Monotonicity and convexity of the ratios of the first kind modified Bessel functions and applications*

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Abstract

Let $I_v(x)$ be modified Bessel functions of the first kind. We prove the monotonicity property of the function $x \mapsto I_u(x) I_v(x) / I_{(u+v)/2}(x)^2$ on $(0, \infty)$. As a direct consequence, it deduces some known results including Turán-type inequalities and log-convexity or log-concavity of I_v in v, as well as it yields some new and interesting monotonicity and convexity concerning the ratios of modified Bessel functions of the first kind. In addition, a few of sharp bounds involving $I_v(x)$ and their ratios are presented.

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1 Introduction

As we know, the modified Bessel functions of the first kind of order v, denoted by $I_v(x)$, is a particular solution of the second-order differential equation [41, p. 77]

$$x^2 y''(x) + x y'(x) - \left(x^2 + v^2\right) y(x) = 0, \tag{1.1}$$

which is explicitly represented by

$$I_v(x) = \frac{(x/2)^v}{\Gamma(v+1)} \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\,(v+1)_n}, \quad x \in \mathbb{R}, \quad v \in \mathbb{R} \backslash \{-1, -2, \cdots\}, \tag{1.2}$$

where $(a)_n$ is the Pochhammer symbol as

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

for $n \in \mathbb{N}$, $(a)_0 = 1$, and $a \neq 0, -1, -2, \cdots$.

For various recurrence formulas and many important properties of modified Bessel functions, readers can refer to the classical book of Watson [41]. In recent decades, Turán type inequalities for special functions including the modified Bessel functions have attracted the attention of many mathematicians (see, for

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example, [26, 20, 6, 3, 8, 9, 25, 13]). In fact, Turán type inequalities for the modified Bessel functions and related results appeared in many problems of probability and statistics [19, 34, 16], chemistry [28], physics [40, 38] and engineering sciences [24].

Turán-type inequalities for the first kind modified Bessel functions states that for all $v > -1$, the double inequality

$$0 < I_v(x)^2 - I_{v-1}(x) I_{v+1}(x) < \frac{1}{v+1} I_v(x)^2 \tag{1.3}$$

holds for $x > 0$, which was proved first by Thiruvenkatachar and Nanjundiah in [39, Sect. 3] by the approach of comparing the coefficients in Cauchy product. The left-hand side of (1.3) was also derived by Amos in [4, p. 243] and Joshi and Bissu in [21, Sect. 3]. Later, Lorch [27] in 1994 showed that the left-hand side of (1.3) also holds for all $v > -1/2$, and he conjectured that there holds the generalized Turánian $I_v(x)^2 - I_{v-a}(x) I_{v+a}(x) > 0$ for all $x > 0$, $a \in (0, 1]$ and $v \in (-1, -1/2]$. On the other hand, the right-hand side of (1.3) for $v \geq 0$ was also proved in [21, Sect. 3]. In 2010, Baricz [10] reconsidered the proof of Joshi and Bissu [21] and pointed out that (1.3) hold true for all $v > -1$ with the best constants 0 and $1/(v+1)$. An alternative proof of (1.3) can see also Segura's paper [35]. Recently, an improvement of (1.3) was derived by Baricz in [14, Theorem 1]. Some new delovements can be found in [22], [14], [29].

Speaking generally, Turán-type inequalities are closely related to log-convexity or log-concavity. As was shown in [27] by Lorch, the function $v \mapsto I_{v+a}(x)/I_v(x)$ is decreasing for each fixed $a \in (0, 2]$ and $x > 0$, while $v > \max\{-1, -(a+1)/2\}$. This implies that the function $v \mapsto I_v(x)$ is log-concave on $(-1, \infty)$. In 2008, Baricz [7, Theorem 1 (a)] showed that function $v \mapsto 2^v \Gamma(v+1) x^{-v} I_v(x)$ is log-convex on $(-1, \infty)$, who got it by two methods: one is to make use of the fact that the sum of log-convex functions is still log-convex, another is to use the expression $I_v(x)$ as an infinite product and concavity of $v \mapsto j_{v,n}$ on $(-n, \infty)$ for all $n \geq 1$, where $j_{v,n}$ denotes the nth positive zero of the Bessel function J_v.

Very recently, Yang and Zheng [43] proved that the function $x \mapsto K_u(x) K_v(x) / K_{(u+v)/2}(x)^2$ is strictly decreasing on $(0, \infty)$, where $K_v(x)$ denotes the modified Bessel functions of the second kind, which not only solves the conjecture posed by Baricz [10, Conjecture 3.2], but also yields various new results for the monotonicity and convexity of the ratios of the modified Bessel functions of the second kind. This idea can also be applied to the modified Bessel functions of the first kind. Indeed, it is easy to check that the log-concavity of function $v \mapsto I_v(x)$ and log-convexity of $v \mapsto 2^v \Gamma(v+1) x^{-v} I_v(x)$ on $(-1, \infty)$ are equivalent to the following double inequality

$$\frac{\Gamma((u+v)/2+1)^2}{\Gamma(u+1)\Gamma(v+1)} < \frac{I_u(x) I_v(x)}{I_{(u+v)/2}(x)^2} < 1 \tag{1.4}$$

for $u, v > -1$ and $x > 0$. Moreover, by using the asymptotic formulas [1, p. 375, 377]

$$I_v(x) \sim \left(\frac{x}{2}\right)^v / \Gamma(v+1) \quad \text{as } x \to 0 \text{ for } v \neq -1, -2, \cdots, \tag{1.5}$$

$$I_v(x) \sim \frac{e^x}{\sqrt{2\pi x}} \quad \text{as } x \to \infty, \tag{1.6}$$

we have

$$\lim_{x \to 0} \frac{I_u(x) I_v(x)}{I_{(u+v)/2}(x)^2} = \frac{\Gamma((u+v)/2+1)^2}{\Gamma(u+1)\Gamma(v+1)} \quad \text{and} \quad \lim_{x \to \infty} \frac{I_u(x) I_v(x)}{I_{(u+v)/2}(x)^2} = 1.$$

Then it is natural for us to claim that the ratio

$$x \mapsto \frac{I_u(x) I_v(x)}{I_{(u+v)/2}(x)^2} := \mathcal{I}_{u,v}(x) \tag{1.7}$$

is strictly increasing from $(0, \infty)$. It is obvious that if our claim is valid, then the log-concavity of function $v \mapsto I_v(x)$ and log-convexity of $v \mapsto 2^v \Gamma(v+1) x^{-v} I_v(x)$ on $(-1, \infty)$ easily follows. Also, replacing (u, v)

by $(v-1, v+1)$ we can see that for $v > -1$, the function $x \mapsto I_{v-1}(x) I_{v+1}(x) / I_v(x)^2$ is strictly increasing from $(0, \infty)$ onto $(v/(v+1), 1)$. Consequently, the double inequality

$$\frac{v}{v+1} < \frac{I_{v-1}(x) I_{v+1}(x)}{I_v(x)^2} < 1 \tag{1.8}$$

holds for $x > 0$ and $v > -1$, which is clearly equivalent to Turán type inequalities (1.3) with the best constants 0 and $1/(v+1)$. In fact, this assertion has been verified by Baricz in [10, Theorem 2.1].

The main purpose of this paper is to prove the above claim of the inequality (1.7), which is precisely showed as Theorem 2.4 in Section 2. As direct consequences of Theorem 2.4, both the convexity of ratios

$$v \mapsto \frac{I_v'(x)}{I_v(x)}, \quad v \mapsto \frac{I_{v-1}(x)}{I_v(x)}, \quad v \mapsto \frac{I_{v+1}(x)}{I_v(x)} \tag{1.9}$$

and the monotonicity of ratios

$$x \mapsto \frac{I_u(x)^p I_v(x)^q}{I_{pu+qv}(x)}, \quad x \mapsto \frac{I_{u+a}(x)}{I_u(x)} \frac{I_v(x)}{I_{v+a}(x)} \tag{1.10}$$

can be readily deduced.

The remainders of the paper are organized as follows. In Section 3, some sharp estimates for $I_v(x)$ are further presented. In the last section, some known bounds for certain ratios listed in (1.9) are reproved in alternating ways, and other new inequalities are established.

2 Main results

Before proving our main results we need two Preliminary lemmas. The first lemma is stated which first appeared in [39, (3.5)].

Lemma 2.1 *For $I_u(x)$ and $I_v(x)$, there holds*

$$I_u(x) I_v(x) = \frac{1}{\Gamma(u+1)\Gamma(v+1)} \sum_{n=0}^{\infty} \frac{(u+v+n+1)_n}{n!\,(u+1)_n (v+1)_n} \left(\frac{x}{2}\right)^{2n+u+v}. \tag{2.11}$$

Remark 2.2 *Setting $u = v$ in Lemma 2.1 we have*

$$I_v(x)^2 = \frac{1}{\Gamma(v+1)^2} \sum_{n=0}^{\infty} \frac{(2v+n+1)_n}{n!\,(v+1)_n^2} \left(\frac{x}{2}\right)^{2n+2v}. \tag{2.12}$$

Lemma 2.3 ([15]) *Let $A(t) = \sum_{k=0}^{\infty} a_k t^k$ and $B(t) = \sum_{k=0}^{\infty} b_k t^k$ be two real power series converging on $(-r, r)$ $(r > 0)$ with $b_k > 0$ for all k. If the sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $t \mapsto A(t)/B(t)$ is also increasing (decreasing) on $(0, r)$.*

We now are in a position to state and prove our main result.

Theorem 2.4 *Let u, v be two distinct real numbers satisfying $\min(u, v) > -2$ and $u + v > -2$ with u, $v \neq -1$. Then the function $x \mapsto \mathcal{I}_{u,v}(x)$ defined by (1.7) is strictly increasing on $(0, \infty)$. Consequently, the double inequality (1.4) holds for $x > 0$.*

Proof. By the identity (2.12) we get

$$I_{(u+v)/2}(x)^2 = \frac{1}{\Gamma((u+v)/2+1)^2}\sum_{n=0}^{\infty}\frac{(u+v+n+1)_n}{n!\,((u+v)/2+1)_n^2}\left(\frac{x}{2}\right)^{2n+u+v},$$

then

$$\begin{aligned}\frac{I_u(x)\,I_v(x)}{I_{(u+v)/2}(x)^2} &= \frac{\frac{1}{\Gamma(u+1)\Gamma(v+1)}\sum_{n=0}^{\infty}\frac{(u+v+n+1)_n}{n!(u+1)_n(v+1)_n}\left(\frac{x}{2}\right)^{2n+u+v}}{\frac{1}{\Gamma((u+v)/2+1)^2}\sum_{n=0}^{\infty}\frac{(u+v+n+1)_n}{n!((u+v)/2+1)_n^2}\left(\frac{x}{2}\right)^{2n+u+v}}\\ &:= \frac{\Gamma((u+v)/2+1)^2}{\Gamma(u+1)\,\Gamma(v+1)}\frac{\sum_{n=0}^{\infty}a_n\left(x^2/4\right)^n}{\sum_{n=0}^{\infty}b_n\left(x^2/4\right)^n},\end{aligned}$$

where

$$a_n = \frac{(u+v+n+1)_n}{n!\,(u+1)_n\,(v+1)_n}, \quad b_n = \frac{(u+v+n+1)_n}{n!\,((u+v)/2+1)_n^2}.$$

Simple computation yields

$$\frac{a_n}{b_n} = \frac{(u+v+n+1)_n}{n!\,(u+1)_n\,(v+1)_n}\Bigg/\frac{(u+v+n+1)_n}{n!\,((u+v)/2+1)_n^2} = \frac{((u+v)/2+1)_n^2}{(u+1)_n\,(v+1)_n},$$

$$\begin{aligned}\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n} &= \frac{((u+v)/2+1)_{n+1}^2}{(u+1)_{n+1}\,(v+1)_{n+1}}-\frac{((u+v)/2+1)_n^2}{(u+1)_n\,(v+1)_n}\\ &= \frac{(u-v)^2}{4}\frac{((u+v)/2+1)_n^2}{(u+1)_{n+1}\,(v+1)_{n+1}}.\end{aligned}$$

Since $u+v>-2$ and $\min(u,v)>-2$ with $u\neq v$, we see that $b_n>0$ for $n\geq 0$, and $\max(u,v)>-1$. In what follows, we discuss it in two different cases:

Case 1: While $\min(u,v)>-1$, the sequence $\{a_n/b_n\}$ is strictly increasing for $n\geq 0$. By Lemma 2.3 it follows that the ratio $\left(\sum_{n=0}^{\infty}a_n\left(x^2/4\right)^n\right)/\left(\sum_{n=0}^{\infty}b_n\left(x^2/4\right)^n\right)$ is strictly increasing on $(0,\infty)$, and so is the function $x\mapsto \mathcal{I}_{u,v}(x)$.

Case 2: While $\min(u,v)\in(-2,-1)$, it is easy to check that $(u+1)_{n+1}(v+1)_{n+1}<0$ for $n\geq 0$ due to $\max(u,v)>-1$. Then the sequence $\{a_n/b_n\}$ is strictly decreasing for $n\geq 0$. So is the ratio $\left(\sum_{n=0}^{\infty}a_n\left(x^2/4\right)^n\right)/\left(\sum_{n=0}^{\infty}b_n\left(x^2/4\right)^n\right)$ on $(0,\infty)$, together with $\Gamma(u+1)\,\Gamma(v+1)<0$ it yields the function $x\mapsto \mathcal{I}_{u,v}(x)$ is strictly increasing on $(0,\infty)$. This completes the proof. □

In virtue of Theorem 2.4, we can deduce some new and interesting further consequences.

Theorem 2.5 *(i) Each of the functions*

$$v\mapsto\frac{I_v'(x)}{I_v(x)},\quad v\mapsto\frac{I_{v-1}(x)}{I_v(x)}\quad\text{and}\quad v\mapsto\frac{I_{v+1}(x)}{I_v(x)}$$

is strictly convex on $(-1,\infty)$.

(ii) For any fixed $a>0$*, each of the functions*

$$v\mapsto\frac{I_{v+a}'(x)}{I_{v+a}(x)}-\frac{I_v'(x)}{I_v(x)},\quad v\mapsto\frac{I_{v+a-1}(x)}{I_{v+a}(x)}-\frac{I_{v-1}(x)}{I_v(x)}\quad\text{and}\quad v\mapsto\frac{I_{v+a+1}(x)}{I_{v+a}(x)}-\frac{I_{v+1}(x)}{I_v(x)}$$

is still strictly increasing on $(-1,\infty)$.

(iii) For any $x>y>0$*, the function* $v\mapsto I_v(x)/I_v(y)$ *is log-convex on* $(-1,\infty)$.

Proof. (i) By Theorem 2.4 we easily see that $x \mapsto \ln \mathcal{I}_{u,v}(x)$ is strictly increasing on $(0,\infty)$ for $u,v > -1$. Then we get

$$(\ln \mathcal{I}_{u,v}(x))' = \frac{I_u'(x)}{I_u(x)} + \frac{I_v'(x)}{I_v(x)} - 2\frac{I_{(u+v)/2}'(x)}{I_{(u+v)/2}(x)} > 0,$$

which implies that the function $v \mapsto I_v'(x)/I_v(x)$ is strictly convex on $(-1,\infty)$.

From the recurrence formulas [41, p. 79]

$$xI_v'(x) + vI_v(x) = xI_{v-1}(x), \tag{2.13}$$
$$xI_v'(x) - vI_v(x) = xI_{v+1}(x), \tag{2.14}$$

we have

$$\frac{I_v'(x)}{I_v(x)} = \frac{I_{v-1}(x)}{I_v(x)} - \frac{v}{x}, \tag{2.15}$$
$$\frac{I_v'(x)}{I_v(x)} = \frac{I_{v+1}(x)}{I_v(x)} + \frac{v}{x}; \tag{2.16}$$

which show that $v \mapsto I_{v-1}(x)/I_v(x)$ and $v \mapsto I_{v+1}(x)/I_v(x)$ have the same convexity as $v \mapsto I_v'(x)/I_v(x)$ on $(-1,\infty)$.

(ii) The increasing property of the above three functions easily follows from the property of convex functions.

(iii) It follows from Theorem 2.4 that the inequality $\mathcal{I}_{u,v}(x) > \mathcal{I}_{u,v}(y)$ for $u,v > -1$ and $x > y > 0$, which is equivalent to

$$\frac{I_u(x)}{I_u(y)}\frac{I_v(x)}{I_v(y)} > \left[\frac{I_{(u+v)/2}(x)}{I_{(u+v)/2}(y)}\right]^2 \text{ if } x > y > 0.$$

This shows that $v \mapsto I_v(x)/I_v(y)$ is log-convex on $(-1,\infty)$, which completes the proof. □

With Theorem 2.5 in hand, we easily obtain the following statement.

Theorem 2.6 *Let real numbers u,v,p,q satisfy $u,v,pu+qv > -1$ with $u \neq v$ and $p+q=1$. Then the function*

$$x \mapsto \frac{I_u(x)^p I_v(x)^q}{I_{pu+qv}(x)} := \Phi_{u,v}(x) \tag{2.17}$$

is strictly increasing on $(0,\infty)$ if $pq > 0$. The function $x \mapsto \Phi_{u,v}(x)$ is strictly decreasing on $(0,\infty)$ if $pq < 0$. Consequently, the double inequality

$$\frac{\Gamma(pu+qv+1)}{\Gamma(u+1)^p\,\Gamma(v+1)^q} < \frac{I_u(x)^p I_v(x)^q}{I_{pu+qv}(x)} < 1 \tag{2.18}$$

holds for $x \in (0,\infty)$ if $pq > 0$, where the lower and upper bounds are the best possible. For $pq < 0$, the double inequality (2.18) is reversed.

Proof. In fact, from Theorem 2.5 we see that the function $v \mapsto I_v'(x)/I_v(x)$ is strictly convex on $(-1,\infty)$. Hence, by the property of convex functions, we get for $u,v,pu+qv > -1$ and $pq > 0$ with $p+q=1$ there holds

$$(\ln \Phi_{u,v}(x))' = p\frac{I_u'(x)}{I_u(x)} + q\frac{I_v'(x)}{I_v(x)} - \frac{I_{pu+qv}'(x)}{I_{pu+qv}(x)} > 0,$$

which indicates that the function $x \mapsto \Phi_{u,v}(x)$ is strictly increasing on $(0,\infty)$ for $pq > 0$.

For $pq < 0$, it implies that $p > 0$ and $q < 0$. Setting $p^* = -q/p$ and $q^* = 1/p$, then p^* and q^* satisfy $p^*, q^* > 0$ with $p^* + q^* = 1$ and $p^* v + q^* (pu + qv) = u$. Therefore, it follows that

$$\begin{aligned} (\ln \Phi_{u,v}(x))' &= -p\left(-\frac{I_u'(x)}{I_u(x)} - \frac{q}{p}\frac{I_v'(x)}{I_v(x)} + \frac{1}{p}\frac{I_{pu+qv}'(x)}{I_{pu+qv}(x)}\right) \\ &= -p\left(p^*\frac{I_v'(x)}{I_v(x)} + q^*\frac{I_{pu+qv}'(x)}{I_{pu+qv}(x)} - \frac{I_u'(x)}{I_u(x)}\right) < 0, \end{aligned}$$

which shows that the function $x \mapsto \Phi_{u,v}(x)$ is strictly decreasing on $(0, \infty)$ for $pq < 0$.

Furthermore, using the asymptotic formulas (1.5) and (1.6) yields

$$\Phi_{u,v}(0) = \frac{\Gamma(pu + qv + 1)}{\Gamma(u + 1)^p \Gamma(v + 1)^q} \text{ and } \Phi_{u,v}(\infty) = 1.$$

Thus we complete the proof. □

It is obvious that Theorem 2.6 can be generalized as the following form.

Theorem 2.7 *Let real numbers $v_k > -1$, $p_k > 0$ for $k = 1, 2, ..., n$ with $\sum_{k=1}^n p_k = 1$ and $\bar{v} = \sum_{k=1}^n p_k v_k$. Then the function*

$$x \mapsto \frac{\prod_{k=1}^n I_{v_k}(x)^{p_k}}{I_{\bar{v}}(x)}$$

is increasing on $(0, \infty)$. Consequently, the double inequality

$$\frac{\Gamma(\bar{v} + 1)}{\prod_{k=1}^n \Gamma(v_k + 1)^{p_k}} \leq \frac{\prod_{k=1}^n I_{v_k}(x)^{p_k}}{I_{\bar{v}}(x)} \leq 1$$

holds for $x \in (0, \infty)$, where the lower and upper bounds are the best possible. The equalities are valid if and only if all v_k are equal.

Theorem 2.8 *Let $u, v > -1$ and $a > 0$. Then for $u > v$, the function*

$$x \mapsto \frac{I_{u+a}(x)}{I_u(x)}\frac{I_v(x)}{I_{v+a}(x)} := \Upsilon_{u,v}(x)$$

is strictly increasing on $(0, \infty)$. Furthermore, we have

$$\frac{\Gamma(u + 1)\Gamma(v + a + 1)}{\Gamma(v + 1)\Gamma(u + a + 1)} < \frac{I_{u+a}(x)}{I_u(x)}\frac{I_v(x)}{I_{v+a}(x)} < 1$$

for $x > 0$, where the lower and upper bounds are sharp.

Proof. By the part (ii) of Theorem 2.5, we have

$$(\ln \Upsilon_{u,v}(x))' = \frac{I_{u+a}'(x)}{I_{u+a}(x)} - \frac{I_u'(x)}{I_u(x)} - \left(\frac{I_{v+a}'(x)}{I_{v+a}(x)} - \frac{I_v'(x)}{I_v(x)}\right) > 0,$$

which implies that the function $x \mapsto \Upsilon_{u,v}(x)$ is strictly increasing on $(0, \infty)$. So, a direct computation gives

$$\Upsilon_{u,v}(0) = \frac{\Gamma(u + 1)\Gamma(v + a + 1)}{\Gamma(v + 1)\Gamma(u + a + 1)} \text{ and } \Upsilon_{u,v}(\infty) = 1,$$

which completes the proof. □

Remark 2.9 *Since*

$$\begin{aligned}\lim_{a\to0^+}\frac{1}{a}\ln\left(\frac{I_{u+a}(x)}{I_u(x)}\frac{I_v(x)}{I_{v+a}(x)}\right) &= \frac{\partial\ln I_u(x)}{\partial u}-\frac{\partial\ln I_v(x)}{\partial v},\\ \lim_{a\to0^+}\frac{1}{a}\ln\frac{\Gamma(u+1)\Gamma(v+a+1)}{\Gamma(v+1)\Gamma(u+a+1)} &= \psi(v+1)-\psi(u+1),\end{aligned}$$

where ψ is the psi function, Theorem 2.8 shows that for $u>v>-1$, the function

$$x\mapsto\frac{\partial\ln I_u(x)}{\partial u}-\frac{\partial\ln I_v(x)}{\partial v}$$

is strictly increasing from $(0,\infty)$ onto $(\psi(v+1)-\psi(u+1),1)$. Further, it implies that for $v>-1$, the function

$$x\mapsto\frac{\partial^2\ln I_v(x)}{\partial v^2}$$

is strictly increasing from $(0,\infty)$ onto $(-\psi'(v+1),0)$.

3 Sharp estimates for $I_v(x)$ in terms of hyperbolic functions

This section is devoted to presenting some sharp bounds for $I_v(x)$ in terms of $I_{n-1/2}(x)$ and $I_{n+1/2}(x)$. As we know, Rayleigh type formula shows (cf. [1, p.445, (10.2.24)]) that for $n=0,1,2,\cdots$,

$$I_{n-1/2}(x)=\sqrt{\frac{2}{\pi}}x^{n-1/2}\left(\frac{1}{x}\frac{d}{dx}\right)^n(\cosh x).$$

In particular,

$$I_{-1/2}(x)=\sqrt{\frac{2}{\pi x}}\cosh x,\;\; I_{1/2}(x)=\sqrt{\frac{2}{\pi x}}\sinh x,\;\; I_{3/2}(x)=\sqrt{\frac{2}{\pi x}}\left(\cosh x-\frac{\sinh x}{x}\right); \tag{3.19}$$

which means that the modified Bessel functions of half-integer order are all elementary functions. Now let us put $(u,v)=(n-1/2,n+1/2)$ and $(p,q)=(n-v+1/2,v-n+1/2)$ in Theorem 2.6, then we obtain

Proposition 3.1 *For $n=0,1,2,\cdots$, the function*

$$x\mapsto\left(\frac{I_{n+1/2}(x)}{I_{n-1/2}(x)}\right)^{n-v}\frac{I_v(x)}{\sqrt{I_{n-1/2}(x)I_{n+1/2}(x)}}$$

is strictly decreasing on $(0,\infty)$ if $v\in(n-1/2,n+1/2)$, and increasing on $(0,\infty)$ if $v\in(-1,n-1/2)\cup(n+1/2,\infty)$. Consequently, while $v\in(n-1/2,n+1/2)$, the double inequality

$$\begin{aligned}&\left(\frac{I_{n-1/2}(x)}{I_{n+1/2}(x)}\right)^{n-v}\sqrt{I_{n-1/2}(x)I_{n+1/2}(x)}<I_v(x)<\\ &\qquad\frac{(n+1/2)^{v-n+1/2}\Gamma(n+1/2)}{\Gamma(v+1)}\left(\frac{I_{n-1/2}(x)}{I_{n+1/2}(x)}\right)^{n-v}\sqrt{I_{n-1/2}(x)I_{n+1/2}(x)}\end{aligned}$$

holds for $x\in(0,\infty)$, where the lower and upper bounds are the best. It is reversed while $v\in(-1,n-1/2)\cup(n+1/2,\infty)$.

Furthermore, if taking $n = 0, 1$ in Proposition 3.1 we immediately conclude the following corollaries.

Corollary 3.2 *Let $v > -1$. Then, for $v \in (-1/2, 1/2)$ the function*

$$x \mapsto \sqrt{\frac{2x}{\sinh 2x}} \left(\coth^{v} x\right) I_{v}(x)$$

is strictly decreasing on $(0, \infty)$; it is strictly increasing on the same interval while $v \in (-1, -1/2) \cup (1/2, \infty)$. Consequently, if $v \in (-1/2, 1/2)$ then the double inequality

$$\sqrt{\frac{2}{\pi}} \sqrt{\frac{\sinh 2x}{2x}} \tanh^{v} x < I_{v}(x) < \frac{1}{2^{v}\Gamma(v+1)} \sqrt{\frac{\sinh 2x}{2x}} \tanh^{v} x \tag{3.20}$$

holds for $x \in (0, \infty)$, where the coefficients $1/\left(2^{v}\Gamma(v+1)\right)$ and $\sqrt{2/\pi}$ are the best. If $v \in (-1, -1/2) \cup (1/2, \infty)$ then the double inequality (3.20) is reversed.

Remark 3.3 *Letting $v = 0$ in the above corollary, we see that $x \mapsto \sqrt{2x/\sinh 2x} I_{0}(x)$ is strictly decreasing on $(0, \infty)$, and we have*

$$\sqrt{\frac{\sinh 2x}{\pi x}} < I_{0}(x) < \sqrt{\frac{\sinh 2x}{2x}}$$

for $x > 0$ with the best constants π and 2. Indeed, this double inequality ever appeared in [42, Theorem 3.2].

Corollary 3.4 *Let $v > -1$. Then, for $v \in (1/2, 3/2)$ the function*

$$x \mapsto \frac{\sqrt{x} I_{v}(x)}{\left(\coth x - 1/x\right)^{v-1/2} \sinh x}$$

is strictly decreasing on $(0, \infty)$; it is strictly increasing on $(0, \infty)$ while $v \in (-1, 1/2) \cup (3/2, \infty)$. Therefore, if $v \in (1/2, 3/2)$ then it holds that

$$\sqrt{\frac{2}{\pi}} \frac{\sinh x}{\sqrt{x}} \left(\coth x - \frac{1}{x}\right)^{v-1/2} < I_{v}(x) < \frac{3^{v-1/2}}{2^{v}\Gamma(v+1)} \frac{\sinh x}{\sqrt{x}} \left(\coth x - \frac{1}{x}\right)^{v-1/2} \tag{3.21}$$

for $x \in (0, \infty)$ with the best coefficients $\sqrt{2/\pi}$ and $2^{-v}3^{v-1/2}/\Gamma(v+1)$. If $v \in (-1, 1/2) \cup (3/2, \infty)$, then the double inequality (3.21) is reversed.

Taking $v = n$ in Proposition 3.1 gives the following statement.

Corollary 3.5 *The function*

$$x \mapsto \frac{I_{n}(x)}{\sqrt{I_{n-1/2}(x) I_{n+1/2}(x)}}$$

is strictly decreasing $(0, \infty)$. Consequently, the double inequality

$$\sqrt{I_{n-1/2}(x) I_{n+1/2}(x)} < I_{n}(x) < \alpha_{n} \sqrt{I_{n-1/2}(x) I_{n+1/2}(x)}$$

holds for $x \in (0, \infty)$ with the best coefficients

$$1 \text{ and } \alpha_{n} = \frac{(n+1/2)^{1/2}\,\Gamma(n+1/2)}{n!}.$$

Remark 3.6 *It is easy to see that*

$$\alpha_n = \sqrt{\pi (n+1/2)}\frac{(2n-1)!!}{(2n)!!} := \sqrt{\pi (n+1/2)}W_n,$$

where W_n denotes the Wallis ratio. From the Wallis inequalities proved first by Kazarinoff (see [23])

$$\frac{1}{\sqrt{\pi (n+1/2)}} < W_n < \frac{1}{\sqrt{\pi (n+1/4)}} \tag{3.22}$$

we have

$$1 < \alpha_n < \sqrt{\frac{n+1/2}{n+1/4}} < 1 + \frac{1}{2(4n+1)}.$$

This shows that

$$\lim_{n\to\infty} \frac{I_n(x)}{\sqrt{I_{n-1/2}(x) I_{n+1/2}(x)}} = 1,$$

or

$$I_n(x) \sim \sqrt{I_{n-1/2}(x) I_{n+1/2}(x)} \quad \text{as } n \to \infty.$$

4 Sharp bounds for certain ratios

In this section, we focus on some sharp bounds for certain ratios. Note that the ratio $xI_\nu(x)/I_{\nu+1}(x) := W_\nu(x)$ play an important role in the finite elasticity [36, 37] and the epidemiological models [30, 31]. It was showed in [5, Theorem 2.2] that W_ν is strictly increasing on $(0,\infty)$ for $\nu > -2$, so one has $W_\nu(x) > \lim_{x\to 0} W_\nu(x) = 2\nu + 2$. In addition, the ratio $I_{\nu+1}(x)/I_\nu(x) := R_\nu(x)$ usually appeared in probability and statistics [34, 16] with applications in chemical kinetics [2, 28], optics [38] and signal processing [24]. As for the ratio $I'_\nu(x)/I_\nu(x)$, it can be explicitly expressed by $I_{\nu-1}(x)/I_\nu(x)$ or $I_{\nu+1}(x)/I_\nu(x)$ as showed in the relations (2.15) and (2.16).

In the sequel we will present some sharp bounds for these ratios above, which is as another application of our Theorems 2.4–2.8.

Proposition 4.1 *Let $\nu \geq 0$. The the following inequalities*

$$\sqrt{\frac{\nu}{\nu+1} + \frac{\nu^2}{x^2}} < \frac{I'_\nu(x)}{I_\nu(x)} < \sqrt{1 + \frac{\nu^2}{x^2}}, \tag{4.23}$$

$$\begin{aligned} \sqrt{\frac{\nu}{\nu+1}x^2 + \nu^2} + \nu < \frac{xI_{\nu-1}(x)}{I_\nu(x)} &< \sqrt{x^2+\nu^2} + \nu \\ < \frac{xI_\nu(x)}{I_{\nu+1}(x)} &< \sqrt{\frac{\nu+1}{\nu}x^2 + (\nu+1)^2} + \nu + 1 \end{aligned} \tag{4.24}$$

hold for $x > 0$, where in particular the right hand side inequality in (4.23) holds for $\nu > -1$.

Proof. Joshi and Bissu [21, (3.6)] showed that

$$\frac{I_{\nu-1}(x) I_{\nu+1}(x)}{I_\nu(x)^2} = 1 - \frac{1}{x}\left(\frac{xI'_\nu(x)}{I_\nu(x)}\right)',$$

which together with (1.8) implies that $\left(xI_\nu'(x)/I_\nu(x)\right)' > 0$, that is, the function $x \mapsto xI_\nu'(x)/I_\nu(x)$ is strictly increasing on $(0,\infty)$ for $\nu > -1$. Hence we have

$$\frac{xI_\nu'(x)}{I_\nu(x)} > \lim_{x\to 0}\frac{xI_\nu'(x)}{I_\nu(x)} = \lim_{x\to 0}\left(\frac{xI_{\nu+1}(x)}{I_\nu(x)} + \nu\right) = \nu,$$

which yields $I_\nu'(x)/I_\nu(x) > \nu/x$ for $\nu > -1$.

Note that from the recurrence relations (2.15) and (2.16) we get

$$\frac{I_{\nu-1}(x)I_{\nu+1}(x)}{I_\nu(x)^2} = \frac{I_{\nu-1}(x)}{I_\nu(x)}\frac{I_{\nu+1}(x)}{I_\nu(x)} = \left(\frac{I_\nu'(x)}{I_\nu(x)}\right)^2 - \frac{\nu^2}{x^2},$$

which implies that $x \mapsto I_{\nu-1}(x)I_{\nu+1}(x)/I_\nu(x)^2$ is strictly increasing on $(0,\infty)$ for $\nu > -1$ due to Theorem 1.4. Therefore we have

$$\frac{\nu}{\nu+1} < \frac{I_{\nu-1}(x)I_{\nu+1}(x)}{I_\nu(x)^2} = \left(\frac{I_\nu'(x)}{I_\nu(x)}\right)^2 - \frac{\nu^2}{x^2} < 1,$$

which yields

$$\frac{\nu}{\nu+1} + \frac{\nu^2}{x^2} < \left(\frac{I_\nu'(x)}{I_\nu(x)}\right)^2 < 1 + \frac{\nu^2}{x^2} \quad \text{for } \nu > -1. \tag{4.25}$$

Since $I_\nu'(x)/I_\nu(x) = I_{\nu+1}(x)/I_\nu(x) + \nu/x > \nu/x$, we have

$$\frac{I_\nu'(x)}{I_\nu(x)} + \sqrt{1 + \frac{\nu^2}{x^2}} > \frac{\nu}{x} + \sqrt{1 + \frac{\nu^2}{x^2}} > \frac{\nu}{x} + \frac{|\nu|}{x} > 0.$$

As a result of the second inequality in (4.25), we get that the second one in (4.23) holds for $x > 0$ and $\nu > -1$.

While $\nu \geq 0$, we have

$$\frac{I_\nu'(x)}{I_\nu(x)} + \sqrt{\frac{\nu}{\nu+1} + \frac{\nu^2}{x^2}} > \frac{\nu}{x} + \sqrt{\frac{\nu}{\nu+1} + \frac{\nu^2}{x^2}} \geq 0,$$

Therefore, the left hand side inequality in (4.23) for $\nu \geq 0$ follows from the left hand side one in (4.25).

Next let us apply the formula (2.15) to (4.23), then the first, second and third inequalities in (4.24) follow immediately.

Finally, utilizing the formula (2.16) to (4.23) it yields

$$\sqrt{\frac{\nu}{\nu+1} + \left(\frac{\nu}{x}\right)^2} < \frac{I_{\nu+1}(x)}{I_\nu(x)} + \frac{\nu}{x} < \sqrt{1 + \left(\frac{\nu}{x}\right)^2},$$

which is equivalent to the third, fourth and fifth inequalities in (4.24). This completes the proof. □

Remark 4.2 *The second inequality in (4.23) can be written as*

$$\sqrt{\frac{\nu}{\nu+1}x^2 + \nu^2} < \frac{xI_\nu'(x)}{I_\nu(x)} < \sqrt{x^2 + \nu^2} \tag{4.26}$$

for $x > 0$. We would like to mention that the right hand side inequality in (4.26) for $\nu > 0$ was first proved by Gronwall [17] in 1932, motivated by a problem in wave mechanics, which was also reproved in [33] by Phillips and Malin for $\nu \in \mathbb{N}$ in different way. Recently, Baricz in [8, 10] further showed that which holds for $\nu \geq -1/2$ and $\nu > -1$, respectively. The left hand side inequality in (4.26) for $\nu \in \mathbb{N}$ was also first proved in [33] by Phillips and Malin, which was established by Baricz in [8] that it holds for all $\nu > 0$. More details to see [35, 25, 18, 12, 14].

Proposition 4.3 *Let $v > -1$. If $v \in (-1/2, 1/2)$, then the double inequality*

$$\frac{I_v'(x)}{I_v(x)} < \frac{2v + \cosh 2x}{\sinh 2x} + \frac{2v-3}{4x}, \tag{4.27}$$

holds for $x \in (0, \infty)$. If $v \in (-1, -1/2) \cup (1/2, \infty)$, then the inequality is reversed.

Proof. Theorems 2.5 tells us that the function $v \mapsto I_v'(x)/I_v(x)$ is convex on $(-1, \infty)$, which implies that for $u, v > -1$, the inequality

$$\frac{I_{pu+qv}'(x)}{I_{pu+qv}(x)} < p\frac{I_u'(x)}{I_u(x)} + q\frac{I_v'(x)}{I_v(x)}$$

holds for $x \in (0, \infty)$, where $pq > 0$ with $p + q = 1$. If $pq < 0$, the inequality is reversed.

Putting $(u, v) = (-1/2, 1/2)$ and $(p, q) = (1/2 - v, 1/2 + v)$ together with the formulas listed in (3.19) it yields that for $v \in (-1/2, 1/2)$,

$$\begin{aligned}\frac{I_v'(x)}{I_v(x)} &< \left(\frac{1}{2} - v\right)\frac{I_{-1/2}'(x)}{I_{-1/2}(x)} + \left(\frac{1}{2} + v\right)\frac{I_{1/2}'(x)}{I_{1/2}(x)} \\ &= \left(\frac{1}{2} - v\right)\left(\frac{\sinh x}{\cosh x} - \frac{1}{2x}\right) + \left(\frac{1}{2} + v\right)\left(\frac{\cosh x}{\sinh x} - \frac{1}{2x}\right) \\ &= \frac{2v + \cosh 2x}{\sinh 2x} + \frac{2v-3}{4x},\end{aligned}$$

which proves the desired inequality. □

Remark 4.4 *Thanks to Proposition 4.27, we can easily get a simple inequality as follows: for $v < (>)\, 1/2$ and $x > 0$,*

$$\frac{I_v'(x)}{I_v(x)} < (>) \coth x - \frac{1}{2x},$$

or

$$\frac{xI_v'(x)}{I_v(x)} < (>)\, x\coth x - \frac{1}{2},$$

which was first appeared in [17]. In fact, for $v < (>)\, 1/2$ and $x > 0$, we have

$$\frac{I_v'(x)}{I_v(x)} < (>) \frac{2v + \cosh 2x}{\sinh 2x} + \frac{2v-3}{4x} < (>) \frac{1 + \cosh 2x}{\sinh 2x} + \frac{1-3}{4x} = \coth x - \frac{1}{2x}.$$

Proposition 4.5 *For $-1 < u < v$ and $x > 0$, there holds the following inequality*

$$\frac{I_v(x)\, I_{v+1}(x)}{I_u(x)\, I_{u+1}(x)} < \frac{I_v(x)^2 - I_{v-1}(x)\, I_{v+1}(x)}{I_u(x)^2 - I_{u-1}(x)\, I_{u+1}(x)}. \tag{4.28}$$

In particular, for $v > -1/2$ it holds

$$I_v(x)^2 - I_{v-1}(x)\, I_{v+1}(x) > \left(\frac{2}{\sinh 2x} + \frac{1}{x}\right) I_v(x)\, I_{v+1}(x), \tag{4.29}$$

which is reversed for $-1 < v < -1/2$.

Proof. Based on Theorems 2.5 we see that for $a > 0$, the function $v \mapsto I_{v+a-1}(x)/I_{v+a}(x) - I_{v-1}(x)/I_v(x)$ is strictly increasing on $(-1, \infty)$. Then, for $a = 1$ and $-1 < u < v$ we have

$$\frac{I_u(x)}{I_{u+1}(x)} - \frac{I_{u-1}(x)}{I_u(x)} < \frac{I_v(x)}{I_{v+1}(x)} - \frac{I_{v-1}(x)}{I_v(x)}, \tag{4.30}$$

which proves the inequality (4.28) for $u, v > -1$ with $u < v$.

Letting $u = -1/2$ in the inequality (4.30) yields

$$\frac{I_v(x)^2 - I_{v-1}(x) I_{v+1}(x)}{I_v(x) I_{v+1}(x)} > \frac{I_{-1/2}(x)}{I_{1/2}(x)} - \frac{I_{-3/2}(x)}{I_{-1/2}(x)} = \frac{2}{\sinh 2x} + \frac{1}{x},$$

which proves (4.29). □

Proposition 4.6 *Let $0 < u < v$ and $x > 0$. Then the inequality*

$$\frac{I_v(x)^2 - I_{v-1}(x) I_{v+1}(x)}{I_u(x)^2 - I_{u-1}(x) I_{u+1}(x)} < \frac{I_v(x) I_{v-1}(x)}{I_u(x) I_{u-1}(x)} \tag{4.31}$$

holds. Particularly, for $v > 1/2$, we have

$$I_v(x)^2 - I_{v-1}(x) I_{v+1}(x) < \frac{\sinh 2x - 2x}{x \sinh 2x} I_{v-1}(x) I_v(x), \tag{4.32}$$

which is reversed for $0 < v < 1/2$.

Proof. Likewise, thanks to the increasing property of the function $v \mapsto I_{v+a+1}(x)/I_{v+a}(x) - I_{v+1}(x)/I_v(x)$ we get that for $-1 < u < v$,

$$\frac{I_{u+1+1}(x)}{I_{u+1}(x)} - \frac{I_{u+1}(x)}{I_u(x)} < \frac{I_{v+1+1}(x)}{I_{v+1}(x)} - \frac{I_{v+1}(x)}{I_v(x)}.$$

Replacing (u, v) with $(u-1, v-1)$ gives (4.31) for $0 < u < v$.

Taking $u = 1/2$ in (4.31) leads to

$$I_v(x)^2 - I_{v-1}(x) I_{v+1}(x) < I_{v-1}(x) I_v(x) \frac{I_{1/2}(x)^2 - I_{-1/2}(x) I_{3/2}(x)}{I_{1/2}(x) I_{-1/2}(x)},$$

which proves (4.32). □

Remark 4.7 *It would be pointed out that Baricz in [14, Theorem 1] gave an improvement of the double inequality (1.3), which shows that for $x > 0$,*

$$\frac{v + 1/2}{v + 1} \frac{I_v(x)^2}{\sqrt{x^2 + (v + 1/2)^2}} < I_v(x)^2 - I_{v-1}(x) I_{v+1}(x) < \frac{I_v(x)^2}{\sqrt{x^2 + v^2 - 1/4}}, \tag{4.33}$$

where the left hand side inequality holds for $v \geq -1/2$ and the right hand side one holds for $v \geq 1/2$.

Numeric computations show that the lower bounds given in inequalities (4.33) and (4.29) are not comparable for all $v \geq -1/2$ and $x > 0$. Analogously, the upper bounds given in inequalities (4.33) and (4.32) are also not comparable for all $v \geq 1/2$ and $x > 0$.

Proposition 4.8 *Let $x > 0$. If $v > (<)\, u > -1$, then the ratio $W_v(x) = xI_v(x)/I_{v+1}(x)$ satisfies*

$$W_v(x) - \frac{x^2}{W_v(x)} - 2v > (<)\, W_u(x) - \frac{x^2}{W_u(x)} - 2u. \tag{4.34}$$

In particular, while $-1 < v > (<) - 1/2$ it holds

$$W_v(x) > (<)\, v + \frac{1}{2} + \frac{x}{\sinh 2x} + \sqrt{\left(v + \frac{1}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2}; \tag{4.35}$$

while $-2 < v > (<) - 3/2$ there holds

$$W_v(x) < (>)\, v + \frac{1}{2} - \frac{x}{\sinh 2x} + \sqrt{\left(v + \frac{3}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2}. \tag{4.36}$$

Proof. By the relations (2.15) and (2.16) we have

$$W_{v-1}(x) = \frac{xI_{v-1}(x)}{I_v(x)} = \frac{xI_{v+1}(x)}{I_v(x)} + 2v = \frac{x^2}{W_v(x)} + 2v, \tag{4.37}$$

and applying it to the inequality (4.30) gives (4.34).

Particularly, while $u = -1/2$ we see that

$$\begin{aligned} W_{-1/2}(x) &= \frac{xI_{-1/2}(x)}{I_{1/2}(x)} = x\coth x, \\ W_{-1/2}(x) - \frac{x^2}{W_{-1/2}(x)} &= x\coth x - \frac{x^2}{x\coth x} = \frac{2x}{\sinh 2x}. \end{aligned}$$

By (4.34) we obtain

$$W_v(x) - \frac{x^2}{W_v(x)} - 2v > (<)\, W_{-1/2}(x) - \frac{x^2}{W_{-1/2}(x)} + 1 = \frac{2x}{\sinh 2x} + 1,$$

or equivalently,

$$\left[W_v(x) - \left(\frac{x}{\sinh 2x} + v + \frac{1}{2}\right)\right]^2 > (<)\, x^2 + \left(\frac{x}{\sinh 2x} + v + \frac{1}{2}\right)^2. \tag{4.38}$$

As mentioned at the beginning of this section, $W_v(x) > 2v + 2$ for $v > -2$, therefore

$$W_v(x) - \left(\frac{x}{\sinh 2x} + v + \frac{1}{2}\right) > v + \frac{3}{2} - \frac{1}{2}\frac{2x}{\sinh(2x)} > v + 1 > 0.$$

Thus, the desired inequality (4.35) follows from (4.38).

Finally, putting the inequality (4.35) together with (4.37) gives that for $-1 < v > (<) - 1/2$,

$$\begin{aligned} W_{v-1}(x) &= \frac{x^2}{W_v(x)} + 2v \\ &< (>)\, \frac{x^2}{v + \frac{1}{2} + \frac{x}{\sinh 2x} + \sqrt{\left(v + \frac{1}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2}} + 2v \\ &= v - \frac{1}{2} - \frac{x}{\sinh 2x} + \sqrt{\left(v + \frac{1}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2}. \end{aligned}$$

Replacing $v - 1$ by v yields (4.36) for $-2 < v > (<) - 3/2$, and the proof is complete. □

Remark 4.9 *Note that [4] showed that for* $v \geq 0$*,*

$$W_v(x) < v + \frac{1}{2} + \sqrt{x^2 + \left(v + \frac{3}{2}\right)^2}, \tag{4.39}$$

which also is valid for $v > -3/2$ *due to*

$$W_v(x)^2 - (2v+1) W_v(x) - \left(x^2 + v + \frac{1}{2}\right) < v + \frac{3}{2}, \tag{4.40}$$

see [32, Proposition 5]. Here, we claim that the upper bound for $W_v(x)$ *given in (4.36) is superior to that given in (4.39) for* $v > -3/2$*. In fact, consider*

$$\begin{aligned} & v + \frac{1}{2} - \frac{x}{\sinh 2x} + \sqrt{\left(v + \frac{3}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2} - \left(v + \frac{1}{2} + \sqrt{x^2 + \left(v + \frac{3}{2}\right)^2}\right) \\ = & \sqrt{\left(v + \frac{3}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2} - \left(\frac{x}{\sinh 2x} + \sqrt{x^2 + \left(v + \frac{3}{2}\right)^2}\right) \end{aligned}$$

and

$$\begin{aligned} & \left(v + \frac{3}{2} + \frac{x}{\sinh 2x}\right)^2 + x^2 - \left(\frac{x}{\sinh 2x} + \sqrt{x^2 + \left(v + \frac{3}{2}\right)^2}\right)^2 \\ = & 2\frac{x}{\sinh 2x}\left[\left(v + \frac{3}{2}\right) - \sqrt{x^2 + \left(v + \frac{3}{2}\right)^2}\right] < 0. \end{aligned}$$

Letting $u = -1/2$ and $a = 1$ in Theorem 2.8, we conclude the following immediately

Proposition 4.10 *The function* $x \mapsto (\coth x) I_{v+1}(x) / I_v(x)$ *is strictly increasing* $(0, \infty)$ *if* $v > -1/2$*, and decreasing on* $(0, \infty)$ *if* $-1 < v < -1/2$*. Moreover, for* $v > -1/2$ *the double inequality*

$$\frac{\tanh x}{2(v+1)} < \frac{I_{v+1}(x)}{I_v(x)} < \tanh x \tag{4.41}$$

holds for $x > 0$*, where* $2(v+1)$ *and* 1 *are the best possible. It is reversed for* $-1 < v < -1/2$*.*

Remark 4.11 *The right hand side inequality for* $v > -1/2$ *in (4.41) and its reverse for* $v < -1/2$ *were proved in [11, Theorem 2.2].*

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