. \tag{5.4}$$

For the proof, see [58, p. 384].

A similar theorem holds also for nonobtuse tetrahedra, but the inequality $<$ in (5.4) must be replaced by $\leq$. For obtuse tetrahedra, the inequality does not hold (cf. Remark 6.8). Such theorems can be used in the construction of acute and nonobtuse partitions of $\mathbf{R}^3$.

Remark 6.13. The first algorithm to partition the whole space $\mathbf{R}^3$ into acute tetrahedra was given in [59]. Later, in [60], four more algorithms were given, together with an acute tetrahedral partition of slabs. Recently, in [61], also the cube was partitioned into acute tetrahedra. Finally, in [62] all other Platonic solids were acutely partitioned.

Remark 6.14. Note that small enough perturbations of acute partitions remain acute. This is the not the case for nonobtuse partitions. Further properties of acute partitions are given in our survey paper [2].

6 Generalizations to Higher Dimensions

Several natural generalizations of the previous geometric results to higher dimensions are presented in below.

A *simplex* S in $\mathbf{R}^d$ is a convex hull of $d+1$ points, $A_1, A_2, \ldots, A_{d+1}$, that do not belong to the same hyperplane. We denote by h_S the length of the longest edge of S. Let F_i be the $(d-1)$-dimensional facet of a simplex S opposite to the vertex A_i and let v_i be the altitude from the vertex A_i to the facet F_i. Formula (2.3) for the radius of the inscribed ball of S can be easily generalized to an arbitrary space dimension, namely

$$r_S = \frac{d\,\mathrm{vol}_d S}{\mathrm{vol}_{d-1}\partial S}. \tag{6.1}$$

By [63, 64], or [65, p. 125], the volume of a d-simplex S can be computed in terms of lengths of its edges using the so-called Cayley-Menger determinant of size $(d+2)\times(d+2)$

$$D_d = (-1)^{d+1} 2^d (d!)^2 (\mathrm{vol}_d S)^2 = \det \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & a_{12}^2 & \cdots & a_{1d}^2 & a_{1,d+1}^2 \\ 1 & a_{21}^2 & 0 & \cdots & a_{2d}^2 & a_{2,d+1}^2 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 1 & a_{d+1,1}^2 & a_{d+1,2}^2 & \cdots & a_{d+1,d}^2 & 0 \end{bmatrix}, \tag{6.2}$$

where a_{ij} is the length of the edge A_iA_j for $i \neq j$.
The radius R_S of the circumscribed ball B satisfies (see [66])

$$R_S^2 = -\frac{1}{2}\frac{\Delta_d}{D_d}, \tag{6.3}$$

where

$$\Delta_d = \det \begin{bmatrix} 0 & a_{12}^2 & \cdots & a_{1d}^2 & a_{1,d+1}^2 \\ a_{21}^2 & 0 & \cdots & a_{2d}^2 & a_{2,d+1}^2 \\ \vdots & \vdots & & \ddots & \vdots \\ a_{d+1,1}^2 & a_{d+1,2}^2 & \cdots & a_{d+1,d}^2 & 0 \end{bmatrix}.$$

Let $\Omega \subset \mathbf{R}^d$ be a domain. If the boundary of the closure $\partial\overline{\Omega}$ of $\overline{\Omega}$ is contained in a finite number of $(d-1)$-dimensional hyperplanes, we say that $\overline{\Omega}$ is *polytopic*. Moreover, if $\overline{\Omega}$ is bounded, it is called a *polytope*; in particular, $\overline{\Omega}$ is called a *polygon* for $d = 2$ and a *polyhedron* for $d = 3$.

We shall again consider only face-to-face simplicial partitions of a polytope $\overline{\Omega}$ and their families $\mathcal{F}$.

Condition $1'$**:** There exists $c_1 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\mathrm{vol}_d\, S \geq c_1 h_S^d. \tag{6.4}$$

Condition $2'$**:** There exists $c_2 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\mathrm{vol}_d\, b \geq c_2 h_S^d, \tag{6.5}$$

where $b \subset S$ is the inscribed ball of S.

Condition $3'$**:** There exists $c_3 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$ and any $S \in \mathcal{T}_h$ we have

$$\mathrm{vol}_d\, S \geq c_3\, \mathrm{vol}_d\, B, \tag{6.6}$$

where $B \supset S$ is the circumscribed ball about S.

Condition $4'$**:** There exists $c_4 > 0$ such that for any $\mathcal{T}_h \in \mathcal{F}$, any $S \in \mathcal{T}_h$, and any $i \in \{1, 2, \ldots, d+1\}$ we have

$$\sin_d(\hat{A}_i | A_1 A_2 \ldots A_{d+1}) \geq c_4, \tag{6.7}$$

where

$$\sin_d(\hat{A}_i | A_1 A_2 \ldots A_{d+1}) = \frac{d^{d-1}\,(\mathrm{vol}_d S)^{d-1}}{(d-1)!\,\Pi_{j=1, j\neq i}^{d+1} \mathrm{vol}_{d-1} F_j}. \tag{6.8}$$

Theorem 6.9. *Conditions* $1'$, $2'$, $3'$, *and* $4'$ *are equivalent.*

For the proof see [67] and [68]. If one of the conditions holds, then the family $\mathcal{F}$ of simplicial partitions is called *regular*.
Formula (5.1) can be rewritten as follows:

$$(\nabla v_i)^\top \nabla v_j = -\frac{\cos\alpha_{ij}}{h_i h_j}, \quad i, j = 1, \ldots, d+1, \quad i \neq j, \tag{6.9}$$

where h_i is the height in S above F_i and α_{ij} are dihedral angles between facets F_i and F_j. Their definition is similar to (2.6).

Many other results from the previous sections have been generalized to any dimension, for instance, local nonobtuse simplicial refinements towards a vertex [5], superconvergence phenomena [30], the maximum angle condition [42], the discrete maximum principle [44–47, 51].

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