From G-Equation to Michelson-Sivashinsky Equation in Turbulent Premixed Combustion Modelling

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Abstract
It is well known that the Michelson-Sivashinky equation describes hydrodynamic instabilities in turbulent premixed combustion. Here a formulation of the flame front propagation based on the G-equation and on stochastic fluctuations imposed to the average flame position is considered to derive the Michelson-Sivashinky equation from a modelling point of view. The same approach was shown to reproduce the G-equation along the motion of the mean flame position, when the stochastic fluctuations are removed, as well ast the Zimont & Lipatnikov model, when a plane front is assumed. The new results here presented support this promising approach as a novel and general stochastic formulation for modelling turbulent premixed combustion.

Introduction
The research presented at the XXXVIII Meeting of the Italian Section of the Combustion Institute [1] is here continued. Further results on a novel promising formulation for modelling turbulent premixed combustion [1,2], see also [3], are derived. In particular, it is here reminded that such approach starts from a G-equation that describes the mean flame position, and when stochastic fluctuations are introduced, a reaction-diffusion equation which describes the effective burned fraction follows under the assumption that the probability density function (PDF) of the random process underlying the random front motion is known [1,2,3].

This formulation shows that two different approaches to model turbulent premixed combustion, and furthermore considered alternatives to each other, i.e. the G-equation and the reaction-diffusion equation, are indeed complementary and they can be reconciled. When stochastic fluctuations are removed, such formulation reduces to the G-equation along the motion of the mean flame position, and when a plane front is assumed, the Zimont & Lipatnikov model [4,5] is recovered. Considered these results other efforts are porsued to develop further this novel and general stochastic formulation for modelling turbulent premixed combustion.

Here this approach is used to derive the Michelson-Sivashinky equation [6,7,8] that describes hydrodynamic instabilities in turbulent premixed combustion.

Model Formulation

The following presentation of the model formulation is based on [1]. Let the scalar function $G(x,t), x \in \mathbb{R}^n$, be a level surface that represents the front which devides burned and unburned domains and the front be denoted by $\Gamma(t), t \geq 0$. Let $x_c$be a point on the level surface G=c at the instant t_0, such that the corresponding front is $\Gamma_0 = \Gamma(t_0) = \{x = x_0 \in S | G(x_0, t_0) = c\}$, where $S \subseteq \mathbb{R}^n$. The level surface propagates with a consumption speed given by the laminar burning velocity $s_L$in the normal direction ***n*** relative to the mixture element and its evolution is described by the following Hamilton-Jacobi equation where the flow velocity field is ***u***

$$\frac{\partial G}{\partial t} + u \cdot \nabla G = s_L \|\nabla G\| \ , \quad n = -\frac{\nabla G}{\|\nabla G\|} \ . \tag{1}$$

Let the front motion be described by the random process $X_c^\omega(\hat{x},t)$where ω labels any independent realization, such that the random contour is

$$\Gamma^\omega(t) = \{x = X_c^\omega(t) \in S | G^\omega(X_c^\omega, t) = c\}. \tag{2}$$

Let the mean value of X_c^ωbe denoted by $\langle X^\omega(\hat{x},t)\rangle = \hat{x}(t)$, then if $P(x_c; t|\hat{x})$ is the corresponding PDF, with initial condition $P(x_c; t_0|\hat{x}) = \delta(x - x_0)$, the mean flame position is given by the integral

$$\langle x_c \rangle = \int_{\mathbb{R}^n} x_c P(x_c; t|\hat{x}) dx_c = \hat{x}(t). \tag{3}$$

Introducing $\breve{G}(\hat{x},t)$, with $\breve{G}(\hat{x},t_0) = \breve{G}(x_0,t_0) = c$, as the implicit formulation of the mean flame position $\hat{x}$, the ensemble averaging of (1) gives [9]

$$\frac{\partial \breve{G}}{\partial t} + \hat{u} \cdot \nabla \breve{G} = -\widehat{s_L n} \cdot \nabla \breve{G}. \tag{4}$$

Since the G-equation can be derived on the basis of considerations about symmetries, there is a unique model for the RHS term of equation (4) providing a relation between the laminar burning velocity $s_L$and the turbulent burning velocity s_T[9], i.e.

$$\widehat{s_L n} = s_T \breve{n} \ , \qquad \breve{n} = -\frac{\nabla \breve{G}}{\|\nabla \breve{G}\|}. \tag{5}$$

Finally, combining equation (4) and model (5), the G-equation that describes the surface motion along the mean flame position results to be

$$\frac{\partial \breve{G}}{\partial t} + \hat{u} \cdot \nabla \breve{G} = s_T \|\nabla \breve{G}\|. \tag{6}$$

Note in (5) that the normal vector of the mean flame front, i.e. $\breve{n}$, is different from the mean of the normal vectors to the random flame front, i.e. $\hat{n}$. In general, the mean of the random level surface $\langle G^{\omega}\rangle$is different from the level surface $\breve{G}$depicted by the mean position of the flame [10].

Applying properties of the Dirac δ-function, it follows

$$G^{\omega}(X_c^{\omega},t) = \int_{\mathbb{R}^n} G(x,t)\delta(x - X_c^{\omega}(\hat{x},t))dx, \qquad (7)$$

as well as a formula including the stochastic fluctuations around the front depicted by the mean flame position, i.e.

$$\phi^{\omega}(x,t) = \int_{\mathbb{R}^n} \breve{G}(\hat{x},t)\delta(x - X_c^{\omega}(\hat{x},t))d\hat{x}. \qquad (8)$$

Given the level surface $\breve{G}$, the inner domain $\breve{\Omega}(t)$enclosed by the front contour $\breve{\Gamma}(t) = \{x \in \partial\breve{\Omega}(t)\}$can be understood as the effective volume occupied by the burned fraction. Then the following indicator function is introduced

$$I_{\breve{\Omega}}(t) = \begin{cases} 1, x \in \breve{\Omega}(t), \\ 0, x \notin \breve{\Omega}(t). \end{cases} \qquad (9)$$

In analogy with (8), the random indicator associated to the surface which enclose the volume of the burned fraction is given by the following formula

$$I_{\breve{\Omega}}^{\omega}(x,t) = \int_{\mathbb{R}^n} I_{\breve{\Omega}}(\hat{x},t)\delta(x - X_c^{\omega}(\hat{x},t))d\hat{x} = \int_{\breve{\Omega}} \delta(x - X_c^{\omega}(\hat{x},t))d\hat{x}. \qquad (10)$$

Finally, ensemble averaging of (10) gives the effective fraction of the burned mass

$$\langle I_{\breve{\Omega}}^{\omega}(x,t)\rangle = \int_{\breve{\Omega}}\langle\delta(x - X_c^{\omega}(\hat{x},t))\rangle d\hat{x} = \int_{\breve{\Omega}(t)} P(x;t|\hat{x})d\hat{x} = V(x,t). \qquad (11)$$

Applying the Reynolds transport theorem to formula (11), a reaction-diffusion equation follows [3]:

$$\frac{\partial V}{\partial t} = \int_{\breve{\Omega}(t)} \frac{\partial P}{\partial t} d\hat{x} + \int_{\breve{\Omega}(t)} \nabla_{\hat{x}} \cdot [s_T \breve{n} P(x;t|\hat{x})]d\hat{x}. \qquad (12)$$

Equation (12) reduces to a Hamilton-Jacobi equation when no diffusion is assumed [1,2,3], and, when a plane front is assumed, it reduces to the same equation derived by Zimont & Lipatnikov [4] and studied in [5], see [1,2,3]. This suggests that equation (12) can be considered as the natural extension of Zimont & Lipatnikov model to the case with non null mean curvature.

Derivation of the Michelson-Sivashinky equation

Let $-(-\nabla^2)^s, s \in (0,1)$be the fractional Laplacian defined by its Fourier symbol $-|k|^{2s}$. When $s = 1$the classical Laplacian is recovered. It is here reminded that the

Lévy stable densities are the Green functions of the space-fractional diffusion equation

$$\frac{\partial P}{\partial t} = -(-\nabla^2)^s P, \qquad (13)$$

and in particular the Green function corresponds to the Gaussian density when $s = 1$and to the Lorentzian density when$s = 1/2$. The Gaussian density is associated to classical diffusion and the Lorentzian density can be associated to a lightly damped linear oscillator.

Consider now equation (12), by setting

$$\frac{\partial P}{\partial t} = \nabla^2 P - [-(-\nabla^2)^{1/2} P], \qquad (14)$$

where the second term on the RHS, because of the sign minus and $s = 1/2$, may be understood as a counter-damping effect of an harmonic oscillator, and by setting

$$s_T = -\frac{(\int_{\breve{\Omega}(t)} \nabla_{\hat{x}} P(x;t|\hat{x}) d\hat{x})^2}{\int_{\breve{\Omega}(t)} \nabla_{\hat{x}} \cdot [\check{n} P(x;t|\hat{x})] d\hat{x}}, \qquad (15)$$

the *multi-dimensional* Michelson-Sivashinky equation is obtained. In fact, in the one-dimensional case it holds

$$\frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial x^2} - (\frac{\partial V}{\partial x})^2 - D_x^1 V, \qquad (16)$$

which is the Michelson-Sivashinky equation [6,7,8], where$D_x^1$is the fractional derivative of order 1 in the Riesz-Feller sense with Fourier symbol is$-|k|$, which differs from the classical first derivative, and is related to the Hilbert transform by the formula

$$D_x^1 V = \frac{1}{\pi} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{V(x\prime)}{(x\prime - x)} dx'. \qquad (17)$$

The solution$V(x,t)$to the Michelson-Sivashinky equation (16), or to its multi-dimensional version, can be obtained computing the integral (11) where the kernel function$P(x;t|\hat{x})$is the Green function of (14) and the domain of integration $\breve{\Omega}(t)$is obtained by the indicator function (9) solving the G-equation (6) with $s_T$defined in (15). This procedure constitutes a practical scheme to compute numerically the solution to the Michelson-Sivashinky equation which is alternative to the pole decomposition method [11].

Summary and Outlook

In the present extended abstract the evolution equation of reaction-diffusion type is briefly derived for an observable that can be understood as the effective burned fraction. When stochastic fluctuations are removed, such equation reduces to the G-equation along the motion of the mean flame position, which suggests that approaches based on reaction-diffusion equations and G-equation are indeed complementary and they can be reconciled. Moreover, when a plane front is assumed, the Zimont & Lipatnikov model is recovered.

This promising approach has been adopted here to derive the Michelson-Sivashinky equation from a modelling point of view. The random process underlying the front motion involves the classical diffusion and a second effect that may be understood as a counter-dumping effect when compared with a lightly damped linear oscillator whose intensity of oscillations follows the Lorentzian function. This second effect can be linked to the mechanism that, a given pressure gradient, accelerates the light products more than the heavier reactants and generates counter-gradient diffusion and turbulent energy production [12]. This physical interpretation will be the investigated in the future.

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