

SHARP WEIGHTED ESTIMATES INVOLVING ONE SUPREMUM

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ABSTRACT. In this note, we study the sharp weighted estimate involving one supremum. In particular, we give a positive answer to an open question raised by Lerner and Moen [13]. We also extend the result to rough homogeneous singular integral operators.

1. INTRODUCTION AND MAIN RESULT

Our main object is the following so-called sparse operator:

$$A_{\mathcal{S}}(f)(x) = \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \chi_Q(x),$$

where $\mathcal{S} \subset \mathcal{D}$ is a sparse family, i.e., for all (dyadic) cubes $Q \in \mathcal{S}$, there exists $E_Q \subset Q$ which are pairwise disjoint and $|E_Q| \geq \gamma|Q|$ with $0 < \gamma < 1$, and $\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f$. We only consider the sparse operator because it dominates Calderón-Zygmund operator pointwisely, see [2, 14, 9, 8, 11].

We are going to study the sharp weighted bounds of $A_{\mathcal{S}}$. Before that, let us recall

$$[w]_{A_p} = \sup_Q A_p(w, Q) := \sup_Q \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1},$$

$$[w]_{A_{\infty}} = \sup_Q A_{\infty}(w, Q) := \sup_Q \frac{\langle M(w\chi_Q) \rangle_Q}{\langle w \rangle_Q}.$$

In [6], Hytönen and Lacey proved the following estimate:

$$(1.1) \quad \|A_{\mathcal{S}}\|_{L^p(w)} \leq c_n [w]_{A_p}^{\frac{1}{p}} ([w]_{A_{\infty}}^{\frac{1}{p'}} + [w^{1-p'}]_{A_{\infty}}^{\frac{1}{p}}),$$

which generalizes the famous A_2 theorem, obtained by Hytönen in [5] (We also remark that when $p = 2$, (1.1) was obtained by Hytönen and Pérez in [7]). It was also conjectured in [6] that

$$\|A_{\mathcal{S}}\|_{L^p(w)} \leq c_n ([w]_{A_p}^{\frac{1}{p}} [w]_{A_{\infty}}^{\frac{1}{p'}} + [w^{1-p'}]_{A_p}^{\frac{1}{p'}} [w]_{A_{\infty}}^{\frac{1}{p}}),$$

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where

$$[w]_{A_p^\alpha A_r^\beta} := \sup_Q A_p(w, Q)^\alpha A_r(w, Q)^\beta.$$

This conjecture, which is usually referred as the one supremum conjecture, is still open. Before this conjecture was formulated, Lerner [10] obtained the following mixed A_p - A_r estimate:

$$\|A_S\|_{L^p(w)} \leq c_{n,p,r}([w]_{A_p^{\frac{1}{p-1}} A_r^{1-\frac{1}{p-1}}} + [w^{1-p'}]_{A_p^{\frac{1}{p'-1}} A_r^{1-\frac{1}{p'-1}}}),$$

which was further extended by Lerner and Moen [13] to the $r = \infty$ case with Hrusčev A_∞ constant:

$$\|A_S\|_{L^p(w)} \leq c_{n,p}([w]_{A_p^{\frac{1}{p-1}} (A_\infty^{\text{exp}})^{1-\frac{1}{p-1}}} + [w^{1-p'}]_{A_p^{\frac{1}{p'-1}} (A_\infty^{\text{exp}})^{1-\frac{1}{p'-1}}}),$$

where $A_\infty^{\text{exp}}(w, Q) = \langle w \rangle_Q \exp(\langle \log w^{-1} \rangle_Q)$. Some further extension can be also found in [15]. Comparing this result with the one supremum conjecture, besides replacing the Fujii-Wilson A_∞ constant by Hrusčev A_∞ constant, the main difference is that the power of A_p constant is larger, leading to a result which is weaker than the one supremum conjecture. However, there is also another idea, which is replacing A_p by A_q , where $q < p$. Our main result follows from this idea and it is formulated as follows

Theorem 1.2. *Let $1 \leq q < p$ and $w \in A_q$. Then*

$$\|A_S\|_{L^p(w)} \leq c_{n,p,q} [w]_{A_q^{\frac{1}{p}} (A_\infty^{\text{exp}})^{\frac{1}{p'}}}.$$

This result was conjectured by Lerner and Moen, see [13, p.251]. It improves the previous result of Duoandikoetxea [3], i.e.,

$$\|A_S\|_{L^p(w)} \leq c_{n,p,q} [w]_{A_q},$$

proved by means of extrapolation. In the next section, we will give a proof for this theorem. Extensions to rough homogeneous singular integrals will be provided in Section 3.

2. PROOF OF THEOREM 1.2

Before we state our proof, we would like to demonstrate our understanding of this A_q condition, which allows us to avoid extrapolation or interpolation completely. We can rewrite the A_q condition in the following form:

$$\begin{aligned} \langle w \rangle_Q \langle w^{1-q'} \rangle_Q^{q-1} &= \langle w \rangle_Q \langle w^{(1-p')\frac{p-1}{q-1}} \rangle_Q^{q-1} \\ &:= \langle w \rangle_Q \langle \sigma^{\frac{1}{p'}} \rangle_{\bar{A}, Q}^p, \end{aligned}$$

where $\bar{A}(t) = t^{p'(p-1)/(q-1)} = t^{\frac{p}{q-1}}$ and as usual, $\sigma = w^{1-p'}$. So we have seen that A_q condition is actually power bumped A_p condition! Now we are

ready to present our proof. Without loss of generality, we can assume $f \geq 0$. By duality, we have

$$\begin{aligned}
\|A_{\mathcal{S}}(f)\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}(w)}=1} \int A_{\mathcal{S}}(f)gw \\
&= \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q \in \mathcal{S}} \langle f \rangle_Q \langle g \rangle_Q^w w(Q) \\
&\leq \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q \in \mathcal{S}} \langle fw^{\frac{1}{p}} \rangle_{A,Q} \langle w^{-\frac{1}{p}} \rangle_{\bar{A},Q} \langle g \rangle_Q^w \langle w \rangle_Q |Q| \\
&\quad \times \exp(\langle \log w^{-1} \rangle_Q)^{\frac{1}{p'}} \exp(\langle \log w \rangle_Q)^{\frac{1}{p'}} \\
&\leq [w]_{A_q^{\frac{1}{p}}(A_{\infty}^{\text{exp}})^{\frac{1}{p'}}} \sup_{\|g\|_{L^{p'}(w)}=1} \left(\sum_{Q \in \mathcal{S}} \langle fw^{\frac{1}{p}} \rangle_{A,Q}^p |Q| \right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{Q \in \mathcal{S}} (\langle g \rangle_Q^w)^{p'} \exp(\langle \log w \rangle_Q) |Q| \right)^{\frac{1}{p'}} \\
&\leq c_n \gamma^{-1} p \|M_A\|_{L^p} [w]_{A_q^{\frac{1}{p}}(A_{\infty}^{\text{exp}})^{\frac{1}{p'}}} \|f\|_{L^p(w)},
\end{aligned}$$

where in the last step, we have used the sparsity and the Carleson embedding theorem.

3. ROUGH HOMOGENEOUS SINGULAR INTEGRAL OPERATORS

Recall that the rough homogeneous singular integral operator T_{Ω} is defined by

$$T_{\Omega}(f)(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y) dy,$$

where $\int_{S^{n-1}} \Omega = 0$. The quantitative weighted bound of T_{Ω} with $\Omega \in L^{\infty}$ has been studied in [8], based on refinement of the ideas in [4], see also a recent paper by the author and Pérez, Rivera-Ríos and Roncal [16], relying upon the sparse domination formula established in [1].

Our main result in this section is stated as follows.

Theorem 3.1. *Let $1 \leq q < p$, $w \in A_q$ and $\Omega \in L^{\infty}(S^{n-1})$. Then*

$$\|T_{\Omega}\|_{L^p(w)} \leq c_{n,p,q} [w]_{A_q^{\frac{1}{p}}(A_{\infty}^{\text{exp}})^{\frac{1}{p'}}}.$$

Proof. The proof is again based on the sparse domination formula [1] (see also a very recent paper by Lerner [12]). It suffices to prove

$$\|A_{r,\mathcal{S}}\|_{L^p(w)} \leq c_{n,p,r,q} [w]_{A_q^{\frac{1}{p}}(A_{\infty}^{\text{exp}})^{\frac{1}{p'}}},$$

where $1 < r < \frac{p}{q}$ and

$$A_{r,\mathcal{S}}(f) = \sum_{Q \in \mathcal{S}} \langle |f|^r \rangle_Q^{\frac{1}{r}} \chi_Q.$$

Denote $\bar{B}(t) = t^{\frac{p'(p-1)}{r(q-1)}} = t^{\frac{p}{r(q-1)}}$. Again, we assume $f \geq 0$. By duality, we have

$$\begin{aligned}
\|A_{r,S}(f)\|_{L^p(w)} &= \sup_{\|g\|_{L^{p'}(w)}=1} \int A_{r,S}(f)gw \\
&= \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q \in \mathcal{S}} \langle f^r \rangle_Q^{\frac{1}{r}} \langle g \rangle_Q^w w(Q) \\
&\leq \sup_{\|g\|_{L^{p'}(w)}=1} \sum_{Q \in \mathcal{S}} \langle f^r w^{\frac{r}{p}} \rangle_{B,Q}^{\frac{1}{r}} \langle w^{-\frac{r}{p}} \rangle_{B,Q}^{\frac{1}{r}} \langle g \rangle_Q^w \langle w \rangle_Q |Q| \\
&\quad \times \exp(\langle \log w^{-1} \rangle_Q)^{\frac{1}{p'}} \exp(\langle \log w \rangle_Q)^{\frac{1}{p'}} \\
&\leq [w]_{A_q^{\frac{1}{p}}(A_\infty^{\text{exp}})^{\frac{1}{p'}}} \sup_{\|g\|_{L^{p'}(w)}=1} \left(\sum_{Q \in \mathcal{S}} \langle f^r w^{\frac{r}{p}} \rangle_{B,Q}^{\frac{p}{r}} |Q| \right)^{\frac{1}{p}} \\
&\quad \times \left(\sum_{Q \in \mathcal{S}} (\langle g \rangle_Q^w)^{p'} \exp(\langle \log w \rangle_Q) |Q| \right)^{\frac{1}{p'}} \\
&\leq c_n \gamma^{-1} p \|M_B\|_{L^{p/r}}^{\frac{1}{r}} [w]_{A_q^{\frac{1}{p}}(A_\infty^{\text{exp}})^{\frac{1}{p'}}} \|f\|_{L^p(w)},
\end{aligned}$$

where again, in the last step we have used the sparsity and the Carleson embedding theorem. \square

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