REPRESENTATION OF SURFACE HOMEOMORPHISMS BY TÊTE-À-TÊTE GRAPHS

J. FERNÁNDEZ DE BOBADILLA, M. PE PEREIRA, AND P. PORTILLA CUADRADO

Abstract. We use tête-à-tête graphs as defined by N. A'campo and extended versions to codify all periodic mapping classes of an orientable surface with non-empty boundary, improving work of N. A'Campo and C. Graf. We also introduce the notion of mixed tête-à-tête graphs to model some pseudo-periodic homeomorphisms. In particular we are able to codify the monodromy of any irreducible plane curve singularity. The work ends with an appendix that studies all the possible combinatorial structures that make a given filtered metric ribbon graph with some regularity conditions into a mixed tête-à-tête graph.

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Contents

1. Introduction

Monodromy of singularities has been extensively studied since the end of the 19-th century. You can find some historical references in the book [BK86]. In particular, the monodromy of a degeneration of Riemann surfaces was well known to be pseudo-periodic. In [A'C73], A'Campo gave an explicit description of the monodromy of an irreducible plane curve singularity.

More recently in [A'C10], A'Campo introduced Tête-à-tête graphs and twists as a generalization of Dehn twists in order to model monodromies of plane curve singularities. Graf pursued their study in [Gra15] with the purpose of modeling periodic mapping classes of orientable surfaces with boundary. In this paper we take this study further and generalize and improve works of A'Campo and Graf.

Now we explain the contents of this paper in connection with these previous works.

Tête-à-tête graphs are metric ribbon graphs (metric spines which are regular retract of an orientable surface with boundary) with additional properties which allow to define an element of the mapping class group of the corresponding surface. There are several versions of tête-à-tête graphs and twists: pure, relative, general, signed and mixed.

The mapping classes in which we find representatives induced by tête-à-tête graphs can be considered up to isotopy leaving the boundary components free (in the case of pure, relative and general tête-à-tête graphs) or up to isotopy that fixes some or all of the boundary components pointwise (in the case of relative, signed and mixed). They always give mapping classes that contain either elements isotopic to periodic homeomorphisms up to isotopy free at the boundary or pseudo- periodic homeomorphisms (see Definition 10.1). The mixed tête-à-tête graphs are required for the pseudo-periodic case.

In [A'C10] A'Campo defined pure and relative tête-à-tête graphs and twists. A pure tête-à-tête graph is a metric ribbon graph Γ without valency 1 vertices which is a regular retract of an orientable surface with boundary Σ, which satisfies the following special metric property: take any point p at the interior of an edge in Γ. Imagine two ants in the same face of an orientable surface walking along the graph starting at p but in different senses. Assume that every time they reach a vertex they take the next edge turning to the right. If after going independently along the graph for a distance exactly π, they meet face to face in a point q, then we say the graph has the tête-à-tête property. The walks they describe are called *safe walks* starting at p.

Given a tête-à-tête graph there is an associated mapping class of Σ fixing the boundary pointwise which is constructed using the tête-à-tête property and generalizes the construction of Dehn twists. More precisely, the homeomorphism associated to a tête-à-tête graph consists in performing a π- right handed twist along the safe walks in each of the cylinders obtained after cutting the surface along the graph (which is supposed to be a regular retract).

Relative tête-à-tête graphs are a generalization made to model mapping classes interchanging some boundary components. Let Σ be an orientable surface with boundary and B a disjoint union of boundary components such that B *is not* the whole boundary. A relative tête-à-tête graph is a pair (Γ, A), where Γ is a metric graph, A is a disjoint subset of circles, and there is an embedding of (Γ, A) in (Σ, B) sending A to B, in such a way (Γ, A) is a regular retract of Σ. The tête-à-tête twist associated with (Γ, A) is a mapping class that fixes pointwise the boundary components of Σ not contained in B, and may permute the boundary components in A. A'Campo showed that the monodromy of any irreducible plane curve singularity with 1 Puiseux pair is a pure tête-à-tête twist.

In the same work, he also introduced a blow up operation which takes pure into relative graphs, and defined a closing and gluing operation which would take relative graphs into a more general version (mixed tête-à-tête graphs), with the aim of modelling the monodromy of any plane curve singularity. In the irreducible case he showed that his construction works in the example $f(x,y) = (x^3 - y^2)^2 - 4x^5y - x^7$, which has 2 Puiseux pairs, but the general case is not worked out. He concludes the paper hinting an approach to the case of higher dimensional singularities

The definitions of pure and relative tête-à-tête graphs admit a completely combinatorial description, which is already present in A'Campo preprint (see Definition 3.6). One of the advantages of this description is to be able to use graph combinatorics in order to study mapping classes.

In [Gra15], Graf explores which mapping classes can be modelled by pure tête-à-tête graphs. In order to do so he modified A'Campo construction by allowing either

- 'multi-speed" paths: the walks starting at q_i above could have different lengths, even negative ones (this means that the path turns to the left instead of to the right).
- by allowing graphs Γ which have valency 1 vertices,

and defining the corresponding generalized tête-à-tête twists. His main theorem (Theorem 3.1 in [Gra15], states that given any orientable surface Σ with boundary and any mapping class fixing the boundary pointwise, such that it is periodic if considered in the mapping class group leaving the boundary free, there is a generalized tête-à-tête graph embedded in Σ as a regular retract whose associated twist is the mapping class fixing the boundary pointwise.

Allowing multi-speed graphs or valency 1 vertices enlarges a priori the set of mapping classes that can be represented, so the question of which mapping classes can be realized with the original A'Campo definition remained open.

This paper has 2 parts. In the first one we explore thoroughly the limits of A'Campo original Definition and characterize the set of boundary fixed mapping classes which arise as pure tête-à-tête twists. Given a boundary fixed mapping class which is periodic viewed in the boundary free mapping class group there is a notion of fixed-boundary rotation number at each boundary component. The simplest version of our main result is (see Theorem 5.4)

Theorem A. *Let Σ be an orientable surface with boundary. Let $[\phi]$ be an element of the boundary fixed mapping class group such that the mapping class of ϕ in the boundary free mapping class group is periodic, and such that the rotation number at each boundary component of Σ is positive. Then there exists a pure tête-à-tête graph in the sense of A'Campo, which is a regular retract of Σ and such that its associated tête-à-tête twist is the mapping class $[\phi]$.*

This improves Graf's result in the sense that we do not need to enlarge the category of tête-à-tête twists allowing multi-speed or valency 1 vertices. In fact, in order to model any element of the boundary fixed mapping class group such that the mapping class of ϕ in the boundary free mapping class group is periodic, we introduce a notion of *signed tête-à-tête graph and twist* (see Definition 9.2) and prove (see Theorem 9.7):

Theorem B. *Let Σ be an orientable surface with boundary. Let $[\phi]$ be an element of the boundary fixed mapping class group such that the mapping class of ϕ in the boundary free mapping class group is periodic. Then there exists a signed tête-à-tête graph, which is a regular retract of Σ and such that its associated tête-à-tête twist is the mapping class $[\phi]$.*

Our definition of signed tête-à-tête graph is more restrictive than Graf's definition of multi-speed tête-à-tête graphs: Graf can transform his definition into a signed one at the expense of adding valency 1 vertices, but we do not need to do so. In this sense our result improves Graf's Theorem.

In the first part we prove also relative versions of the above theorems: mapping classes $[\phi]$ of the mapping class group that fixes pointwise at least one boundary component are represented by A'Campo relative tête-à-tête twists if the rotation number at the components fixed pointwise are positive, and by signed relative tête-à-tête twists in the general case (see Theorem 6.2 and Theorem 9.12).

All the definitions and theorems above do not cover homeomorphisms that permute all boundary components. For this purpose we define *general tête-à-tête graphs* and model with them mapping classes which may not leave invariant any boundary component. See Definition 7.4 and Theorem 7.6.

In Part 2 of the paper which owes much to the third author, we focus on pseudo-periodic homeomorphisms. We noticed that A'Campo's proposal of mixed tête-à-tête graph does not model irreducible plane curve singularities with 3 or more Puiseux pairs. We give a definition of *mixed tête-à-tête graph* based on a filtration of a metric ribbon graph. We use it to model a class of pseudo-periodic mapping classes which are large enough to contain the monodromies of any irreducible plane curve singularity.

A *mixed tête-à-tête graph* is a *filtered* metric ribbon graph Definition 11.4. Given a mixed tête-à-tête graph there is a notion of mixed safe walk. It is a generalization of the previous notion of safe walk: a mixed safe walk is a concatenation of usual safe walks of prescribed lengths, with the property that the i-th path at the iteration has to remain in the level i of the filtration of the graph. See Definition 11.1. Associated with a mixed tête-à-tête graph there is a thickening surface. Using this notion one can define mixed tête-à-tête twists (see Section 12), which are a generalization of pure tête-à-tête twists.

The filtration in a mixed tête-à-tête graph induces a set of simple closed curves in the thickening surface Σ, such that the corresponding mixed tête-à-tête twist is pseudo-periodic, and the periodic pieces are obtained by cutting Σ by this system of simple closed curves (see Theorem 12.12).

We have a notion of screw number at each separating simple closed curve.

The simplest version of our main result in Part 2 is the following Realization Theorem (see Theorem 13.3).

Theorem C. *Let ϕ be a pseudo-periodic homeomorphism of an oriented surface with boundary Σ such that:*

- *it fixes pointwise the boundary and all the fixed-boundary rotation numbers are positive ((see definitions 8.5-8.6 and Lemma 8.8)).*

- *Let $\mathcal{C}$ be a minimal system of simple closed curves splitting Σ in pieces where ϕ is periodic up to isotopy. Consider the graph obtained by assigning a vertex to each connected component of $\Sigma \setminus \mathcal{C}$ and an edge connecting two vertices for each simple closed curve separating them. Suppose that the graph is a tree.*
- *The screw numbers are all negative (see Definition 10.8).*

Then there is a mixed tête-à-tête graph whose thickening is Σ and whose associated mixed tête-à-tête twist is the mapping class of ϕ in the mapping class group fixing poinwise the boundary.

It is well known that the monodromy of an isolated plane curve singularity is pseudo-periodic, and it is periodic only when the singularity is irreducible and has exactly 1 Puiseux pair. As we said before, in [A'C73], A'Campo gave an explicit construction of the Milnor fiber and the monodromy of an isolated plane curve singularity that includes the computation of the screw numbers. It follows from his construction that it fits in the hypothesis of the previous Theorem and so we obtain the following

Corollary D. *Monodromies of irreducible plane curve singularities are mixed tête-à-tête twists.*

In fact we introduce the more general notion of *relative mixed tête-à-tête graphs*, which play the role of relative tête-à-tête graph. In particular they model pseudo-periodic automorphism of surfaces fixing pointwise at least one boundary component, and having the properties stated in Theorem C (this is the real content of Theorem 13.3). Moreover, these are not the only homeomorphisms up to isotopy that these mixed tête-à-tête graphs modelize, as we show in Example 13.17. It is an open question to characterize the mapping classes that are induced by them.

There are many examples completely worked out in the paper, which should help the reader to grasp the ideas.

Sections 4, 8 and 10 are about general theory of the different mapping classes that appear in the text.

The paper ends up with an appendix, entirely due to the third author, in which given a metric filtered ribbon graph, analizes the set of possible lengths for the mixed safe walks that give a structure of mixed tête-à-tête graph to it. We feel that this study is very convenient for the further study of pseudo-periodic automorphisms using these ideas.

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PART I

In this part we focus on the study of periodic mapping classes of homeomorphisms of an oriented surface Σ with non-empty boundary. The main tool will be ribbon graphs (introduced in Section 2) with an extra metric structure (introduced in Section 3) with some property. In Section 4 we deal with the relevant general theory of periodic mapping clases. In Sections 5,6 and 7 we see how to find for every periodic mapping class a tête-à-tête graph that codifies it. In Section 5, Theorem 5.4 we see how *pure tête-à-tête graphs* allow to codify homeomorphisms that leave all the boundary components invariant. In Section 6, Theorem 6.2 we see how to codify also homeomorphism that leave at least one boundary component invariant by means of *relative tête-à-tête graphs*. In Section 7, Theorem 7.6 we do the general case introducing *the general tête-à-tête graphs*.

In the sections 8 and 9 we deal with mapping classes up to isotopy fixing the boundary pointwise. In Section 8 we make the needed general theory and in Section 9 we introduce the *signed tête-à-tête graphs* and prove, in the theorems 9.7 and 9.12, that they codify up isotopy fixing the boundary all possible representatives fixing the boundary pointwise of periodic mapping classes.

2. Graphs, spines and regular thickenings

A *graph* Γ is a 1-dimensional finite CW-complex. We denote by $v(\Gamma)$ the 0-skeleton and refer to it as the *set of vertices*. We denote by $e(\Gamma)$ the disjoint union of the 1-cells and refer to it as the *set of edges*. With this defintion we allow *loops*, which correspond to 1-cells whose attaching map is constant, and we also allow several edges connecting two vertices. For a vertex v we denote by $e(v)$ the set of edges adjacent to v, where an edge e appears twice in case it is a loop joining v with v. The *valency* of a vertex is the cardinality of $e(v)$ (a loop counts twice). Unless we state the contrary we assume that there are no vertices of valency 1.

We consider graphs that appear as regular retracts of oriented surfaces, or in other words, we consider regular thickenings of graphs.

A *regular thickening* of a graph Γ is described by a cyclic order of $e(v)$ for every $v \in v(\Gamma)$ in the following way.

Assume that for every $v \in v(\Gamma)$ the set $e(v)$ is cyclically ordered:

$$e(v) = \{e_1, ..., e_k\}.$$

For each vertex $v \in v(\Gamma)$ we draw in $\mathbb{R}^2$ a star-shaped graph consisting of v and as many arms as elements in $e(v) = \{e_1, \dots, e_k\}$ which are drawn counterclockwise. Now, we thicken it constructing a star-shaped planar piece as in Figure 2.1, which comes equipped with a unique orientation if the valency of the vertex is at least 3. In the case of valency-2-vertices choose one of the possible orientations.

Whenever we have an edge e joining vertices v_i and v_j, we glue the corresponding thickened arms of the star-shaped pieces in the way that produces an oriented surface, and identifying points of the corresponding edges. If there is at least a vertex in the graph with valency at least 3, this procedure gives a unique orientation in the surface. If there are only valency-2-vertices, then the surface is an oriented annulus, and the orientation depends on the choice of orientations of the 2-star shaped pieces. We choose one of the possible orientations.

Then we obtain an oriented surface Σ with boundary and with the graph Γ embedded in it. We say Σ is the *thickening* of Γ and we write (Σ, Γ).

Reciprocally, every oriented surface with finite topology and non-empty boundary has a *spine* Γ (i.e. an embedded graph in $\Sigma \setminus \partial\Sigma$ that is a *regular retract* of Σ) and certain cyclic orderings in the subsets of edges adjacent to every vertex such that Σ is homeomorphic to the thickening of Γ.

In this way, we can model the topology of every connected oriented surface with non-empty boundary except the disk.

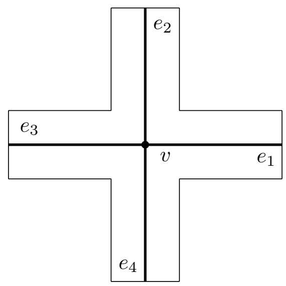

Figure 2.1. Star-shaped piece for some vertex v of valency 4.

Definition 2.1. *A* ribbon graph *is a graph* Γ *such that for every vertex we fix a cyclic order in the set of edges* $e(v)$. *The surface* Σ *constructed above is the* thickening *of* Γ.

Example 2.2. Let $K_{p,q}$ be the bipartite graph (p, q). The set of vertices is the union of two sets A and B of p and q vertices respectively. The edges are exactly all the possible non-ordered pairs of points one in A and one in B.

Now we fix cyclic orderings in A and B. These give cyclic orderings in the sets of edges adjacent to vertices in B and A respectively.

One can check that the tickening surface has as many boundary components as $gcd(p, q)$ and genus equal to $\frac{1}{2}[(p-1)(q-1) - gcd(p, q) + 1]$.

In Figure 2.2 we have the example of $K_{2,3}$.

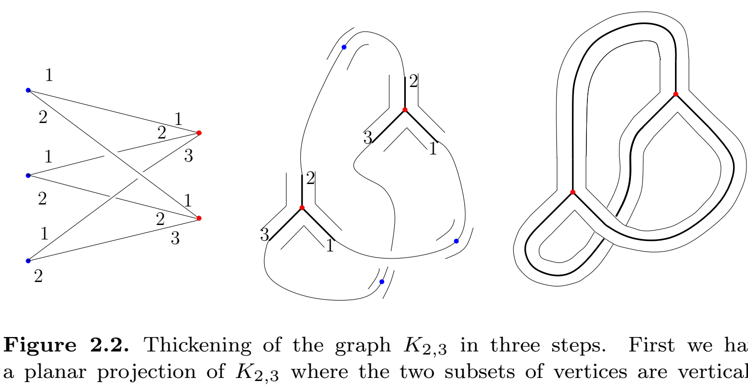

Figure 2.2. Thickening of the graph $K_{2,3}$ in three steps. First we have a planar projection of $K_{2,3}$ where the two subsets of vertices are vertically ordered in different parallel lines. Then we thicken a neighbourhood of every vertex and finally we glue the pieces. The resulting surface is homeomorphic to the once-punctured torus.

Following A'Campo we introduce a generalization of the notion of spine of a surface with boundary Σ, which treats in a special way a certain union of boundary components. Let us start by the corresponding graph theoretic notion.

Let (Γ, A) be a pair formed by a graph Γ and an oriented subgraph A, such that each of its connected components A_i is homeomorphic to the oriented circle $\mathbb{S}^1$. Let v be any vertex of Γ. If we have $v \in A$, since the connected components of A are circles, there are at most two elements in $e(v)$ that belong to A, and in this case they belong to the same component A_i. We say that a cyclic ordering of $e(v)$ is *compatible* with the orientation of A if the elements of $e(v)$ can be enumerated as $e(v) = \{e_1, ..., e_k\}$ in such a way that

- $e_i < e_{i+1}$ and $e_k < e_1$,
- the edges belonging to A_i are e_1 and e_k,
- if we consider a small interval in A_i around v and we parametrize it in the direction induced by the orientation of A_i, then we pass first by e_1 and after by e_k.

Definition 2.3. *Let (Γ, A) be a pair formed by a graph Γ and a oriented subgraph A as above. The pair is a* relative ribbon graph *if for any vertex v the set of incident edges $e(v)$ is endowed with a cyclic ordering compatible with the orientation of A.*

We define the *thickening surface of a relative ribbon graph* (Γ, A) as follows.

Definition 2.4. *We consider star shaped pieces as in Figure 2.1 for every vertex in $\Gamma \setminus A$. For every component A_i we consider a partial thickening of A_i with as many arms as edges with a vertex in A_i as in Figure 2.3. We glue arms of different pieces corresponding to the same edges analogously to Section 2, in such a way that the resulting surface admits an orientation that induces on A its opposite orientation. This is the orientation that we had previously fixed on Σ. The output of this procedure is a pair (Σ, A) given by an oriented surface and a union A of its boundary components.*

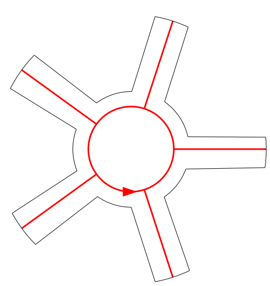

Figure 2.3. Partial thickening of A_i with 5 arms, this corresponds to a collar of a boundary component of a surface.

It is clear that Γ is a regular retract of the thickening surface. Reciprocally: for any pair (Σ, A) given by an oriented surface and a union of some boundary components, if there is a graph Γ embedded in Σ and containing A, such that Γ is a regular retract of Σ, then (Σ, A) is the thickening of (Γ, A).

Notation 2.5. From now on the letter I denotes an interval, unless is otherwise specified, it will denote the unit interval I. Let (Γ, A) be a relative ribbon graph, whose thickening is (Σ, A). There is a connected component of $\Sigma \setminus \Gamma$ for each boundary component C_i of Σ not contained in A, this component is homeomorphic to $C_i \times (0,1]$. We denote by $\widetilde{\Sigma}_i$ the compactification of $C_i \times (0,1]$ to $C_i \times I$. We denote by Σ_Γ the surface obtained by cutting Σ along Γ, that is taking the disjoint union of the $\widetilde{\Sigma}_i$. Let

$$g_\Gamma : \Sigma_\Gamma \to \Sigma$$

be the gluing map. We denote by $\widetilde{\Gamma}_i$ the boundary component of the cylinder $\widetilde{\Sigma}_i$ that comes from the graph (that is $g_\Gamma(\widetilde{\Gamma}_i) \subset \Gamma$) and by C_i the one coming from a boundary component of Σ (that is $g_\Gamma(C_i) \subset \partial\Sigma$). From now on, we take the convention that C_i is identified with $C_i \times \{1\}$ and that $\widetilde{\Gamma}_i$ is identified with $C_i \times \{0\}$. We set $\Sigma_i := g_\Gamma(\widetilde{\Sigma}_i)$ and $\Gamma_i := g_\Gamma(\widetilde{\Gamma}_i)$. Finally we denote $g_\Gamma(C_i)$ also by C_i since $g_\Gamma|_{C_i}$ is bijective. The orientation of Σ induces an orientation on every cylinder $\widetilde{\Sigma}_i$ and on its boundary components.

3. Tête -à-tête graphs.

We now consider metric relative ribbon graphs (Γ, A, d). We denote by Σ its thickening surface. A metric graph is given by a graph Γ and lengths $l(e) \in \mathbb{R}$ for every edge $e \in e(\Gamma)$. In an edge e, we take a homogeneous metric that gives e total length $l(e)$. We consider the distance $d(x, y)$ on Γ given by the minimum of the lengths of the paths joining x and y. It is a complete metric space.

Definition 3.1. *A* walk *in a graph Γ is a continuous mapping*

$$\gamma : I \to \Gamma,$$

from an interval I, possibly infinite, and such that for any $t \in I$ there exists a neighbourhood around t where γ is injective.

Now we give the first definitions, following essentially [A'C10]. The notion of safe walk is central in this paper. We start by a purely graph theoretical definition.

Definition 3.2 (Safe walk)**.** *Let (Γ, A) be a metric relative ribbon graph. A safe walk for a point p in the interior of some edge is a walk $\gamma_p : \mathbb{R}_{\geq 0} \to \Gamma$ with $\gamma_p(0) = p$ and such that:*

1. *The absolute value of the speed $|\gamma_p'|$ measured with the metric of Γ is constant and equal to 1. Equivalently, the safe walk is parametrized by arc length, i.e. for s small enough $d(p, \gamma_p(s)) = s$.*
2. *when γ_p gets to a vertex, it continues along the next edge in the given cyclic order.*
3. *If p is in an edge of A, the walk γ_p starts running in the direction prescribed by the orientation of A.*

An ℓ-safe walk is the restriction of a safe walk to the interval $[0, \ell]$. If a length is not specified when refering to a safe walk, we will understand that its length is π.

The notion in (2) of *continuing along the next edge in the order* of $e(v)$ is equivalent to the notion of *turning to the right* in every vertex for paths parallel to Γ in Σ in A'Campo's words in [A'C10].

Following A'Campo notation, we treat mainly with π-safe walks. In Section 9 we will need safe walks of different lengths, that is why we introduce it in such a generality.

Remark 3.3 (Safe walk via cylinder decomposition)**.** Condition (2) in the previous definition is equivalent to:

(2) the path γ admits a lifting $\tilde{\gamma}_p : \mathbb{R} \to \Sigma_\Gamma$ in the cylinder decomposition of Σ_Γ (see Notation 2.5), which runs in the opposite direction to the one indicated by the orientation induced at the boundary of the cylinder.

Remark 3.4. In fact, from this viewpoint, a safe walk starting from $p \in \Gamma$ is nothing else that the image by g_Γ of a negative arc-lenth parametrization of a circle $\widetilde{\Gamma}_i$ (see Notation 2.5) starting from a preimage of p. This extends the previous definition to safe walks starting at any point p in the graph Γ.

Each of the equivalent formulations of Property (2) of safe walks has its own virtues. The first is purely described in terms of graphs and is more convenient for defining *General Tête-à-tête graphs* in Section 7. The second is more convenient to define *Signed Tête-à-tête graphs* in Section 8 and *Mixed Tête-à-tête graphs in Section 11*.

Remark 3.5. Given a relative ribbon graph (Γ, A) with thickening (Σ, A), we make four observations in order to help fixing ideas:

(1) every vertex $v \in v(\Gamma)$ has as many preimages by g_Γ as its valency $e(v)$. These preimages belong to certain $\widetilde{\Gamma}_i \subseteq \widetilde{\Sigma}_i$ for certain cylinders $\widetilde{\Sigma}_i$ which could occasionally be the same. An interior point of an edge not included in A has always two preimages. An interior point of an edge included in A has always one preimage.

(2) for every point $p \in \Gamma$ and every oriented direction from p along Γ compatible with the orientation of A there is a safe walk starting on p following that direction. This safe walk admits a lifting to one of the cylinders $\widetilde{\Sigma}_i$.

In particular:

(a) For p an interior point of an edge not belonging to A, that is for $p \in \Gamma \setminus (v(\Gamma) \cup A)$, only 2 starting directions for a safe walk are possible, corresponding to the two different preimages of p by g_Γ. We will denote the corresponding safe walks by γ_p and ω_p. If p is at the interior of an edge contained in A only one starting direction for a safe walk at p is possible.

(b) For a vertex v, not belonging to A there are as many starting directions as edges in $e(v)$, and for any vertex v belonging to A, there are as many starting directions as edges in $e(v)$ minus 1 (the edge in A whose orientation arrives to v does not count).

Definition 3.6 (Tête-à-tête property)**.** *Let (Γ, A, d) be a metric relative ribbon graph without univalent vertices. We say that Γ satisfies the* ℓ-tête-à-tête property*, or that Γ is an* ℓ-tête-à-tête graph *if*

- *For any point $p \in \Gamma \setminus (A \cup v(\Gamma))$ the two different ℓ-safe walks starting at p (see Remark 3.5), that we denote by γ_p, ω_p, satisfy $\gamma_p(\ell) = \omega_p(\ell)$.*
- *for a point p in $A \setminus v(\Gamma)$, the end point of the unique ℓ-safe walk starting at p belongs to A.*

If Σ is the regular thickening of the graph Γ and A denotes the corresponding union of boundary components, we say that (Γ, A) gives a relative ℓ-tête-à-tête structure to $(\Sigma, (\Gamma, A))$ *or that* (Γ, A) *is a* relative ℓ-tête-à-tête graph or spine for $(\Sigma, (\Gamma, A))$.

If $A = \emptyset$, we call it a pure *ℓ-tête-à-tête structure or graph.*

Remark 3.7. Since by Remark 3.4 there are safe walks starting at any vertex the seemingly stronger notion of the *ℓ-tête-à-tête property* may be defined:

- For any point $p \in \Gamma$ all the ℓ- safe walks starting at p end at the same point.
- If p belongs to A, the end point of the unique ℓ-safe walk starting at p belongs to A.

Lemma 3.8 (Lemma and Definition)**.** *For an ℓ-tête-à-tête graph (Γ, A, d), the following are true:*

(1) the conditions stated in Remark 3.7 hold true.
(2) The mapping $\sigma_\Gamma : \Gamma \to \Gamma$ defined by $\sigma_\Gamma(p) = \gamma_p(\ell)$ is a homeomorphism.

Proof. A proof in terms of the description of safe walks via images of parametrizations of boundaries of cylinders as in Remark 3.4 is easy: let

$$\sigma : \coprod_i \widetilde{\Gamma}_i \to \coprod_i \widetilde{\Gamma}_i$$

be the homeomorphism which restricts to the metric circle $\widetilde{\Gamma}_i$ to the negative rotation of amplitude ℓ (move each point to a point which is at distance l in the negative sense with respect to the orientation). The tête-à-tête property implies that σ is compatible with the gluing g_Γ at any point which is not the preimage of a vertex. By continuity the compatibility extends to all the points. The mapping σ descends to the mapping σ_Γ. This proves simultaneously both assertions.

However, for later use in the definition of *general tête-à-tête homeomorphisms* we give below a proof using only the combinatorial description of a safe walk.

We note first that if $q = \gamma_p(s)$ is an interior point of an edge, then we have the equality $\gamma_q(\ell) = \gamma_{\gamma_p(\ell)}(s)$.

Take a vertex $v \in v(\Gamma)$. Let $\epsilon > 0$ be smaller than ℓ and than half the length of any edge. For any edge $e_i \in e(v) = \{e_1, ..., e_k\}$ take a sequence of points $\{y_i^n\}_n$ of e_i such that $d(y_i^n, v) = \epsilon/n$. We have that $d(y_i^n, y_j^m) = \epsilon/n + \epsilon/m$ if $i \neq j$ because we have chosen ϵ small enough. We also have that $\gamma_{y_i^n}(\epsilon/n + \epsilon/m) = y_{i+1}^m$. Thus $d(\gamma_{y_i^n}(\ell), \gamma_{y_{i+1}^n}(\ell)) \leq \epsilon/n + \epsilon/m$. Then for every $i = 1, ..., k$ the Cauchy sequences $\{\gamma_{y_i^n}(\ell)\}_n$ converge to the same point u. Simmilarly the Cauchy sequences $\{\omega_{y_i^n}(\ell)\}_n$ converge to the same point u'. By the tête-à-tête property we have the equality $u = u'$.

It is easy to observe that u is the image of all the safe walks starting at v. This proves the first assertion.

We define $\sigma_\Gamma(v) =: u$. It is clear that with this definition σ_Γ is continuous and that u is a common vertex of the images by σ_Γ of the edges e_i.

The inverse of σ_Γ for a point $q \in \sigma_\Gamma(\Gamma \setminus v(\Gamma))$ is the end of a path of length ℓ starting at q that when approaching a vertex v turns to the previous edge in the cyclic ordering of $e(v)$. Note that the tête-à-tête property of Definition 3.6 also holds for this type of paths whenever they start in $\sigma_\Gamma(\Gamma \setminus v(\Gamma))$. So, we can do the same as in the first part of the proof and see that it extends to the whole Γ and that it is continuous. Then, σ_Γ has a continuous inverse. □

Remark 3.9. There is a special and easy case for π-tête-à-tête graphs: when Γ is homeomorphic to $\mathbb{S}^1$. The thickening surface is in this case the cylinder.

If Γ is $\mathbb{S}^1$, then the only possibilities for σ_Γ are the identity or the π rotation (for the homogenous metric). Then Γ has total length of $2\pi/n$ for some $n \in \mathbb{N}$.

Corollary 3.10. *The homeomorphism σ_Γ has the following properties:*

1. *it is an isometry,*
2. *it preserves the cyclic orders of $e(v)$ for every $v \in v(\Gamma)$,*
3. *it takes vertices of valency $k > 2$ to vertices of the same valency,*
4. *it has finite order.*

Proof. Point (1) follows from the proof of Lemma 3.8 because σ_Γ is a homeomorphism that is an isometry restricted to the edges. Point (2) follows also from the proof. Point (3) is immediate since σ_Γ is a homeomorphism.

To see that σ_Γ has finite order when it is not $\mathbb{S}^1$, we observe that σ_Γ induces a permutation between edges and vertices of Γ' and is an isometry. Then, it has finite order. When the graph is homeomorphic to $\mathbb{S}^1$, it follows from Remark 3.9. □

Corollary 3.11. *The following assertions hold:*

1. *If $\sigma_\Gamma|_e = id$ for some edge e, then σ_Γ is the identity.*
2. *for every $m \in \mathbb{N}$ the homeomorphism σ_Γ^m is also induced by a tête-à-tête graph,*
3. *If $\sigma_\Gamma^m|e = id$ for some edge e, then σ_Γ^m is the identity.*

Proof. Given σ_Γ as in (1), since it preserves the cyclic order at every v, then it fixes all the edges adjacent to the vertices of e. Since the graph is connected, this argument extends to the whole graph and the statement follows.

To see (2) and find a tête-à-tête graph for σ_Γ^m one can take the same combinatorial graph Γ with edge lengths equal to the ones of Γ divided by m.

For (3), we have by (2) that the homeomorphism σ_Γ^m is also induced by a tête-à-tête graph and then we are in the case of point (1). □

Lemma 3.12. *If (Γ, A, d) is a tête-à-tête graph, only modifying the underlying combinatorics (without changing the topological type of Γ), we can ensure we are in one of the following cases:*

1. *unless Γ is either homeomorphic to $\mathbb{S}^1$ or contractible, all the vertices have valency ≥ 3,*
2. *there are no loops (edges joining a vertex with itself) and there is at the most one edge joining two vertices. In this case the restriction $\sigma_\Gamma|_{v(\Gamma)}$ determines σ_Γ.*
3. *all the edges have the same length,*
4. *the graph satisfies properties in (2) and (3) simultaneously.*

Proof. If Γ is either homeomorphic to $\mathbb{S}^1$ or contractible, after Remark 3.9, the proof is trivial.

Let's see the case where Γ is neither homeomorphic to $\mathbb{S}^1$ nor contractible. To get a graph as in (1) we can consider the graph Γ' forgetting the valency-2-vertices of Γ but keeping distances. It is clearly a tête-à-tête graph.

To get a graph as in (2) we consider the graph Γ as in (1). We add as vertices some mid-point-edges q_1,...,q_m to Γ' in order the new graph has no loops and no more than one edge between any pair of vertices. Now, we have to add as vertices any other point p for which $\gamma_p(\pi)$, the end of the safe walk for Γ, is one of these new vertices q_i. Since σ_Γ takes isometrically edges to edges, it will take midpoint edges to midpoint edges of Γ. Then we have to add at the most all the mid point edges of Γ as vertices to reach the desired graph.

Moreover, we note that in a graph as in (2), the image $\sigma_\Gamma(e)$ of an edge e joining v_i and v_j, has to be the only edge joining $\sigma_\Gamma(v_i)$ and $\sigma_\Gamma(v_j)$. Then, $\sigma_\Gamma|_{v(\Gamma)}$ determines σ_Γ.

To find a tête-à-tête graph as in (3), we start with a graph Γ as in (1). The homeomorphism σ_Γ permutes edges. Moreover the tête-à-tête condition says that certain summations of the lengths $\{l(e)\}_{e\in e(\Gamma)}$ are equal to π. We consider only the summations that come from measuring the lengths of the safe walks that start in vertices of Γ, which are a finite number. We collect all these linear equations in the variables $l(e)$ in a system S. We consider the system of equations S' by replacing the independent term π in the equations of S by 1. It is clear that there exist positive rational solutions $l(e)$ of the system S'. Let N be a common denominator. We consider the graph Γ' by subdividing every edge e of Γ into $N \cdot l(e)$ edges of length π/N obtaining the desired graph.

If Γ' does not satisfy properties in (2), taking $N = 2N$ you get it. You can also add the middle points of all the edges as vertices and finish as in the proof of (2). □

A way to obtain relative tête-à-tête graphs from pure ones is A'Campo notion of ϵ-blow up.

Definition 3.13 (**ϵ-Blow up of (Σ, Γ) at a vertex of Γ**). *Let Γ be a pure ℓ-tête-à-tête graph and Σ be its thickening surface. Let v be a vertex of valency p. We consider the real blow up of Σ at v. We denote by Σ' and Γ' the trasformations of Σ and Γ. Note that Σ has one more boundary component and Γ' has changed the vertex v by a circle $A \cong \mathbb{R}P^1$ with p edges attached. Away from v, we consider the metric in Γ' as in Γ. We assign the length 2ϵ to the new edges in Γ' along $\mathbb{R}P^1$ and redefine the length of the edges corresponding to each $e \in e(v)$ by $length(e) - \epsilon$. (See figure below Figure 3.1). We do this at every vertex on the orbit of v by the σ_Γ and denote the resulting space by $Bl_v(\Gamma, \epsilon)$. We say that it is the result of performing the ϵ-blowing up of Γ at v.*

It is immediate to check that (Γ', A) is a relative tête-à-tête graph and that Σ' is its thickening. we denote it by $(\Sigma', (\Gamma', A))$.

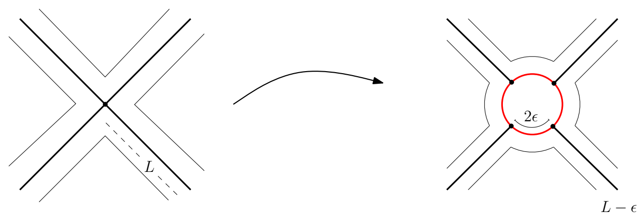

Figure 3.1. Blow-up some vertex v of valency 4.

4. Periodic homeomorphisms up to isotopy

In this section we recall some known facts about periodic homeomorphisms up to isotopy.

Definition 4.1. *We say that two homeomorphisms ϕ and ψ of Σ are* boundary free isotopic *or simply* isotopic *if there exists a family of homeomorphisms of Σ, namely, a continuous map*

$$\theta : \Sigma \times I \to \Sigma$$

such that $\theta_s(x) := \theta(x, s)$ is a homeomorphism of Σ for each $s \in I$, and such that $\theta_0 = \phi$ and $\theta_1 = \psi$. We write $[\psi] = [\phi]$. The Mapping Class Group $MCG(\Sigma)$ is the group of equivalence classes with the operation induced by composition of homeomorphisms. We denote by $MCG^+(\Sigma)$ the restriction to homeomorphisms preserving orientation.

A homeomorphism of a surface is periodic in $MCG(\Sigma)$ *or* periodic up to boundary free isotopy *if there exists $n \in \mathbb{N}$ such that $[\phi^n] = [id]$.*

In the next section we will also use isotopies whose diffeomorphisms remain fixed along certain subvarieties of Σ. We introduce the notion here:

Definition 4.2. *Given two homeomorphisms ϕ and ψ of Σ that both leave invariant some subset $B \subset \Sigma$ such that $\phi|_B = \psi|_B$, we say they are* isotopic relative to the action $\phi|_B$ *if there exists a family of homeomorphisms of Σ that isotope them as before and such that any homeomorphism of the family has the same restriction to B as ϕ and ψ. We write $[\phi]_{B,\phi|_B} = [\psi]_{B,\phi|_B}$. We denote by $MCG(\Sigma, B, \phi|_B)$ the set of classes $[\phi]_{B,\phi|_B}$ with respect to this equivalence relation. We denote by $MCG^+(\Sigma, B, \phi|_B)$ if we restrict to homeomorphisms preserving orientation.*

If the action is the identity on B, we omit the action in the notation and recover the classical notion of isotopy relative to B*, that means that all the homeomorphisms in the isotopy fix B pointwise. We write these classes simply by $[\phi]_B$ and we denote by $MCG^+(\Sigma, B)$ the set of homeomorphisms up to isotopy relative to B that preserve orientation.*

In the case $B = \partial\Sigma$ we will simply write $[\phi]_{\partial,\phi|_\partial}$ or $[\phi]_\partial$.

Remark 4.3. We will mainly work in the case B is contained in $\partial\Sigma$.

If B is the whole boundary $\partial\Sigma$ we recover the classical notion of mapping classes fixing pointwise the boundary. If B is empty we are in the case of Definition 4.1 and we recover the Mapping Class Group up to *boundary free isotopy*. These two extreme cases are the most important, but we will need the more general notion when we develop later relative tête-à-tête homeomorphisms fixing pointwise the union of some boundary components that we denote by $\partial^1\Sigma$.

Remark 4.4. Observe that $MCG(\Sigma, \partial, \phi|_\partial)$ is not a group. However, the group $MCG(\Sigma, \partial)$ acts transitive and freely on it.

In this section we assume all isotopies are boundary free. We focus on periodic elements of $MCG^+(\Sigma)$.

A key result, only true in dimension 2 (see [RS77]), is the following classical theorem:

Theorem 4.5 (Nielsen's Realization Theorem [Nie43], also see Theorem 7.1 in [FM12])**.** *If ϕ^n is isotopic to the identity, then there exists $\hat{\phi} \in [\phi]$ such that $\hat{\phi}^n = Id$. Moreover, there exists a metric on Σ such that $\hat{\phi}$ is an isometry.*

We will use the following well-known fact:

Lemma 4.6. *Let $\phi : \Sigma \to \Sigma$ be an orientation-preserving isometry of Σ. Then either the fixed points are isolated or ϕ is the identity. Moreover, if ϕ is a periodic homeomorphism, then the ramification points are also isolated.*

Notation 4.7. Let ϕ be a periodic orientation preserving homeomorphism of Σ. We denote by Σ^ϕ the orbit space (which is a surface) and by

$$p : \Sigma \to \Sigma^\phi$$

the quotient mapping. The mapping p is a Galois ramified covering map. The set of points in Σ whose orbit has cardinality strictly smaller than the order of ϕ are called *ramification* points. Its images by p are called *branching* points.

Remark 4.8. Since the covering map p is Galois, any point at the preimge by p of a branching point is a ramification point.

Definition 4.9. *Let ϕ be a homeomorphism of Σ that leaves a boundary component $C_i \subseteq \partial\Sigma$ invariant. We cap this boundary component C_i with a disk D^2 obtaining a new surface Σ'. We extend ϕ to a periodic orientation-preserving homeomorphism of Σ' as follows: if θ is the angular and r the radial coordinates for D^2 then we define $\Phi : D^2 \to D^2, (\theta, r) \mapsto (r, \phi(\theta))$. The homeomorphisms Φ and ϕ glue along C_i. We call this extension procedure the* Alexander trick.

Lemma 4.10. *Let $\phi : \Sigma \to \Sigma$ be an orientation-preserving periodic homeomorphism of Σ. Then all the ramification points of $p : \Sigma \to \Sigma^\phi$ are in the interior of Σ.*

Proof. A ramification point is a point such that $\phi^k(p) = p$ but $\phi^k \neq id$. Hence, replacing ϕ by ϕ^k it is enough to prove that there are no fixed points at the boundary. If there is a fixed point p at the boundary, since ϕ leaves invariant the boundary and preserves orientation, it has to fix the boundary near p, but since ramification points are isolated, then ϕ has to be the identity. □

Corollary 4.11. *Given a periodic homeomorphism ϕ of a surface that leaves all the boundary components invariant, the restriction to any boundary component has the same order than ϕ.*

Remark 4.12. Recall that given ϕ and ψ two homeomorphisms of Σ that both leave a spine Γ invariant, if $\phi|_\Gamma$ and $\psi|_\Gamma$ are isotopic, then ϕ and ψ are isotopic. In other words, the isotopy type of the restriction of a homeomorphisms to an invariant spine determines the isotopy type of the homeomorphism of Σ.

Lemma 4.13. *Let Σ be a surface with $\partial\Sigma \neq \emptyset$ which is not a disk or a cylinder. Let $\phi : \Sigma \to \Sigma$ be an orientation preserving homeomorphism. Then ϕ is periodic up to isotopy if and only if there exists $\hat{\phi} \in [\phi]$ such that there exists a spine Γ of Σ which is invariant by $\hat{\phi}$.*

Proof. Assume ϕ is periodic up to isotopy. By Nielsen's realization Theorem we can assume that ϕ is periodic. Let Σ^ϕ be the orbit space of ϕ.

The quotient map $p : \Sigma \to \Sigma^\phi$ is a branched covering map whose ramification points are isolated and are contained in the interior of Σ by lemmas Lemma 4.6-Lemma 4.10. Pick any spine Γ^ϕ for Σ^ϕ containing all the branch points. We claim that $\Gamma := p^{-1}(\Gamma^\phi)$ is an invariant spine for Σ. Indeed, it is invariant by construction since the preimage of every point is precisely the orbit of that point. Since Γ^ϕ is a spine of Σ^ϕ then there exists a regular retraction $\Sigma^\phi \times I \to \Gamma^\phi$. Since Γ^ϕ contains all ramification points, that retraction lifts to a retraction from Σ to Γ.

Conversely, assume that an invariant spine Γ exists for some $\hat{\phi} \in [\phi]$. Since $\hat{\phi}$ leaves the spine invariant. We consider the spine with a graph structure only with vertices of valency greater than 2. Then $\hat{\phi}$ acts as a permutation on edges and the vertices. Then, there is a power of $\hat{\phi}$, say $\hat{\phi}^m$ that leaves all the edges and vertices invariant. Then, we just observe that given any two intervals I, I', any two orientation preserving homeomorphisms $h, g : I \to I'$ are isotopic relative to the boundary of I. Thus, $\hat{\phi}^m|_\Gamma$ is isotopic to the identity. Since Γ is a spine, by the previous remark we have that $\hat{\phi}^m$ is isotopic to the identity.

□

Remark 4.14. In the theorem above we excluded the cases when Σ is a cylinder or a disk for being trivial. In this case every homeomorphism is isotopic to a periodic one.

Not every spine obtained in the proof of the previous lemma accepts a tête-à-tête structure such that $\sigma_\Gamma = \phi|_\Gamma$ (see Example 5.13). In Theorem 5.4 we will see show how to find one that accepts it.

Notation 4.15. If a homeomorphism ϕ of Σ leaves a spine Γ invariant, then the homeomorphism lifts to a homeomorphism of Σ_Γ that we denote by $\widetilde{\phi}$.

For a periodic homeomorphism that leaves a boundary component invariant we have a notion of rotation number associated to that boundary component using the following:

Definition 4.16. *Let $\phi : \mathbb{S}^1 \to \mathbb{S}^1$ be an orientation preserving periodic homeomorphism of the circle of order q. Let $x \in \mathbb{S}^1$ and let x^ϕ be the orbit of x by ϕ and let p be the number of points in x^ϕ that lie on the arc connecting x with $\phi(x)$ in the positive direction. We define the rotation number of ϕ as the quotient $\frac{p}{q}$ and we denote it by $rot(\phi)$. The fraction $\frac{p}{q}$ is always reduced and the number p/q is in the interval $(0, 1]$.*

The above definition corresponds with Poincare's classical notion of rotation number of a homeomorphism of a circle into itself. Poincare's rotation number takes values in $\mathbb{R}/\mathbb{Z}$, and in $\mathbb{Q}/\mathbb{Z}$ if the homeomorphism is periodic. We are taking the unique representative in the interval $(0, 1]$.

The following is well-known:

Lemma 4.17. *Let ϕ be an orientation preserving periodic homeomorphism of the circle such that $rot(\phi) = p/q$, then ϕ is conjugate to the rotation of $2\pi\frac{p}{q}$ radians.*

Corollary 4.18. *An orientation preserving periodic homeomorphism of the cylinder which leaves invariant each boundary component is conjugate to a rotation.*

5. Tête-à-tête graphs and periodic homeomorphisms leaving every boundary component invariant.

Given a tête-à-tête graph Γ, we define a periodic homeomorphism of the thickening surface that we denote by ϕ_Γ. We prove in Theorem 5.4 that all orientation preserving periodic homeomorphisms leaving invariant each of the the boundary components of Σ are both conjugate and isotopic to one of this type.

Let Γ be a tête-à-tête graph. Let's define ϕ_Γ. We use Notation 2.5. Consider the homeomorphism $\sigma_\Gamma : \Gamma \to \Gamma$ (see Lemma 3.8). By the definition of σ_Γ and that of the safe walk, we have that σ_Γ leaves each $\Gamma_i \subseteq \Gamma$ invariant and that moreover σ_Γ lifts to a periodic homeomorphism of $\bigcup \widetilde{\Gamma}_i$ that we denote by $\widetilde{\sigma}_\Gamma$.

For every i we define a homeomorphism $\phi_{\widetilde{\Sigma}_i}$ on $\widetilde{\Sigma}_i$ as follows: choose a product structure $\widetilde{\Sigma}_i \approx \widetilde{\Gamma}_i \times I$ for each cylinder $\widetilde{\Sigma}_i$. Consistently with Notation 2.5 we identify $\widetilde{\Gamma}_i$ with $\widetilde{\Gamma}_i \times \{0\}$ and C_i with $\widetilde{\Gamma}_i \times \{1\}$. For each $(x,t) \in \widetilde{\Sigma}_i \approx \widetilde{\Gamma}_i \times I$ we define $\phi_{\widetilde{\Sigma}_i}(x,t) := (\widetilde{\sigma}_\Gamma(x), t)$. The homeomorphisms $\{\phi_{\widetilde{\Sigma}_i}\}$ glue well by definition to give a periodic homeomorphism of Σ that we denote ϕ_Γ. If p is in Σ_i, this homeomorphism is defined by $\phi_\Gamma(p) = g_\Gamma(\widetilde{\sigma}(x), t)$ for any $(x,t) \in g_\Gamma^{-1}(p)$ expressed in a trivialization $\widetilde{\Sigma}_i = \widetilde{\Gamma}_i \times I$.

The homeomorphism ϕ_Γ depends on the chosen product structures, but its boundary-free isotopy class does not. The restriction of ϕ_Γ to Γ is also independent since it coincides with $\sigma_\Gamma|_\Gamma$. Since any two choices of product structures are conjugate the conjugation class of ϕ_Γ is also independent of the choice.

Definition 5.1. *Given a tête-à-tête graph Γ we denote by $[\phi_\Gamma]$ the boundary free isotopy class of the periodic homeomorphism of Σ defined above. And we denote by ϕ_Γ a periodic representative for some choice of product structure on each $\widetilde{\Sigma}_i$.*

Remark 5.2. A choice of product structures for the cylinders $\widetilde{\Sigma}_i$ associated to the definition of some ϕ_Γ as above determines:

(1) A retraction from $\Sigma \to \Gamma$ such that ϕ_Γ takes retraction lines to retraction lines.
(2) The restriction of the retraction to the boundary component C_i induces homeomorphisms from C_i to $\widetilde{\Gamma}_i$. Considering the pullback metric in C_i, the restriction of ϕ_Γ to $\partial\Sigma$ is a rigid rotation of each of its boundary components.

Moreover, if ℓ is the length for the tête-à-tête structure on Γ, the induced metric on $\partial\Sigma$ assigns length $\ell/rot(\phi|_{C_i})$ to $C_i \subseteq \partial\Sigma$. Hence this length does not depend on the particular product structure.

Remark 5.3. Consider the pullback metric on the boundary $\partial\Sigma$ as in the previous remark. Not for every spine Γ of Σ there exists a metric on Γ such that there exists a regular retraction $r : \Sigma \to \Gamma$ satisfying that the restriction $r|_{\partial\Sigma}$ is a local isometry. See Example 5.13. The proof of the following theorem consists in finding the right spine.

Theorem 5.4. *Let ϕ be a periodic orientation preserving homeomorphism of a surface Σ with r boundary components (different from the disk and the cylinder) which leaves each boundary component invariant. Then,*

(i) there exists an invariant spine Γ and a metric on it such that Γ is a π-tête-à-tête graph with this metric and $\phi|_\Gamma$ is equal to the tête-à-tête homeomorphism σ_Γ of Γ.
(ii) We have the equality $[\phi] = [\phi_\Gamma]$ as boundary-free isotopy classes.
(iii) the automorphism ϕ is conjugate to ϕ_Γ for any choice of product structures of the cylinders $\widetilde{\Sigma}_i$ by a homeomorphism that restricts to the identity on Γ (and hence it is isotopic to the identity).

Proof. We will use the notations Notation 2.5, Notation 4.7 and Notation 4.15.

We will construct the invariant graph Γ as in the proof of Lemma 4.13, as the preimage by the quotient map $p : \Sigma \to \Sigma^\phi$ of an appropriate spine Γ^ϕ for Σ^ϕ.

Let us prove first the case $g := genus(\Sigma^\phi) \geq 1$. To choose a spine in Σ^ϕ, we use a planar representation of Σ^ϕ as a convex $4g$-gon in $\mathbb{R}^2$ with r disjoint open disks removed from its convex hull. The sides of the $4g$-gon are labeled clockwise like $a_1b_1a_1^{-1}b_1^{-1}a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$, where edges labeled with the same letter (but different exponent) are identified by an orientation reversing homeomorphism. The number r is the number of boundary components. We number the boundary components $C_i \subseteq \partial\Sigma$, $1 \leq i \leq r$. We denote by d the arc $a_2b_2a_2^{-1}b_2^{-1}\dots a_gb_ga_g^{-1}b_g^{-1}$. We consider l_1,...,l_{r-1} arcs as in Figure 5.1. We denote by c_1,...,c_r the edges in which a_1^{-1} (and a_1) is subdivided, numbered according to the component $p(C_i)$ they enclose. We consider the spine Γ^ϕ of Σ^ϕ given by the union of d, $a_1b_1a_1^{-1}b_1^{-1}$ and the l_i's. We construct Γ^ϕ so that it passes by all the branching points of ϕ. Then the retraction of Σ^ϕ to Γ^ϕ lifts to a retraction of Σ to the preimage $\Gamma := p^{-1}(\Gamma^\phi)$. Hence Γ is a spine of Σ.

Let us see what conditions on the lengths of the edges of Γ^ϕ have to be imposed such that the pullback-metric in Γ defines a tête-à-tête metric adapted to ϕ, that is, so that we have the equality $\gamma_p(\pi) = \phi(p)$ for the safe walks along Γ with that metric.

We consider the rotation numbers $R_i := rot(\widetilde{\phi}|_{C_i}) \in (0, 1] \cap \mathbb{Q}$ of the lifting of $\phi|_\Gamma$. The tête-à-tête structure of Γ has to satisfy the equality $\gamma_p(\pi) = \phi(p)$ and, in particular, the safe walk has to follow a rotation of $2\pi R_i$ up to conjugation. We want that for every i we have

$$R_i \cdot length(\widetilde{\Gamma}_i) = \pi. \tag{5.5}$$

Moreover, we want the metric on Γ to be the pullback of a metric on Γ^ϕ. We denote by $\Sigma^\phi_{\Gamma^\phi}$ the surface obtained by cutting Σ^ϕ along Γ^ϕ and consider the gluing map $g_{\Gamma^\phi} : \Sigma^\phi_{\Gamma^\phi} \to \Sigma^\phi$ analogously to Notation 2.5. We consider the lifting of $p : \Sigma \to \Sigma^\phi$ to the cut surfaces and we denote it by $\widetilde{p} : \Sigma_\Gamma \to \Sigma^\phi_{\Gamma^\phi}$. We denote by $\widetilde{p(\Gamma_i)}$ the preimage of $p(\Gamma_i)$ by g_{Γ^ϕ}. Since $p|_{\Gamma_i} : \Gamma_i \to p(\Gamma_i)$ and also $\widetilde{p}|_{\widetilde{\Gamma}_i} : \widetilde{\Gamma}_i \to \widetilde{p(\Gamma_i)}$ are an $n : 1$ covering mappings, we have the equality

$$length(\widetilde{\Gamma}_i) = n \cdot length(\widetilde{p(\Gamma_i)}). \tag{5.6}$$

Note that one can easily read $length(\widetilde{p(\Gamma_i)})$ looking at the lengths of the edges of $p(\Gamma_i) \subseteq \Gamma^\phi$.

Putting (5.5)-(5.6) together, we have that what we need is that the equality

$$length(\widetilde{p(\Gamma_i)}) = \frac{\pi}{n \cdot R_i} \tag{5.7}$$

holds for all i.

Notice that our choice of representative of Poincare's rotation number has been important here: since R_i belongs to $(0, 1]$ we did not have to divide by 0.

We denote by D, B_1 and C_i the lengths of d, b_1 and c_i. We will assume that all the l_i and b_1 of the same length L. Then the system (5.7) for this case can be expressed as follows:

$$\begin{gathered} 2L + 2C_1 = \frac{\pi}{n \cdot R_1} \\ 2L + 2C_i = \frac{\pi}{n \cdot R_i} \quad for\ i = 2, ..., r-1 \\ 2L + 2C_r + D = \frac{\pi}{n \cdot R_r}, \end{gathered} \tag{5.8}$$

which has obviously positive solutions C_i, D after choosing for example $L = min\{\frac{\pi}{4\cdot n\cdot R_i}\}$. We assign $length(a_i) = length(b_i) = D/4(g-1)$ for $i > 1$ to get a metric on Γ^ϕ. We consider the pullback-metric in Γ. This induces by construction a tête-à-tête structure in Γ.

This tête-à-tête structure on Γ induces by construction a rigid rotation $\widetilde{\phi}_\Gamma|_{\widetilde{\Gamma}_i}$ of rotation number R_i in each $\widetilde{\Gamma}_i$. It is conjugate to $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$ since they have the same rotation number. The orbits of $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$ are the fibres of $\widetilde{p}|_{\widetilde{\Gamma}_i}$. By the choice of lengths, the orbits of the tête-à-tête rotation in $\widetilde{\Gamma}_i$ are also the fibres of $\widetilde{p}|_{\widetilde{\Gamma}_i}$. Conjugation between homeomorphisms of $\mathbb{S}^1$ preserves the cyclic order in $\mathbb{S}^1$ of a point p and its iterations. Then, since $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$ and $\widetilde{\phi}_\Gamma|_{\widetilde{\Gamma}_i}$ are conjugate with the same orbits, they coincide. Then $\phi|_\Gamma$ and $\phi_\Gamma|_\Gamma$ coincide.

By Remark 4.12 we have that ϕ and ϕ_Γ are isotopic since they coincide on Γ.

Since $\widetilde{\phi}|_{\widetilde{\Sigma}_i}$ and $\widetilde{\phi}_\Gamma|_{\widetilde{\Sigma}_i}$ are both periodic and its restriction to $\widetilde{\Gamma}_i$ coincide, then they are conjugate by Corollary 4.18. Note that the conjugation homeomorphisms of each $\widetilde{\Sigma}_i$ are the identity on $\widetilde{\Gamma}_i$. Then they glue to a conjugation homeomorphism of the whole Σ. It is isotopic to the identity because it is the identity on Γ. So we conclude.

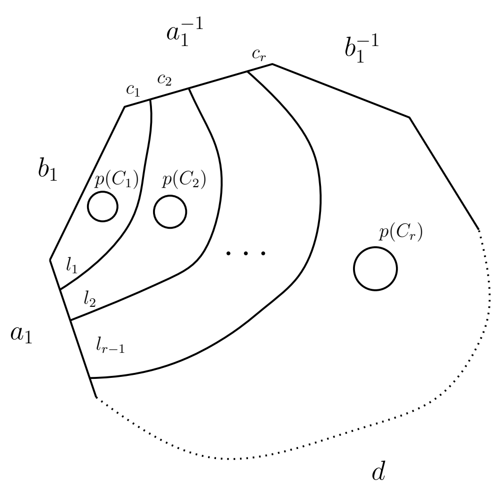

Figure 5.1. Planar representation of the surfaceΣ^ϕ of genus≥ 1 and r boundary components $p(C_1)$, ...,$p(C_r)$. Drawing of $l_1, ..., l_r$ and $c_1, .., c_r$.

For the case $genus(\Sigma^\phi) = 0$ we proceed a bit differently to choose the spine Γ^ϕ. The surface Σ^ϕ is a disk with $r-1$ smaller disjoint disks removed. We cut the surface along an embedded segment that we call c as we can see in the first image of Figure 5.2. Cutting along c we get another planar representation of Σ^ϕ as in the second image. The exterior boundary corresponds to cc^{-1}; we call the exterior boundary P and denote by q_1 and q_2 the points in P that come from the two extremes of c. We look at the graph of the third picture in Figure 5.2. We have drawn $r-1$ vertical segments l_1,...,l_{r-1} so that P union with them contains all branch points and is a regular retract of the disk enclosed by P minus the r disks.

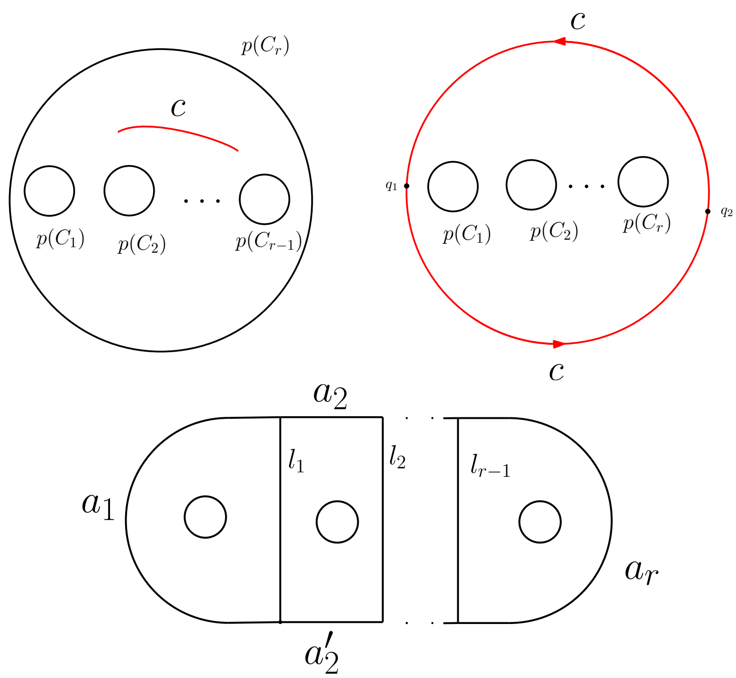

Figure 5.2. In the first two pictures we have the disk with $r-1$ smaller disks removed. In the third one we forget the identification along the exterior and we draw a spine.

The rest of the proof follows by cases on the number of boundary components r and the number of branch points.

If $r = 1$ and there are no branch points or 1 branch point, then Σ is a disk (this follows from the Hurwitz formula) which is not covered by the statement of the theorem. If there are at least 2 branch points, we can get that two branch points lie in q_1 and q_2 so that Γ has no univalent vertices. In this case we set $length(c) = \frac{\pi}{2nR_1}$.

Suppose now $r = 2$. If there are no branch points, then Σ is a cylinder which is not included in the statement of this theorem.

If there is at least 1 branch point we consider two cases, namely $R_1 = R_2$ and $R_1 > R_2$.

In the case $R_1 = R_2$, we choose the graph depicted on the right hand side of Figure 5.3. That is, q_1 and q_2 are exactly $a_1 \cap a_2$. In this case we do not care about the location of the branch point as long as it is contained in the graph. In this case we set $lenght(c) = length(l_1) = \frac{\pi}{2nR_1}$

In the case $R_1 < R_2$, we choose the graph depicted on the left hand side of Figure 5.3 and we choose the branch point to lie on q_1, this way, since q_1 is the only vertex of valency 1, we get that the preimage of this graph by p does not have univalent vertices. We set the lengths, $length(l_1) = \frac{\pi}{nR_1}$ and $length(c) = (\frac{\pi}{nR_2} - \frac{\pi}{nR_1})/2$.

Suppose now $r > 2$.

We are going to assign lengths to every edge in Figure 5.2 and decide how to divide and glue P in order to recover Σ^{ϕ}. This means that we are going to decide

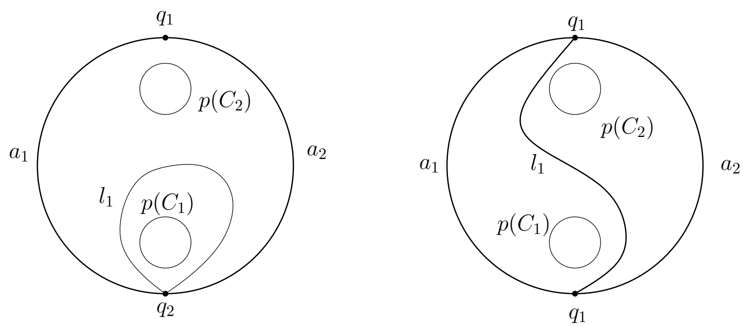

Figure 5.3. On the left, the case $A_1 < A_2$. On the right the case when $R_1 = R_2$.

the position of q_1 and q_2 in P, relative to the position of the ends of the l_i's, in order to get a suitable metric spine of Σ^ϕ.

To every vertical interior segment l_j we assign the same length

$$L < min\{\frac{\pi}{2(n \cdot R_i)}\}.$$

We look at the segments a_1, a_2, a_2',...,a_{r-1}, a_{r-1}', a_r in which P is divided by the vertical segments (see Figure 5.2) and give lengths A_1, A_2, A_2',...,A_{r-1}, A_{r-1}', A_r. The following system corresponds to (5.7) for this case:

$$\begin{aligned} A_1 + L &= \frac{\pi}{n \cdot R_1} \\ A_i + A_i' + 2L &= \frac{\pi}{n \cdot R_i} \quad for \quad i = 2, ..., r-1 \\ A_r + L &= \frac{\pi}{n \cdot R_r}. \end{aligned} \tag{5.9}$$

It has obviously positive solutions A_i. We choose $A_i = A_i'$ for $i = 2, ..., r-1$.

In order this distances can be pullback to the original graph Σ^ϕ, there is one equation left: we have to impose equal length to the two paths c and c^{-1}, or in other words to place q_1 and q_2 dividing P in two segments of equal length.

If ϕ has at least two branching points, we choose q_1 and q_2 to be any two of the branching points. Then, we can choose the metric and the vertical segments so that q_1 and q_2 are the middle points of a_1 and a_r. If we identify the two paths c and c^{-1} joining q_1 and q_2 then we recover Σ^ϕ, and we get a metric graph on it. Then, the preimage by p of the resulting metric graph gives a metric graph. We claim that this graph has no univalent vertices. Indeed, a univalent vertex of this graph has to be the preimage of univalent vertices of the graph below, which are only q_1 and q_2, which are branching points. By Remark 4.8 all their preimages are ramification points and then they are not univalent vertices. The metric induces a tête-à-tête structure in the graph by construction. Now, we finish the proof as in the case $genus(\Sigma^\phi) \geq 1$.

If ϕ has no ramification points or only one, in the previous construction we could obtain univalent vertices at the preimages of q_1 and q_2. So we need to do some changes in the assignations of lengths and in the positions of the vertical segments relative to q_1 and q_2 in order that the extremes q_1 and q_2 of c coincide with vertices of the graph in Σ^ϕ.

Assume that $A_1 \geq A_2 \geq \cdots \geq A_r$.

We choose $q_1 := a_1 \cap a_2$. Let q_2 be the antipodal point (so q_1 and q_2 divides P in two paths of equal length). If q_2 is a vertex we have finished. If it is not, then it is on a segment a'_i, for some $i = 2, ..., r_1$. We redefine $A'_i := d(a'_i \cap a'_{i-1}, q_2)$ and $A_i := A_i + d(q_2, a'_i \cap a'_{i+1})$. Now the antipodal point of q_1 is $a_i \cap a_{i+1}$. We redefine $q_2 := a_i \cap a_{i+1}$ and identify orientation reversing the two paths joining q_1 and q_2 to recover Σ^ϕ.

Now the pullback of the resulting metric graph has no univalent vertices and gives a tête-à-tête structure by construction. We can finish the proof as in the previous cases.

□

Remark 5.10. Note that the spine $p(\Gamma)$ of Σ^ϕ chosen in the proof is the same for all homeomorphisms ϕ of any surface Σ with the same quotient surface Σ^ϕ, whenever Σ^ϕ has genus different from 0.

Remark 5.11. We continue Remark 5.2. Let ϕ_Γ be the homeomorphism constructed in the previous proof. In order to find a product structure in every cylinder of Σ_Γ preserved by the lifting homeomorphism $\widetilde{\phi} : \Sigma_\Gamma \to \Sigma_\Gamma$ (or equivalently, a retraction $r_\Gamma : \Sigma \to \Gamma$ such that ϕ takes retraction lines to retraction lines), we can lift any product structure of the cylinder decomposition of $\Sigma^\phi_{p(\Gamma)}$ (or any retraction from Σ^ϕ to Γ^ϕ).

Example 5.12. Consider the tête-à-tête -structure of the $K_{p,q}$ of the Example 2.2. There we have the formulas for the genus and the boundary components. The induced homeomorphism ϕ_Γ has order $M = lcm(p, q)$ (and leaves the boundary components invariant by definition). There are two special orbits, the one of p vertices that are ramification points of order M/q and another one of q vertices of ramification order M/p.

Then, knowing that Σ^{ϕ_Γ} has also $D = \gcd(p, q)$ boundary components and using Hurwitz formula we get that the genus of the quotient surface Σ^{ϕ_Γ} is 0.

If we look at the image of $\Gamma = K_{p,q}$ by the quotient map $p : \Sigma \to \Sigma^{\phi_\Gamma}$ map we obtain the graph in Σ^{ϕ_Γ} of Figure 5.6.

Example 5.13. We provide with an example of a spine of a surface Σ which is invariant by a periodic homeomorphism ϕ and such that it does not admit a metric modelling ϕ as a tête-à-tête homeomorphism.

Consider the complete bipartite graph $K_{2,4}$ with the cyclic ordering at each vertex given by the projection on the plane depicted on the left part of Figure 5.4. The thickening Σ of this graph is the surface of genus 1 and 2 boundary components.

Giving a length of $\pi/2$ to each edge, we provide the graph with a tête-à-tête structure such that the corresponding periodic homeomorphism has order 4. We denote a periodic representative of the induced mapping class by $\phi_{K_{2,4}}$. We get that the red vertices have trivial isotropy group while the blue vertices have isotropy group of order 2.

The orbit surface $\Sigma^{\phi_{K_{2,4}}}$ has genus 0 and 2 boundary components. It is depicted on the right side of the figure together with the image by the projection map $p : \Sigma \to \Sigma^{\phi_{K_{2,4}}}$ of the graph $K_{2,4}$.

We perturb this graph as in the right side of Figure 5.5. Observe that the vertex of valency one on this graph corresponds with a ramification point of the map p.

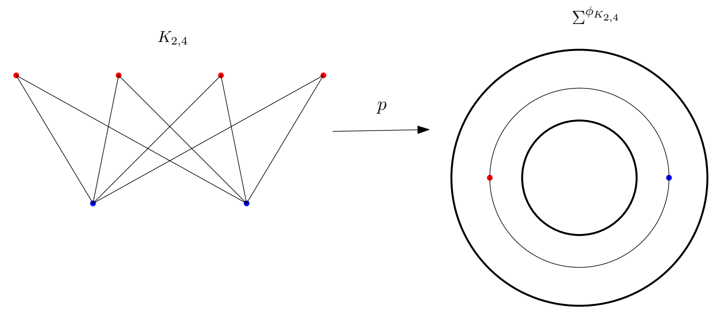

Figure 5.4. On the left we see $K_{2,3}$. On the right we see the orbit surface $\Sigma^{\phi_{K_{2,4}}}$.

Now we take the preimage by p of this graph. This is depicted on the left part of Figure 5.5.

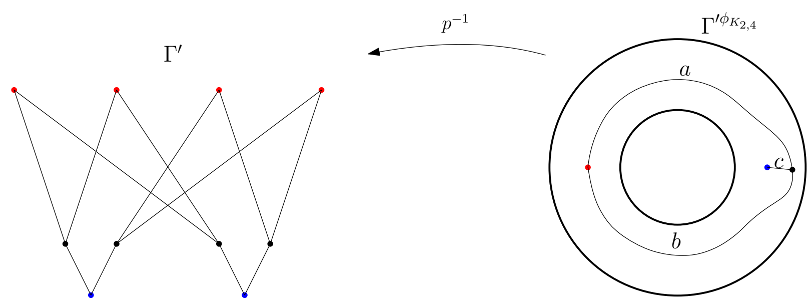

Figure 5.5. On the left we see $K_{2,3}$. On the right we see the orbit surface $\Sigma^{\phi_{K_{2,4}}}$.

We make the following observation. The length of all the edges in Γ' is determined by the length of the edges of $\Gamma'^{\phi_{K_{2,4}}}$. The rotation number of ϕ on each of the two boundary components is $1/4$. If $l(a), l(b)$ and $l(c)$ denotes the length of the edges of $\Gamma'^{\phi_{K_{2,4}}}$, by the observation made in this paragraph, we must have

$$l(a) + l(b) = l(a) + l(b) + 2l(c)$$

which is impossible since there can be no edges with length 0.

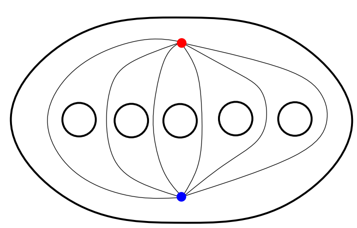

Figure 5.6. Drawing of $p(K_{pq})$ which consists of two vertices (red and blue) corresponding to the two special fibres and r edges corresponding to the D boundary components, in the picture $D = 6$.

6. Relative tête-à-tête graphs and periodic homeomorphisms leaving at least one boundary component invariant.

Now we improve the results in previous section and model, in terms of relative tête-à-tête graphs, periodic homeomorphisms leaving at least one boundary component invariant. Our strategy is to use Alexander trick to reduce to the case of periodic homeomorphisms leaving all boundary components invariant as treated in previous Section 5.

Definition 6.1. *Let (Γ, A) be a relative tête-à-tête graph and (Σ, A) be its thickening. Definition 5.1 defines a homeomorphism*

$$\phi_{(\Gamma,A)} : (\Sigma, A) \to (\Sigma, A)$$

associated with (Γ, A).

Also, note that as in Remark 4.12, if ϕ and ψ are two homeomorphisms of a thickening surface Σ of Γ relative to A which both leave Γ invariant and whose restrictions to Γ are isotopic, then ϕ and ψ are isotopic.

Let ϕ be an orientation preserving homeomorphism of Σ leaving at least one boundary component invariant. Let A be the union of all boundary components that are not invariant by ϕ, say $A = C_1 \cup \ldots \cup C_k$. The homeomorphism ϕ induces a permutation on this set of boundary components. We glue a disk capping each of those boundary components and extend ϕ to the interior of such disks using Alexander's trick. By doing this we get a new surface $\hat{\Sigma}$ and a homeomorphism $\hat{\phi}$ which is an extension of ϕ. Denote by D_i the disk cupping C_i and by t_i the center of D_i. The extension $\hat{\phi}$ leaves the set $\bigcup_i D_i$ invariant and we can assume that leaves also the set $\bigcup_i \{t_i\}_i$ invariant. Moreover the only possible ramification points of $\hat{\phi}|_{\bigcup_i D_i}$ are the t_i's.

The homeomorphism $\hat{\phi}$ leaves each of the boundary components of $\hat{\Sigma}$ invariant and then we can use Theorem 5.4 to produce a tête-à-tête graph $\hat{\Gamma}$ such that $\phi_{\hat{\Gamma}}$ is conjugate to $\hat{\phi}$. Following the proof, we see that we can assume that $\hat{\Gamma}$ passes by all the t_i and that $\hat{\Gamma} \cap D_i$ is a union of segments with vertex in t_i, all of them transverse to $C_i \hookrightarrow \hat{\Sigma}$. Indeed, let

$$p : \hat{\Sigma} \to \hat{\Sigma}^{\hat{\phi}}$$

be the quotient map. All we have to do is to appropriately choose $\Gamma^{\hat{\phi}}$ passing through $p(t_i)$ and transverse to $p(C_i)$. Furthermore we can assume that the distance in $\hat{\Gamma}$ from t_i to C_i along any of these portions contained in D_i of the edges of $e(t_i)$ is equal to a small enough and fixed $\epsilon > 0$.

Now we define a metric in the graph $\Gamma := (\hat{\Gamma} \cap \Sigma) \cup A$. We take the induced metric in $\hat{\Gamma} \cap \Sigma$ and the metric on C_i that gives length 2ϵ to every edge joining two consecutive vertices on $C_i \subseteq A$ (we choose as vertices of Γ on C_i only the points $C_i \cap \hat{\Gamma}$). We claim that the pair (Γ, A) with the given metric is a relative tête-à-tête graph in the sense of Definition 3.6. The proof is straightforward using the fact that Γ is a tête-à-tête graph and taking into account the way of choosing the metric.

If we observe that the homeomorphism $\phi_{(\Gamma,A)}$ associated with (Γ, A) coincides with the restriction of $\hat{\phi}$ to Σ, then we note that we have proven the following theorem, which models any periodic homeomorphism leaving at least one boundary component invariant in terms of a relative tête-à-tête graph:

Theorem 6.2. *Let Σ be a connected surface with non-empty boundary which is not a disk or a cylinder. And let ϕ be an orientation preserving periodic homeomorphism of Σ that leaves (at least) one boundary component invariant. Let A be the set containing all boundary components that are not invariant by ϕ. Then*

- *(i) there exists a relative tête-à-tête graph (Γ, A) embedded in (Σ, A), which is invariant by ϕ, such that we have the equality of restrictions $\phi_{(\Gamma,A)}|_\Gamma = \phi|_\Gamma$*
- *(ii) We have the equality of boundary-free isotopy classes $[\phi_{(\Gamma,A)}] = [\phi]$ and moreover we have the equality $[\phi_{(\Gamma,A)}]_{A,\phi|_A} = [\phi]_{A,\phi|_A}$ (see Definition 4.2).*
- *(iii) Moreover the homeomorphisms $\phi_{\Gamma,A}$ and ϕ are conjugate for any choice of product structures of the cylinders $\widetilde{\Sigma}_i$.*

Remark 6.3. The procedure we did to get $\Gamma \hookrightarrow \Sigma$ from $\hat{\Gamma} \hookrightarrow \hat{\Sigma}$ can also be reinterpreted as the ϵ blow up of $(\hat{\Sigma}, \hat{\Gamma})$ at the t_i's (see Definition 3.13).

Remark 6.4. Note that we have the same result as Theorem 6.2 changing A for any union of connected components of $\partial\Sigma$ that does not include at least one that is invariant by ϕ.

7. General tête-à-tête structures

In this section we study any orientation preserving periodic homeomorphism. This includes homeomorphisms of non-connected surfaces that do not leave any boundary component invariant and even exchange connected components of the surface itself. Let ϕ be such a homeomorphism. We realize its boundary-free isotopy type and its conjugacy class by a generalization of tête-à-tête graphs, using a technique that reduces to the case of homeomorphisms of a larger surface that leave all boundary components invariant.

First we extend our definition of ribbon graphs (see Definition 2.1). *In this subsection, we allow graphs with some special univalent vertices.*

Definition 7.1. *A ribbon graph with boundary is a pair $(\Gamma, \mathcal{P})$ where Γ is a ribbon graph, and $\mathcal{P}$ is the set of univalent vertices, with the following additional property: given any vertex v of valency greater than 1 in the cyclic ordering of adjacent edges $e(v)$ there are no two consecutive edges connecting v with vertices in $\mathcal{P}$.*

In order to define the thickening of a ribbon graph with boundary we need the following construction:

Let Γ' be a ribbon graph and let Σ be its thickening. Let

$$g_{\Gamma'} : \Sigma_{\Gamma'} \to \Sigma$$

be the gluing map. The surface $\Sigma_{\Gamma'}$ splits as a disjoint union of cylinders $\coprod_i \widetilde{\Sigma}_i$. Let w be a vertex of Γ'. The cylinders $\widetilde{\Sigma}_i$ such that w belongs to $g_{\Gamma'}(\widetilde{\Sigma}_i)$ are in a natural bijection with the pairs of consecutive edges (e', e'') in the cyclic order of the set $e(w)$ of adjacent edges to w.

Let $(\Gamma, \mathcal{P})$ be a ribbon graph with boundary. The graph Γ' obtained by erasing from Γ the set E of all vertices in $\mathcal{P}$ and its adjacent edges is a ribbon graph. Consider the thickening surface Σ of Γ'. Let e be an edge connecting a vertex $v \in \mathcal{P}$ with another vertex w, let e' and e'' be the inmediate predecesor and succesor of e in the cyclic order of $e(w)$. By the defining property of ribbon graphs with boundary they are consecutive edges in $e(w) \setminus E$, and hence determine a unique associated cylinder which will be denoted by $\widetilde{\Sigma}_i(v)$.

Each cylinder $\widetilde{\Sigma}_i$ has two boundary components, one, denoted by $\widetilde{\Gamma}_i$ corresponds to the boundary component obtained by cutting the graph, and the other, called C_i, corresponds to a boundary component of Σ. Fix a cylinder $\widetilde{\Sigma}_i$. Let $\{v_1, ..., v_k\}$ be the vertices of $\mathcal{P}$ whose associated cylinder is $\widetilde{\Sigma}_i$. Let $\{e_1, ..., e_k\}$ be the corresponding edges, let $\{w_1, ..., w_k\}$ be the corresponding vertices at Γ', and let $\{w'_1, ..., w'_k\}$ be the set of preimages by $g_{\Gamma'}$ contained in $\widetilde{\Sigma}_i$. The defining property of ribbon graphs with boundary imply that w'_i and w'_j are pairwise different if $i \neq j$. Furthermore, since $\{w'_1, ..., w'_k\}$ is included in the circle $\widetilde{\Gamma}_i$, which has an orientation inherited from Σ, the set $\{w'_1, ..., w'_k\}$, and hence also $\{e_1, ..., e_k\}$ and $\{v_1, ..., v_k\}$ has a cyclic order. We assume that our indexing respects it.

Fix a product structure $\mathbb{S}^1 \times I$ for each cylinder $\widetilde{\Sigma}_i$, where $\mathbb{S}^1 \times \{0\}$ corresponds to the boundary component $\widetilde{\Gamma}_i$, and $\mathbb{S}^1 \times \{1\}$ corresponds to the boundary component of C_i.

Using this product structure we can embedd Γ in Σ: for each vertex $v \in \mathcal{P}$ consider the corresponding cylinder $\widetilde{\Sigma}_{i(v)}$, let w' be the point in $\widetilde{\Gamma}_{i(v)}$ determined above. We embedd the segment $g_{\Gamma'}(w' \times I)$ in Σ.

Doing this for any vertex v we obtain an embedding of Γ in Σ such that all the vertices $\mathcal{P}$ belong to the boundary $\partial\Sigma$, and such that Σ admits Γ as a regular deformation retract.

Definition 7.2. *Let $(\Gamma, \mathcal{P})$ be a ribbon graph with boundary. We define the thickening surface Σ of $(\Gamma, \mathcal{P})$ to be the thickening surface of Γ' toghether with the embeding $(\Gamma, \mathcal{P}) \subset (\Sigma, \partial\Sigma)$ constructed above. We say that $(\Gamma, \mathcal{P})$ is a* general spine *of $(\Sigma, \partial\Sigma)$.*

Definition 7.3 (General safe walk)**.** *Let $(\Gamma, \mathcal{P})$ be a metric ribbon graph with boundary. Let σ be a permutation of $\mathcal{P}$.*

We define a general safe walk *in $(\Gamma, \mathcal{P}, \sigma)$ starting at a point $p \in \Gamma \setminus v(\Gamma)$ to be a map $\gamma_p : [0, \pi] \to \Gamma$ such that*

1) *$\gamma_p(0) = p$ and $|\gamma'_p| = 1$ at all times.*
2) *when γ_p gets to a vertex of valency ≥ 2 it continues along the next edge in the cyclic order.*
3) *when γ gets to a vertex in $\partial\Sigma$, it continues along the edge indicated by the permutation σ.*

Definition 7.4 (General tête-à-tête graph)**.** *Let $(\Gamma, \mathcal{P}, \sigma)$ be as in the previous definition. Let γ_p, ω_p be the two safe walks starting at a point p in $\Gamma \setminus v(\Gamma)$. We say Γ has the* general tête-à-tête property *if*

- *for any $p \in \Gamma \setminus v(\Gamma)$ we have $\gamma_p(\pi) = \omega_p(\pi)$*

Moreover we say that $(\Gamma, \mathcal{P}, \sigma)$ gives a general tête-à-tête structure *for $(\Sigma, \partial\Sigma)$ if $(\Sigma, \partial\Sigma)$ is the thickening of $(\Gamma, \mathcal{P})$.*

In the following construction we associate with a general tête-à-tête graph $(\Gamma, \mathcal{P}, \sigma)$ a homeomorphism of $(\Gamma, \mathcal{P})$ which restricts to the permutation σ in $\mathcal{P}$; we call it the *general tête-à-tête homeomorphism of* $(\Gamma, \mathcal{P}, \sigma)$. We construct also a homeomorphism of the thickening surface which leaves the Γ invariant and restricts on Γ to the general tête-à-tête homeomorphism of $(\Gamma, \mathcal{P}, \sigma)$. We construct the homeomorphism on the graph and on its thickening simultaneously.

Consider the homeomorphism of $\Gamma' \setminus v(\Gamma)$ defined by

$$p \mapsto \gamma_p(\pi).$$

The second proof of Lemma 3.8 shows that there is an extension to homeomorphism

$$\sigma_\Gamma : \Gamma \to \Gamma.$$

The restriction of the general tête-à-tête homeomorphism that we are constructing to Γ' coincides with σ_Γ. The mapping σ_Γ leaves Γ' invariant for being a homeomorphism.

The homeomorphism $\sigma_\Gamma|_{\Gamma'}$ lifts to a periodic homeomorphism

$$\tilde{\sigma} : g_{\Gamma'}^{-1}(\overline{\Gamma'}) \to g_{\Gamma'}^{-1}(\overline{\Gamma'}),$$

which may exchange circles in the following way. For any $p \in \overline{\Gamma'}$, the points in $g_{\Gamma'}^{-1}(p)$ corresponds to the safe walks in $\overline{\Gamma'}$ starting at p (see Remark 3.5). A safe walk starting at p is determined by the point p and an starting direction at an edge containg p. An inspection of the definition of σ_Γ via the proof of Lemma 3.8 yields a definition of a continuous mapping $\tilde{\sigma}$ taking points with starting directions to points with starting directions.

As we have seen, if $(\Gamma, \mathcal{P}, \sigma)$ is a general tête-à-tête structure for $(\Sigma, \partial\Sigma)$ then the surface $\Sigma_{\Gamma'}$ is a disjoint union of cylinders. The lifting $\tilde{\sigma}$ extends to $\Sigma_{\Gamma'}$ similarly to Definition 5.1. This extension interchanges some cylinders $\widetilde{\Sigma}_i$ and goes down to an homeomorphism of Σ. We denote it by $\phi_{(\Gamma,\mathcal{P},\sigma)}$. If necessary, we change the embedding of the part of Γ not contained in Γ' in Σ such that it is invariant by $\phi_{(\Gamma,\mathcal{P},\sigma)}$. This is done by an adequate choice of the trivilizations of the cylinders.

Definition 7.5. *The homeomorphism $\phi_{(\Gamma,\mathcal{P},\sigma)}$ is by definition the homeomorphism of the thickening, and its restriction to Γ the* general tête-à-tête homeomorphism of $(\Gamma, \mathcal{P}, \sigma)$.

Theorem 7.6. *Given a periodic homeomorphism ϕ of a surface with boundary $(\Sigma, \partial\Sigma)$ which is not a disk or a cylinder, the following assertions hold:*

(i) There is a general tête-à-tête graph $(\Gamma, \mathcal{P}, \sigma)$ such that the thickening of $(\Gamma, \mathcal{P})$ is $(\Sigma, \partial\Sigma)$, the homeomorphism ϕ leaves Γ invariant and we have the equality $\phi|_\Gamma = \phi_{(\Gamma,\mathcal{P},\sigma)}|_\Gamma$.

(ii) We have the equality of boundary-free isotopy classes $[\phi|_\Gamma] = [\phi_{(\Gamma,\mathcal{P},\sigma)}]$.

(iii) The homeomorphisms ϕ and $\phi_{(\Gamma,\mathcal{P},\sigma)}$ are conjugate.

Proof. In the first part of the proof we extend the homeomorphism ϕ to a homeomorphism $\hat{\hat{\phi}}$ of a bigger surface $\hat{\hat{\Sigma}}$ that leaves all the boundary components invariant. Then, we find a tête-à-tête graph $\hat{\hat{\Gamma}}$ for $\hat{\hat{\phi}}$ such that $\hat{\hat{\Gamma}} \cap \Sigma$, with a small modification in the metric and a suitable permutation, is a general tête-à-tête graph for ϕ.

Let n be the order of the homeomorphism.

Consider the permutation induced by ϕ in the set of boundary components. Let $\{C_1, ..., C_m\}$ be an orbit of cardinality strictly bigger than 1, numbered such that $\phi(C_i) = C_{i+1}$ and $\phi(C_m) = C_1$. Take an arc $\alpha \subseteq C_1$ small enough so that it is disjoint from all its iterations by ϕ. Define the arcs $\alpha^i := \phi^i(\alpha)$ for $i \in \{0, ..., n-1\}$, which are contained in $\cup_i C_i$. Obviously we have the equalities $\alpha^{i+1} = \phi(\alpha^i)$ and $\phi(\alpha^{n-1}) = \alpha^0 = \alpha$.

We consider a star-shaped piece S of n arms as in Figure 7.1. We denote by D the central boundary component. Let $a^0, \ldots, a^{n-1}$ be the boundary of the arms of the

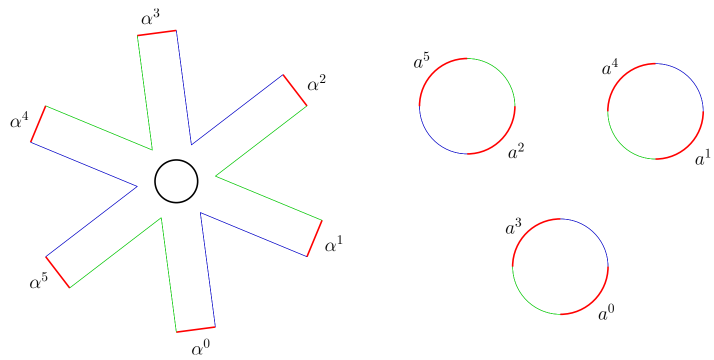

Figure 7.1. Example of a star-shaped piece S with 6 arms on the left and boundary components components on the right. The arcs along which the two pieces are glued, are marked in red. In blue and red are the boundaries of the two disks that we used to cap off the new boundaries.

star-shaped piece labelled in the picture, oriented counterclockwise. We consider the rotation r of order n acting on this piece such that $r(a^i) = a^{i+1}$. Note that this rotation leaves D invariant.

We consider the surface $\hat{\Sigma}$ obtained by gluing Σ and S identifying a^i with α^i reversing the orientation, and such that ϕ and the rotation r glue to a periodic homeomorphism $\hat{\phi}$ in the resulting surface.

The boundary components of the new surface are the boundary components of Σ different from $\{C_1, ..., C_m\}$, the new boundary component D, and the boundary components C'_1,...,C'_k that contain the part of the C_i's not included in the union $\cup_{i=0}^{n-1} \alpha^i$.

The homeomorphism $\hat{\phi}$ leaves D invariant and may interchange the new boundary components C'_1,...,C'_k. We cup each component C'_i with a disk D_i and extend the homeomorphism by the Alexander trick, obtaining a homeomorphism $\hat{\hat{\phi}}$ of a bigger surface $\hat{\hat{\Sigma}}$. The only new ramification points that the action of $\hat{\hat{\phi}}$ may induce are the centers t_i of these disks. We claim that, in fact, each of the t_i's is a ramification point.

Denote the quotient map by

$$p : \hat{\hat{\Sigma}} \to \hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}.$$

In order to prove the claim notice that the difference $\hat{\hat{\Sigma}} \setminus \Sigma$ is homeomorphic to a closed surface with $m+1$ disks removed. On the other hand the difference of quotient surfaces $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}} \setminus \Sigma^{\phi}$ is homeomorphic to a cylinder. Since m is strictly bigger than 1, Hurwitz formula for p forces the existence of ramification points. Since p is a Galois cover each t_i is a ramification point.

The new boundary component of $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$ corresponds to $p(D)$, where D is invariant by $\hat{\hat{\phi}}$. The point $q_1 := p(t_i)$ is then a branch point of p.

We do this operation for every orbit of boundary components in Σ of cardinality greater than 1. Then we get a surface $\hat{\hat{\Sigma}}$ and an extension $\hat{\hat{\phi}}$ of ϕ that leaves all the boundary components invariant. The quotient surface $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$ is obtained from Σ^{ϕ}

attaching some cylinders $\mathcal{C}_j$ to some boundary components. Let

$$p : \hat{\hat{\Sigma}} \to \hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$$

denote the quotient map. Comparing $p|_\Sigma$ and $p|_{\hat{\Sigma}}$, we see that we have only one new branching point q_j in every cylinder $\mathcal{C}_j$.

Now we construct a tête-à-tête graph for $\hat{\hat{\phi}}$ modifying slightly the construction of Theorem 5.4.

To fix ideas we consider the case in which the genus of the quotient $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$ is positive. The modification of the genus 0 case is exactly the same. As in Theorem 5.4 we use a planar representation of Σ^ϕ as a convex $4g$-gon in $\mathbb{R}^2$ with r disjoint open disks removed from its convex hull and whose edges are labeled clockwise like $a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, we number the boundary components $C_i \subseteq \partial\Sigma$, $1 \leq i \leq r$, we denote by d the arc $a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, and we consider l_1,...,l_{r-1} arcs as in Figure 7.2. We denote by c_1,...,c_r the edges in which a_1^{-1} (and a_1) is subdivided according to the component $p(C_i)$ they enclose.

We impose the further condition that each of the regions in which the polygon is subdivided by the l_i's encloses not only a component $p(C_i)$, but also the branching point q_i that appears in the cylinder $\mathcal{C}_i$. We assume that the union of d, $a_1 b_1 a_1^{-1} b_1^{-1}$ and the l_i's contains all the branching points of p except the q_i's.

In order to be able to lift the retraction we need that the spine that we draw in the quotient contains all branching points. In order to achieve this we add an edge s_i joining q_i and some interior point q_i' of l_i for $i = 1, \dots, r-1$ and joining q_r with some interior point q_r' of l_{r-1}. We may assume that q_i' is not a branching point. We consider the circle $p(C_i)$ and ask s_i to meet it transversely to it at only 1 point. See Figure 7.3. We consider the graph Γ' as the union of the previous segments and the s_i's. Clearly the quotient surface retracts to it. Since it contains all branching points its preimage $\hat{\hat{\Gamma}}$ is a spine for $\hat{\hat{\Sigma}}$. It has no univalent vertices since the q_i's are branching points of a Galois cover.

In order to give a metric in the graph we proceed as follows. We give the segments d and C_i's the same length they had in the proof of Theorem 5.4. We impose every s_i to have length some small enough ϵ and the part of s_i inside the cylinder $\mathcal{C}_i$ to have length $\epsilon/2$ (see Figure 7.3). We give each segment l_i length $L - 2\epsilon$. It is easy to check that the preimage graph $\hat{\hat{\Gamma}}$ with the pullback metric is tête-à-tête .

Now we consider the graph $\Gamma := \hat{\hat{\Gamma}} \cap \Sigma$ with the restriction metric, except on the edges meeting $\partial\Sigma$ whose length is redefined to be ϵ. Along the lines of the proof of Theorem 5.4 we get that $\phi_{(\Gamma,\mathcal{P},\sigma)}$ and ϕ are isotopic and conjugate. If we denote by $\mathcal{P}$ the set of univalent vertices of Γ, it is an immediate consequence of the construction that $(\Gamma, \mathcal{P}, \sigma)$ with the obvious permutation σ of $\mathcal{P}$ is a general tête-à-tête graph with $\phi_{(\Gamma,\mathcal{P},\sigma)}|_\Gamma = \phi|_\Gamma$. □

Example 7.7. We show an example that illustrates these ideas. Let Σ be surface of genus 1 and 3 boundary components C_0, C_1, C_2 embedded in $\mathbb{R}^3$ as in the picture 7.4. Let $\phi : \Sigma \to \Sigma$ be the restriction of the space rotation of order 3 that exchanges the 3 boundary components. We observe that in particular $\phi^3|_{C_i} = id$ for $i = 0, 1, 2$.

We consider the star-shaped piece S with 3 arms together with the order 3 rotation r that exchanges the arms (see the picture Figure 7.4).

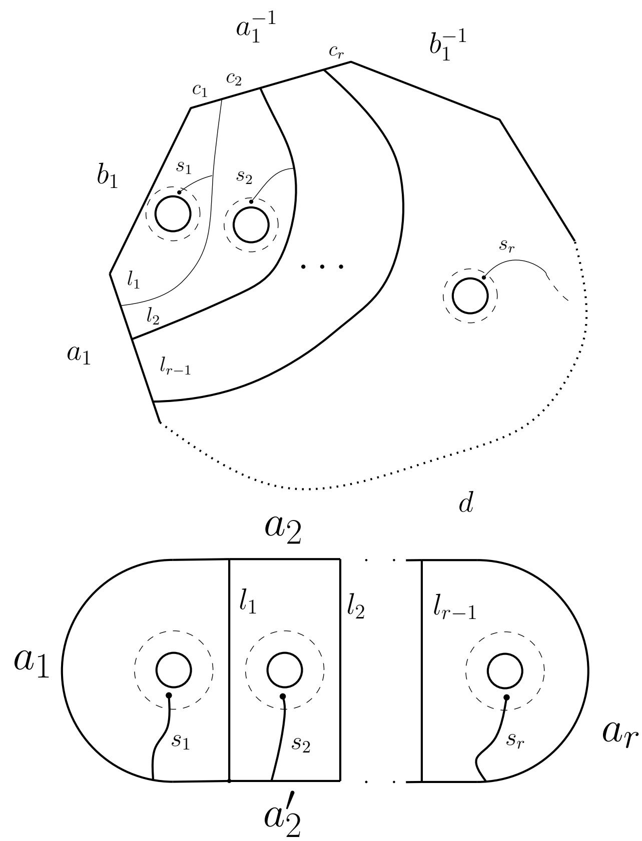

Figure 7.2. Drawing of Γ' for the case $genus(\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}) \geq 1$ in the firts image and $genus(\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}) = 0$ in the second.

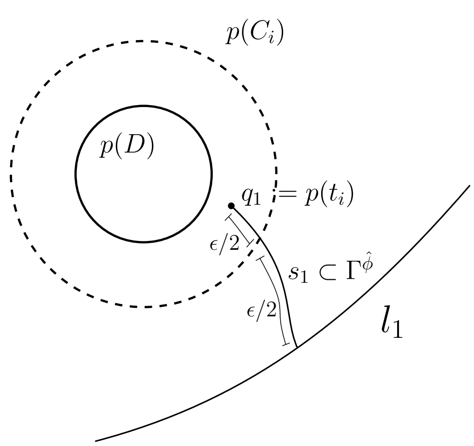

Figure 7.3. Neighbourhood of $q_1 = p(t_i)$ in $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$ and edge s_1 joining q_1 and l_1.

We glue S to Σ as the theorem indicates: we mark a small arc $\alpha^0 \subset C_2$ and all its iterated images by the rotation. Then we glue $\alpha^0, \alpha^1, \alpha^2$ to a^0, a^1, a^2 respectively

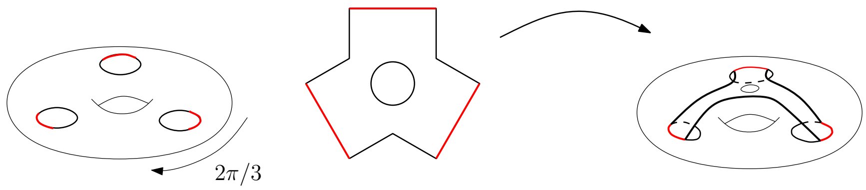

Figure 7.4. On the left, the torus Σ with 3 disks removed and the orbit of an arc marked, that is, 3 arcs in red. In the center, the star-shaped piece S with 3 arms to be glued to the torus along those arcs. On the right, the surface we get after gluing, with 2 boundary components, one of them invariant by the induced homeomorphism.

by orientation reversing homeomorphisms. We get a new surface $\hat{\Sigma} := \Sigma \cup S$ with 2 boundary components. We cap the boundary component that intersects $C_0 \cup C_1 \cup C_2$ with a disk D^2 and extend the homeomorphism to the interior of the disk getting a new surface $\hat{\hat{\Sigma}}$ and a homeomorphism $\hat{\hat{\phi}}$.

Using Hurwitz formula $2 - 2g - 4 = 0 - 2$ we get that the surface we are gluing to Σ has genus 0 and hence it is a sphere with 4 boundary components. See picture Figure 7.5. Three of them are identified with C_0, C_1, C_2, and the 4-th is called C and is the only boundary component of $\hat{\hat{\Sigma}}$.

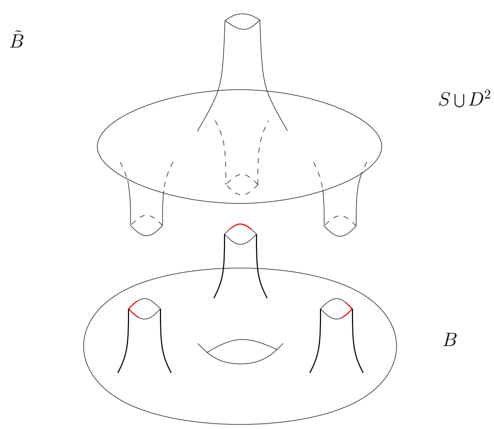

Figure 7.5. On the left, the torus with 3 disks removed and 3 the orbit of an arc marked. On the right, the star-shaped piece with 3 arms to be glued to the torus along those arcs.

We compute the orbit space $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$ by the extended homeomorphism $\hat{\hat{\phi}}$ and get a torus with 1 boundary component. We consider the graph Γ' as in picture Figure 7.6. We put a metric in this graph. We set every edge of the hexagon to be $\pi/6 - \epsilon/3$ long and the path joining the hexagon with the branch point to be ϵ long. In this way, if we look at the result of cutting $\hat{\hat{\Sigma}}^{\hat{\hat{\phi}}}$ along the graph $\hat{\hat{\phi}}$ we see that the only boundary component that maps to the graph by the gluing map has length $6(\pi/6 - \epsilon/3) + 2\epsilon = \pi$.

The preimage $\hat{\hat{\Gamma}}$ of Γ' by the quotient map is a tête-à-tête graph whose thickening is $\hat{\Sigma}$. Its associated homeomorphism $\hat{\hat{\phi}}$ leaves Σ invariant and its restriction to it

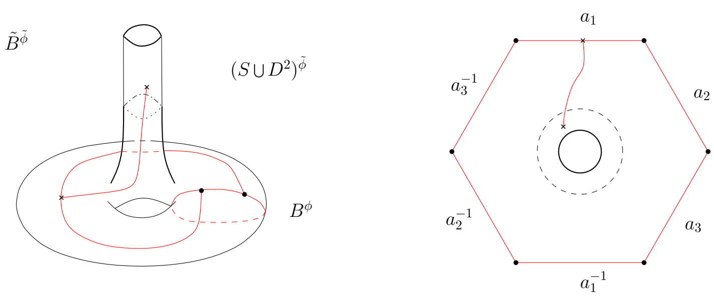

Figure 7.6. On the lower part we have the original surface. On the upper part we have the surface that we attach, in this case a sphere with 4 holes removed.

coincides with the rotation ϕ. Moreover $(\hat{\hat{\Gamma}} \cap \Sigma, \hat{\hat{\Gamma}} \cap \partial\Sigma)$ is a general spine of $(\Sigma, \partial\Sigma)$. Modifying the induced metric in $\hat{\hat{\Gamma}} \cap \Sigma$ as in the proof of the Theorem and adding the order 3 cyclic permutation to the valency 1 vertices we obtain a tête-à-tête graph whose associated homeomorphism equals ϕ.

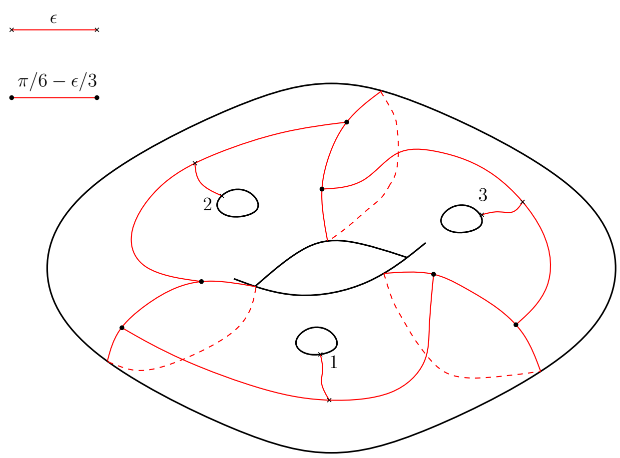

Figure 7.7. On the left, the torus with 3 disks removed and 3 the orbit of an arc marked. On the right, the star-shaped piece with 3 arms to be glued to the torus along those arcs.

8. Homeomorphisms up to isotopy fixing pointwise at least a component of the boundary.

Now we take a look at the Mapping Class Group where homeomorphisms and isotopies fixed pointwise the union of some boundary components that we denote by $\partial^1\Sigma \subset \partial\Sigma$ (recall Definition 4.2).

Consider a non-empty union $\partial^1\Sigma$ of the boundary components. In this section we study the elements of $MCG^+(\Sigma, \partial^1\Sigma)$ (see Definition 4.2) that are boundary-free isotopic to a periodic one. We simply write $[\phi]_{\partial^1}$.

If Σ is the disk, it is clear that $MCG^+(\Sigma, \partial\Sigma) \approx 0$. If Σ is the cylinder, then $MCG^+(\Sigma, \partial\Sigma) \approx \mathbb{Z}$ and it is generated by the right (or left) Denh twist along a curve that is parallel to the boundary components. All its elements are boundary free isotopic to the identity.

Remark 8.1. In this work we take the convention that *negative* Dehn twists are *right-handed* Dehn twists. See Figure 8.1.

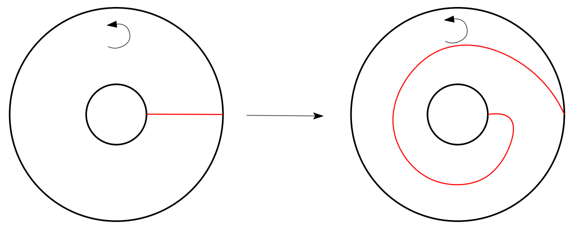

Figure 8.1. We see an oriented annulus on the left and the image of the red curve by a right-handed Dehn twist on the right. The little curve on the top of each picture represents the orientation on the annulus.

Let Σ be a surface with non empty boundary $\partial\Sigma$. Consider a non-empty union $\partial^1\Sigma$ of boundary components. Let ϕ be an orientation preserving authmorphism of Σ, fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one $\hat{\phi}$. Our next aim is to define a notion of fixed-boundary rotation number at each connected component of $\partial^1\Sigma$, which reduces modulo $\mathbb{Z}$ to the usual notion of rotation number of $\hat{\phi}$ at the corresponding component.

We start recalling some facts about Dehn twists. If we do not say the contrary, the letter D with a subindex, denotes a negative (right-handed) Dehn twist some curve that will be clear from the context.

Lemma 8.2. *Let $\alpha_1, \dots, \alpha_m$ be a set of non-trivial, pairwise disjoint, and pairwise non-homotopic closed curves. Then the group generated by Dehn twists along those curves $< \mathcal{D}_{\alpha_1}, \dots, \mathcal{D}_{\alpha_m} >$ is free abelian of rank m.*

Proof. Lemma 3.17 in [FM12]. □

Corollary 8.3. *Let Σ be a surface with $r > 0$ boundary components that is not a disk or an annulus. Then the group generated by the Dehn twists $D_1, \dots D_r$ along curves parallel to each boundary component is free abelian of rank r.*

Lemma 8.4. *Let Σ be a surface that is neither a disk nor a cylinder. Let $\partial^1\Sigma$ be non-empty union of r boundary components of Σ. Let ϕ be an orientation preserving homeomorphism of Σ fixing $\partial^1\Sigma$ pointwise. If ϕ is boundary-free isotopic to the identity then there exist unique integers $n_1, \dots, n_r$ such that we have the equality $[\phi]_{\partial^1} = [D_1^{n_1}]_{\partial^1} \cdot \dots \cdot [D_r^{n_r}]_{\partial^1}$.*

Proof. We can assume that ϕ is the identity outside a collar neighbourhood $U = \bigsqcup U_i$ of $\partial^1\Sigma$ where U_i is a collar for the boundary component C_i. The restrictions $[\phi|_{U_i}]_\partial$ are elements in $MCG^+(U_i, \partial U_i)$. Since the U_i are cylinders, the group $MCG^+(U_i, \partial U_i)$ is generated by the boundary Dehn twist $[D_i]_\partial$ and we can use Corollary 8.3 and get that $[\phi]_{\partial^1} = [D_1^{n_1}]_{\partial^1} \cdot [D_2^{n_2}]_{\partial^1} \cdot \dots \cdot [D_r^{n_r}]_{\partial^1}$ and the numbers $n_1, ..., n_r$ are unique. □

In the next definition we introduce the concept of *rational rotation number* at each boundary component of a homeomorphism fixing pointwise some components of the boundary and boudary-free isotopic to a periodic one that we are seeking for.

Definition 8.5. *Let Σ be a surface that is neither a disk nor a cylinder. Let $\partial^1\Sigma$ be non-empty union of r boundary components of Σ. Let $\phi : \Sigma \to \Sigma$ be a homeomorphism fixing pointwise $\partial^1\Sigma$ and boudary-free isotopic to a periodic one. Let $m \in \mathbb{N}$ such that $[\phi^m] = [id]$. Let $t_1, \ldots, t_r$ be integers such that $[\phi^m]_{\partial^1} = [D_1^{t_1}]_{\partial^1} \cdot \ldots \cdot [D_r^{t_r}]_{\partial^1}$. We define the* fixed-boundary rotation number *at C_i by*

$$rot_{\partial^1}(\phi, C_i) := t_i/m.$$

Note that the fixed-boundary rotation numbers do not depend on the number m we choose to compute them or the representative $\phi \in [\phi]_{\partial^1}$.

Now we describe how to compute the fixed-boundary rotation number in terms of a invariant spine.

Let ϕ be an orientation preserving homeomorphism of Σ that fixes a non-empty union $\partial^1\Sigma$ of components of the boundary and which is boundary-free isotopic to a periodic one. Let A the union of the remaining components of the boundary. Suppose that there exists a relative spine (Γ, A) in $(\Sigma, (\Gamma, A))$ which is invariant by ϕ (recall 2.4).

We cut Σ along Γ into a dijoint union of cylinders, one for each component C_i of $\partial^1\Sigma$. We use notations Notation 2.5 and Notation 4.15. We lift the retraction $\Sigma \to \Gamma$ to a retraction $\Sigma_\Gamma \to \widetilde{\Gamma}$ and the homeomorphisms ϕ to a homeomorphism $\widetilde{\phi}$ of Σ_Γ. Let $\frac{p_i}{n}$ be the rotation number of $\widetilde{\phi}|_{\widetilde{\Gamma}_i}$. Choose in the cylinder $\widetilde{\Sigma}_i$ an oriented retraction line L_i from C_i to $\widetilde{\Gamma}_i$. Consider the orientation in $\widetilde{\Sigma}_i$ inherited from the orientation in Σ. We take the classes $[L_i]$ and $[\phi(L_i)]$ in $H_1(\widetilde{\Sigma}_i, \partial\widetilde{\Sigma}_i)$. The class

$$\widetilde{\phi}|^n_{\widetilde{\Gamma}_i}([L_i]) - [L_i]$$

belongs to $H_1(\widetilde{\Sigma}_i)$ since $\widetilde{\phi}|^n_{\widetilde{\Gamma}_i}$ is the identity at the boundary. Let

$$k_i := (\widetilde{\phi}|^n_{\widetilde{\Gamma}_i}([L_i]) - [L_i]) \cdot [L_i],$$

that is the oriented intersection number of the two homology classes.

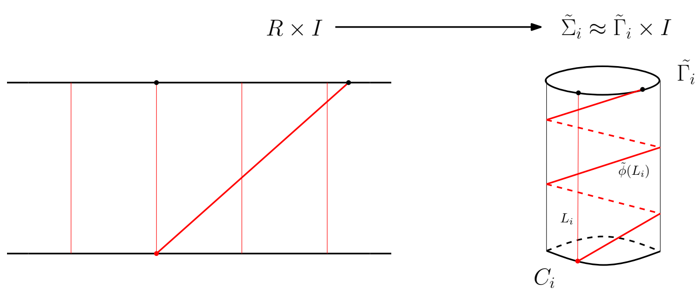

Figure 8.2. Construction of tête-à-tête homeomorphism leaving fixed the boundary components.

Definition 8.6. *According with the previous procedure we define*

$$rot(\phi, \Gamma, C_i) := k_i/n$$

In the following lemma we see in the isotopy class fixing $\partial^1\Sigma$ pointwise there is always a representative fixing a spine.

Lemma 8.7. *Let ϕ be an orientation preserving homeomorphism of a surface Σ which fixes pointwise a non-empty union $\partial^1\Sigma$ of boundary components, and which is boundary-free isotopic to a periodic homeomorphism $\hat{\phi}$. Then there there exists a collar U of $\partial^1\Sigma$, a homeomorphism $f:\Sigma\to\overline{\Sigma\setminus U}$, and a homeomorphism ψ of Σ such that: ψ is isotopic relative to $\partial^1\Sigma$ to ϕ and the restriction $\psi|_{\overline{\Sigma\setminus U}}$ is periodic and equal to $f\circ\hat{\phi}\circ f^{-1}$.*

In particular ψ leaves a spine Γ invariant and $\psi|_\Gamma$ is periodic.

Proof. Let $\Phi:\Sigma\times I\to\Sigma$ be the boundary-free isotopy between $\Phi_0=\hat{\phi}$ and $\Phi_1=\phi$. Let $f:\Sigma\to\overline{\Sigma\setminus U}$ be some homeomorphism. Consider a trivialization of a collar U of $\partial^1\Sigma$ as $U\cong\partial^1\Sigma\times I$ where $\partial^1\Sigma\cap U$ corresponds to $\partial^1\Sigma\times\{1\}$. We define ψ as follows:

$$\psi|_{\overline{\Sigma\setminus U}}=f\circ\hat{\phi}\circ f^{-1},\qquad \psi|_U:=\Phi|_{\partial^1\Sigma\times I}.$$

It is easy to give an isotopy relative to the boundary between Φ and Ψ.

The last assertion is Theorem 6.2 applied to $\psi|_{\overline{\Sigma\setminus U}}$. □

In the next lemma we see that the two definitions of fixed-boundary rotation number coincide:

Lemma 8.8. *Let Σ be a oriented surface with non-empty boundary that is neither a disk nor a cylinder. Let $\phi:\Sigma\to\Sigma$ be an orientation preserving homeomorphism that fixes a non-empty union $\partial^1\Sigma$ of boundary components. Denote by A the union of the remaining boundary components. Assume that ϕ fixes a relative spine (Γ,A) and that its restriction to it is periodic. Let C_i be a boundary component contained in $\partial^1\Sigma$. Then we have the equality $rot_{\partial^1}(\phi,C_i)=rot(\phi,\Gamma,C_i)$. In particular $rot(\phi,\Gamma,C_i)$ does not depend on the chosen spine Γ or even on the representative of $[\phi]_\partial$.*

Proof. Let $rot(\phi,\Gamma,C_i)=k_i/n$ be the fixed-boundary rotation numbers as in Definition 8.6. Note that ϕ^n fixes Γ and that the lifting $\widetilde{\phi}^n|_{\widetilde{\Sigma}_i}$ is isotopic relative to the boundary to the composition of k_i right boundary Dehn twist if k_i is positive (and $-k_i$ left Dehn twists if k_i is negative) around the boundary component C_i. Then $[\phi^n]_{\partial^1}=[D_1]^{k_1}\cdot\ldots\cdot[D_r]^{k_r}$ and the result follows. □

Corollary 8.9. *Let $g,h:\Sigma\to\Sigma$ be two homeomorphisms that fix pointwise a non empty union $\partial^1\Sigma$ of components of the boundary $\partial\Sigma$. Let A be the union of the remaining boundary components. Assume that both preserve a common relative spine (Γ,A) and that they coincide and are periodic at it. Then the equality $rot_\partial(g,C_i)=rot_\partial(h,C_i)$ holds for every i if and only if h and g are isotopic relative to $\partial^1\Sigma$.*

Proof. The two properties we want to prove equivalent are straightforward equivalent to the equality $rot_\partial(f,\Gamma,C_i)=rot_\partial(g,\Gamma,C_i)$. □

Corollary 8.10. *Let $\phi:\Sigma\to\Sigma$ be a homeomorphism that fixes pointwise a non-empty union $\partial^1\Sigma$ of components of the boundary, and that is isotopic to a periodic homeomorphism $\hat{\phi}$. Let C_i be a component in $\partial^1\Sigma$. Then the usual rotation number up to an integer $rot(\hat{\phi}|_{C_i})$ equals $|rot_\partial(\phi,C_i)-\lfloor rot_\partial(\phi,C_i)\rfloor|$ with $\lfloor x\rfloor$ is the biggest integer less that x.*

Remark 8.11. We observe that by our convention on Remark 8.1, negative (or equivalently right-handed Dehn twists) produce positive fixed-boundary rotation numbers.

9. Signed tête-à-tête graphs and homeomorphisms which fix the boundary pointwise and which are periodic up to boundary-free isotopy.

Our objective in this section is to characterize homeomorphisms which fix pointwise some components of the boundary and that are boundary-free isotopic to a periodic one, as generalized tête-à-tête twists. In order to model by tête-à-tête structures both right and left Dehn twists we need to enlarge the definition of a tête-à-tête structure. We will work directly for the class of relative graphs in order to avoid repetitions.

Let (Γ, A) be a metric relative ribbon graph. Let (Σ, A) be its thickening, let

$$g_\Gamma : \Sigma_\Gamma \to \Sigma$$

be the gluing mapping as in Notation 2.5.

We start making an extension of Remark 3.5 adding point (3'):

Remark 9.1. [Remark and notation]

(3') for every point $p \in \Gamma \setminus v(G)$ and every of the two possible oriented direction from p along Γ, there is a walk starting on p following each of this directions, such that in every vertex v, the walk continues along the previous edge in the cyclic order of $e(v)$. Each of the oriented directions at p corresponds to a point in $q \in g_\Gamma^{-1}(p)$, which lives in a cylinder $\widetilde{\Sigma}_i$. The walk considered in this remark is the image of the negative sense parametrization of the boundary of $\widetilde{\Sigma}_i$ starting at q.

We denote by γ_p^+ and ω_p^+ the usual safe walks (see Definition 3.2) which are walks as in (2) of Remark 3.5 of length π and speed 1. We denote by γ_p^- and ω_p^- the walks as in (2') of length π and speed 1. We call them *positive and negative safe walks* respectively.

In the case of points in A, since A is oriented, we have also a positive and negative sense for a parametrization. Then, for $p \in A$, we define γ_p^+ (respectively γ_p^-) as the parametrization from p that starts along A in the positive (respectively negative) sense and that when reaching a vertex v takes the next (respectively previous) edge in the order of $e(v)$).

We also define a *safe constant walk* $\gamma_p^0 := p$.

Before stating next Definition recall that there is a bijection between a set $\mathcal{C}$ of boundary components of Σ and the cylinders $\widetilde{\Sigma}_i$'s. Given a "sign" mapping $\iota : \mathcal{C} \to \{0, +, -\}$, we denote by $\iota(i)$ the image by ι of the component that corresponds to $\widetilde{\Sigma}_i$ under the bijection.

Definition 9.2. *Let (Γ, A) be a metric relative ribbon graph and let $(\Sigma, (\Gamma, A))$ be its thickening. Let $\mathcal{C}$ denote the set of boundary components of Σ which do not belong to A. Fix a mapping*

$$\iota : \mathcal{C} \to \{0, +, -\}.$$

We say (Γ, A) satisfies the signed tête-à-tête property *for ι or that (Γ, A, ι) is a* signed relative tête-à-tête graph *if given any point p contained at the interior of an edge the following properties are satisfied*

(1) *if p does not belong to A and $p \in g_\Gamma(\widetilde{\Sigma}_i) \cap g_\Gamma(\widetilde{\Sigma}_j)$ for some i, j, given $\gamma_p^{\iota(i)}$ and $\omega_p^{\iota(j)}$, the two safe walks starting at p and determined by the sign ι, we have the equality*
$$\gamma_p^{\iota(i)}(\pi) = \omega_p^{\iota(j)}(\pi).$$
In other words: the ending points of the two corresponding signed safe walks starting at p coincide.

(2) *if p belongs to A then p belongs to $g_\Gamma(\widetilde{\Sigma}_i)$ for a unique cylinder. The ending point $\gamma_p^{\iota(i)}(\pi)$ of the unique signed safe walk starting at p belongs to A.*

Notation 9.3 (Remark and Notation)**.** Observe that the mapping $\Gamma \setminus v(\Gamma) \to \Gamma$ that sends $p \in \Gamma \setminus v(\Gamma)$ to $\gamma_p^{\iota(i)}(\pi)$ extends to $v(\Gamma)$ and define a homeomorphism of Γ that we denote by $\sigma_{(\Gamma,\iota)}$. The proof is as the one of Lemma 3.8.

Definition 9.4. *Let (Γ, A, ι) be a signed relative tête-à-tête graph. For every choice of product structure $\widetilde{\Sigma}_i \approx \widetilde{\Gamma}_i \times I$ we consider the homeomorphism*
$$\psi_i : \widetilde{\Gamma}_i \times I \longrightarrow \widetilde{\Gamma}_i \times I$$
$$(p, s) \mapsto (\tilde{\gamma}_p^{\iota(i)}(s \cdot \pi), s)$$
where $\tilde{\gamma}_p^{\iota(i)}$ is the lifting of the safe walk to $\widetilde{\Gamma}_i$. The homeomorphism ψ_i of the cylinder can be visualized very easily using the universal covering of the cylinder as in Figure 8.2.

These homeomorphisms glue well due to the properties of the signed relative tête-à-tête graph (Γ, ι) and define a homeomorphism of (Σ, Γ, A) that leaves $\partial^1\Sigma$ fixed pointwise and A invariant. We denote by $\phi_{(\Gamma,A,\iota)}$ the resulting homeomorphism of (Σ, A).

Any 2 choices of product structures produce homeomorphisms that are conjugate by a homeomorphism fixing $\partial^1\Sigma$ pointwise.

The homeomorphism $\phi_{(\Gamma,A,\iota)}$ leaves Γ invariant and has obviously the following fixed-boundary rotation numbers:

$$rot^\partial(\phi_{(\Gamma,A,\iota)}, C_i) = \iota(i) \cdot \frac{\pi}{length(\widetilde{\Gamma}_i)}. \tag{9.5}$$

Observe that when $\iota(i) = 0$, then the homeomorphism ψ_i is the identity and $rot^\partial(\phi_{(\Gamma,A,\iota)}, C_i) = 0$.

Remark 9.6. Signed tête-à-tête homeomorphisms are a generalisation of Dehn twists. We will call them also *signed tête-à-tête twists*. If all the signs are positive this notion coincides with A'Campo original notion of tête-à-tête twists.

In the next Theorem we characterize homeomorphisms fixing pointwise and boundary-free isotopic to a periodic one as signed tête-à-tête homeomorphisms. We state first the non-relative case, in order to emphasize its importance.

Theorem 9.7. *Let Σ be an oriented surface with non-empty boundary. Let ϕ be an orientation preserving homeomorphism fixing pointwise the boundary, and boundary-free isotopic to a periodic one $\hat{\phi}$. Then,*

(i) *there exists a signed tête-à-tête spine (Γ, ι) embedded in Σ and invariant by ϕ such that the restriction of ϕ to Γ coincides with $\phi_{\Gamma,\iota}$.*

(ii) *The isotopy classes relative to the boundary $[\phi]_\partial$ and $[\phi_{\Gamma,\iota}]_\partial$ coincide.*

(iii) *the homeomorphisms* ϕ *and* $\phi_{\Gamma,\iota}$ *are conjugate by a homeomorphism that fixes the boundary pointwise, fixes* Γ *and is isotopic to the identity in* $MCG(\Sigma, \partial\Sigma)$.

(iv) *Moreover, if* $genus(\Sigma^{\hat{\phi}}) \neq 0$ *or the quotient map by* $\hat{\phi}$ *has at least two branching points, then there exists a spine* Γ *of* Σ *such that for any homeomorphism* ψ *which fixes pointwise the boundary and is boundary-free isotopic to* $\hat{\phi}$*, there is a signed tête-à-tête structure on* Γ *(that is, a metric and a sign function* ι*) such that* ψ *is isotopic relative to the boundary to the corresponding signed tête-à-tête homeomorphism. In other words, there is a universal spine which may be endowed of signed tête-à-tête structures representing all boundary fixed isotopy classes of homeomorphisms which are boundary free isotopic to* ϕ.

Remark 9.8. Note that the spine $p(\Gamma)$ of Σ^{ϕ} chosen in the proof is the same for all homeomorphisms ϕ of any surface Σ with the same quotient surface Σ^{ϕ} whenever Σ^{ϕ} has genus different from 0.

Corollary 9.9. *This theorem characterises the originally defined A'Campo tête-à-tête twists as: orientation preserving homeomorphism fixing pointwise the boundary and boundary-free isotopic to a periodic one, with strictly positive fixed-boundary rotation numbers.*

Proof of Theorem 9.7. By Lemma 8.7 we can assume that there exist a collar for $\partial\Sigma$ such that $\phi' = \phi|_{\Sigma\setminus U}$ is periodic. Let $\Sigma^{\phi'}$ be the quotient surface.

The proof consists, as in Theorem 5.4, in giving a metric in a spine for $\Sigma^{\phi'}$ such that the pullback metric in the corresponding invariant graph in Σ solves the problem.

We will see that the graphs used in the proof of Theorem 5.4 are also enough here. We denote them by $\Gamma^{\phi'}$. We redefine

$$R_i := |rot^{\partial}(\phi, C_i)|.$$

Now, the system that we need to solve comes also from (5.5) by the following computation:

$$rot^{\partial}(\phi, C_i) \cdot length(\widetilde{\Gamma}_i) \cdot \iota(i) = \pi, \tag{9.10}$$

if $rot_{\partial}(\phi, C_i)$ is different from 0.

If $rot^{\partial}(\phi, C_i)$ equals 0, by the definition of constant safe walk (see the end of Remark 9.1), we obtain no condition. By the definition of R_i and the fact that both $rot_{\partial}(\phi, C_i)$ and $\iota(i)$ have the same sign, this equation becomes:

$$R_i \cdot length(\widetilde{\Gamma}_i) = \pi, \tag{9.11}$$

From this point, following word by word the derivation of the systems of equations (5.6), (5.7) and (5.8) we obtain the system (5.8) with some equations deleted. It has positive solutions by the same kind of arguments.

For $genus(\Sigma^{\phi'}) = 0$ we arrive to a subsystem of system (5.9) by the same arguments. We assign coherent lengths to all the edges of $\Gamma^{\phi'}$ as in that proof.

It is easy to convince oneself that the graph $\Gamma := p^{-1}(\Gamma^{\hat{\phi}})$ is a tête-à-tête spine for ι defined as $\iota(i) = sign(rot^{\partial}(\phi, C_i))$ and that $rot_{\partial}(\phi_{(\Gamma,\iota)}, C_i) = rot_{\partial}(\phi, C_i)$. As in the proof of Theorem 5.4 we can see that $\phi|_{\Gamma}$ equals $\phi_{(\Gamma,\iota)}|_{\Gamma}$. Moreover, since ϕ and $\phi_{\Gamma,\iota}$ coincide on a spine with a periodic homeomorphisms and have the same fixed-boundary rotation numbers we can conclude by Corollary 8.9 that they are isotopic relative to the boundary and conjugate by a homeomorphism fixing the boundary. The part (*iii*) on the conjugation follows because $\phi|_\Gamma$ and $\phi_{\Gamma,\iota}|_\Gamma$ coincide and because the fixed-boundary rotation numbers also coincide. The second part of the theorem is immediate by the way of constructing the graph after quotienting by ϕ'. □

Now we state the relative case in a simple way, skipping the obvious strengthenings similar to the previous Theorem:

Theorem 9.12. *Let Σ be an oriented surface with non-empty boundary. Let $\partial^1\Sigma$ be a non empty union of boundary components. Let A be the union of the boundary components not contained in $\partial^1\Sigma$. Let ϕ be an orientation preserving homeomorphism fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one $\hat{\phi}$. Then, there exists a signed tête-à-tête spine (Γ, A, ι) such that $\phi_{\Gamma,A,\iota}$ is isotopic relative to $\partial^1\Sigma$ to ϕ. Moreover, if ϕ is periodic outside a collar of $\partial^1\Sigma$, we have also that $[\phi]_{\partial,\phi|_\partial} = [\phi_{\Gamma,A,\iota}]_{\partial,\phi|_\partial}$.*

Proof. Apply Alexander's trick to the boundary components in A in order to obtain a larger surface and a homeomorphism fixing pointwise the boundary. Construct a signed tête-à-tête graph inducing this homeomorphism like in the proof of Theorem 9.7 and modify it to get a signed relative tête-à-tête graph following the proof of Theorem 6.2. □

Corollary 9.13. *This theorem characterises the originally defined A'Campo relative tête-à-tête twists as: orientation preserving homeomorphism fixing pointwise $\partial^1\Sigma$ and boundary-free isotopic to a periodic one $\hat{\phi}$ with strictly positive fixed-boundary rotation numbers.*

PART II

In this part we focus on pseudo-periodic homeomorphisms of oriented surfaces with non-empty boundary. In Section 10 we summarize the needed facts about the general theory. The main tool in this part are *mixed tête-à-tête graphs* which are introduced in Section 11. In Section 12 we see how they can codify a pseudo-periodic homeomorphism. In Section 13, Theorem 13.3, we see that a wide class of homeomorphisms can be modelled by them. In particular, monodromy of plane branches are of this type (see Corollary 13.14). In Example 13.17 we see that mixed tête-à-tête graphs allow to model further cases.

10. Pseudo-periodic homeomorphisms.

We recall some definitions and fix some conventions on pseudo-periodic homeomorphisms of surfaces with boundary.

Definition 10.1. *A homeomorphism $\phi : \Sigma \to \Sigma$ is pseudo-periodic if it is isotopic to a homeomorphism satisfying that there exists a finite collection of disjoint simple closed curves $\mathcal{C}$ such that*

(1) $\phi(\mathcal{C}) = \mathcal{C}$
(2) $\phi|_{\Sigma\setminus\mathcal{C}}$ is boundary free isotopic to a periodic homeomorphism.

Assuming that none of the connected components of $\Sigma \setminus \mathcal{C}$ is neither a disk nor an annulus and that the set of curves is minimal, which is always possible, we name $\mathcal{C}$ an admissible set of curves for ϕ.

The following theorem is a particularization on pseudo-periodic homeomorphisms of the more general Corollary 13.3 in [FM12] that describes a *canonical form* for every homeomorphism of a surface.

Theorem 10.2 (Canonical Form)**.** *Let Σ be a surface with $\partial\Sigma \neq \emptyset$. Any pseudo-periodic map of Σ is isotopic to a homeomorphism in* canonical form*, that means a homeomorphism ϕ which has an admissible set of curves $\mathcal{C} = \{\mathcal{C}_i\}$ and annular neighbourhoods $\mathcal{A} = \{\mathcal{A}_i\}$ with $\mathcal{C}_i \subset \mathcal{A}_i$ such that*

(1) $\phi(\mathcal{A}) = \mathcal{A}$,
(2) the map $\phi|_{\overline{\Sigma\setminus\mathcal{A}}}$ is periodic.

Remark 10.3. In the case we have a pseudo-periodic homeomorphism of Σ that fixes pointwise some components $\partial^1\Sigma$ of the boundary $\partial\Sigma$ we can always find *a canonical form* as follows. We can find an isotopic homeomorphism ϕ relative to $\partial^1\Sigma$ that coincides with a canonical form as in the previous theorem outside a collar neighborhood U of $\partial^1\Sigma$. We may assume that there exists an isotopy connecting the homeomorphism and its canonical form relative to $\partial^1\Sigma$.

Notation 10.4. Let $s, c \in \mathbb{R}$. We denote by $\mathcal{D}_{s,c}$ the homeomorphism of $\mathbb{S}^1 \times I$ induced by $(x,t) \mapsto (x + st + c, t)$ (we are taking $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$). Observe that

$$\mathcal{D}_{s,c} \circ \mathcal{D}_{s',c'} = \mathcal{D}_{s+s',c+c'},$$
$$\mathcal{D}_{s,c}^{-1} = \mathcal{D}_{-s,-c}.$$

However, in this work we will always have $s \in \mathbb{Q}$.

Definition 10.5. *Let $\mathcal{C} \subset \Sigma$ be a simple closed curve embedded in an oriented surface Σ. And let $\mathcal{A}$ be a tubular neighborhood of $\mathcal{C}$. Let $\mathcal{D} : \Sigma \to \Sigma$ be a homeomorphism of the surface with $\mathcal{D}|_{\Sigma\setminus\mathcal{A}} = id$. We say that $\mathcal{D}$ is a positive Dehn twist around*

$\mathcal{C}$ or a positive Dehn twist on $\mathcal{A}$ if there exist a parametrization $\eta : \mathbb{S}^1 \times I \to \mathcal{A}$ preserving orientation such that

$$\mathcal{D} = \eta \circ \mathcal{D}_{1,0} \circ \eta^{-1}.$$

A negative Dehn twist is defined similarly changing $\mathcal{D}_{1,0}$ by $\mathcal{D}_{-1,0}$ in the formula above, equivalently, a negative Dehn twist is $\mathcal{D}^{-1}$. Compare with Remark 8.1 to see that they are equivalent.

Lemma 10.6 (Linearization. Lemma 2.1 in [MMA11])**.** *Let $\mathcal{A}_i$ be an annulus and let $\phi : \mathcal{A}_i \to \mathcal{A}_i$ be a homeomorphism that does not exchange boundary components. Suppose that $\phi|_{\partial \mathcal{A}_i}$ is periodic. Then there exists a parametrization $\eta : \mathbb{S}^1 \times I \to \mathcal{A}_i$ such that*

$$\phi = \eta \circ \mathcal{D}_{-s,-c} \circ \eta^{-1}$$

for some $s \in \mathbb{Q}$, some $c \in \mathbb{R}$.

Remark 10.7. In the case $\phi|_{\partial \mathcal{A}_i}$ is the identity, we have that

$$\phi = \eta \circ \mathcal{D}_{-s,0} \circ \eta^{-1}$$

for some $s \in \mathbb{Z}$, that is $\phi = \mathcal{D}^s$ for some Dehn twist as in Definition 10.5.

Definition 10.8 (Screw number)**.** *Let ϕ be a pseudo-periodic homeomorphism as in Theorem 10.2. Let n be the order of $\phi|_{\Sigma \setminus \mathcal{A}}$. By Remark 10.7, $\phi^n|_{\mathcal{A}_i}$ equals $\mathcal{D}|_{\mathcal{A}_i}^{s_i}$ for a certain $s_i \in \mathbb{Z}$. Let α be the length of the orbit in which $\mathcal{A}_i$ lies and suppose that $\phi^{\alpha_i}|_{\mathcal{A}_i}$ does not interchange boundary components. We define*

$$s(\mathcal{A}_i) := \frac{s_i}{n}\alpha.$$

We call $s(\mathcal{A}_i)$ the screw number *of ϕ at $\mathcal{A}_i$ or at $\mathcal{C}_i$.*

Remark 10.9. Compare Definition 10.8 with [[MMA11] p.4] and with Definition 2.4 in [MMA11]. The original definition is due to Nielsen [[Nie44]. Sect. 12] and it does not depend on a canonical form for ϕ. Since we are restricting to homeomorphisms that do not exchange boundary components of the annuli $\mathcal{A}$, our definition is a bit simpler.

We start with an easy lemma that is important for Theorem 13.3

Lemma 10.10. *Let ϕ be a homeomorphism as in Theorem 10.2 and let $\{\mathcal{A}_i\} \subset \mathcal{A}$ be a set of annuli cyclically permuted by ϕ, i.e. $\phi(\mathcal{A}_i) = \mathcal{A}_{i+1}$. Then there exist coordinates*

$$\eta_i : \mathbb{S}^1 \times I \to \mathcal{A}_i$$

for the annuli in the orbit such that

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \mathcal{D}_{-s/\alpha,-c/\alpha}$$

where s and c are associated to $\mathcal{A}_1$ as in Lemma 10.6.

Proof. Take a parametrization of $\mathcal{A}_1$ as in Lemma 10.10, say $\eta_1 : \mathbb{S}^1 \times I \to \mathcal{A}_1$.

Define recursively $\eta_j := \phi \circ \eta_{j-1} \circ \mathcal{D}_{s/\alpha,c/\alpha}$ (see Notation 10.4). Then, we have

$$\eta_{j+1}^{-1} \circ \phi \circ \eta_j = \mathcal{D}_{-s/\alpha,-c/\alpha}.$$

Since for every j we have that $\eta_j = \phi^{j-1} \circ \eta_1 \circ \mathcal{D}_{s(j-1)/\alpha, c(j-1)/\alpha}$ we have also that

$$\eta_1^{-1} \circ \phi \circ \eta_\alpha = \eta_1^{-1} \circ \phi \circ \phi^{\alpha-1} \circ \eta_1 \circ \mathcal{D}_{s(\alpha-1)/\alpha, c(\alpha-1)/\alpha} = \mathcal{D}_{-s/\alpha,-c/\alpha}.$$

□

Remark 10.11. After this proof we can check that $\eta_k^{-1} \circ \phi^\alpha \circ \eta_k = \mathcal{D}_{-s,-c}$ to see that the screw number $s = s(\mathcal{A}_i)$ and the parameter c modulo $\mathbb{Z}$ of Lemma 10.10 only depend on the orbit of $\mathcal{A}_i$.

We observe also that the numbers s and c of Lemma 10.10 satisfy

- s equals $s(\mathcal{A}_i)$ and
- c is only determined modulo $\mathbb{Z}$ and equals $rot(\phi^{\alpha_i}|_{\eta(\mathbb{S}^1\times\{0\})}) \in (0,1]$.

This was previously observed in Corollary 2.2 in [MMA11].

10.1. **Gluings and boundary Denh twists.** In this section we introduce some notions and examples that will be used in the rest of the paper. We begin with two different easy remarks:

Remark 10.12. Given a homeomorphism ϕ of a surface Σ with $\partial\Sigma \neq \emptyset$. Let C be a connected component of $\partial\Sigma$. Let A be a cylinder parametrized by $\eta : \mathbb{S}^1 \times I \to A$. We glue A with Σ by an identification $f : \mathbb{S}^1 \times \{0\} \to C$. Then, we can extend ϕ *trivially* along A by defining a homeomorphism $\bar{\phi}$ of $\Sigma \cup_f A$ as $\bar{\phi}|_\Sigma = \phi$ and for any $p \in A$ we set $\bar{\phi}(p) = \phi(f \circ p_1 \circ \eta^{-1}(p))$ where p_1 is the canonical projection from $\mathbb{S}^1 \times I$ to $\mathbb{S}^1 = \mathbb{S}^1 \times \{0\}$.

Remark 10.13. Given a homeomorphism ϕ of a surface Σ with $\partial\Sigma \neq \emptyset$. Let C be a connected component of $\partial\Sigma$. Let A be a compact collar neighbourhood of C (isomorphic to $I \times C$) in Σ. Let $\eta : \mathbb{S}^1 \times I \to A$ be a parametrization[1] of A, with $\phi(\mathbb{S}^1 \times \{1\}) = C$.

Then, there exists a homeomorphism ϕ' isotopic to ϕ relative to the boundary such that

- the restriction to A satisfies $p_2 \circ \eta^{-1} \circ \phi'|_A \circ \eta(t,x) = p_2 \circ \eta^{-1} \circ \phi'|_A \circ \eta(t',x)$ for all $t,t' \in I$, where $p_2 : \mathbb{S}^1 \times I \to \mathbb{S}^1$ is the canonical projection.

Definition 10.14. *Let C be a component of $\partial\Sigma$ and let A be a compact collar neighbourhood of C in Σ. Suppose that C has a metric and total length is equal to ℓ. Let $\eta : \mathbb{S}^1 \times I \to A$ be a parametrization of A, such that $\eta|_{\mathbb{S}^1\times\{1\}} : \mathbb{S}^1 \times \{1\} \to C$ is an isometry. Suppose that $\mathbb{S}^1$ has the metric induced from taking $\mathbb{S}^1 = \mathbb{R}/\ell\mathbb{Z}$ with $\ell \in \mathbb{R}_{>0}$ and the standard metric on $\mathbb{R}$. A* boundary Denh twist *of length $r \in \mathbb{R}_{>0}$ along C is a homeomorphism $\mathcal{D}_r^\eta(C)$ of Σ such that:*

(1) it is the identity outside A
(2) the restriction of $\mathcal{D}_r^\eta(C)$ to A in the coordinates given by η is given by $(x,t) \mapsto (x + r \cdot t, t)$.

The isotopy type of $\mathcal{D}_r^\eta(C)$ by isotopies fixing the action on $\partial\Sigma$ does not depend on the parametrization η. When we write just $\mathcal{D}_r(C)$, it means that we are considering a boundary Dehn twist with respect to some *parametrization η.*

Example 10.15. We can restate the $\ell-$tête-à-tête property in terms of boundary Dehn twists of length ℓ as follows. Let (Σ, Γ) be a pair of a surface and a metric ribbon graph. Let $g_\Gamma : \Sigma_\Gamma \to \Sigma$ be the gluing map. Consider the pull back metric on $g^{-1}(\Gamma)$. Denote by $\mathcal{D}_{\underline{\ell}}$ the composition of the boundary Denh twists $\mathcal{D}_\ell$ along each $\widetilde{\Gamma}_j \subset \widetilde{\Gamma}$. Then (Σ, Γ) holds the ℓ-tête-à-tête property if and only if $\mathcal{D}_{\underline{\ell}}$ is compatible with the gluing g_Γ. We see from this that the lenghts of $\widetilde{\Gamma}_j$ are in $\ell\mathbb{Q}_+$.

[1]The parametrization is needed if ϕ is not the identity on C.

11. Mixed tête-à-tête graphs.

With pure, relative and general tête-à-tête graphs we model periodic homeomorphisms. Now we extend the notion of tête-à-tête graph to be able to model some pseudo-periodic homeomorphisms.

Let $(\Gamma^\bullet, A^\bullet)$ be a decreasing filtration on a connected relative metric ribbon graph (Γ, A). That is

$$(\Gamma, A) = (\Gamma^0, A^0) \supset (\Gamma^1, A^1) \supset \cdots \supset (\Gamma^d, A^d)$$

where $\supset$ between pairs means $\Gamma^i \supset \Gamma^{i+1}$ and $A^i \supset A^{i+1}$, and where (Γ^i, A^i) is a (possibly disconnected) relative metric ribbon graph for each $i = 0, \dots, d$. We say that d is the depth of the filtration $\Gamma^\bullet$. We assume each Γ^i does not have univalent vertices and is a subgraph of Γ in the usual terminology in Graph Theory. We observe that since each (Γ^i, A^i) is a relative metric ribbon graph, we have that $A^i \setminus A^{i+1}$ is a disjoint union of connected components homeomorphic to $\mathbb{S}^1$.

For each $i = 0, \dots, d$, let

$$\delta_i : \Gamma^i \to \mathbb{R}_{\geq 0}$$

be a locally constant map (so it is a map constant on each connected component). We put the restriction that $\delta_0(\Gamma^0) > 0$. We denote the collection of all these maps by $\delta_\bullet$.

Let $p \in \Gamma$, we define c_p as the largest natural number such that $p \in \Gamma^{c_p}$.

Definition 11.1 (Mixed safe walk)**.** *Let $(\Gamma^\bullet, A^\bullet)$ be a filtered relative metric ribbon graph. Let $p \in \Gamma \setminus A \setminus v(\Gamma)$. We define a mixed safe walk γ_p starting at p as a concatenation of paths defined iteratively by the following properties*

i) *γ_p^0 is a safe walk of length $\delta_0(p)$ starting at $p_0^\gamma := p$. Let $p_1^\gamma := \gamma^0(\delta_0)$ be its endpoint.*
ii) *Suppose that γ_p^{i-1} is defined and let p_i^γ be its endpoint.*
   - *If $i > c_p$ or $p_i^\gamma \notin \Gamma^i$ we stop the algorithm.*
   - *If $i \leq c_p$ and $p_i^\gamma \in \Gamma^i$ then define $\gamma_p^i : [0, \delta_i(p_i)] \to \Gamma^i$ to be a safe walk of length $\delta_i(p_i^\gamma)$ starting at p_i^γ and going in the same direction as γ_p^{i-1}.*
iii) *Repeat step ii) until algorithm stops.*

Finally, define $\gamma_p := \gamma_p^k \star \cdots \star \gamma_p^0$, that is, the mixed safe walk starting at p is the concatenation of all the safe walks defined in the inductive process above.

As in the pure case, there are two safe walks starting at each point on $\Gamma \setminus (A \cup v(\Gamma))$. We denote them by γ_p and ω_p.

Definition 11.2 (Boundary mixed safe walk)**.** *Let $(\Gamma^\bullet, A^\bullet)$ be a filtered relative metric ribbon graph and let $p \in A$. We define a boundary mixed safe walk b_p starting at p as a concatenation of a collection of paths defined iteratively by the following properties*

i) *$b_{p_0}^0$ is a boundary safe walk of length $\delta_0(p)$ starting at $p_0 := p$ and going in the direction indicated by A (as in the relative tête-à-tête case). Let $p_1 := b_p^0(\delta_0)$ be its endpoint.*
ii) *Suppose that $b_{p_{i-1}}^{i-1}$ is defined and let p_i be its endpoint.*
   - *If $i > c_p$ or $p_i \notin \Gamma^i$ we stop the algorithm.*
   - *If $i \leq c(p)$ and $p_i \in \Gamma^i$ then define $b_{p_i}^i : [0, \delta_i(p_i)] \to \Gamma^i$ to be a safe walk of length $\delta_i(p_i)$ starting at p_i and going in the same direction as $b_{p_{i-1}}^{i-1}$.*

iii) *Repeat step ii) until algorithm stops.*

Finally, define $b_p := b^k_{p_k} \star \cdots \star b^0_{p_0}$*, that is, the boundary mixed safe walk starting at* p *is the concatenation of all the safe walks defined in the inductive process.*

Notation 11.3. We call the number k in Definition 11.1 (resp. Definition 11.2), the *order* of the mixed safe walk (resp. boundary mixed safe walk) and denote it by $o(\gamma_p)$ (resp. $o(b_p)$).

We denote by $l(\gamma_p)$ the *length* of the mixed safe walk γ_p which is the sum $\sum_{j=0}^{o(\gamma_p)} \delta_j(p^\gamma_j)$ of the lengths of all the walks involved. We consider the analogous definition $l(b_p)$.

As in the pure case, two mixed safe walks starting at $p \in \Gamma \setminus v(\Gamma)$ exist. We denote by ω_p the mixed safe walk that starts at p but in the opposite direction to the starting direction of γ_p.

Observe that since the safe walk $b^0_{p_0}$ is completely determined by p, for a point in A there exists only one boundary safe walk.

Now we define the relative mixed tête-à-tête property.

Definition 11.4 (Relative mixed tête-à-tête property)**.** *Let* $(\Gamma^\bullet, A^\bullet)$ *be a filtered relative metric ribbon graph and let* $\delta_\bullet$ *be a set of locally constant mappings* $\delta_k : \Gamma^k \to \mathbb{R}_{\geq 0}$*. We say that* $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ *satisfies the relative mixed tête-à-tête property or that it is a relative mixed tête-à-tête graph if for every* $p \in \Gamma - (v(\Gamma) \cup A)$

I) *The endpoints of* γ_p *and* ω_p *coincide.*

II) $c_{\gamma_p(l(\gamma_p))} = c_p$

and for every $p \in A$*, we have that*

III) $b_p(l(b_p)) \in A^{c_p}$

As a consequence of the two previous definitions we have:

Lemma 11.5. *Let* $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ *be a mixed relative tête-à-tête graph, then*

a) $o(\omega_p) = o(\gamma_p) = c_p$

b) $l(\gamma_p) = l(\omega_p)$ *for every* $p \in \Gamma \setminus v(\Gamma)$.

Proof. *a*)By Definition 11.1 ii), we have that $o(\gamma_p) \leq c_p$ for all $p \in \Gamma \setminus v(\Gamma)$. Suppose that for some p, we have that $o(\gamma_p) = k < c_p$. This means, that while constructing the mixed safe walk we stopped after constructing the path γ^k_p either because $k > c_p$ which contradicts the supposition, or because the endpoint p^γ_k of γ^k_p is not in Γ^k which contradicts that $c_p = c_{\gamma_p(l(\gamma_p))}$. This proves the equality $o(\gamma_p) = c_p$. In order to prove $o(\omega_p) = c_p$ use the equality $\gamma_p(l(\gamma_p)) = \omega_p(l(\omega_p))$ and repeat the same argument.

b) Let q be the endpoint of γ_p and ω_p. Since the image of the safe walks $\gamma^{c_p}_p$ and $\omega^{c_p}_p$ lies on the same connected component of Γ^{c_p} we have that their starting points $p^\gamma_{c_p}$ and $p^\omega_{c_p}$ also lie on that same connected component. Therefore $\delta_{c_p}(p^\gamma_{c_p}) = \delta_{c_p}(p^\omega_{c_p})$.

Suppose now that p^γ_i and p^ω_i lie on the same connected component of Γ^i (and so $\delta_i(p^\gamma_i) = \delta_i(p^\omega_i)$). Then the image of the safe walks γ^{i-1}_p and ω^{i-1}_p lies on the same connected component of Γ^{i-1} and we have that their starting points p^γ_{i-1} and p^ω_{i-1} also lie on that same connected component. So $\delta_{i-1}(p^\gamma_{i-1}) = \delta_{i-1}(p^\omega_{i-1})$.

We conclude that $\delta_j(p^\gamma_j) = \delta_j(p^\omega_j)$ for all $j = 0, \ldots, d$ which concludes the proof. □

Remark 11.6. Note that for mixed tête-à-tête graphs it is not true that $p \mapsto \gamma_p(\delta(p))$ gives a continuous mapping from Γ to Γ.

12. Mixed tête-à-tête homeomorphisms.

Let $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ be a a mixed tête-à-tête graph. Let Σ be the relative thickening of (Γ, A) and let $\partial^1\Sigma$ be the union of the boundary components of Σ not contained in A. In this section we define, up to isotopy fixing $\partial^1\Sigma$, and also relative to the action on A, a homeomorphism ϕ_Γ of Σ which will be a pseudo-periodic homeomorphism associated to $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ (see Theorem 12.12).

Remark 12.1. Note that in particular a homeomorphism induced by a mixed tête-à-tête graph also represents a class in the Mapping Class Group up to boundary free isotopy.

For the sake of simplicity in notation we assume that $A^\bullet = \emptyset$ during the construction. The general case is analogous.

Notation 12.2. Let $g_{\Gamma^i} : \Sigma_{\Gamma^i} \to \Sigma$ be the gluing map as in Notation 2.5. Let Γ_{Γ^i} be the preimage of Γ by g_{Γ^i}. We also denote by g_{Γ^i} its restriction $g_{\Gamma^i} : \Gamma_{\Gamma^i} \to \Gamma$. The union of the boundary components of Σ_{Γ^i} that come from Γ^i is denoted by $\widetilde{\Gamma}^i$. Observe that a single connected component of Γ^i might produce more than one boundary component in Σ_{Γ^i}.

It's clear that g_{Γ^i} factorizes as follows:

$$\Sigma_{\Gamma^i} \to \Sigma_{\Gamma^{i+1}} \to ... \to \Sigma.$$

We denote these mappings by $g_j : \Sigma_{\Gamma^j} \to \Sigma_{\Gamma^{j+1}}$ for $j = 0, ..., d-1$ and also their restrictions $g_j : \Gamma_{\Gamma^j} \to \Gamma_{\Gamma^{j+1}}$.

Remark 12.3. Observe that by Definition 11.4, each connected component of the relative metric ribbon graph $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$ has the relative tête-à-tête property for safe walks of length $\delta_0(\Gamma)$.

Let $\phi_{\Gamma,0} : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^1}$ be the tête-à-tête homeomorphism fixing each boundary component that is not in $\widetilde{\Gamma}^1$ as in Definition 9.4. It is the homeomorphism induced by the relative tête-à-tête property of each connected component of $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$ for some choice of product structures on $(\Sigma_{\Gamma^1})_{\Gamma_{\Gamma^1}}$.

Also according to Definition 9.4, observe that since we do not specify anything, we assume that the sign ι is constant $+1$.

Now we continue to define inductively the homeomorphism ϕ_Γ.

Notation 12.4. Let

$$\mathcal{D}_{\delta_i} : \Sigma_{\Gamma^i} \to \Sigma_{\Gamma^i}$$

be the homeomorphism consisting of the composition of all the boundary Dehn twists $\mathcal{D}_{\delta_i(g_{\Gamma^i}(C))}(C)$ for all components in $\widetilde{\Gamma}^i$. Recall Definition 10.14.

Lemma 12.5. *The homeomorphism*

$$\widetilde{\phi}_{\Gamma,1} := \mathcal{D}_{\delta_1} \circ \phi_{\Gamma,0} : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^1}$$

is compatible with the gluing $g_1 : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^2}$.

Proof. We use the notation introduced in Definition 11.1.

Since g_1 only identifies points in $\widetilde{\Gamma}^1$, we must show that if x, y are different points in $\widetilde{\Gamma}^1$ such that $g_1(x) = g_1(y) \in \Gamma^1$, then $g_1(\widetilde{\phi}_{\Gamma,1}(x)) = g_1(\widetilde{\phi}_{\Gamma,1}(y))$.

So let $x, y \in \widetilde{\Gamma}^1$ be such that $g_1(x) = g_1(y) = p$. In particular, $c_p = 1$ and we have that $\gamma_p = \gamma_p^1 \star \gamma_p^0$ and $\omega_p = \omega_p^1 \star \omega_p^0$ by Lemma 11.5 a). So the mixed safe walks end in a connected component of Γ^1. Denote by $\hat{p}$ their endpoint. By Definition 11.4 ii) we have that $c_{\hat{p}} = 1$.

First observe that $\phi_{\Gamma,0}(x) = b_x(\delta_0(p))$ where $b_x : [0, \delta_0(p)] \to \Gamma_{\Gamma^1}$ is the boundary safe walk of length $\delta_0(p)$ given by the relative tête-à-tête structure on $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$. Analogously $\phi_{\Gamma,0}(y) = b_y(\delta_0(y))$. So we have $g_1(\phi_{\Gamma,0}(x)) = \gamma_p(\delta_0(p))$ and $g_1(\phi_{\Gamma,0}(y)) = \omega_p(\delta_0(p))$.

It is clear that $(\mathcal{D}_{\delta_1}(\phi_{\Gamma,0}(x))) = \widetilde{\gamma}_p^0(\delta_1(\phi_{\Gamma,0}(x)) + \delta_0(p))$ with $\widetilde{\gamma}_p$ the safe walk in $\widetilde{\Gamma}^1$. It is also clear that $\widetilde{\gamma}_p$ is the actual lifting of γ_p along Γ. Then,

$$g_1(\mathcal{D}_{\delta_1} \circ \phi_{\Gamma,0}(x)) = \gamma_p(\delta_1(\phi_{\Gamma,0}(x)) + \delta_0(p)) = \gamma_p(\delta_1(\gamma_p^0(\delta_0(p))) + \delta_0(p)) = \gamma_p(l(\gamma_p))$$

and analogously $g_1(\mathcal{D}_{\delta_1} \circ \phi_{\Gamma,0}(y)) = \omega_p(l(\omega_p))$.

By property i) of a mixed tête-à-tête graph, we can conclude. □

Now, we consider the homeomorphism induced by $\widetilde{\phi}_{\Gamma,1}$ and we denote it by

$$\phi_{\Gamma,1} : \Sigma_{\Gamma^2} \to \Sigma_{\Gamma^2}.$$

The same argument applies inductively to prove that each map

$$\widetilde{\phi}_{\Gamma,i} := \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1} : \Sigma_{\Gamma^i} \to \Sigma_{\Gamma^i} \tag{12.6}$$

is compatible with the gluing g_i and hence it induces a homeomorphism

$$\phi_{\Gamma,i} : \Sigma_{\Gamma^{i+1}} \to \Sigma_{\Gamma^{i+1}}. \tag{12.7}$$

In the end we get a map

$$\phi_{\Gamma} := \phi_{\Gamma,d} : \Sigma \to \Sigma \tag{12.8}$$

which we call the mixed tête-à-tête homeomorphism induced by $(\Gamma^\bullet, \delta_\bullet)$.

Notation 12.9. We can extend the notation introduced before by defining $\phi_{\Gamma,-1} := id$ and $\widetilde{\phi}_{\Gamma,0} := \mathcal{D}_{\delta_0} \circ \phi_{\Gamma,-1} = \mathcal{D}_{\delta_0}$. Then we can restate Remark 12.3 by saying that $\widetilde{\phi}_{\Gamma,0}$ is compatible with the gluing g_0 and induces the homeomorphism $\phi_{\Gamma,0}$.

Remark 12.10. After the description of the construction of the mixed tête-à-tête map above and the diagram 12.11, we observe that satisfying I) and II) of the mixed tête-à-tête property in Definition 11.4 is equivalent to satisfying:

I') For all $i = 0, \dots, d-1$, the homeomorphism $\widetilde{\phi}_{\Gamma,i} = \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1}$ is compatible with the gluing g_i, that is,

$$g_i(x) = g_i(y) \Rightarrow g_i(\widetilde{\phi}_{\Gamma,i}(x)) = g_i(\widetilde{\phi}_{\Gamma,i}(y)).$$

Below we see the diagram which shows the construction of ϕ_Γ.

(12.11)

$$
\begin{array}{ccccccccc}
\Sigma_{\Gamma^0} & \xrightarrow{\phi_{\Gamma,-1}} & \Sigma_{\Gamma^0} & \xrightarrow{\mathcal{D}_{\delta_0}} & \Sigma_{\Gamma^0} & & & & \\
\downarrow g_0 & & & & \downarrow g_0 & & & & \\
\Sigma_{\Gamma^1} & & \xrightarrow{\phi_{\Gamma,0}} & & \Sigma_{\Gamma^1} & \xrightarrow{\mathcal{D}_{\delta_1}} & \Sigma_{\Gamma^1} & & \\
\downarrow g_1 & & & & & & \downarrow g_1 & & \\
\Sigma_{\Gamma^2} & & & \xrightarrow{\phi_{\Gamma,1}} & & & \Sigma_{\Gamma^2} & \xrightarrow{\mathcal{D}_{\delta_2}} & \Sigma_{\Gamma^2} \\
\downarrow g_2 & & & & & & & & \downarrow g_2 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\Sigma_{\Gamma^d} & & & \xrightarrow{\phi_{\Gamma,d-1}} & & & \Sigma_{\Gamma^d} & \xrightarrow{\mathcal{D}_{\delta_d}} & \Sigma_{\Gamma^d} \\
\downarrow g_d & & & & & & & & \downarrow g_d \\
\Sigma & & & & \xrightarrow{\phi_\Gamma = \phi_{\Gamma,d}} & & & & \Sigma
\end{array}
$$

We prove the following:

Theorem 12.12. *The homeomorphism $\phi_{\Gamma,i}$ is pseudo-periodic for all $i = 0, ..., d$. In particular, ϕ_Γ is pseudo-periodic.*

Proof. The mapping $\phi_{\Gamma,0}$ is periodic. Assume $\phi_{\Gamma,i-1}$ is pseudo-periodic. Let's see so is $\phi_{\Gamma,i}$. Choose a collar neighbourhood $\mathcal{A}^i$ of $\widetilde{\Gamma}^i$, union of annuli $\mathcal{A}^i_{j,k}$, such that $\phi_{\Gamma,i-1}(\mathcal{A}^i_{j,k}) = \mathcal{A}^i_{j,k+1}$. Similarly to Remark 10.13 we can assume that, up to isotopy fixing the action on the boundary, the homeomorphism $\phi_{\Gamma,i-1}$ satisfies that for some parametrizations $\eta^i_{j,k} : \mathbb{S}^1 \times I \to \mathcal{A}^i_{j,k}$ we have that

$$p_2 \circ (\eta^i_{j,k+1})^{-1} \circ \phi_{\Gamma,i-1}|_{\mathcal{A}^i_{j,k}} \circ \eta^i_{j,k}(t, x) = p_2 \circ (\eta^i_{j,k+1})^{-1} \circ \phi_{\Gamma,i-1}|_{\mathcal{A}^i_{j,k}} \circ \eta^i_{j,k}(t', x)$$

for every $t, t' \in I$ where $p_2 : \mathbb{S}^1 \times I \to \mathbb{S}^1$ is the canonical projection.

Now we consider $\widetilde{\phi}_{\Gamma,i} := \mathcal{D}_{\delta_i} \circ \phi_{\Gamma,i-1}$ as in (12.6). The curves $\mathcal{C}^i_{j,k} = \eta^i_{j,k}(\mathbb{S}^1 \times \{0\})$ are invariant by $\widetilde{\phi}_{\Gamma,i}$. These curves separate Σ_{Γ^i} in two pieces: $\mathcal{A}^i$ and its complementary that we call $\mathcal{B}$. After quotienting by g_i, we get a $\phi_{\Gamma,i}$-invariant piece that is $g_i(\mathcal{B}) \approx \mathcal{B}$ and another one that is $g_i(\mathcal{A}^i)$. The restriction of $\phi_{\Gamma,i}$ to $g_i(\mathcal{B})$ is conjugate to $\phi_{\Gamma,i-1}$ and then pseudo-periodic. The restriction to $g_i(\mathcal{A}^i)$ has an invariant spine that is $g_i(\widetilde{\Gamma}^i)$ and then, by Lemma 4.13, it is boundary free isotopic to a periodic one. Then, we have seen that $\phi_{\Gamma,i}$ is pseudo-periodic. □

Remark 12.13. We make the following observation that will be used in Theorem 13.3. Let $(\Gamma^\bullet, A^\bullet, \delta_\bullet)$ be a relative mixed tête-à-tête graph.

For any choice of product structures on $(\Sigma_{\Gamma^1})_{\Gamma_{\Gamma^1}}$, the relative tête-à-tête homeomorphism $\phi_{\Gamma,0} : \Sigma_{\Gamma^1} \to \Sigma_{\Gamma^1}$ is, by definition, an isometry restricted to $\widetilde{\Gamma}^1 \cup (A^0 \setminus A^1)$. Recall that in the construction, for simplicity of notation, we assumed $A = \emptyset$ but it is not always the case.

Observe now that $\mathcal{D}_{\delta_1}|_{\widetilde{\Gamma}^1}$ is also an isometry by definition of boundary Dehn twist. Also $\mathcal{D}_{\delta_1}|_{A^0 \setminus A^1} = id$. We can conclude that the induced $\phi_{\Gamma,1}|_{A^0 \setminus A^2}$ is an isometry. The same argument extends to the rest of the filtered graph, so we conclude that for any choice of product structure, the mixed tête-à-tête homeomorphism ϕ_Γ satisfies that its restriction to the relative boundaries $\phi_\Gamma|_A$ is an isometry.

13. A restricted type of pseudo-periodic homeomorphisms.

In this section, we work with a special type of pseudo-periodic homeomorphisms that we are able to codify, up to isotopy, with a mixed tête-à-tête graph.

Let ϕ be a pseudo-periodic homeomorphism of a surface Σ with $\partial\Sigma \neq \emptyset$.

Notation 13.1. Let $\mathcal{C}$ be an admissible set of curves for ϕ as in Definition 10.1.

Let $G(\phi, \Sigma)$ be a graph constructed as follows:

1. It has a vertex for each connected component of $\Sigma \setminus \mathcal{C}$.
2. There are as many edges joining two vertices as curves in $\mathcal{C}$ intersect the two surfaces corresponding to those vertices.

Now we impose our restrictions on the homeomorphism $\phi : \Sigma \to \Sigma$ for this subsection.

Note that we can extend the definition of the fixed-boundary rotation number of a homeomorphism ϕ with respect to a boundary component C (see 8.5-8.6 and Lemma 8.8) to pseudo-periodic homeomorphisms that fix C by considering the restriction to the connected component in $\Sigma \setminus \mathcal{A}$ that contains C.

Assumptions on ϕ:

1. the graph $G(\phi, \Sigma)$ is a tree and
2. the screw numbers are all negative.
3. We assume that
   - (3') it leaves at least one boundary component pointwise fixed,
   - (3") the fixed-boundary rotation numbers along at least one of these fixed-boundary components is positive.

Remark 13.2. Note that if ϕ satisfies (3), there is always an isotopic homeomorphism satisfying $(3') - (3'')$.

Theorem 13.3. *Let ϕ be a pseudo-periodic homeomorphism satisfying assumptions (1)-(3). Let $\partial^1\Sigma$ be a union of some, at least one, connected components of $\partial\Sigma$ that:*

- *are fixed pointwise by ϕ,*
- *are contained in a single connected component of $\Sigma \setminus \mathcal{C}$ and*
- *have positive fixed-boundary rotation number.*

Assume ϕ is in the canonical form of Theorem 10.2 with respect to $\partial^1\Sigma$. Then, there exists a mixed tête-à-tête graph (Γ, B, δ) with $B = \partial\Sigma \setminus \partial^1\Sigma$ and an embedding $\Gamma \hookrightarrow \Sigma$ such that:

$$[\phi_\Gamma]_{\partial^1\Sigma} = [\phi]_{\partial^1\Sigma}.$$

Moreover,

$$[\phi_\Gamma]_{\partial,\phi|_\partial} = [\phi]_{\partial,\phi|_\partial}.$$

We start fixing the notation for the proof.

Notation 13.4. As in the statement we assume ϕ is in the canonical form of Theorem 10.2 with respect to $\partial^1\Sigma$. We denote by $\hat{\Sigma}$ the closure of $\Sigma \setminus \mathcal{A}$ in Σ.

Let $v(G(\phi, \Sigma))$ be the set of vertices of $G(\phi, \Sigma)$. We choose as root of $G(\phi, \Sigma)$ the vertex $v \in v(G(\phi, \Sigma))$ corresponding to the connected component of $\Sigma \setminus \mathcal{A}$ in the statement that contains $\partial^1\Sigma$. We say that $G(\phi,\Sigma)$ is rooted at v. Since $\hat{\phi}$ permutes the surfaces in $\hat{\Sigma}$, it induces a permutation of the set $v(G(\phi,\Sigma))$ which we denote by σ_ϕ.

We label the set $v(G(\phi,\Sigma))$, as well as the connected components of $\hat{\Sigma}$ and the connected components of $\mathcal{A}$ in the following way:

(1) Denote the vertex chosen as the root by $v^0_{1,1}$. Let $V^0 := \{v^0_{1,1}\}$.
(2) Let $d : G(\phi,\Sigma) \to \mathbb{Z}_{\geq 0}$ be the distance function to V^0, that is, $d(v)$ is the number of edges of the smallest bamboo in $G(\phi,\Sigma)$ that joins v with V^0. Let $V^i := d^{-1}(i)$. Observe that the permutation σ_ϕ leaves the set V^i invariant. There is a labeling of V^i induced by the orbits of σ_ϕ: suppose it has β_i different orbits. For each $j = 1,\dots,\beta_i$, we label the vertices in that orbit by $v^i_{j,k}$ with $k = 1,\dots,\alpha_j$ so that $\sigma_\phi(v^i_{j,k}) = v^i_{j,k+1}$ and $\sigma_\phi(v^i_{j,\alpha_j}) = v^i_{j,1}$.

Denote by $\Sigma^i_{j,k}$ the surface in $\hat{\Sigma}$ corresponding to the vertex $v^i_{j,k}$. Denote by Σ^i the union of the surfaces corresponding to the vertices in V^i. Note that Σ^0 equals $\Sigma^0_{1,1}$.

Denote by $\mathcal{A}^{i+1}_{j,k}$ the only annulus in $\mathcal{A}$ that intersects both $\Sigma^{i+1}_{j,k}$ and Σ^i. Observe that if there were more than one such annulus, $G(\phi,\Sigma)$ would not be a tree. Denote by $\mathcal{C}'^{i+1}_{j,k}$ the boundary component of $\mathcal{A}^{i+1}_{j,k}$ that lies in $\partial\Sigma^i$ and by $\mathcal{C}''^{i+1}_{j,k}$ the other boundary component. Denote $\mathcal{A}^{i+1}$ the union of all the annuli that intersect Σ^{i+1} and Σ^i and define analogously $\mathcal{C}'^{i+1}$ and $\mathcal{C}''^{i+1}$. We also define recursively

$$\Sigma^{\leq 0} := \Sigma^0, \quad \Sigma^{\leq i+1} := \Sigma^{\leq i} \cup \mathcal{A}^{i+1} \cup \Sigma^{i+1}.$$

We recall that α_j is the smallest positive number such that $\phi^{\alpha_j}(\mathcal{A}^{i+1}_{j,k}) = \mathcal{A}^{i+1}_{j,k}$, and in consequence the least such that $\phi^{\alpha_j}(\mathcal{C}'^{i+1}_{j,k}) = \mathcal{C}'^{i+1}_{j,k}$ and $\phi^{\alpha_j}(\mathcal{C}''^{i+1}_{j,k}) = \mathcal{C}''^{i+1}_{j,k}$.

Example 13.15 shows an explicit example of the construction depicted in the following proof.

Proof of Theorem 13.3. Recall that by hypothesis ϕ is in the canonical form of Remark 10.3 relative to $\partial^1\Sigma$. We follow Notation 13.4.

Observe that the mapping class $[\phi|_{\Sigma^0}]$ is periodic and $\phi|_{\Sigma^0}$ fixes $\partial^1\Sigma \neq \emptyset$. So by Theorem 9.12, we can choose a π-relative tête-à-tête graph (Γ^0, B^0) embedded in Σ^0 inducing a homeomorphism ϕ_{Γ^0} such that

$(1)_0$ $[\phi|_{\Sigma^0}]_{\partial^1\Sigma} = [\phi_{\Gamma^0}]_{\partial^1\Sigma}$
$(2)_0$ $B^0 = \partial\Sigma^0 \setminus \partial^1\Sigma$
$(3)_0$ $\phi|_{B^0} = \phi_{\Gamma^0}|_{B^0}$, (in particular $\phi|_{B^0}$ is an isometry, recall 12.13).
$(4)_0$ All the vertices in B^0 have valency 3.

Assume we have constructed a mixed tête-à-tête graph $(\Gamma^{\leq i}, B^{\leq i}, \delta_\bullet)$ with $\Gamma^{\leq i}$ embedded in $\Sigma^{\leq i}$ and with filtration of depth i

$$(\Gamma^{\leq i}, B^{\leq i}) \supset (F^1, B^1) \supset \cdots \supset (F^i, B^i). \tag{13.5}$$

such that, being $\phi_{\Gamma^{\leq i}}$ the induced homeomorphism, we have that:

$(1)_i$ $[\phi|_{\Sigma^{\leq i}}]_{\partial^1\Sigma} = [\phi_{\Gamma^{\leq i}}]_{\partial^1\Sigma}$
$(2)_i$ $B^i = \partial\Sigma^{\leq i} \cap \Sigma^i$.
$(3)_i$ $\phi|_{B^i} = \phi_{\Gamma^{\leq i}}|_{B^i}$. (in particular the restriction of ϕ to B^i is an isometry, recall 12.13.)
$(4)_i$ all the vertices in $B^{\leq i}$ have valency 3.

Observation: the relative components B^i may contain boundary components from the surface Σ as well as other boundary components that may be glued to give further levels of the filtration in the next step.

We will construct a mixed tête-à-tête graph $(\Gamma^{\leq i+1}, B^{\leq i+1}, \delta_\bullet)$ for $\Sigma^{\leq i+1}$ from $(\Gamma^{\leq i}, B^{\leq i}, \delta_\bullet)$.This graph will satisfy conditions $(1)_{i+1} - (4)_{i+1}$ above.

We start with some preparation steps (a)-(e):

(a) We consider for each $j = 1, \dots, \beta_{i+1}$ and for each $k = 1, \dots, \alpha_j$, parametrizations $\{\eta^{i+1}_{j,k}\}$ of the annuli $\{\mathcal{A}^{i+1}_{j,k}\}$ as in Lemma 10.10. We choose them such that $\eta^{i+1}_{j,k}(\mathbb{S}^1 \times \{0\}) = \mathcal{C}'^{i+1}_{j,k}$.

(b) The metric of the graph $\Gamma^{\leq i}$ assigns a metric on $\mathcal{C}'^{i+1}_{j,k}$ for every j, k. We use the natural identification from $\mathcal{C}'^{i+1}_{j,k}$ to $\mathcal{C}''^{i+1}_{j,k}$ by $\eta^{i+1}_{j,k}$ (i.e. $\eta^{i+1}_{j,k}(x, 0) \mapsto \eta^{i+1}_{j,k}(x, 1)$) to put a metric on $\mathcal{C}''^{i+1}_{j,k}$.

(c) For each $j = 1, \dots, \beta_{i+1}$ we do the following. Choose a relative spine $(\Gamma^{i+1}_{j,1}, B^{i+1}_{j,1})$ for $\Sigma^{i+1}_{j,1}$ invariant by $\phi^{\alpha_j}|_{\Sigma^{i+1}_{j,1}}$ such that

- (i) $\Gamma^{i+1}_{j,1}$ contains all points whose isotropy subgroup by the action of the group generated by $\phi^{\alpha_j}|_{\Sigma^{i+1}_{j,1}}$ is non-trivial.
- (ii) $B^{i+1}_{j,1}$ contains all the boundary components of $\Sigma^{i+1}_{j,1}$ except $\mathcal{C}''^{i+1}_{j,1}$ (which is invariant by $\phi^{\alpha_j}|_{\Sigma^{i+1}_{j,1}}$).

Condition (ii) implies that the surface obtained by cutting $\Sigma^{i+1}_{j,1}$ along $\Gamma^{i+1}_{j,1}$ is a unique cylinder $\widetilde{\Sigma}^{i+1}_{j,1}$. Its boundary components are $\mathcal{C}''^{i+1}_{j,1}$ and the one coming from $\Gamma^{i+1}_{j,1}$ that we denote by $\widetilde{\Gamma}^{i+1}_{j,1}$.

We take a product structure for the cylinder $\widetilde{\Sigma}^{i+1}_{j,1}$

$$\tilde{r}^{i+1}_{j,1} : \widetilde{\Gamma}^{i+1}_{j,1} \times I \to \widetilde{\Sigma}^{i+1}_{j,1}$$

with $\widetilde{\Gamma}^{i+1}_{j,1} \times \{0\}$ sent to $\mathcal{C}''^{i+1}_{j,1}$ as in Remark 5.11 (that is, a product structure invariant by the lifting of $\phi^{\alpha_j}|_{\Sigma^{i+1}_{1,j}}$ to $\widetilde{\Sigma}^{i+1}_{j,1}$).

To find the spine $(\Gamma^{i+1}_{j,1}, B^{i+1}_{j,1})$ and the product structure $\tilde{r}^{i+1}_{j,1}$ we consider the quotient map by the action of ϕ^{α_j} in $\Sigma^{i+1}_{j,1}$ as in the proof of Theorem 5.4 or 6.2. By condition (i) we can lift regular retractions (or product structures in the cylinder decomposition) Then, one sees that we can also assume that

- (iii) $\tilde{r}^{i+1}_{j,1}(x, 1)$ does not correspond to a vertex of $\Gamma^{i+1}_{j,1}$ whenever x is such that $\tilde{r}^{i+1}_{j,1}(x, 0)$ is a vertex of $\Gamma^{\leq i}$ or the image of a vertex of $\Gamma^{\leq i}$ by any power of ϕ.

(d) For each $j = 1, \dots, \beta_{i+1}$ we put a metric on $\widetilde{\Gamma}^{i+1}_{j,1}$ by pull back with the mapping given by $\tilde{r}^{i+1}_{j,1}(x, 1) \mapsto \tilde{r}^{i+1}_{j,1}(x, 0)$ and the metric in $\mathcal{C}''^{i+1}_{j,1}$ from (b).

Let $g_{i+1}|_{\widetilde{\Gamma}^{i+1}_{j,1}} : \widetilde{\Gamma}^{i+1}_{j,1} \to \Sigma^{i+1}_{j,1}$ be the gluing map. The metric on $\widetilde{\Gamma}^{i+1}_{j,1}$ is compatible with the gluing because $\phi^{\alpha_j}|_{\mathcal{C}''^{i+1}_{j,1}}$ is an isometry and ϕ^{α_j} respects retraction lines. In particular, the metric on $\Gamma^{i+1}_{j,1}$ is invariant by ϕ^{α_j}.

(e) For each $j = 1, \dots, \beta_{i+1}$, we copy constructions in steps (c)-(d), using ϕ, to every $\Sigma^{i+1}_{j,k}$ for every $k = 1, ..., \alpha_j - 1$. More concretely, we define $\Gamma^{i+1}_{j,k+1} := \phi^k(\Gamma^{i+1}_{j,1})$ with their induced metric. Also, define product structures

$$\tilde{r}^{i+1}_{j,k+1} := \widetilde{\phi}^k \circ \tilde{r}^{i+1}_{j,1} : \widetilde{\Gamma}^{i+1}_{j,1} \times I \to \widetilde{\Sigma}^{i+1}_{j,k}$$

where $\widetilde{\phi}$ is the lifting of ϕ to the union of cylinders $\tilde{\Sigma}^{i+1}_{j,k}$.

We consider metrics on every $\widetilde{\Gamma}^{i+1}_{j,k}$ and $\Gamma^{i+1}_{j,k}$ by pull back by ϕ^k of the ones considered in (d).

After these preparation steps, we denote by Γ^{i+1} the union of the graphs $\Gamma^{i+1}_{j,k}$ for all j,k. Note that the metric we have put on it makes ϕ an isometry.

We consider the surface $(\Sigma^{\leq i+1})_{\Gamma^{i+1}}$ obtained by cutting $\Sigma^{\leq i+1 i+1}$ along Γ^{i+1}. It consists in the union of $\Sigma^{\leq i}$, the annuli $\mathcal{A}^{i+1}$ and the disjoint union of cylinders that is $(\Sigma^{i+1})_{\Gamma^{i+1}}$. Note that this surface is connected. We denote by $g_{i+1} : (\Sigma^{\leq i+1})_{\Gamma^{i+1}} \to \Sigma^{\leq i+1}$ the gluing map. Denote by $\widetilde{\phi}_{i+1}$ the lifting of ϕ to $(\Sigma^{\leq i+1})_{\Gamma^{i+1}}$. Note that the metric we put n step (e) on $\widetilde{\Gamma}^{i+1}$ is invariant by $\widetilde{\phi}_{i+1}$ and it glues well with the mapping g_{i+1}.

Now, we can define the homeomorphism $\bar{\phi}_i$ as the homeomorphism of $(\Sigma^{\leq i+1})_{\Gamma^{i+1}}$ that coincides with $\phi|_{\Sigma^{\leq i}}$ in $\Sigma^{\leq i}$ and that extends *trivially* as in Remark 10.12 using the concatenation of the parametrization $\eta^{i+1}_{j,k}$ and $r^{i+1}_{j,k}$. Then, it is clear that

$$[\mathcal{D}_{\delta_{i+1}} \circ \bar{\phi}_i]_{\partial,\bar{\phi}_i|_\partial} = [\widetilde{\phi}_{i+1}]_{\partial,\bar{\phi}_i|_\partial}. \tag{13.6}$$

where $\mathcal{D}_{\delta_{i+1}}$ is the composition of the Dehn twists of length $-s_j/\alpha_j \cdot length(\mathcal{C}''^{i+1}_{j,1})$ along each $\widetilde{\Gamma}^{i+1}_{j,k}$ (see Definition 10.14) with s_j the screw number of ϕ on $\mathcal{A}^{i+1}_{j,k}$.

Now, we find a mixed tête-à-tête graph embedded in $(\Sigma^{\leq i+1})_{\Gamma^{i+1}}$, which we will denote by $(\widetilde{\Gamma}^{\leq i+1}, \widetilde{B}^{\leq i+1}, \delta_\bullet)$, that induces $\bar{\phi}_i$ up to isotopy relative to the action on $\partial(\Sigma^{\leq i+1})_{\Gamma^{i+1}}$.

In order to find $(\widetilde{\Gamma}^{\leq i+1}, \widetilde{B}^{\leq i+1}, \delta_\bullet)$ we start with $\Gamma^{\leq i} \subseteq \Sigma^{\leq i}$ that we see embedded in $\Sigma^{\leq i+1}$.

- We remove $\mathcal{C}'^{i+1}_{j,k}$ from $\Gamma^{\leq i}$ for every j and k.
- For every edge e in $\Gamma^{\leq i} \setminus B^i$ containing a vertex in $\mathcal{C}'^{i+1}_{j,k}$, if L is its length in $\Gamma^{\leq i}$, then, we redefine its metric to $L - \epsilon$ (to simplify we take ϵ smaller than the lengths of every edge).
- We add the embedded segments $\eta^{i+1}_{j,k}(I \times \{x\})$ for all x such that $\eta^{i+1}_{j,k}(0,x)$ is a vertex of B^i. We set the length of each of this segments to be $\epsilon/2$.
- We add the embedded segments $\widetilde{r}^{i+1}_{j,k}(I \times \{x\})$ that concatenate with the ones added in the previous step. We set the length of each of this segments to be $\epsilon/2$.
- We add $\widetilde{\Gamma}^{i+1}$ with the metric defined in (e).

In this way we have obtained a metric ribbon graph which is naturally isometric to $(\Gamma^{\leq i}, B^{\leq i})$ (we are using hypothesis $(4)_i$ here). We denote this metric ribbon graph by $(\widetilde{\Gamma}^{\leq i+1}, \widetilde{B}^{\leq i+1})$. We copy the filtration from $(\Gamma^{\leq i}, B^{\leq i})$ by using the natural isometry. This endows $(\widetilde{\Gamma}^{\leq i+1}, \widetilde{B}^{\leq i+1})$ with the structure of a mixed tête-à-tête graph. We denote its filtration by:

$$(\widetilde{\Gamma}^{\leq i+1}, \widetilde{B}^{\leq i+1}) \supset (F'^1, B'^1) \supset \cdots \supset (F'^i, B'^i). \tag{13.7}$$

We denote by $\phi_{\widetilde{\Gamma}^{\leq i+1}}$ the induced mixed tête-à-tête homeomorphism on the surface $(\Sigma^{\leq i+1})_{\Gamma^{i+1}}$. It is obvious that

$$[\phi_{\widetilde{\Gamma}^{\leq i+1}}]_{\partial,\bar{\phi}_i|_\partial} = [\bar{\phi}_i]_{\partial,\bar{\phi}_i|_\partial}. \tag{13.8}$$

Then, after (13.6), we have that

$$[\mathcal{D}_{\delta_{i+1}} \circ \phi_{\widetilde{\Gamma}^{\leq i+1}}]_{\partial,\bar{\phi}_i|_\partial} = [\widetilde{\phi}_{i+1}]_{\partial,\bar{\phi}_i|_\partial}. \tag{13.9}$$

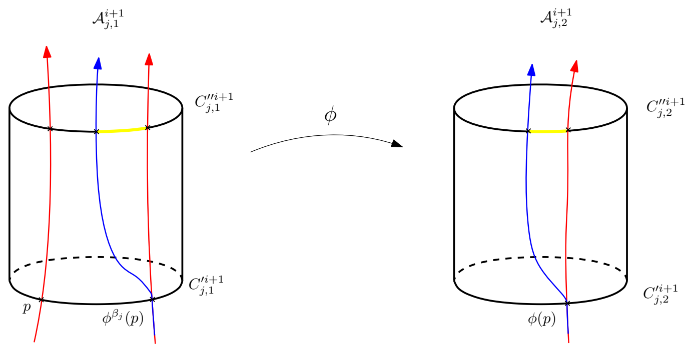

Figure 13.1. In red, the edges of $\widetilde{\Gamma}^{\leq i+1}$ passing by a point $p \in \Sigma^{\leq i} \cap \mathcal{A}_1$, by $\phi(p)$ and by $\phi^{\beta_{i+1}}(p)$. In black, the image by $\phi_{\Gamma^{i+1}}$ or $\hat{\phi}^{i+1}$ of the first one, and the image through the point $\phi^{\beta}(p)$ of an edge through the point $\phi^{\beta-1}(p)$. In yellow, a curve in $\mathcal{C}''_1$ and $\mathcal{C}''_2$ that has length $-s \cdot length(\mathcal{C}''_1)$ and $-s/\alpha \cdot length(\mathcal{C}''_1)$ respectively.

Thus, since $\widetilde{\phi}_{i+1}$ is compatible with the gluing g_{i+1} by definition, we have that

$$\mathcal{D}_{\delta_{i+1}} \circ \phi_{\widetilde{\Gamma}_{\leq i+1}} \tag{13.10}$$

is compatible with the gluing g_{i+1}. Consequently, the induced mapping in $\Sigma^{\leq i+1}$ coincides up to isotopy relative to the action on the boundary with the induced mapping by $\widetilde{\phi}_{i+1}$ that is $\phi|_{\Sigma^{\leq i+1}}$.

Finally, we see that the induced mapping by $\mathcal{D}_{\delta_{i+1}} \circ \phi_{\widetilde{\Gamma}_{\leq i+1}}$ in $\Sigma^{\leq i+1}$ is induced by a mixed tête-à-tête graph, precisely:

$$(\Gamma^{\leq i+1}, B^{\leq i+1}) := (g_{i+1}(\widetilde{\Gamma}^{\leq i+1}), g_{i+1}(\widetilde{B}^{\leq i+1}) \cap \partial\Sigma^{\leq i+1}) \tag{13.11}$$

with the filtrations:

$$\Gamma^{\leq i+1} \supset \hat{F}^1 := g_{i+1}(F'^1) \supset \cdots \supset \hat{F}^i := g_{i+1}(F'^i) \supset \hat{F}^{i+1} := g_{i+1}(\Gamma^{i+1}),$$

and

$$B^{\leq i+1} \supset \hat{B}^1 := g_{i+1}(B'^1) \cap \partial\Sigma^{i+1} \supset \cdots$$
$$\supset \hat{B}^i := g_{i+1}(B'^i) \cap \partial\Sigma^{i+1} \supset \hat{B}^{i+1} := g_{i+1}(\Gamma^{i+1}) \cap \partial\Sigma^{i+1}.$$

This filtered metric ribbon graph has depth equal to the depth of $(\Gamma^{\leq i}, B^{\leq i})$ plus one. For $a \leq i$, the functions δ_a are the same as for $\Gamma^{\leq i}$. We define

$$\delta_{i+1}(\Gamma^{i+1}_{j,k}) := -s_j/\alpha_j \cdot length(\mathcal{C}''^{i+1}_{j,1}).$$

The fact that it is a mixed tête-à-tête graph follows from $I')$ in Remark 12.10 and from the homeomorphism (13.10) being compatible with the gluing g_{i+1}.

By the construction in Section 12 of the homeomorphism associated to $\Gamma^{\leq i+1}$, we see that $\phi_{\Gamma^{\leq i+1}}$ is the induced mapping by (13.10) after gluing. As we explain after (13.10), this coincides up to isotopy relative to the boundary with $\phi|_{\Sigma^{\leq i+1}}$. This completes the proof of $(1)_{i+1}$. Condition $(2)_{i+1}$ follows from (iii), condition $(4)_{i+1}$ from (i) and condition $(3)_{i+1}$ is also clear from the construction.

□

Remark 13.12. We could broaden the definition of mixed tête-à-tête homeomorphism allowing $\delta_i : \Gamma^i \to \mathbb{R}$. In this way, we allow turning in the other direction along the separating annuli. We would find a *signed* mixed tête-à-tête homeomorphism that would model pseudo-periodic homeomorphisms with positive screw numbers as well.

Remark 13.13. In Example 13.17 we show an example of mixed tête-à-tête homeomorphism that does not satisfy the assumptions at the beginning of the section. It is an open question to characterize the mapping classes modelizable by mixed tête-à-tête graphs.

Corollary 13.14. *Let $f : \mathbb{C}^2 \to \mathbb{C}$ be an irreducible polynomial with an isolated singularity at 0. Let Σ be the corresponding Milnor fiber and let $h : \Sigma \to \Sigma$ be a representative of the monodromy that fixes $\partial\Sigma$ pointwise. Then, there exists a mixed tête-à-tête graph $(\Gamma^\bullet, \delta_\bullet)$ (with no relative boundaries) such that $[\phi_\Gamma]_{\partial\Sigma} = [h]_{\partial\Sigma}$.*

Proof. In [A'C73] it is given a description of the Milnor fiber and the monodromy, which, in particular shows that:

(0) h is pseudo-periodic,
(1) $G(h, \Sigma)$ is a tree,
(2) it has all screw numbers negative,
(3) the Milnor fiber has 1 boundary component and the fixed-boundary rotation number with respect to it is positive.

hence we conclude by Theorem 13.3. □

Example 13.15. Let Σ be the surface of Figure 13.2. Suppose it is embedded in $\mathbb{R}^3$ with its boundary component being the unit circle in the xy-plane. Consider the rotation of π radians around the z-axis and denote it by R_π. By the symmetric embedding of the surface, it leaves the surface invariant. Isotope the rotation so that it is the identity on $z \leq 0$. More concretely, let $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(r,\theta,z) := \begin{cases} (re^{i(\theta+\pi)}, z) & \text{if} \quad z \leq \epsilon \\ (re^{\theta + \frac{z}{\epsilon}\pi}, z) & \text{if} \quad 0 \leq z \leq \epsilon \\ id & \text{if} \quad z \leq 0 \end{cases} \tag{13.16}$$

With (r, θ) polar coordinates on the xy-plane and $\epsilon > 0$ small.

Let D_i be a full positive Dehn twist on $\mathcal{A}^1_{1,k}$, $k = 1, 2$. We define the homeomorphism

$$\phi := D_2 \circ D_1^{-2} \circ T|_\Sigma.$$

We apply Theorem 13.3 to construct a mixed tête-à-tête graph embedded in Σ modelling ϕ.

It is clear that ϕ is a pseudo-periodic homeomorphism and it is already in the form of Lemma 10.10. Also we observe that $[\phi|_{\Sigma\setminus\mathcal{A}}]$ has order 2.

Clearly, $G(\phi, \Sigma)$ is the graph depicted in Figure 13.2 and is a tree. Also, since Σ has only 1 boundary component, there is only one possible root of $G(\phi, \Sigma)$. So we root the graph and label the corresponding parts as in the cited figure.

On Figure 13.3 we see the relative tête-à-tête graph (Γ^0, B^0) (in blue). This corresponds to the first iteration of the induction process. In the figure it is indicated the lengths of the edges. The lengths that are not indicated can be deduced knowing that the metric is invariant by ϕ.

Clearly, we only need to iterate once more to construct the final mixed tête-à-tête graph.

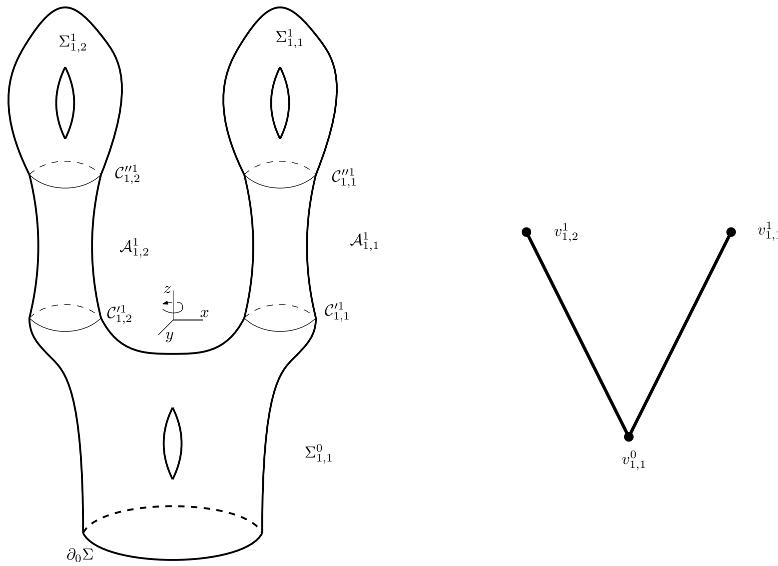

Figure 13.2. The surface Σ. The axis are also depicted in the figure. The rotation around the z-axis is isotoped near the boundary of the surface so that it leaves the boundary fixed. On the right, the corresponding $G(\phi, \Sigma)$ rooted at the only vertex at which is possible to root it.

On Figure 13.4 we see a choice of parametrization $\eta^1_{1,1}$ of $\mathcal{A}^1_{1,1}$ on the right and we also see in green $\eta^1_{1,1}(I \times \{p\})$ and $\eta^1_{1,1}(I \times \{q\})$. On the left we see two copies of $\mathcal{A}^1_{1,2}$: on the upper copy we can see the image by ϕ of the two retraction lines in $\mathcal{A}^1_{1,1}$; on the lower copy we can see the two retraction lines given by $\eta^1_{1,2}$. The latter ones are the ones that are going to be part of the final mixed tête-à-tête graph.

On Figure 13.6 we see the following:

- on the upper part, we see Σ^1 and the graphs $\Gamma^1_{1,1}$ and $\Gamma^1_{1,2} = \phi(\Gamma^1_{1,1})$ (in red).
- on the lower part we can see Σ_{Γ^1}. Also 4 (in green) retraction lines have been added concatenating with the previous 4 added segments.

By the lengths chosen on $\Gamma^1_{1,1}$ which are $\pi/36$ for each of the two edges, we have that $l(\widetilde{\Gamma}^1_{1,1}) = 4 \cdot \pi/36 = \pi/9$ which coindes with $l(\mathcal{C}''^1_{1,1}) = 2 \cdot \pi/18 = \pi/9$.

By the construction on Theorem 13.3, we set the length of each of the orange and green lines to be $\epsilon/2$ and we also redefine the length of the blue edges where the green lines are attached to be $\pi/9 - \epsilon$.

On Figure 13.7 we see the whole graph.

Now we compute the δ numbers. In our case we have that the only screw number is -1. Since $\alpha_1 = 2$, we have that δ_1 is the constant function $1/2$. Also, we have by definition that $\delta_0 = \pi$.

This completes the construction.

Example 13.17. This is an example of mixed tête-à-tête homeomorphism that does not satisfy the assumptions at the beginning of the section.

Now we describe an example of a mixed tête-à-tête graph with a filtration of depth 2:

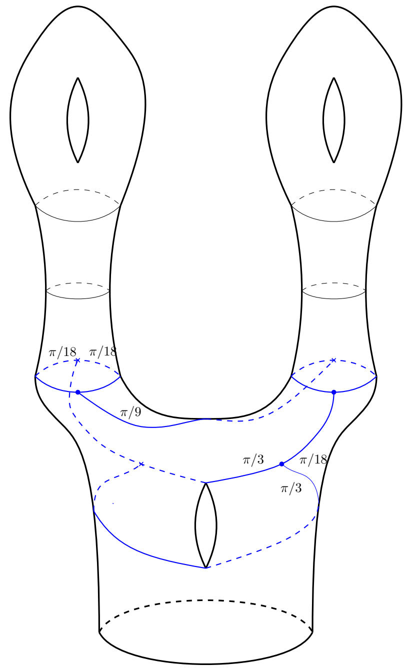

Figure 13.3. In blue, we see the relative tête-à-tête graph (Γ^0, B^0) for $\phi|_{\Sigma^0}$. The graph is embedded in Σ.

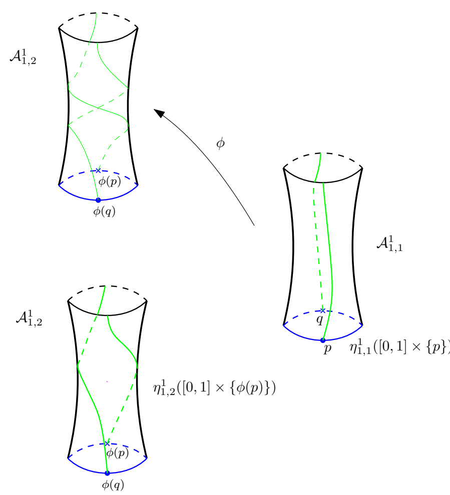

Figure 13.4. Choosing the retraction lines in Step 1.

$$\Gamma \supset \Gamma^1 \supset \Gamma^2.$$

The thickening of Γ is a surface of genus 7, see Figure 13.12 and 1 boundary component.

We start by describing first Γ_{Γ^1}. Consider the complete bipartite graph of type 2, 3 that we denote by $K_{2,3}$. By putting the length of each edge to be $\pi/2$, we make

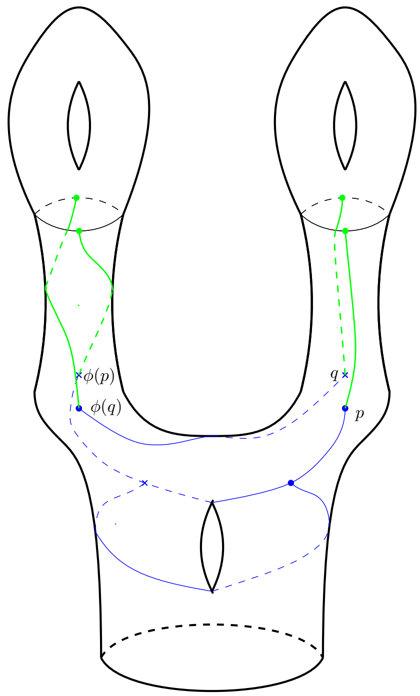

Figure 13.5. The retraction lines chosen embedded in the surface Σ.

it into a tête-à-tête graph. Now perform a blow up of length $\epsilon_1 < \pi/2$ (in the sense of A'Campo) on the orbit formed by the vertices of valency 2. See Figure 13.8. Summarizing, we define $\Gamma_{\Gamma^1} := Bl_v(K_{2,3}, \epsilon)$.

Let's describe $\Gamma^1_{\Gamma^2}$. It consists of three copies of the same relative metric ribbon graph. The graph and the lengths are given in Figure 13.9.

In Figure 13.10 below, we see one of the connected components of $\widetilde{\Gamma}^1$.

We observe that the length of each connected component of $\widetilde{\Gamma}^1$ is $4\epsilon_1$ which coincides with the length of the relative boundary components of Figure 13.8 that come from the blowing up. Therefore, we can pick an isometry from one connected component of $\widetilde{\Gamma}^1$ to one of the boundary components of Figure 13.8. We do so as indicated in Figure 13.8 by the marked green arcs.

Now we describe Γ^2. It is exactly the graph $K_{3,3}$ where each edge is of length $2\epsilon_2/3$. See Figure 13.11. On that picture we also observe its thickening and the result of cutting its thickening along $K_{3,3}$.

We note that the length of each of the boundary components of $\Sigma^3_{K_{3,3}}$ that come from cutting, has length $4\epsilon_2$ which coincides with the lengths of the boundary components of $\Sigma^1_{\Gamma^2}$ contained in the graph (see Figure 13.9). So we can pick isometries that identify the three boundary components with the three boundary components of $\Gamma^1_{\Gamma^2}$. We do so as depicted in the lower part of figure Figure 13.11

With this information we have constructed a filtrated metric ribbon graph (see Figure 13.12).

We observe (see Figure 13.12) that Γ^0, Γ^1 and Γ^2 are connected. Hence, the functions δ_0, δ_1 and δ_2 take only one value each. We define them by

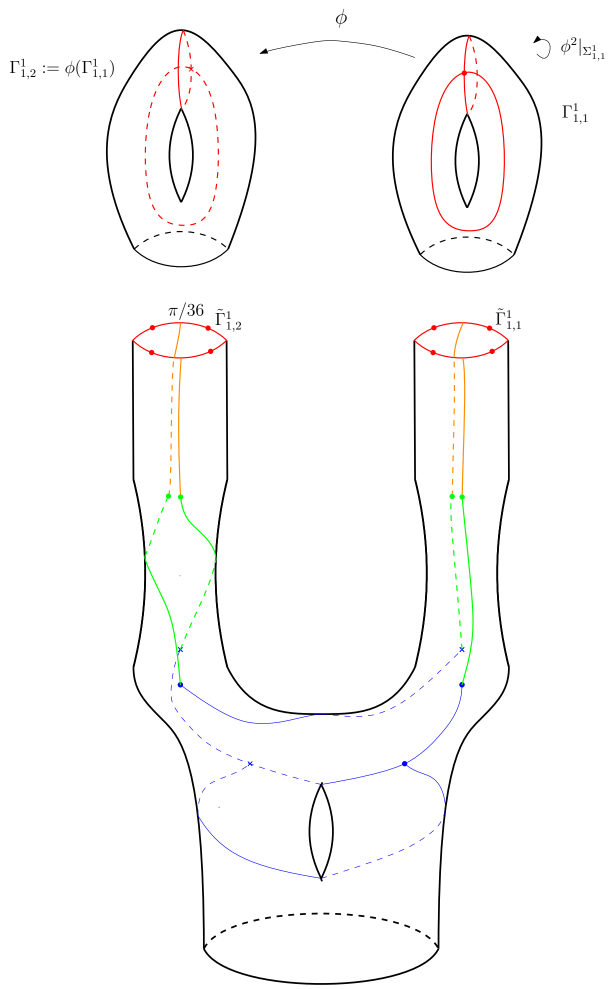

Figure 13.6. On the upper part of the figure we see $\Gamma^1_{1,1}$ and $\Gamma^1_{1,2}$ in red. The surface on the lower part of the figure is Γ_{Γ^1}. We see in blue the relative tête-à-tête graph of the first iteration of the induction process minus its relative boundary components. We see in green the retraction lines added on STEP 1. In orange we see the retraction lines contained in $\Sigma^1_{\Gamma^1}$ that we add on STEP 2. In red we see $\widetilde{\Gamma}^1$.

$$\begin{aligned} \delta_0 &:= \pi \\ \delta_1 &:= 2\epsilon_1 \\ \delta_1 &:= 2\epsilon_2 \end{aligned} \tag{13.18}$$

It can be checked easily that with these values $(\Gamma^\bullet, \delta_\bullet)$ is a mixed tête-à-tête graph.

We can picture the construction of the mixed tête-à-tête homeomorphism. We pick a point p with $c_p = 2$ so that the corresponding mixed safe walk is formed by the concatenation of three safe walks. Let p' be one of the preimages of p by $\widetilde{g}_{\Gamma,1}$.

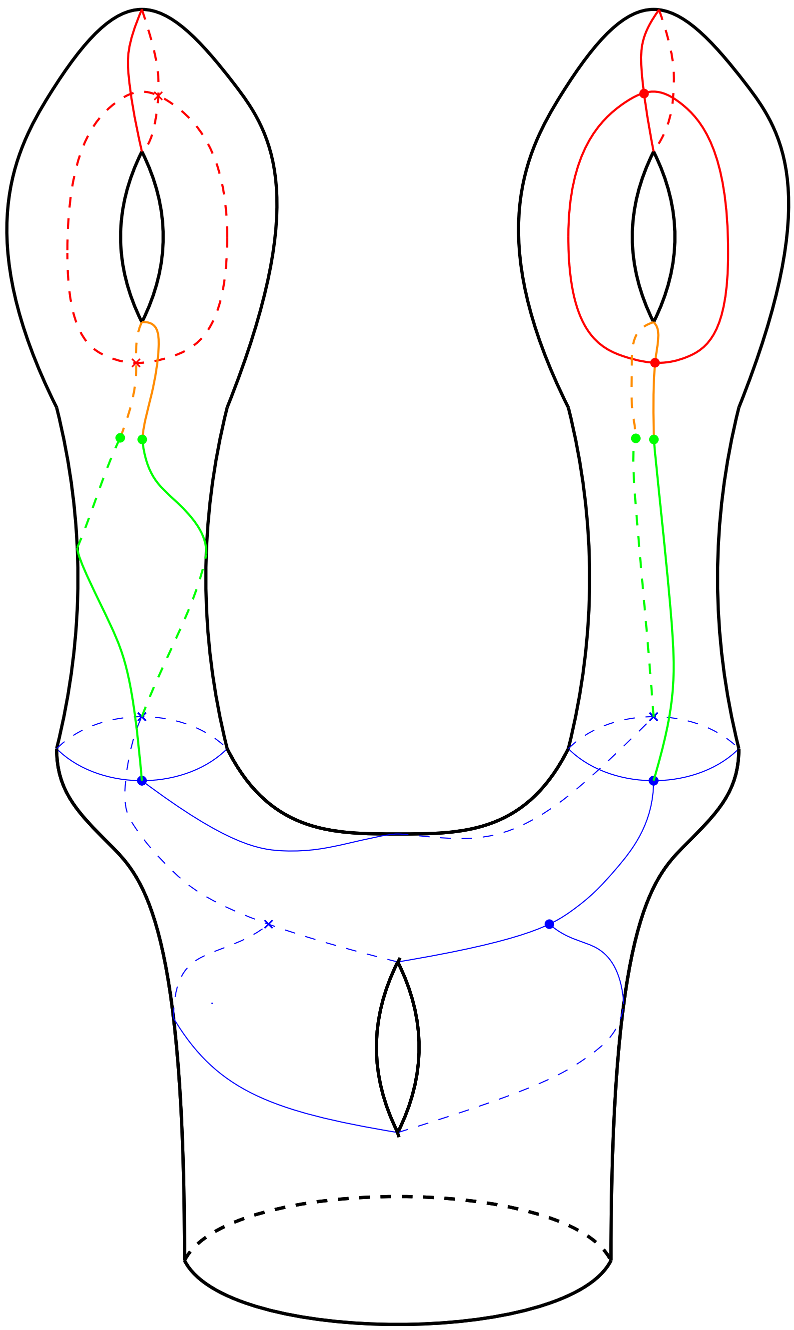

Figure 13.7. The whole graph $\Gamma \supset \Gamma^1$ whose associated mixed tête-à-tête homeomorphisms models T. The red part corresponds to Γ^1.

In Figure 13.13, we see the action of the homeomorphism $\mathcal{D}_{\delta_1} \circ \phi_{\Gamma,0}$ on p'.

Finally, in figures Figure 13.14 and Figure 13.15 we see the two mixed safe walks starting at a point p with $c_p = 2$. One can easily check, using the metric put on

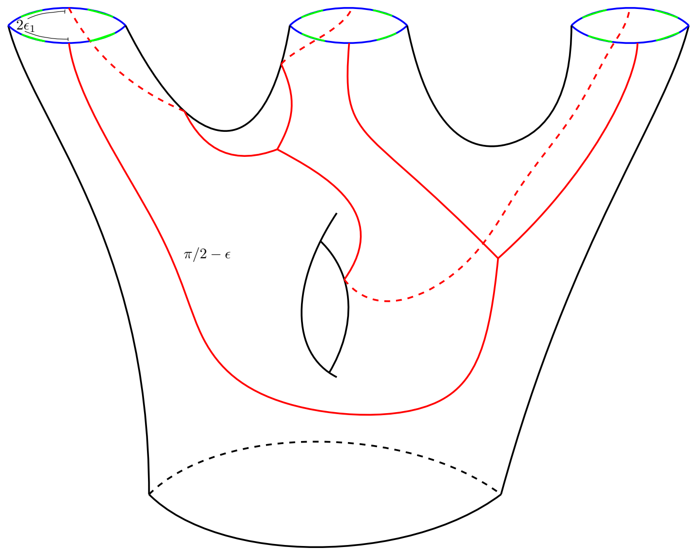

Figure 13.8. The figure corresponds with Γ_{Γ^1}. It equals to $Bl_v(K_{2,3},\epsilon_1)$ where v is any of the three vertices of valency 2 and $\epsilon_1 < \pi/2$. It is a relative tête-à-tête graph whose tête-à-tête homeomorphism has order 6. The three boundary components that come from the blowing-up correspond to $\widetilde{\Gamma}^1$ (when Γ^1 is defined). The four green arcs in each of these bounday components correspond with the part in $\widetilde{\Gamma}^1$ that is sent to $\widetilde{\Gamma}^2$ by $g_{\Gamma,1}$ when these are defined (see the following figures).

the graph, that these are actually the mixed safe walks and that they end at the same point.

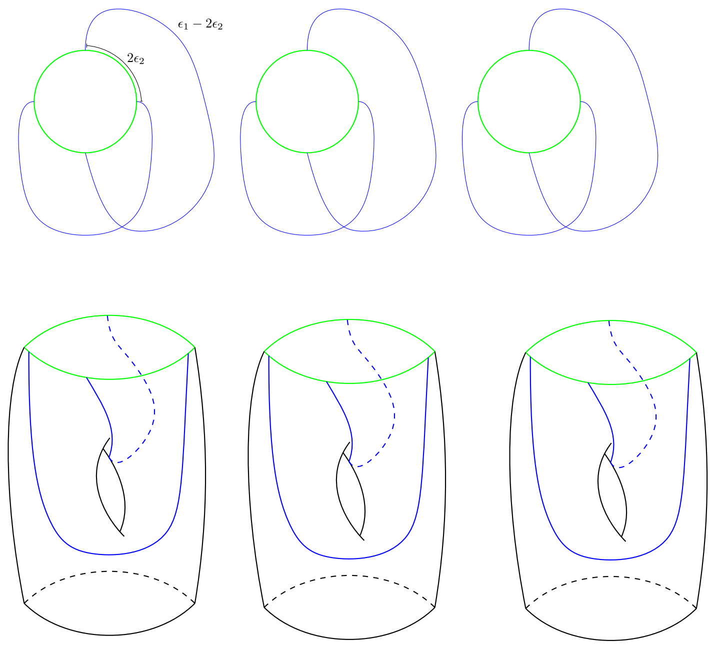

Figure 13.9. The upper part of the picture is $\Gamma^1_{\Gamma^2}$. The lower part of the picture is $\Sigma^1_{\Gamma^2}$. The three green circles will correspond with $\widetilde{\Gamma}^2$ (when they are defined).

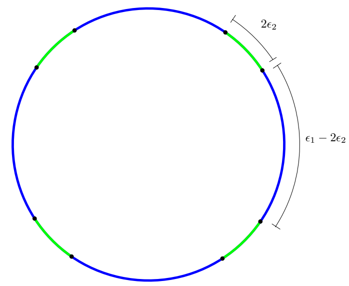

Figure 13.10. We see one of the connected components of $\widetilde{\Gamma}^1$. The green part corresponds with the part of $\Gamma^1_{\Gamma^2}$ that is in $\widetilde{\Gamma}^2$.

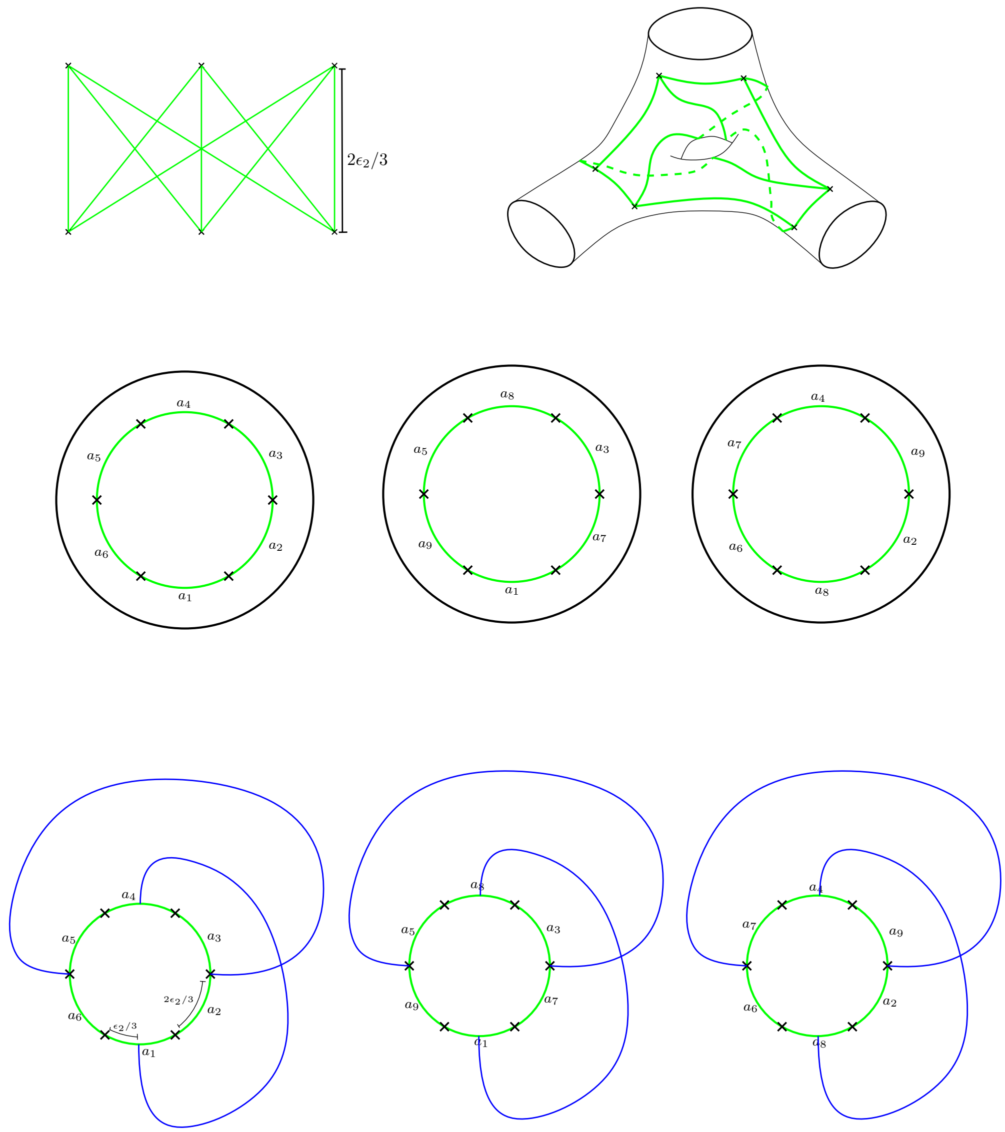

Figure 13.11. We see the graph $\Gamma^2 := K_{3,3}$ and its thickening. On the lower part of the figure we see the three cylinders of $\Sigma^2_{K_{3,3}}$. The labels on the edges indicate that two edges with the same label should be glued by an orientation reversing isometry to recover $K_{3,3}$. On the lower part of the picture, we see the chosen isometries with the boundary components of $\Gamma^1_{\Gamma^2}$.

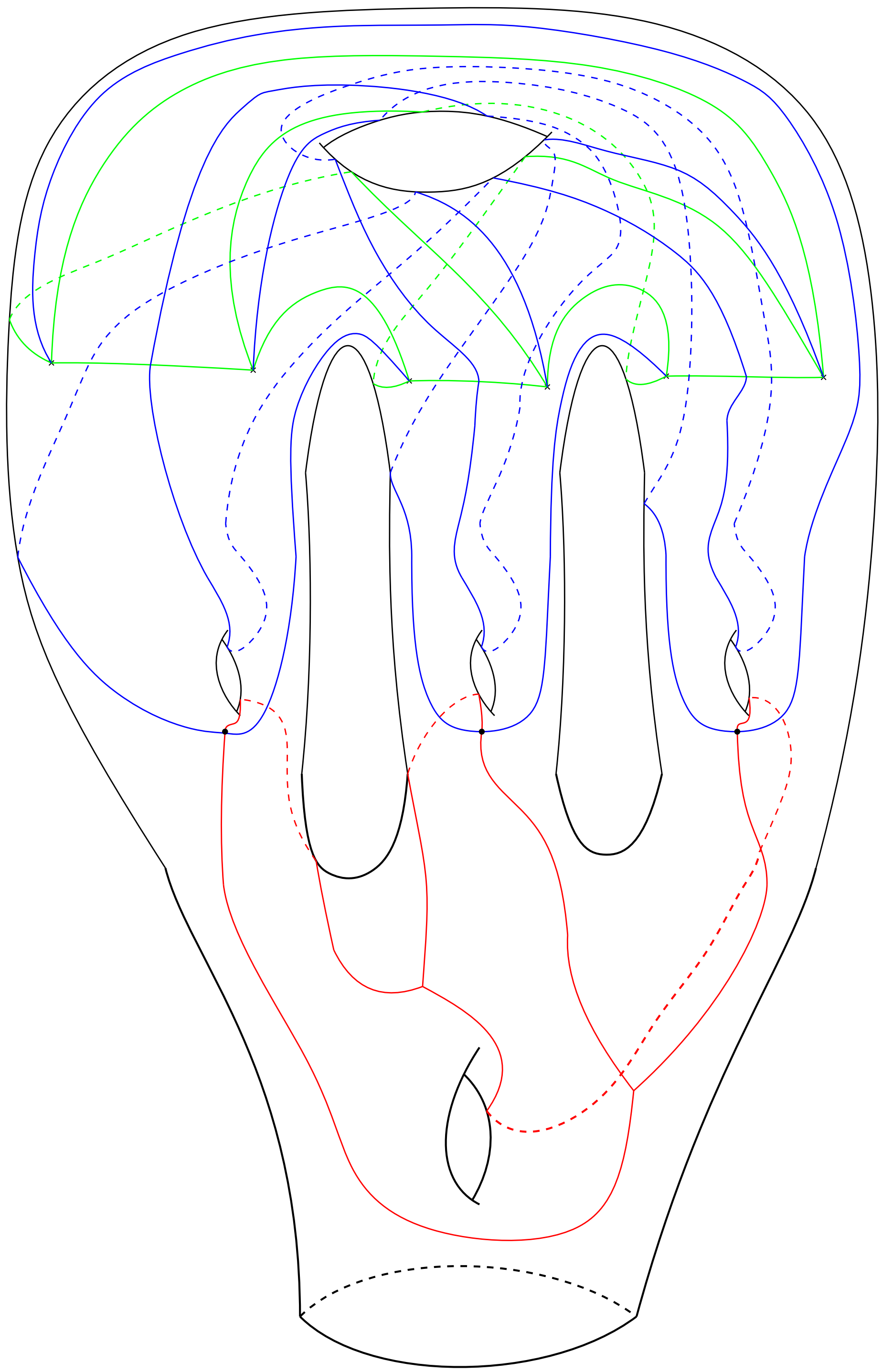

Figure 13.12. We see the graph Γ. In red we see $\Gamma \setminus \Gamma^1$. In blue we see $\Gamma^1 \setminus \Gamma^2$ and in green we see Γ^2. We observe that, since Γ^2 is connected, so is Γ^1, unlike $\Gamma^1_{\Gamma^2}$ that has 3 connected components.

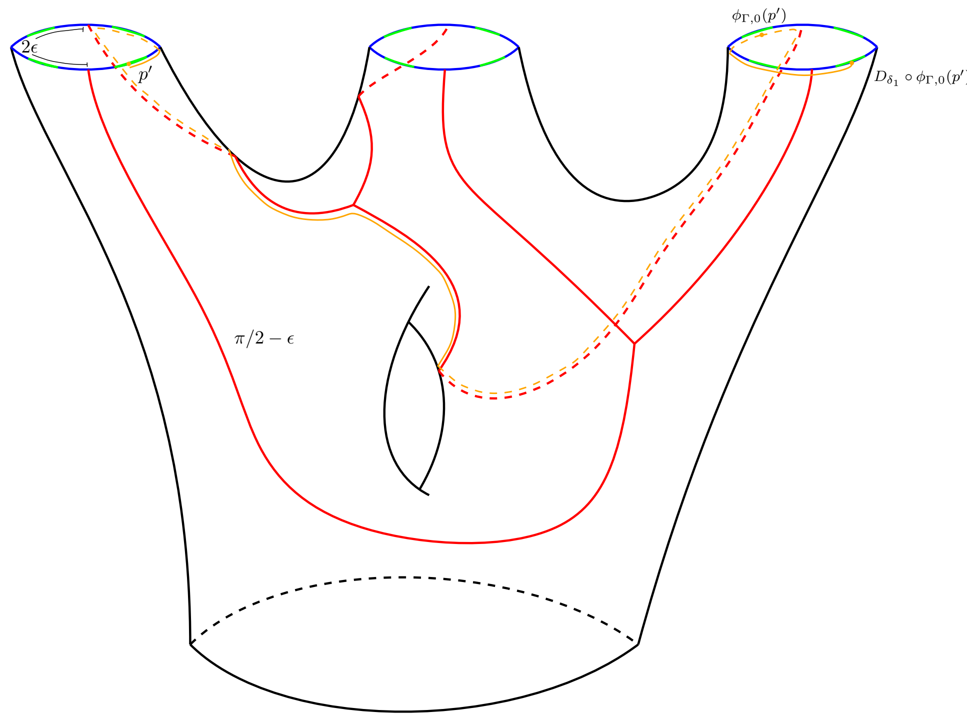

Figure 13.13. Let p' be one of the preimages by $\widetilde{g}_{\Gamma,1}$ of p. We see its image by the relative tête-à-tête homeomorphism $\phi_{\Gamma,0}$ and we also see its image after composing the relative tête-à-tête homeomorphism with $\mathcal{D}_{\delta_1}$.

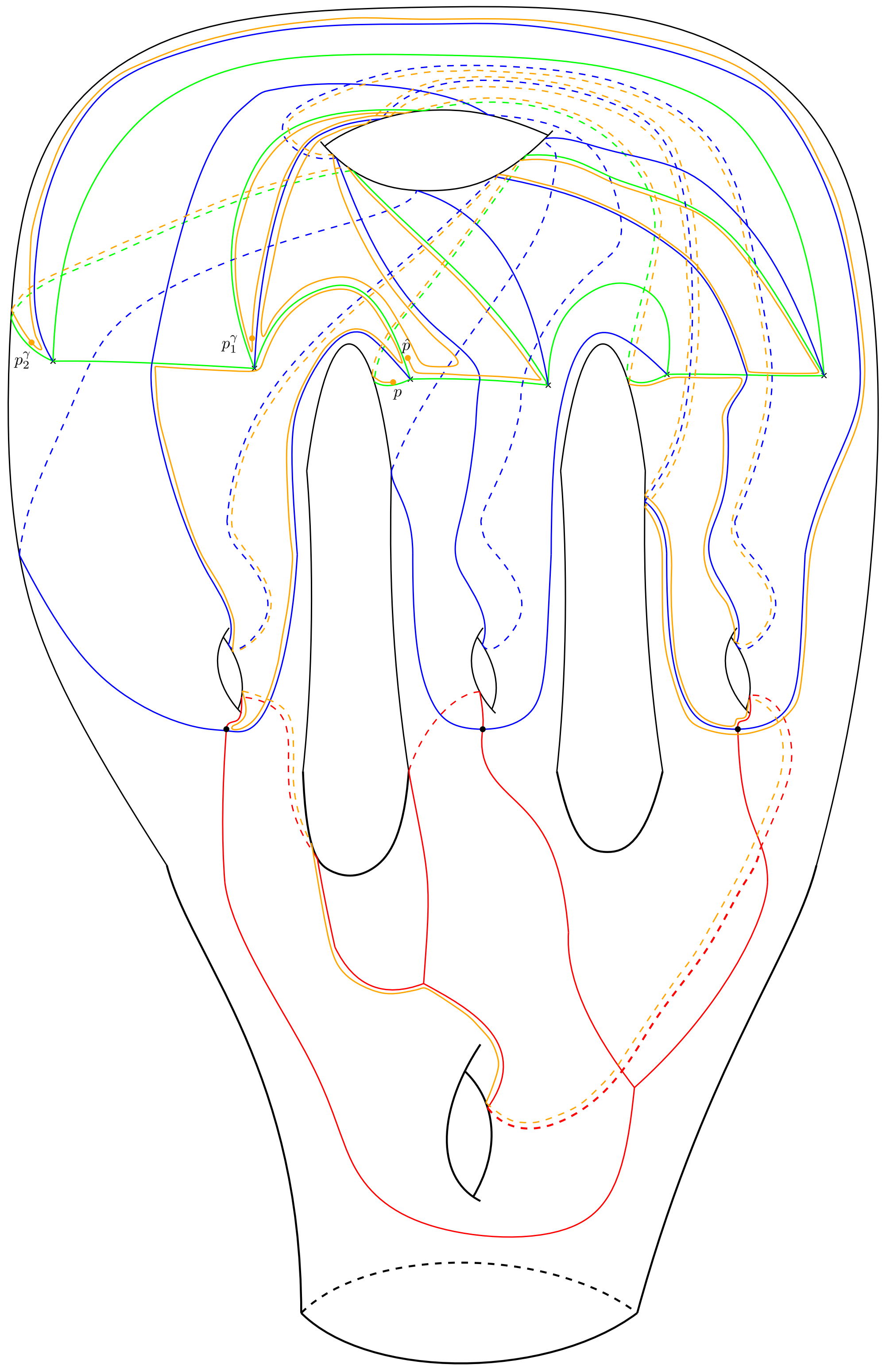

Figure 13.14. We the see Σ with the embedded graph Γ. In orange we see the mixed safe walk γ_p. We have denoted by $\hat{p}$ the end of the mixed safe walk.

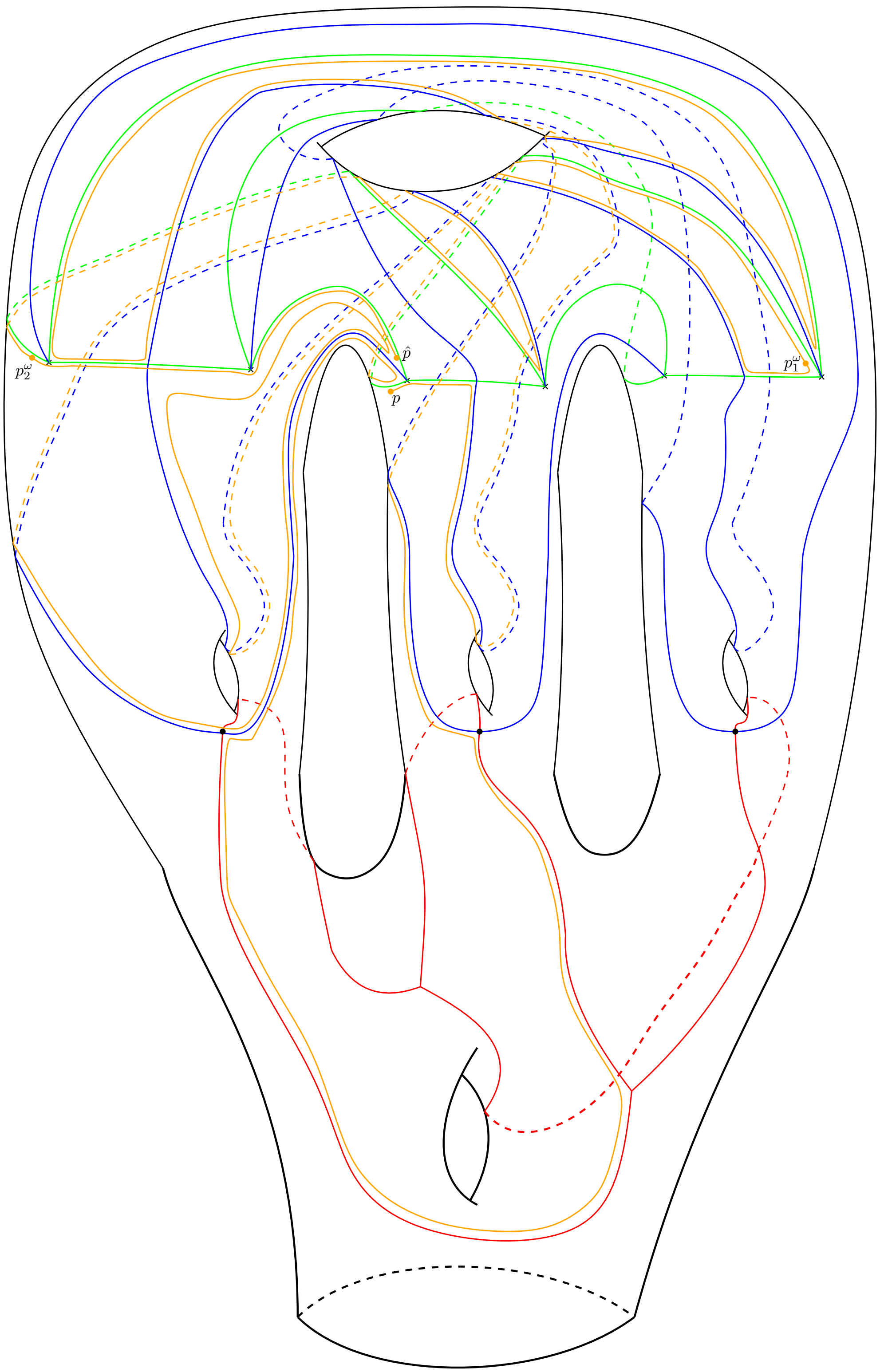

Figure 13.15. We the see Σ with the embedded graph Γ. In orange we see the mixed safe walk ω_p.

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Appendix A. Further study of tête-à-tête structures.

Given a filtered metric ribbon graph $\Gamma^\bullet$ (with some *regularity* condition) we give a complete description in Proposition A.19 of all the possible $\delta_\bullet$ functions that make it a mixed tête-à-tête graph $(\Gamma^\bullet, \delta_\bullet)$. For that, we will define a computable number, the τ number, that codifies the obstruction to extend a homeomorphism on Σ_Γ^{i+1} to $\Sigma_{\Gamma^{i+2}}$ to the next level of the filtration. The appendix ends with an example of a filtered metric ribbon graph of depth 1 in which we apply this result.

We start studying all the possible ℓ-tête-à-tête structures for a ribbon graph. From now on, we only consider metric ribbon graphs where the length of each edge is in $\pi\mathbb{Q}_+$ where $\mathbb{Q}_+$ denotes the positive rational numbers. This does not restrict us in the set of elements in $MCG(\Sigma, \partial\Sigma)$ that we can model since by the proof of Theorem 5.4 and by the proof of Theorem 13.3, we can always get that the lengths of the edges of the constructed graphs lie in $\pi\mathbb{Q}_+$.

We fix a natural notion of isomorphism between two metric ribbon graphs.

Definition A.1. *Let Γ and Γ' be two metric ribbon graphs. We say that Γ and Γ' are isomorphic as metric ribbon graphs if there exists a map $f : \Gamma \to \Gamma'$ such that*

i) *f is an isometry.*
ii) *f preserves the cyclic order at each vertex.*

Similarly, given two relative metric ribbon graphs (Γ, A) and (Γ', A') we say that they are isomorphic as relative metric ribbon graphs if there exists a map $f : \Gamma \to \Gamma'$ such that i) and ii) hold plus

iii) *$f(A) = A'$.*

Clearly not every metric ribbon graph is a pure π-tête-à-tête graph.

Definition A.2. *Let Γ be a metric ribbon graph. We define π_Γ as the smallest number in $\pi\mathbb{Q}_+$ such that Γ satisfies the π_Γ-tête-à-tête property.*

Lemma A.3. *The number π_Γ is well defined for every metric ribbon graph Γ.*

Proof. We prove that the set

$$\mathcal{R} := \{r \in \pi\mathbb{Q}_+ : \Gamma \text{ is a tête-à-tête graph for safe walks of length r}\}$$

is nonempty and discrete.

Let $\widetilde{\Gamma} = \{\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_s\}$ be the set of boundary components in Σ_Γ that come from cutting Σ (the thickening of Γ) along Γ. And let $l(\widetilde{\Gamma}_i)$ be the length of $\widetilde{\Gamma}_i$ for $i = 1, \dots, s$. Since $l(\widetilde{\Gamma}_i) \in \pi\mathbb{Q}_+$, we have that $l(\widetilde{\Gamma}_i)/\pi = p_i/q_i$. The number

$$r := \operatorname{lcm}(p_1, \dots, p_s) \cdot \pi$$

is clearly in $\mathcal{R}$ and the homeomorphism that it induces is the identity on the graph. So the set is nonempty.

The set is discrete: take $m(\Gamma) = \min_{e \in e(\Gamma)} l(e)$. Clearly, if $r \in \mathcal{R}$, no number x with $|x - r| < m(\Gamma)/2$ can be in $\mathcal{R}$.

Note that $\mathcal{R}$ is bounded below by 0, so its minimum exists and, by definition, coincides with π_Γ. □

Let Γ be tête-à-tête graph for safe walks of length r and Σ its thickening. Let $\widetilde{\Gamma} = \{\widetilde{\Gamma}_1, \dots, \widetilde{\Gamma}_b\}$ be the collection of boundary components of Σ_Γ that come from cutting along Γ. Recall Notation 2.5. Denote by ϕ_Γ the tête-à-tête map that fixes the boundary pointwise (for some choice of retraction lines) and by $\widetilde{\phi}_\Gamma$ the homeomorphism induced on Σ_Γ. We always orient the connected components of $\widetilde{\Gamma}$ by setting the opposite orientation to the one that they inherit as boundary of Σ_Γ.

Let $t \in \mathbb{Q}_+$. Denote by $t\Gamma$ the metric ribbon graph resulting from multiplying the lengths of every edge in Γ by t.

In the next lemma we list some easy properties derived from the definitions in this section.

Lemma A.4. *Suppose that Γ is a r-tête-à-tête graph. The following properties hold:*

(1) $r/l(\widetilde{\Gamma}_i) = rot(\widetilde{\phi}_\Gamma|_{\widetilde{\Gamma}_i}) + m$ *with* $m \in \mathbb{N} \cup \{0\}$.
(2) If Γ is also a tête-à-tête graph for safe walks of length R, then it is a tête-à-tête graph for safe walks of length $|mR + nr|$ for any $m, n \in \mathbb{Z}$.
(3) There exists a natural number $m \in \mathbb{N}$ such that $r = m\pi_\Gamma$.
(4) $t \cdot \pi_\Gamma = \pi_{t\Gamma}$
(5) $\frac{\pi}{\pi_\Gamma} \cdot \Gamma$ *is a pure π-tête-à-tête graph.*
(6) Suppose now that Σ has only one boundary component. Let $f : \Gamma \to \Gamma$ be any isomorphism of Γ and $\widetilde{f} : \widetilde{\Gamma} \to \widetilde{\Gamma}$ the induced map. Then there exists $m \in \mathbb{N}$ such that

$$rot(\widetilde{f}|_{\widetilde{\Gamma}}) \cdot l(\widetilde{\Gamma}) = m\pi_\Gamma$$

Proof. (1), (2) and (4) are direct consequences from their respective definitions. (5) is a direct application of (4).

For (3) we use (2): If $r = \pi_\Gamma$, then $m = 1$ and we have finished. Suppose that $r > \pi_\Gamma$. Then Γ satisfies the tête-à-tête property for safe walks of length $r_1 := r - \pi_\Gamma$. In general, define $r_k := r_{k-1} - \pi_\Gamma$. At each step we have that $r_j < r_{j-1}$. By definition it can never happen that $0 < r_j < \pi_\Gamma$ so there exists m such that $r_m = 0$. Then $r = m\pi_\Gamma$.

Now we prove (6). Suppose that $rot(\widetilde{f}|_{\widetilde{\Gamma}}) = p/q$. Then Γ is a tête-à-tête graph for safe walks of length $\frac{p}{q} \cdot l(\widetilde{\Gamma})$. Conclude by (3). □

And as a consequence:

Corollary A.5. *Every ribbon graph admits a metric that makes it a π-tête-à-tête graph.*

Observe that the metric given by the Corollary above might give a mapping class that is the identity in $MCG(\Sigma)$, however it is never the identity in $MCG(\Sigma, \partial\Sigma)$.

A.1. **The τ number.**

Remark A.6. Let $a, b \in \mathbb{R}$ with $b > 0$. We denote by $a \mod b$ the only number in $[0, b)$ which is congruent with a modulo integer multiples of b. For example, with Definition 4.16 of rotation number we have

$$rot(f \circ g) \mod 1 = (rot(f) + rot(g)) \mod 1$$

and

$$(-rot(f)) \mod 1 = (rot(f^{-1})) \mod 1.$$

Definition A.7. *Let Γ be a metric ribbon graph and let Σ be its thickening. Suppose that Σ has only 1 boundary component. Let $\widetilde{\Gamma}$ be the boundary component in $\Sigma_{\widetilde{\Gamma}}$ that comes from cutting along Γ. Let $\tilde{f} : \widetilde{\Gamma} \to \widetilde{\Gamma}$ be an orientation preserving periodic* *isometry. We define the number*

$$\tau(\tilde{f},\Gamma) := \left(-rot(\tilde{f})\cdot l(\widetilde{\Gamma})\right) \mod \pi_\Gamma$$

and we call it the tau number of $\tilde{f}$ with respect to Γ.

Note that if $\tilde{f}$ and $\tilde{g}$ are both orientation preserving periodic isometries of $\tilde{\Gamma} \approx \mathbb{S}^1$, by Remark A.6 we have that

$$\tau(\tilde{g}\circ\tilde{f},\Gamma) - \tau(\tilde{f},\Gamma) - \tau(\tilde{g},\Gamma) \in \pi_\Gamma \cdot \mathbb{Z} \tag{A.8}$$

Lemma A.9. *Let Γ be a metric ribbon graph and let Σ be its thickening. Suppose that Σ has only 1 boundary component. Let $\widetilde{\Gamma}$ be the boundary component in Σ_Γ that comes from cutting along Γ. Let $\tilde{f} : \widetilde{\Gamma} \to \widetilde{\Gamma}$ be an orientation preserving periodic isometry. Then $\tilde{f}$ is compatible with the gluing g_Γ if and only if*

$$\tau(\tilde{f},\Gamma) = 0.$$

Moreover, the homeomorphism

$$\tilde{f}' := \mathcal{D}_{\tau(\tilde{f},\Gamma)}(\widetilde{\Gamma}) \circ \tilde{f} : (\Sigma)_\Gamma \to (\Sigma)_\Gamma$$

is compatible with the gluing g_Γ.

Proof. If $\tau(\tilde{f},\Gamma) = 0$, then $-rot(\widetilde{f})\cdot l(\widetilde{\Gamma}) = m\pi_\Gamma$ for some $m \in \mathbb{Z}$. Since $rot(\widetilde{f}) > 0$ we have that $m < 0$. By definition of π_Γ, we have that Γ is a tête-à-tête graph for safe walks of length $-m\pi_\Gamma$ and we conclude that f is compatible with the gluing g_Γ.

Suppose now that $\tilde{f}$ is compatible with the gluing. This means that it it induces an isomorphism f of Γ. So by (Lemma A.4 (6)) we have that $-rot(\tilde{f})\cdot l(\widetilde{\Gamma}) = m\pi_\Gamma$ for some $m \in \mathbb{Z}$ and hence $\tau(\tilde{f},\Gamma) = 0$.

For the second part, just observe that by (A.8) we have that $\tau(\tilde{f}',\Gamma) = 0$. □

As a consequence, we have that if $\tilde{f}$ and $\tilde{g}$ are both orientation preserving periodic isometries of $\tilde{\Gamma}$, and $\tilde{g}$ is compatible with the gluing, then

$$\tau(\tilde{g}\circ\tilde{f},\tilde{\Gamma}) = \tau(\tilde{f},\tilde{\Gamma}) \;\; and \;\; \tau(\tilde{f}\circ\tilde{g},\tilde{\Gamma}) = \tau(\tilde{f},\tilde{\Gamma}). \tag{A.10}$$

Then, the following definition makes sense:

Definition A.11. *Let Γ_1 and Γ_2 be two isomorphic metric ribbon graphs whose thickenings have 1 boundary component. Let $\widetilde{\phi} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ be an orientation preserving isometry such that there exists at least one isomorphism $g : \Gamma_2 \to \Gamma_1$ with $\tilde{g}\circ\widetilde{\phi}$ periodic. Then we define*

$$\tau(\widetilde{\phi},\Gamma_2) := \tau(\widetilde{g}\circ\widetilde{\phi},\Gamma_1)$$

To check it is well defined, one has only to see that given g and g' as in the definition, we have that $\tilde{g}'\circ\tilde{g}^{-1}$ is a periodic isometry and apply (A.10) to check

$$\tau(\widetilde{g}\circ\widetilde{\phi},\Gamma_1) = \tau((\tilde{g}'\circ\tilde{g}^{-1})\circ\widetilde{g}\circ\widetilde{\phi},\Gamma_1) = \tau(\tilde{g}'\circ\widetilde{\phi},\Gamma_1).$$

The following corollary shows the setting in which we use the previously defined τ number. It is the key part in the proof of Proposition A.19.

Corollary A.12. *Let Γ_1 and Γ_2 be two isomorphic metric ribbon graphs whose thickenings have 1 boundary components. Let $\widetilde{\phi} : (\Sigma_1)_{\Gamma_1} \to (\Sigma_2)_{\Gamma_2}$ be an orientation preserving homeomorphism that restricts as an isometry $\widetilde{\phi}|_{\widetilde{\Gamma}_1} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ and such* *that there exists at least one isomorphism* $g : \Gamma_2 \to \Gamma_1$ *with* $\tilde{g} \circ \widetilde{\phi}|_{\widetilde{\Gamma}^1}$ *periodic. Then* $\widetilde{\phi}$ *is compatible with the gluings* g_{Γ_1} *and* g_{Γ_2} *if and only if*

$$\tau(\widetilde{\phi}|_{\widetilde{\Gamma}_1}, \Gamma_2) = 0. \tag{A.13}$$

Moreover, the homeomorphism

$$\widetilde{\phi}' := \mathcal{D}_{\tau(\widetilde{\phi}|_{\widetilde{\Gamma}_1}, \Gamma_2)}(\widetilde{\Gamma}_1) \circ \widetilde{\phi} : (\Sigma_1)_{\Gamma_1} \to (\Sigma_2)_{\Gamma_2}$$

is compatible with the gluings g_{Γ_1} *and* g_{Γ_2}.

Proof. If $\tilde{\phi}|_{\Gamma_1}$ is compatible, then $\tilde{g} \circ \tilde{\phi}|_{\Gamma_1}$ is compatible for any isomorphism $g : \Gamma_2 \to \Gamma_1$. Then by Lemma A.9, we have (A.13).

Suppose now that (A.13) holds, then there exists some isomorphism $g : \Gamma_2 \to \Gamma_1$ with $\tilde{g} \circ \widetilde{\phi}|_{\widetilde{\Gamma}^1}$ periodic such that $\tilde{g} \circ \tilde{\phi}|_{\Gamma_1}$ is compatible with the gluing g_{Γ_1}. Let $\phi_g : \Gamma_1 \to \Gamma_1$ be the induced mapping. Then it is clear that the lifting of $g^{-1} \circ \phi_g$ coincides with $\tilde{\phi}|_{\Gamma_1}$. Then, $\tilde{\phi}|_{\Gamma_1}$ induces a mapping from Γ_1 to Γ_2 and is compatible with both g_{Γ_1} and g_{Γ_2} as desired.

For the last statement, it is clear that $\tau(\tilde{\phi}', \tilde{\Gamma}) = 0$ and then ϕ' is compatible. □

We illustrate the use of these definitions and properties in the following example.

Example A.14. Let Σ be the one-holed torus which is the thickening of the depicted embedded graph in Figure A.1.

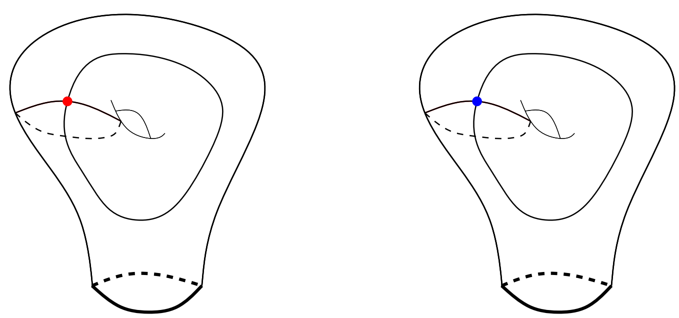

Figure A.1. We see two copies of Σ with an embedded metric ribbon graph. In one copy the only vertex is in red and in the other in blue.

We set that the length of each edge is π. With this metric, the graph satisfies the pure π-tête-à-tête property and the induced homeomorphism has order 4 when restricted to the graph.

In the Figure A.2 is depicted the action of a homeomorphism $\widetilde{\phi} : (\Sigma)_\Gamma \to (\Sigma)_\Gamma$ which restricts as an isometry $\widetilde{\phi}|_{\widetilde{\Gamma}} : \widetilde{\Gamma} \to \widetilde{\Gamma}$.

We pick any isomorphism $f : \Gamma \to \Gamma$. For example, in the picture the induced map $\widetilde{f} : \widetilde{\Gamma} \to \widetilde{\Gamma}$ sends $\hat{v}_i$ to v_i and is an isometry. The map $\widetilde{f} \circ \widetilde{\phi}$ is a rotation with rotation number equal to 7/12.

Now we compute the corresponding tau number.

$$\tau(\widetilde{\phi}|_{\widetilde{\Gamma}}, \Gamma) = \left(-\frac{7}{12} \cdot 4\pi \mod \pi\right) = \frac{2}{3}\pi$$

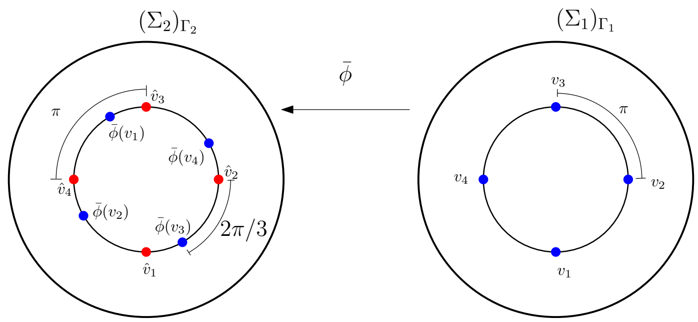

Figure A.2. In blue we see the four preimages (by the gluing g_Γ) of the only vertex in Γ. On the left side of the figure we see in red the preimages of the vertex in Γ and in blue the images by $\widetilde{\phi}$ of the points in $\widetilde{\Gamma}$.

We observe also in the picture that $\mathcal{D}_{2\pi/3} \circ \widetilde{\phi}$ is compatible with the gluings.

Remark A.15 (Relative tête-à-tête versions)**.** Observe that for relative metric ribbon graphs there exists a version for each of the results and definitions in this section. Just substitute metric ribbon graph Γ for relative metric ribbon graph (Γ, A) in each statement.

We define the number defined in Definition A.2 for a relative metric ribbon graph (Γ, A) in the same way and we will denote it by π_Γ. It will always be clear from the context that Γ is a relative tête-à-tête graph without specifying the relative part.

For the statements in Lemma A.4 (6), Definition A.7 change the condition that imposes that Σ has only one boundary component and instead impose that Σ has only one boundary component apart from the boundary components in A.

A.2. **Mixed tête-à-tête structures on filtered metric ribbon graphs.** Let $\Gamma^\bullet$ be a filtration of depth d on a metric ribbon graph. We analyze which functions $\delta_\bullet$ make it a mixed tête-à-tête graph.

We address this question in the case of a particular class of filtered metric ribbon graphs that satisfy a certain "regularity" condition.

Definition A.16. *We say that a filtered metric ribbon graph $\Gamma^\bullet$ of depth d is regular if for every $i = 0, \dots, d$, the graph Γ^i has the same number of connected components as $\widetilde{\Gamma}^i$. We will say that $(\Gamma^\bullet, \delta_\bullet)$ is a regular mixed tête-à-tête graph if it is a regular filtered metric ribbon graph that satisfies the mixed tête-à-tête property.*

Remark A.17. We observe that a filtered metric ribbon graph $\Gamma^\bullet$ of depth d is regular if and only if for every $i = 0, \dots, d$ it happens that when we cut Σ along a connected component of Γ^i only one boundary component appears.

For example, the graph constructed in the proof of Theorem 13.3 and the graph of Example 13.15 are regular.

For each Γ^i we enumerate its connected components $\Gamma^i_1, \dots, \Gamma^i_{d_i}$. We denote by $(\Gamma^i_j)_{\Gamma^{i+1}}$ the result of cutting Γ^i_j along $\Gamma^{i+1} \cap \Gamma^i_j$. By regularity of the graph it makes sense the following notation for the connected components of $\Gamma^i_{\Gamma^{i+1}}$

$$\{(\Gamma^i_1)_{\Gamma^{i+1}}, \dots, (\Gamma^i_{d_i})_{\Gamma^{i+1}}\},$$

and if we denote by $\widetilde{\Gamma}^i_j$ the boundary component in Γ_{Γ^i} that comes from cutting along Γ^i_j we can write as well

$$\{\widetilde{\Gamma}^i_1, \ldots, \widetilde{\Gamma}^i_{d_i}\}.$$

for the connected components of $\widetilde{\Gamma}^i$

Definition A.18. *We say that a permutation of $\mathcal{C}(\widetilde{\Gamma}^i)$ given by a permutation λ on the indices $\{1, \ldots, d_i\}$ is admissible if*

$$\left((\Gamma^i_j)_{\Gamma^{i+1}}, \widetilde{\Gamma}^{i+1} \cap (\Gamma^i_j)_{\Gamma^{i+1}}\right) \simeq \left((\Gamma^i_{\lambda(j)})_{\Gamma^{i+1}}, \widetilde{\Gamma}^{i+1} \cap (\Gamma^i_{\lambda(j)})_{\Gamma^{i+1}}\right)$$

for every $j = 1, \ldots, d_i$. Where $\simeq$ denotes isomorphism as relative metric ribbon graphs.

Let $\Gamma^\bullet$ be a filtration of depth d on a metric ribbon graph. Let $\hat{\delta}_\bullet$ be a set of functions such that $(\Gamma^\bullet_{\Gamma^i}, (\widetilde{\Gamma}^i)^\bullet, \hat{\delta}_\bullet)$ is a relative mixed tête-à-tête graph with relative mixed tête-à-tête map $\phi_{\Gamma_{\Gamma^i}}$. And let λ_i be the permutation induced on $\{1, \ldots, d_i\}$ by $\phi_{\Gamma_{\Gamma^i}}|_{\widetilde{\Gamma}^i}$.

Proposition A.19. *We can extend the collection $\hat{\delta}_\bullet$ to a collection of functions $\delta_\bullet$ such that*

$$(\Gamma^\bullet_{\Gamma^{i+1}}, (\widetilde{\Gamma}^{i+1})^\bullet, \delta_\bullet)$$

is a relative mixed tête-à-tête graph if and only if λ_i is admissible.

Moreover, if λ_i is admissible then all the possible values of δ_i for each connected component of Γ^i are

$$\delta_i|_{(\Gamma^i_j)_{\Gamma^{i+1}}} = \tau\left(\phi_{\Gamma_{\Gamma^i}}|_{\widetilde{\Gamma}^i_{\lambda_i^{-1}(j)}}, (\Gamma^i_j)_{\Gamma^{i+1}}\right) + n\pi_{(\Gamma^i_j)_{\Gamma^{i+1}}} \tag{A.20}$$

where $\pi_{(\Gamma^i_j)_{\Gamma^{i+1}}}$ is the number in Definition A.2 (see also Remark A.15) of the relative metric ribbon graph $((\Gamma^i_j)_{\Gamma^{i+1}}, \widetilde{\Gamma}^{i+1} \cap (\Gamma^i_j)_{\Gamma^{i+1}})$.

Proof. Suppose that we can extend the collection $\hat{\delta}_\bullet$ to a collection $\delta_\bullet$ so that

$$(\Gamma^\bullet_{\Gamma^{i+1}}, (\widetilde{\Gamma}^{i+1})^\bullet, \delta_\bullet)$$

is a relative mixed tête-à-tête graph. This means that $\mathcal{D}_{\delta_i} \circ \phi_{\Gamma_{\Gamma^i}}$ is compatible with g_i, so in particular λ_i was admissible.

Suppose now that λ_i is admissible. In particular it means that $\phi_{\Gamma_{\Gamma^i}}$ restricts as an isometry between $\widetilde{\Gamma}^i_j$ and $\widetilde{\Gamma}^i_{\lambda_i(j)}$ whose gluings by g_i are isomorphic relative metric ribbon graphs.

The first term of the right hand side of A.20

$$\tau\left(\phi_{\Gamma_{\Gamma^i}}|_{\widetilde{\Gamma}^i_{\lambda_i^{-1}(j)}}, (\Gamma^i_j)_{\Gamma^{i+1}}\right)$$

is the length of a boundary Dehn twist that makes the map

$$\mathcal{D}_{\tau(\phi_{\Gamma_{\Gamma^i}}|_{\widetilde{\Gamma}^i_{\lambda_i^{-1}(j)}}, \Gamma^i_{\Gamma^{i+1}})} \circ \phi_{\Gamma_{\Gamma^i}}$$

compatible with the gluing g_i (this follows from Corollary A.12).

The second term $n\pi_{(\Gamma^i_j)_{\Gamma^{i+1}}}$ of the right hand side of equation tells us all the possible lengths of safe walks that make $((\Gamma^i_j)_{\Gamma^{i+1}}, \widetilde{\Gamma}^{i+1} \cap (\Gamma^i_j)_{\Gamma^{i+1}})$ a relative tête-à-tête graph (this follows from Lemma A.4 (6) and from the Definition A.2).

So we can conclude that equation A.20 describes all possible values of $\delta_i|_{(\Gamma^i_j)_{\Gamma^{i+1}}}$ such that the map $\mathcal{D}_{\delta_i|_{(\Gamma^i_j)_{\Gamma^{i+1}}}} \circ \phi_{i-1}$ is compatible with the gluings. $\square$

We use the previous lemma to analyze the situation in an example.

Example A.21. Fix $0 < \epsilon < \pi/4$ rational multiple of π. First consider two non-isomorphic metric ribbon graphs with diffeomorphic thickenings as in Figure A.3.

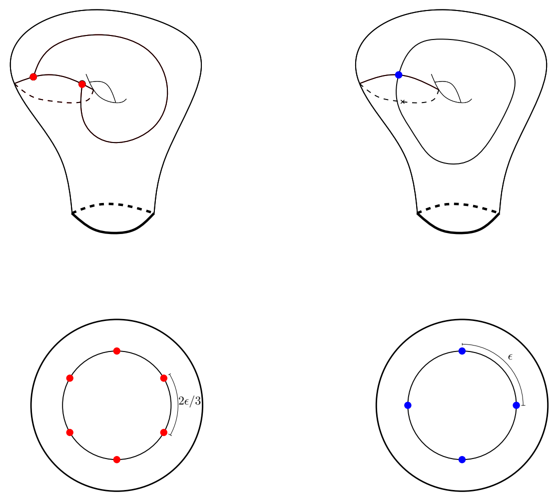

Figure A.3. We see two metric ribbon graphs and their corresponding thickenings. On the lower part of the figure we can see $\Sigma_{\Gamma^1_1}$ and $\Sigma_{\Gamma^1_2}$ together with the induced markings on $\widetilde{\Gamma}^1_1$ and $\widetilde{\Gamma}^1_2$.

Consider the following relative metric ribbon graph with the lengths of each edge indicated in the picture Figure A.4 and the cyclic orientation at each vertex induced by the given projection on the plane. The relative boundaries are the four circles. The markings on the circles mean that they should be glued according to Figure A.3, that is, by taking each of the arcs between two contiguous red vertices (resp. blue vertices) and identifying them with opposite arcs in the same circle by an orientation reversing isometry. When we glue the four circles we get a metric ribbon graph. We denote by Γ^1 the union of the image by the gluing of the four circles. The graph is then filtrated by

$$\Gamma = \Gamma^0 \supset \Gamma^1$$

Denote by $g_1 : \Gamma_{\Gamma^1} \to \Gamma$ be the gluing function. Now we proceed to find possible δ_0 numbers. A priori, the possible values are those that make $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$ a relative tête-à-tête graph which by Lemma A.4 (3) are

$$\delta_0(\Gamma) = n\pi_{\Gamma_{\Gamma^1}}$$

for $n \in \mathbb{N}$.

Let $K_{2,4}$ be the complete bipartite graph of type $2, 4$ where each edge is set to be of length $\pi/2$. We know that this makes it a tête-à-tête graph. Now observe that $(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)$ is $Bl_v(K_{2,4}, \epsilon)$ (recall Definition 3.13) where v is one of the 4 vertices of valency 2 in $K_{2,4}$. This tells us that $\pi_{\Gamma_{\Gamma^1}} = \pi$. We note that the homeomorphism induced on $Bl_v(K_{2,4}, \epsilon)$ by safe walks of length π has order $\mathrm{lcm}(2, 4) = 4$.

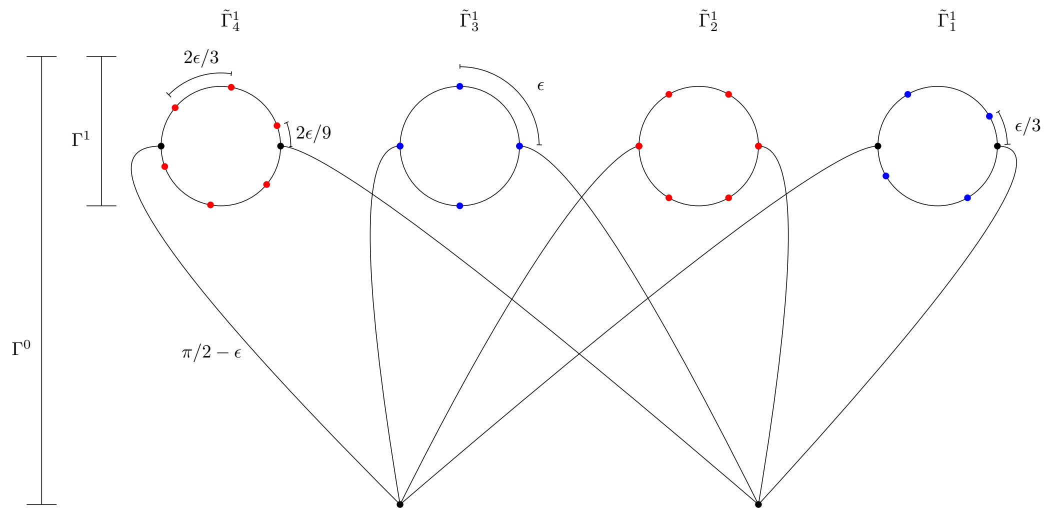

Figure A.4. We see a regular filtered metric ribbon graph of depth 1. The graph Γ^0 is the whole graph and the graph Γ^1 consits of 4 connected components that are the closings of the four circles in the picture. Actually, on the figure we are seeing a planar projection of Γ_{Γ^1}.

Enumerate the circles in the picture from right to left by $\{1,2,3,4\}$ and denote by $\lambda_{0,n}$ the permutation induced by setting $\delta_0(\Gamma)=n\pi$. We see that

$$\lambda_{0,n}=\begin{cases}(1)(2)(3)(4) & \text{if } n\equiv 0 \mod 4\\ (1\,2\,3\,4) & \text{if } n\equiv 1 \mod 4\\ (1\,3)(2\,4) & \text{if } n\equiv 2 \mod 4\\ (1\,4\,3\,2) & \text{if } n\equiv 3 \mod 4\end{cases}$$

Where the notation $(a_1\, a_2\, \ldots\, a_k)$ indicates $\lambda_{0,n}(a_i)=a_{i+1}$ and $\lambda_{0,n}(a_k)=a_1$. And the permutations are always written as product of disjoint cycles.

Let $\Gamma^1_i := g_1(\widetilde{\Gamma}^1_i)$. The graphs Γ^1_1 and Γ^1_3 are isomorphic, and Γ^1_2 and Γ^1_4 are isomorphic and there are no more isomorphism classes in Γ^1. Therefore, by Proposition A.19, we can exclude the cases when $n\equiv 1 \mod 4$ and $n\equiv 3 \mod 4$. The other two are admissible permutations.

Case 1. Suppose $n\equiv 0 \mod 4$ and hence $\lambda_{0,n}=(1)(2)(3)(4)=id$. We look at the formula A.20 specialized at $i=1$ and observe that Γ^2 is empty in this case since the filtration has depth 1. So the possible values of δ_1 in each connected component of Γ^1 are

$$\delta_1|_{\Gamma^1_j}=\tau(\phi_{\Gamma,0}|_{\widetilde{\Gamma}^1_j},\Gamma^1_j)+m_j\pi_{\Gamma^1_j} \tag{A.22}$$

for $j=1,2,3,4$ and $m_j\in\mathbb{N}$.

Since the relative tête-à-tête map induced by the graph $K_{2,4}$ has order 4 we see that in all these cases the map induced by $\phi_{\Gamma,0}:\Sigma_{\Gamma^1}\to\Sigma_{\Gamma^1}$ on $\widetilde{\Gamma}^1$ is the identity. So $\phi_{\Gamma,0}|_{\widetilde{\Gamma}^1_j}=id$, we have that

$$\tau(\phi_{\Gamma,0}|_{\widetilde{\Gamma}^1_j},\Gamma^1_j)=0$$

for all $j=1,2,3,4$.

We compute $\pi_{\Gamma_1^1} = \pi_{\Gamma_3^1} = \epsilon$ and $\pi_{\Gamma_2^1} = \pi_{\Gamma_4^1} = 2\epsilon/3$. This is clear by observing the Figure A.3.

So, substituting in A.22 we have that

$$\delta_1|_{\Gamma_j^1} = \begin{cases} m_1\epsilon & \text{if } j=1 \\ m_2\frac{2\epsilon}{3} & \text{if } j=2 \\ m_3\epsilon & \text{if } j=3 \\ m_4\frac{2\epsilon}{3} & \text{if } j=4 \end{cases}$$

are valid values for the δ_1 function for all $m_1, m_2, m_3, m_4 \in \mathbb{N} \cup \{0\}$. And by Proposition A.19 these are all.

Case 2. Suppose now that $n \equiv 2 \mod 4$ and hence $\lambda_{0,n} = (13)(24)$. Now the permutation is not the identity. We have

$$\delta_1|_{\Gamma_j^1} = \tau(\phi_{\Gamma,0}|_{\widetilde{\Gamma}^1_{\lambda_{0,n}^{-1}(j)}}, \Gamma_j^1) + m\pi_{\Gamma_j^1} \tag{A.23}$$

We observe that now $\phi_{\Gamma,0}$ is different from **Case 1.**. In these cases, $\phi_{\Gamma,0}|_{\Gamma_{\Gamma^1}}$ corresponds with $\phi^2_{(\Gamma_{\Gamma^1}, \widetilde{\Gamma}^1)}$. We compute in the corresponding tau numbers

$$\tau(\phi_{\Gamma,0}|_{\widetilde{\Gamma}^1_{\lambda_{0,n}^{-1}(j)}}, \Gamma_j^1) = \begin{cases} 2\epsilon/3 & \text{if } j=1 \\ 2\epsilon/9 & \text{if } j=2 \\ \epsilon/3 & \text{if } j=3 \\ 4\epsilon/3 & \text{if } j=4 \end{cases}$$

A handy recipe for computing the above numbers is the following: pick a blue vertex in $\widetilde{\Gamma}_1^1$. Follow a boundary safe walk of length 2π; the endpoint of this boundary safe walk is in $\widetilde{\Gamma}_3^1$. Then $\tau(\phi_{\Gamma,0}|_{\widetilde{\Gamma}^1_{\lambda_{0,n}^{-1}(j)}}, \Gamma_j^1)$ is the length of the arc from this endpoint to the next blue vertex in $\widetilde{\Gamma}_3^1$ (in the direction indicated by the boundary safe walk). Do similar for each connected component in Γ^1.

So, substituting in A.23 we have that

$$\delta_1|_{\Gamma_j^1} = \begin{cases} \epsilon/3 + m_1\epsilon & \text{if } j=1 \\ 4\epsilon/3 + m_2 2\epsilon/3 & \text{if } j=2 \\ 2\epsilon/3 + m_3\epsilon & \text{if } j=3 \\ 2\epsilon/9 + m_4 2\epsilon/3 & \text{if } j=4 \end{cases}$$

are valid values for the δ_1 function for all $m_1, m_2, m_3, m_4 \in \mathbb{N} \cup \{0\}$.

And by Proposition A.19 these are all the possible values that δ_0 and δ_1 may take in order to make $(\Gamma^\bullet, \delta_\bullet)$ a mixed tête-à-tête graph.

(1) IKERBASQUE, Basque Foundation for Science, Maria Diaz de Haro 3, 48013, Bilbao, Bizkaia, Spain (2) BCAM âĂŞ Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Basque Country, Spain

E-mail address: jbobadilla@bcamath.org

Facultad de Ciencias Matematicas- UCM, Plaza de Ciencias, 3, Ciudad Universitaria, 28040 MADRID

E-mail address: maria.pe@mat.ucm.es

(1)ICMAT, Campus Cantoblanco UAM, C/ Nicolás Cabrera, 13-15, 28049 Madrid, Spain (2) BCAM, Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Basque Country, Spain

E-mail address: p.portilla89@gmail.com