

UNIQUENESS PROPERTIES FOR DISCRETE EQUATIONS AND CARLEMAN ESTIMATES

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ABSTRACT. Using Carleman estimates, we give a lower bound for solutions to the discrete Schrödinger equation in both dynamic and stationary settings that allows us to prove uniqueness results, under some assumptions on the decay of the solutions.

1. INTRODUCTION

The aim of this paper is to continue the study started in [9, 10] to prove uniqueness properties for functions $u \in C^1([0, 1], \ell^2(\mathbb{Z}^d))$ such that they satisfy the property

$$(1) \quad |i\partial_t u_j + \Delta_d u_j| \leq |V_j u_j|, \quad t \in [0, 1], \quad j \in \mathbb{Z}^d.$$

which bounded potential V , under the assumptions that the function u has fast decay at times $t = 0$ and $t = 1$. Here Δ_d stands for the discrete Laplace operator

$$\Delta_d u_j = \sum_{k=1}^d (u_{j+e_k} + u_{j-e_k} - 2u_j), \quad j \in \mathbb{Z}^d.$$

In particular, all the results we give can be written in terms of solutions to the discrete Schrödinger equation

$$(2) \quad i\partial_t u_j + \Delta_d u_j + V_j u_j = 0, \quad t \in [0, 1], \quad j \in \mathbb{Z}^d.$$

In the continuous case, these results are related to Hardy uncertainty principle for the Fourier transform:

$$|f(x)| \leq C e^{-|x|^2/\beta^2}, \quad |\hat{f}(\xi)| \leq C e^{-4|\xi|^2/\alpha^2}, \quad \text{and } 1/\alpha\beta > 1/4 \implies f \equiv 0.$$

$$\text{If } 1/\alpha\beta = \frac{1}{4}, \text{ then } f(x) = c e^{-|x|^2/\beta^2}.$$

The relation comes from the fact that basically the solution to the free Schrödinger equation, $i\partial_t u + \Delta u = 0$, has the same size as the Fourier transform of an appropriately modulated initial datum, so then Hardy uncertainty principle can be stated, in an L^2 setting, as follows:

$$\|e^{\alpha|x|^2} u(0)\|_{L^2(\mathbb{R}^d)} + \|e^{\beta|x|^2} u(1)\|_{L^2(\mathbb{R}^d)} < +\infty, \quad \alpha\beta > \frac{1}{16} \implies u \equiv 0.$$

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The classical proof of Hardy uncertainty principle is based on complex analysis arguments (Phragmén-Lindelöf principle and properties of entire functions), but in the dynamical context there is a series of papers, [2, 3, 5–7], where the authors prove Hardy uncertainty principle using real variable methods. Furthermore, not only do they prove their results for the free evolution, but they also include a potential term Vu to the Schrödinger equation, under some size constraints for the potential V and without any regularity assumption on the potential needed. The main techniques in the proof of their results are log-convexity properties for solutions with Gaussian decay and Carleman estimates.

In the discrete setting, the first thing we have to understand is how to replace the Gaussian decay, so in [9] we give an analogous version of Hardy uncertainty principle by using complex analysis arguments that suggests that the discrete version of the Gaussian we have to consider is the product of modified Bessel functions, given by the following integral representation,

$$I_m(x) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos(m\theta), \quad m \in \mathbb{Z}.$$

This product of modified Bessel functions appears naturally if we understand the Gaussian as the minimizer of Heisenberg uncertainty principle. When we take discrete versions of the position and momentum operators to give a discrete Heisenberg principle, it turns out that the minimizer is precisely the product of modified Bessel functions. On the other hand, we can also understand the Gaussian as the fundamental solution to the heat equation, and, again, the fundamental solution in the discrete setting is given in terms of modified Bessel functions, whose decay is, for n large,

$$(3) \quad I_n(z) \sim \frac{1}{\sqrt{2\pi n}} \left(\frac{ez}{2}\right)^n e^{-n \log n},$$

and we see here that for n large, depending on the argument z $I_n(z)$ behaves like $e^{-n \log n}$.

Once we have a discrete version of Hardy uncertainty principle proved by complex analysis, we proved in [10] some log-convexity properties for solutions to the discrete Schrödinger equation with discrete Gaussian decay, where not only did we understand the Gaussian decay as above, but we also use other discrete versions of the Gaussian function. Then, by using Carleman estimates we can only give a preliminary result, that says that a solution cannot decay faster than $e^{-\mu|j|^2}$ at two different times. Looking at the behavior of modified Bessel functions, it is clear that this is far from the sharp result. Actually, independently in [11] it has recently been proved, for bounded and real-valued potentials, that in the one-dimensional case a solution cannot decay faster than $e^{-\mu|j| \log |j|}$ for $\mu > \frac{3+\sqrt{3}}{2}$.

In this paper we try a different approach, exploited in [3, 4] in the continuous setting for Schrödinger and KdV equations and also based in log-convexity properties and Carleman estimates, in order to improve the result in [10]. The main difference comes from the fact that first we prove the following lower bound for the solution:

Theorem 1.1 (Lower bound for solutions to Schrödinger equation). *Let $u \in C^1([0, 1] : \ell^2(\mathbb{Z}^d))$ satisfying (1) such that*

$$\int_0^1 \sum_{j \in \mathbb{Z}^d} |u_j(t)|^2 dt \leq A^2, \quad \int_{1/2-1/8}^{1/2+1/8} |u(0, t)|^2 dt \geq 1,$$

and

$$\|V\|_\infty = \sup_{t \in [0, 1], j \in \mathbb{Z}^d} \{|V_j(t)|\} \leq L,$$

then there exists $R_0 = R_0(d, A, L) > 0$ and $c = c(d)$ such that for $R \geq R_0$ it follows that

$$\lambda(R) \equiv \left(\int_0^1 \sum_{R-2 \leq |j| \leq R+1} |u(j, t)|^2 \right)^{1/2} \geq ce^{-cR \log R}.$$

Then, by the use of similar log-convexity properties to those proved in [10] we deduce the following result:

Theorem 1.2 (Uniqueness result). *Let $u \in C^1([0, 1] : \ell^2(\mathbb{Z}^d))$ satisfying (1) with V a bounded potential. Then there exists $\mu_0 = \mu_0(d)$ such that if, for $\mu > \mu_0$*

$$\sum_{j \in \mathbb{Z}^d} e^{2\mu|j| \log(|j|+1)} (|u_j(0)|^2 + |u_j(1)|^2) < +\infty,$$

then $u \equiv 0$.

Notice that in the one-dimensional setting this result agrees with the one in [11], although here we obtain worst constants. On the other hand, the good behavior of the function $e^{\alpha|j|^2}$ in \mathbb{Z}^d allows us to give a similar result in higher dimensions.

However, the approach we use here is not suitable if one wants to relate this result to the continuous Hardy uncertainty principle. In order to do that, we should include the mesh step of the lattice, typically denoted by h , and study what happens when h tends to zero. In that case, the relevant region that relates the discrete and continuous results is of the type $|j|h \lesssim 1$. Now, the Carleman inequality (see Lemma 2.1 below), which is the key element in this approach holds as long as one assumes that $\frac{|j|}{R} \geq 1$ with R large enough. If we identify the role of $\frac{1}{R}$ with the role of h , we clearly see that this result is giving information in the region that is not related to the continuous setting, and, therefore, this result is purely discrete. In order to give a relation between the discrete and the continuous settings, the appropriate weights for the Carleman estimates and log-convexity properties should be related to the modified Bessel function, and, as in [5], some interior estimates on the gradient should be required.

On the other hand, all the arguments can be adapted to the stationary case, where now we consider functions $u \in \ell^2(\mathbb{Z}^d)$ such that

$$(4) \quad |\Delta_d u_j| \leq V_j u_j, \quad j \in \mathbb{Z}^d,$$

so we also have a lower bound in the ring $\{R-2 < |j| < R+1\}$ and a uniqueness result. In this case, it seems that the result obtained (see Corollary 3.1) can be improved, although we will see here that with this method this is not the case. Furthermore, in [1] it is proved a lower bound for solutions to the continuous elliptic problem but instead of the ring $\{R-2 < |j| < R+1\}$, the lower bound is attained in a ball of radius 1 centered at some point of the sphere of radius R . In that paper it is pointed out that such a lower bound is not known in the discrete setting and it is most likely to be false, since one can extend the identically 0 function in a ball in such a way that its discrete Laplacian is zero but the extended function is not zero. For the sake of completeness, we will construct an example of a $\ell^2(\mathbb{Z}^2)$ solution to $\Delta_d u_j + V_j u_j = 0$ with bounded potential that is 1 at the origin but vanishes on a ball of fixed radius centered at a sphere of radius R , for R large enough.

Nevertheless, the same method explained here can be used in the continuous setting, and as an easy application one can get uniqueness properties for solutions to the elliptic problem

$\Delta u + Vu = 0$, assuming that the potential is bounded and that a solution decays faster than $e^{-\eta|x|^{4/3}}$ for some $\eta > 0$, which coincides with the sharp result proved in [13].

The paper is organized as follows: In Section 2 we prove the main results of the paper, the lower bound and the uniqueness property for the time-dependant problem (1). In Section 3 we see that, as opposed to the continuous case, using this method we cannot improve the rate of decay from the evolution problem to the stationary one, and we also give an example of solution which shows that the behavior explained in [1] is not possible in the discrete setting.

2. UNIQUENESS FOR SOLUTIONS TO DISCRETE SCHRÖDINGER EQUATIONS

Before proving the main results, we need a discrete Carleman inequality for discrete Schrödinger evolutions in the spirit of the method developed in [3,4]. In the case of Schrödinger, the condition in the Carleman parameter is $\alpha \geq cR^2$, which leads to the Gaussian decay as the sharp rate of decay. In the discrete setting, the analogous of this condition is $\alpha \geq cR \log R$, so in this case we should look for solutions with this rate of decay, instead of solutions with Gaussian decay.

In order to simplify the notation, we write the notation $\|\cdot\|_2 = \|\cdot\|_{L^2([0,1], \ell^2(\mathbb{Z}^d))}$.

Lemma 2.1 (Carleman inequality for Schrödinger evolutions). *Assume that $R > 0$ and $\varphi : [0,1] \rightarrow \mathbb{R}$ is a smooth function. Then, there exists $c = c(d, \|\varphi'\|_\infty + \|\varphi''\|_\infty)$ such that, the inequality*

$$\sqrt{\sinh(2\alpha/R^2)} \sinh(2\alpha/\sqrt{d}R) \|e^{\alpha|\frac{j}{R} + \varphi(t)e_1}|^2 g\|_2 \leq c \|e^{\alpha|\frac{j}{R} + \varphi(t)e_1}|^2 (i\partial_t + \Delta_d)g\|_2$$

holds, when $\alpha \geq cR \log R$ and $g \in C_0^1([0,1], \ell^2(\mathbb{Z}^d))$ has its support contained in the set

$$\{(j, t) : |j/R + \varphi(t)e_1|^2 \geq 1\}.$$

Proof. Let $f_j = e^{\alpha|\frac{j}{R} + \varphi(t)e_1}|^2 g_j$. If we write

$$e^{\alpha|\frac{j}{R} + \varphi(t)e_1}|^2 (i\partial_t + \Delta_d)g_j = Sf_j + Af_j,$$

with S and A symmetric and skew-symmetric respectively, it turns out that (δ_{mn} denotes Kronecker's delta function)

$$(5) \quad \begin{aligned} Sf_j &= i\partial_t f_j - 2df_j + \sum_{k=1}^d \cosh\left(\frac{2\alpha}{R}\left(\frac{j_k + 1/2}{R} + \varphi\delta_{1k}\right)\right) f_{j+e_k} \\ &\quad + \sum_{k=1}^d \cosh\left(\frac{2\alpha}{R}\left(\frac{j_k - 1/2}{R} + \varphi\delta_{1k}\right)\right) f_{j-e_k}, \end{aligned}$$

$$(6) \quad \begin{aligned} Af_j &= -2i\alpha\left(\frac{j_1}{R} + \varphi\right)\varphi'f_j - \sum_{k=1}^d \sinh\left(\frac{2\alpha}{R}\left(\frac{j_k + 1/2}{R} + \varphi\delta_{1k}\right)\right) f_{j+e_k} \\ &\quad + \sum_{k=1}^d \sinh\left(\frac{2\alpha}{R}\left(\frac{j_k - 1/2}{R} + \varphi\delta_{1k}\right)\right) f_{j-e_k}. \end{aligned}$$

Moreover, an easy computation shows that

$$\|e^{\alpha|\frac{j}{R} + \varphi(t)e_1}|^2 (i\partial_t + \Delta_d)g\|_2^2 = \langle Sf + Af, Sf + Af \rangle \geq \langle [S, A]f, f \rangle.$$

After some calculations, we have that the commutator is given so that

$$(7) \quad \begin{aligned} \langle [S, A]f, f \rangle = & 4 \sinh\left(\frac{2\alpha}{R^2}\right) \int \sum_{j \in \mathbb{Z}^d} \sum_{k=1}^d \sinh^2\left(\frac{2\alpha}{R}\left(\frac{j_k}{R} + \varphi \delta_{1k}\right)\right) |f_j|^2 \\ & + 4 \sinh\left(\frac{2\alpha}{R^2}\right) \int \sum_{j \in \mathbb{Z}^d} \sum_{k=1}^d \left| \frac{f_{j+e_k} - f_{j-e_k}}{2} \right|^2 + 2\alpha \int \sum_{j \in \mathbb{Z}^d} \left[\left(\frac{j_1}{R} + \varphi\right) \varphi'' + (\varphi')^2 \right] |f_j|^2. \\ & + \frac{8\alpha}{R} \int \sum_{j \in \mathbb{Z}^d} \varphi' \cosh\left(\frac{2\alpha}{R}\left(\frac{j_1 + 1/2}{R} + \varphi\right)\right) \Im(f_{j+e_1} \overline{f_j}). \end{aligned}$$

We want to hide the third and fourth term in the last expression in a fraction of the positive terms. We note first that this is something that only depends on the first component of j , so we may assume that $d = 1$. For the last term we are going to see that

$$\sinh\left(\frac{2\alpha}{R^2}\right) \sinh^2\left(\frac{2\alpha}{R}\left(\frac{j}{R} + \varphi\right)\right) \geq \frac{8\alpha \|\varphi'\|_\infty}{R} \cosh\left(\frac{\alpha}{R^2}\right) \cosh\left(\frac{2\alpha}{R}\left(\frac{j}{R} + \varphi\right)\right),$$

and then using Cauchy-Schwarz inequality we can hide this term. First, if $\alpha \geq R^2$ then the inequality follows from the equivalent inequality

$$e^{\frac{2\alpha}{R^2} + \frac{4\alpha}{R} \left|\frac{j}{R} + \varphi\right|} \geq \frac{16\alpha \|\varphi'\|_\infty}{R} e^{\frac{\alpha}{R^2} + \frac{2\alpha}{R} \left|\frac{j}{R} + \varphi\right|} \iff e^{\frac{\alpha}{R^2} + \frac{2\alpha}{R} \left|\frac{j}{R} + \varphi\right|} \geq c \frac{\alpha}{R},$$

and looking at the behavior of both sides, when $\left|\frac{j}{R} + \varphi\right| \geq 1$ this holds. On the other hand if $\alpha \leq R^2$, then the behavior of $\sinh\left(\frac{2\alpha}{R^2}\right)$ and $\cosh\left(\frac{2\alpha}{R^2}\right)$ is different from before and what we need in order to hide the term is

$$e^{\frac{2\alpha}{R} \left|\frac{j}{R} + \varphi\right|} \geq 8R \|\varphi'\|_\infty,$$

and this holds when $\alpha \geq cR \log R$ for some c depending on the dimension and $\|\varphi'\|_\infty$.

For the third sum in the expression of the commutator, we need to see that

$$\sinh\left(\frac{2\alpha}{R^2}\right) \sinh^2\left(\frac{2\alpha}{R}\left(\frac{j}{R} + \varphi\right)\right) \geq 2\alpha \|\varphi''\|_\infty \left|\frac{j}{R} + \varphi\right|.$$

In this case, if $\alpha \geq R^2$ it is clear that this holds. On the other hand, if $cR \log R \leq \alpha \leq R^2$ what we really need is

$$R^{4c} e^{\frac{4\alpha}{R} \left|\frac{j}{R} + \varphi\right|} \geq R^2 \|\varphi''\|_\infty \left|\frac{j}{R} + \varphi\right|,$$

and this holds since $\left|\frac{j}{R} + \varphi\right| \geq 1$ and we can choose c large enough (depending only on the allowed parameters) so that the inequality, and therefore the lemma, holds. In higher dimensions, notice that the condition on the support tells us that there is $k \in \{1, \dots, d\}$ such that $\left|\frac{j_k}{R} + \varphi \delta_{1k}\right| \geq \frac{1}{\sqrt{d}}$ and this explains the factor $\sinh(2\alpha/\sqrt{d}R)$ in the statement. \square

Now we are going to use this lemma to prove the lower bound for a nonzero solution to the discrete Schrödinger equation. We recall that the fundamental solution to this equation is related to the Bessel function, $J_n(z) = I_n(-iz)i^n$, whose decay, for fixed z , is of the type $e^{-n \log n}$ when $n \rightarrow +\infty$, as in formula (3).

Proof of Theorem 1.1. We define the $C^\infty(\mathbb{R}^d)$ cut-off functions $\theta^R(x), \mu(x)$ ($0 \leq \theta^R, \mu \leq 1$) and the $C^\infty([0, 1])$ function φ ($0 \leq \varphi \leq 3$) in the following way.

$$(8) \quad \theta^R(x) = \begin{cases} 1, & |x| \leq R-1, \\ 0, & |x| \geq R, \end{cases} \quad \mu(x) = \begin{cases} 1, & |x| \geq 2, \\ 0, & |x| \geq 1, \end{cases} \quad \varphi(t) = \begin{cases} 3, & t \in [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}], \\ 0, & t \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]. \end{cases}$$

We are going to apply the previous lemma to

$$g_j(t) = \theta_j^R \mu \left(\frac{j}{R} + \varphi(t)e_1 \right) u_j(t),$$

where $\theta_j^R = \theta^R(j)$. Notice that the evolution of g is given by the expression

(9)

$$\begin{aligned} (i\partial_t + \Delta_d)g_j &= i\varphi' \theta_j^R \partial_{x_1} \mu \left(\frac{j}{R} + \varphi e_1 \right) u_j + \theta_j^R \mu \left(\frac{j}{R} + \varphi e_1 \right) (i\partial_t u_j + \Delta_d u_j) \\ &+ \sum_{k=1}^d \left[\theta_{j+e_k}^R \left(\mu \left(\frac{j+e_k}{R} + \varphi e_1 \right) - \mu \left(\frac{j}{R} + \varphi e_1 \right) \right) + \mu \left(\frac{j}{R} + \varphi e_1 \right) (\theta_{j+e_k}^R - \theta_j^R) \right] u_{j+e_k} \\ &+ \sum_{k=1}^d \left[\theta_{j-e_k}^R \left(\mu \left(\frac{j-e_k}{R} + \varphi e_1 \right) - \mu \left(\frac{j}{R} + \varphi e_1 \right) \right) + \mu \left(\frac{j}{R} + \varphi e_1 \right) (\theta_{j-e_k}^R - \theta_j^R) \right] u_{j-e_k}. \end{aligned}$$

Thus, by paying with a dimensional constant c_d and using (1) we have

(10)

$$\begin{aligned} \sqrt{\sinh(2\alpha/R^2) \sinh(2\alpha/\sqrt{d}R)} \|e^{\alpha|\frac{j}{R} + \varphi e_1}|^2 g\|_2 &\leq \|e^{\alpha|\frac{j}{R} + \varphi e_1}|^2 (i\partial_t + \Delta_d)g\|_2 \\ &\leq L \|e^{\alpha|\frac{j}{R} + \varphi e_1}|^2 g\|_2 + c_d \left(\int_0^1 \sum_{j \in \mathbb{Z}^d} e^{2\alpha|\frac{j}{R} + \varphi e_1}|^2 \left| \partial_{x_1} \mu \left(\frac{j}{R} + \varphi e_1 \right) \right|^2 |u_j|^2 dt \right)^{1/2} \\ &+ c_d \left(\int_0^1 \sum_{j \in \mathbb{Z}^d} \sum_{k=1}^d e^{2\alpha|\frac{j}{R} + \varphi e_1}|^2 \left| \mu \left(\frac{j+e_k}{R} + \varphi e_1 \right) - \mu \left(\frac{j}{R} + \varphi e_1 \right) \right|^2 |u_{j+e_k}|^2 dt \right)^{1/2} \\ &+ c_d \left(\int_0^1 \sum_{j \in \mathbb{Z}^d} \sum_{k=1}^d e^{2\alpha|\frac{j}{R} + \varphi e_1}|^2 \left| \mu \left(\frac{j-e_k}{R} + \varphi e_1 \right) - \mu \left(\frac{j}{R} + \varphi e_1 \right) \right|^2 |u_{j-e_k}|^2 dt \right)^{1/2} \\ &+ c_d \left(\int_0^1 \sum_{j \in \mathbb{Z}^d} \sum_{k=1}^d e^{2\alpha|\frac{j}{R} + \varphi e_1}|^2 |\theta_{j+e_k}^R - \theta_j^R|^2 |u_{j+e_k}|^2 dt \right)^{1/2} \\ &+ c_d \left(\int_0^1 \sum_{j \in \mathbb{Z}^d} \sum_{k=1}^d e^{2\alpha|\frac{j}{R} + \varphi e_1}|^2 |\theta_{j-e_k}^R - \theta_j^R|^2 |u_{j-e_k}|^2 dt \right)^{1/2}. \end{aligned}$$

Now we study carefully the support of each term and we finish the proof taking $\alpha = cR \log R$ with $c = c(d)$ a constant satisfying the statement of the Carleman inequality. Indeed, with this choice of α the product of sinh functions in the left-hand side takes the form, for R large,

$$(11) \quad \sqrt{2c \log RR} R^{\frac{2c}{\sqrt{d}} - \frac{1}{2}},$$

and, therefore, if c is large enough, only depending on the dimension, this grows with R , so we can absorb the term that comes from the potential taking R large enough, depending on L .

Now, by the definition of θ^R and μ , we see that if $|j| \leq R - 1$ and $t \in [1/2 - 1/8, 1/2 + 1/8]$ then $|\frac{j}{R} + \varphi_{e_1}| \geq 3 - \frac{R-1}{R} = 2 + \frac{1}{R}$, so the cut-off functions are 1 and $g_j = u_j$. This allows us to bound the left-hand side of the Carleman inequality of the lemma by

$$\|e^{\alpha|\frac{j}{R} + \varphi_{e_1}|^2} g\|_2 \geq e^{\alpha(2 + \frac{1}{R})^2},$$

since $\int_{1/2-1/8}^{1/2+1/8} |u(0, t)|^2 \geq 1$.

On the other hand, we can use again the support of the cut-off functions to bound each term of the right-hand side in (10). In the first term, the one which involves a derivative of the function μ , we see that in its support $|\frac{j}{R} + \varphi_{e_1}| \leq 2$. For the terms involving the difference of θ functions, for each $k \in \{1, \dots, d\}$ we need to compute the coefficients where $\theta_{j \pm e_k}^R \neq \theta_j^R$, that is, we have to distinguish three different cases. First, when $\theta_{j \pm e_k}^R = 1$, but $\theta_j^R \neq 1$, then when $\theta_{j \pm e_k}^R = 0$, but $\theta_j^R \neq 0$, and finally when $0 < \theta_{j \pm e_k}^R < 1$, where since $|j \pm e_k| \neq |j|$ we have that $\theta_{j \pm e_k}^R \neq \theta_j^R$. After doing a shift in each terms, it is easy to see that the difference is not zero in a region included in $\{R - 2 < |j| < R + 1\}$, and we have that $|\frac{j}{R} + \varphi_{e_1}| \leq 4 + \frac{1}{R}$. Observe that when we get rid of the terms where this difference of θ functions vanishes and we bound the corresponding term in the Carleman inequality we get the term $\lambda(R)$ of the statement.

Finally, we have to see what happens with the terms involving the difference of μ . In this case, we will see that if $|\frac{j}{R} + \varphi_{e_1}| \geq 2 + \frac{1}{R}$, then the function μ takes the same value at both points, so the difference is 0. Indeed, in this case we have that $\mu(\frac{j}{R} + \varphi_{e_1}) = 1$ and $|\frac{j}{R} + \varphi_{e_1}| - \frac{1}{R} > 0$.

Therefore,

$$\left| \frac{j \pm e_k}{R} + \varphi_{e_1} \right| \geq \left| \frac{j}{R} + \varphi_{e_1} \right| - \frac{1}{R} \geq 2 + \frac{1}{R} - \frac{1}{R} = 2,$$

so $\mu(\frac{j \pm e_k}{R} + \varphi_{e_1}) = 1$ as well. Gathering all these results we have, when $\alpha = cR \log R$,

$$(12) \quad \sqrt{\sinh\left(\frac{2c \log R}{R}\right)} \sinh\left(\frac{2c \log R}{\sqrt{d}}\right) e^{cR \log R(2 + \frac{1}{R})^2} \leq c_d \left(e^{cR \log R(4 + \frac{1}{R})^2} \lambda(R) + e^{cR \log R(2 + \frac{1}{R})^2} A \right).$$

So for R large enough, depending on A (recall that before we showed that R depends on L as well) we can absorb the second term in the right-hand side in the left-hand side and conclude

$$1 \leq \sqrt{2c \log R R^{\frac{2c}{\sqrt{d}} - \frac{1}{2}}} \leq c_d e^{13cR \log R} \lambda(R).$$

□

We finish this section by proving the uniqueness result Theorem 1.2. For the sake of completeness, we recall the following result which proof for solutions to the discrete Schrödinger equation can be found in [10]. We recall that its proof is based in an abstract argument (see [5, Lemma 2]), and it is clear that the same proof works for functions satisfying (1).

Lemma 2.2. *Assume that u is satisfies (1) where V is a time-dependent bounded potential. Then, for $t \in [0, 1]$ and $\beta \in \mathbb{R}^d$ we have*

$$(13) \quad \sum_{j \in \mathbb{Z}^d} e^{2\beta \cdot j} |u_j(t)|^2 \leq e^{C\|V\|_\infty} \sum_{j \in \mathbb{Z}^d} e^{2\beta \cdot j} (|u_j(0)|^2 + |u_j(1)|^2),$$

where C is independent of β , provided the left-hand side is finite.

From this result we can get a large variety of log-convexity properties for different weights, just by multiplication of (13) with a proper function and integrating with respect to β , as it is explained in [8]. For example, if we multiply it by the function $\exp(-2 \cosh(\beta/\mu)/e)$, we have that

$$\int_{\mathbb{R}} e^{j\beta - 2 \cosh(\beta/\mu)/e} d\beta = K_{\mu j} \left(\frac{2}{e} \right) \sim c \sqrt{2\mu} |j| e^{\mu|j| \log |j| + \mu|j| \log \mu},$$

so the growth of this function is given by $e^{\mu|j| \log |j|}$ and we can adapt this to the multidimensional case to end up with a function that grows as $e^{\mu \|j\|_{\star}}$, where $\|j\|_{\star} = \sum_{k=1}^d |j_k| \log(|j_k| + 1)$. On the other hand, it is easy to check that there is a dimensional constant c_d such that

$$\frac{\|j\|_{\star}}{c_d} \leq |j| \log(|j| + 1) \leq c_d \|j\|_{\star},$$

hence, combining these two facts we have the following corollary.

Corollary 2.1. *Assume that $u = (u_j)_{j \in \mathbb{Z}^d}$ satisfies (1) where V is a time-dependent bounded potential. Then, for $\mu > 0$ and $t \in [0, 1]$, there are constants $c_0 = c_0(d)$ and $c > 0$ independent of μ and t such that*

$$\sum_{j \in \mathbb{Z}^d} e^{2\mu c_0 |j| \log(|j|+1)} |u_j(t)|^2 \leq e^{c \|V\|_{\infty}} \sum_{j \in \mathbb{Z}^d} e^{2\mu |j| \log(|j|+1)} (|u_j(0)|^2 + |u_j(1)|^2),$$

provided that the right-hand side is finite.

Proof of Theorem 1.2. If u is not zero, by translation and dilation, we may assume that u satisfies

$$\int_{1/2-1/8}^{1/2+1/8} |u(0, t)|^2 dt \geq 1,$$

so that we can apply the previous theorem to find a lower bound for $\lambda(R)$. On the other hand, using the previous corollary we have that

$$\sup_{t \in [0, 1]} \sum_{j \in \mathbb{Z}^d} |u(j, t)|^2 e^{2\mu c_0 |j| \log |j|} < +\infty.$$

Hence, from this property for the solution, we prove the following upper bound for the quantity $\lambda(R)$ defined above,

$$\lambda(R) \leq c e^{-\mu c_0 R \log R},$$

while, by the previous theorem we know that $\lambda(R) \geq c e^{-cR \log R}$ for some c depending on the dimension. Therefore, if μ is large enough (we need μ to be larger than the quotient $\frac{c}{c_0}$, so it only depends on the dimension) by letting $R \rightarrow \infty$ we reach a contradiction, so $u \equiv 0$. \square

3. UNIQUENESS FOR THE STATIONARY PROBLEM

Now we turn to the stationary problem in (1), and we get uniqueness from Theorem 1.2 as an immediate consequence, since, obviously, a stationary function satisfying (1) satisfies $|\Delta_d u_j| \leq |V_j u_j|$. Moreover, we can give a stationary version of Lemma 2.1, just by taking $\varphi \equiv 3$, which makes the commutator positive.

As a result we have the following corollary:

Corollary 3.1. *Let $u \in \ell^2$ such that $|\Delta_d u_j| \leq |V_j u_j|$,*

$$\|u\|_2 \leq A, \quad |u(0)| \geq 1, \quad \text{and } \|V\|_\infty \leq L.$$

Then there exist $R_0 = R_0(d, L, A)$ and $c = c(d)$ such that for $R \geq R_0$

$$\lambda(R) \equiv \left(\sum_{R-2 \leq |j| \leq R+1} |u_j|^2 \right)^{1/2} \geq c e^{-cR \log R}.$$

Furthermore, there is η_0 depending on the dimension such that if

$$\sum_{j \in \mathbb{Z}^d} e^{2\eta_0 |j| \log(|j|+1)} |u_j|^2 < +\infty,$$

then $u \equiv 0$.

Remark 3.1. In this section, as opposed to the previous one, we consider $\|\cdot\|_2$ and $\|\cdot\|_\infty$ as the ℓ^2 and ℓ^∞ norms of a sequence.

As we have pointed out above, since the commutator in the Carleman inequality for the discrete Laplacian is positive, we do not need the extra assumption $\alpha \geq cR \log R$ required in the time-dependent case. Moreover, remember that in the application of the Carleman inequality, if we consider $\alpha = cR \log R$ the behavior of the left-hand side (11) grows with R , so one could think that this result is most likely not sharp, and that we could consider $\alpha = cR\phi(R)$ with $1 < \phi(R) < \log R$ and such that the product of the sinh functions in the Carleman inequality still grows with R or we can make it independent of R but as large as we want. However, in order to absorb the term that comes from the potential V we require that

$$\sinh(2\alpha/R^2) \sinh^2(2\alpha/\sqrt{d}R) \geq L^2,$$

so, if $\alpha = cR\phi(R)$ with $1 < \phi(R) < \log(R)$ we have that this is equivalent to

$$\log(2c) + \log(\phi(R)) + \frac{4c}{\sqrt{d}}\phi(R) - \log(R) \geq L^2,$$

and, in order to satisfy this inequality, we need $\phi(R)$ to behave as $\log(R)$ when R is large.

This result shows that the discrete and continuous settings exhibit different behaviors. On the one hand in the discrete setting the rate of decay is given by the exponential $e^{-c|j| \log(|j|+1)}$ while in the continuous case the solutions can decay faster. On the other hand, whereas in the discrete setting we get the same rate of the decay in our results, in the continuous setting the evolution and the stationary problems exhibit different rates of decay. In the case of Schrödinger, the decay is given in terms of Gaussians, while for the elliptic problem in [13] is stated that the sharp rate of decay is of the type $e^{-cR^{4/3}}$. Actually using the same method we have used here in the discrete setting and Lemma 3.1 below, one can prove a lower bound for solutions to the problem $(\Delta + V)u = 0$ and the lower bound is precisely given in terms of $e^{-cR^{4/3}}$.

Lemma 3.1. *The inequality*

$$\frac{\alpha^{3/2}}{R^2} \|e^{\alpha|\frac{x}{R}+3e_1}|^2 g\|_2 + \frac{\alpha^{1/2}}{R} \|e^{\alpha|\frac{x}{R}+3e_1}|^2 \nabla g\|_2 \leq \|e^{\alpha|\frac{x}{R}+3e_1}|^2 \Delta g\|_2$$

holds, when $g \in H^2(\mathbb{R}^d)$ has its (compact) support contained in the set $\{|x/R + 3e_1|^2 \geq 1\}$.

Theorem 3.1. *Let $u \in H^2$ solution to $\Delta u + Vu = 0$ such that*

$$\|V\|_\infty \leq L, \quad \int_{|x|<1} |u(x)|^2 dx \geq 1, \quad \int (|u(x)|^2 + |\nabla u(x)|^2) dx \leq A^2,$$

then, there is $c = c(L, A)$ such that

$$\lambda(R) \equiv \left(\int_{R-1 < |x| < R} (|u(x)|^2 + |\nabla u(x)|^2) dx \right)^{1/2} \geq ce^{-cR^{4/3}}.$$

Furthermore, there is η_0 depending on the dimension such that if

$$\int_{\mathbb{R}^d} e^{2\eta_0|x|^{4/3}} |u(x, t)|^2 dx < +\infty,$$

then $u \equiv 0$.

Moreover, in [13] is constructed a counterexample of solution with this rate of decay, which implies that the decay is sharp. It would be interesting to know if the sharp decay in the discrete setting is $e^{-|j|\log(|j|+1)}$ by constructing a similar counterexample.

Nevertheless, in [1] the lower bound given in Theorem 3.1 is improved in the sense that it is given in a smaller region. Instead of the ring $R-1 < |x| < R$, they proved, for bounded solutions that for $x_0 \in \mathbb{R}^d$ such that $|x_0| = R$,

$$\max_{|x-x_0| \leq 1} |u(x)| > ce^{-c \log RR^{4/3}},$$

that is, they consider a ball of radius 1 centered at a point on the sphere of radius R centered at the origin. This suggests that the discrete setting could have the same behavior. However, as was pointed out in [12] this is not the case, due to the existence of a counterexample (unpublished) of D. Jerison and C. Kenig who kindly sent us the details of it. For the sake of completeness we include below the counterexample appropriately modified for our needs. More concretely we will construct a function u satisfying the following properties:

- $u \in \ell^2(\mathbb{Z}^2)$,
- $(\Delta_d + V_j)u_j = 0$ with $V \in \ell^\infty(\mathbb{Z}^2)$,
- $u(0, 0) = 1$,
- $u(j_1, j_2) = 0$ in the region $\{|j_1| + |j_2 - R| \leq 2\}$, for $R > 0$ (large).

The only thing we have to care about is that, whenever $u(j_1, j_2) = 0$, then the discrete Laplacian at that point must be zero, in order to have a solution to the problem at that point, defining $V(j_1, j_2) = 0$.

We define $u(j_1, j_2)$ at the points such that $|j_1| + |j_2 - R| > 2$ as

$$u(j_1, j_2) = \begin{cases} 2^{-(|j_1|+|j_2|)}, & j_1 \in \mathbb{Z}, & j_2 \leq R-3, \\ -2^{-(|j_1|+R-5)}, & j_1 \in \mathbb{Z} \setminus \{0\}, & j_2 = R \pm 2, \\ 2^{-(|j_1|+R-6)}, & j_1 \in \mathbb{Z} \setminus \{0, \pm 1\}, & j_2 = R \pm 1, \\ -2^{-(|j_1|+R-6)}, & j_1 \in \mathbb{Z} \setminus \{0, \pm 1, \pm 2\}, & j_2 = R, \\ 2^{-(|j_1|+|j_2|-6)}, & j_1 \in \mathbb{Z}, & j_2 \geq R+3. \end{cases}$$

It is clear that $u \in \ell^2(\mathbb{Z}^2)$ and now let us see that at the region where u vanishes, its discrete Laplacian also vanishes. We remind that the discrete Laplacian of a function at the point (j_1, j_2)

$$\begin{array}{ccccccc}
& & & & & & 2^{-k} \\
& & & & & & \\
& & & & & & -2^{-(k-1)} & 0 & -2^{-(k-1)} \\
& & & & & & \\
& & & & & & 2^{-(k-1)} & 0 & 0 & 0 & 2^{-(k-1)} \\
& & & & & & \\
-2^{-k} & 0 & 0 & 0 & 0 & 0 & 0 & -2^{-k} \\
& & & & & & \\
& & & & & & 2^{-(k-1)} & 0 & 0 & 0 & 2^{-(k-1)} \\
& & & & & & \\
& & & & & & -2^{-(k-1)} & 0 & -2^{-(k-1)} \\
& & & & & & \\
& & & & & & 2^{-k}
\end{array}$$

FIGURE 1. Graphic representation of the region $|j_1| + |j_2 - R| \leq 3$. By hypothesis, in the interior u vanishes and due to the construction of u its discrete Laplacian at any point in the interior also vanishes. Due to the construction, $k = R - 3$.

depends on the function at the points $(j_1 \pm 1, j_2)$, $(j_1, j_2 \pm 1)$ and (j_1, j_2) , so graphically it depends on the nearest neighbors of (j_1, j_2) . In Figure 1 it is showed the region where u vanishes and it is clear that if we pick a point such that u vanishes at that point, the sum of its four nearest neighbors is zero, so the discrete Laplacian of u at that point vanishes, as we want.

For the rest of the points, since u is not zero, we define $V_j = -\frac{\Delta_d u_j}{u_j}$ so that the equation is satisfied and the potential V is bounded by a constant independent of R , since the ratios $\frac{u_{j \pm e_k}}{u_j}$ are bounded by a fixed constant when $j \notin \{|j_1| + |j_2 - R| \leq 2\}$.

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REFERENCES

- [1] J. BOURGAIN, C.E. KENIG, *On localization in the continuous Anderson-Bernoulli model in higher dimension*, Invent. Math. 161, no. 2 (2005), pp. 389–426.
- [2] M. COWLING, L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *The Hardy uncertainty principle revisited*, Indiana U. Math. J. 59 no. 6 (2010), pp. 2007–2026.
- [3] L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *On uniqueness properties of solutions of Schrödinger equations*, Comm. Partial Diff. Eq. 31 no.10-12 (2006), pp. 1811–1823.
- [4] L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *On uniqueness properties of solutions of the k -Generalized KdV equations*, J. Func. Anal. 244 (2007), pp. 504–535.

- [5] L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *Hardy's uncertainty principle, convexity and Schrödinger equations*, Journal European Math. Soc. 10 (2008), pp. 883–907.
- [6] L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *The sharp Hardy uncertainty principle for Schrödinger evolutions.*, Duke Math. J. 155 no.1 (2010), pp. 163–187.
- [7] L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *Uniqueness properties of solutions to Schrödinger equations*, Bull. of Amer. Math. Soc. 49 (2012), pp. 415–422.
- [8] L. ESCAURIAZA, C.E. KENIG, G. PONCE, L. VEGA, *Uncertainty principle of Morgan type and Schrödinger evolutions*, J. Long. Math. Soc. (2) 83 no.1 (2011), pp. 187–207.
- [9] A. FERNÁNDEZ-BERTOLIN, *A discrete Hardy's uncertainty principle and discrete evolutions*, *arXiv:1506.00119 [math.AP]*.
- [10] A. FERNÁNDEZ-BERTOLIN, *Convexity properties of discrete Schrödinger evolutions and Hardy's Uncertainty principle*, *arXiv:1506.03717 [math. AP]*.
- [11] P. JAMING, Y. LYUBARSKII, E. MALINNIKOVA, K-M. PERFEKT, *Uniqueness for discrete Schrödinger evolutions*, *arXiv:1505.05398 [math.AP]*.
- [12] C.E. KENIG, *Some recent applications of unique continuation*, BOOKTITLE = Recent developments in nonlinear partial differential equations, Contemp. Math., 439, 25–56, Amer. Math. Soc., Providence, RI, 2007,
- [13] V. Z. MESHKOV, *On the possible rate of decay at infinity of solutions of second-order partial differential equations*, Math. USSR Sbornik 72 (1992), pp. 343–361.

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