Universal Bounds for Large Determinants from Non–Commutative Hölder Inequalities in Fermionic Constructive Quantum Field Theory

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Abstract

Efficiently bounding large determinants is an essential step in non–relativistic fermionic constructive quantum field theory to prove the absolute convergence of the perturbation expansion of correlation functions in terms of powers of the strength $u \in \mathbb{R}$ of the interparticle interaction. We provide, for large determinants of fermionic convariances, *sharp* bounds which hold for *all* (bounded and unbounded, the latter not being limited to semi-bounded) one–particle Hamiltonians. We find the smallest *universal determinant bound* to be exactly 1. In particular, the convergence of perturbation series at $u = 0$ of any fermionic quantum field theory is ensured if the matrix entries, with respect to some fixed orthonormal basis, of the covariance and the interparticle interaction decay sufficiently fast. Our proofs use Hölder inequalities for general non–commutative L^p–spaces derived by Araki and Masuda [AM].

Keywords: determinant bounds; Hölder inequalities for non–commutative L^p-spaces; interacting fermions, constructive quantum field theory.

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Contents

1 Setup of the Problem

The convergence of perturbation expansions in non–relativistic fermionic constructive quantum field theory at weak coupling is ensured if the matrix entries, with respect to some fixed orthonormal basis, of the covariance and the interparticle interaction decay sufficiently fast and if certain determinants arising in the expansion can be bounded efficiently. For *any* one–particle Hamiltonian we show here how to get such bounds on determinants from non–commutative Hölder inequalities. To our knowledge, such estimates are unknown for the unbounded case, even for semibounded (one–particle) Hamiltonians. The unbounded case is important, for instance, in the context of fermionic theories in the continuum. See also Remarks 1.3 and 1.4.

The bounds on determinants (of fermionic covariances) obtained in this way turn out to be *universal* and *sharp*, in a sense to be made precise below (cf. (12) and Corollary 2.4). A consequence of these estimates is that the convergence of perturbation expansions in non–relativistic fermionic quantum field theory is implied by decay properties of interaction and covariance alone. Similar to [dSPS], we give bounds which do not impose cutoffs on the Matsubara frequency, but the results obtained here are stronger than those of [dSPS] on determinants of fermionic covariances.

The paper is organized as follows: Definitions and notation are fixed in Sections 1.1–1.2. The problem of bounding large determinants and the importance of our results in the context of constructive quantum field theory are discussed in Section 1.3. Our main results are Theorem 2.2 and Corollaries 2.3–2.4 of Section 2. Our approach uses Hölder inequalities for general non–commutative L^p–spaces. See, e.g., [AM]. The main lines of the proofs are explained in Section 2, while the technical details are postponed to Section 3.

Notation 1.1
A norm on a generic vector space $\mathcal{Y}$ is denoted by $\|\cdot\|_{\mathcal{Y}}$ and the identity map of $\mathcal{Y}$ by $\mathbf{1}_{\mathcal{Y}}$. The space of all bounded linear operators on $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ is denoted by $\mathcal{B}(\mathcal{Y})$. If $\mathcal{Y}$ is a Hilbert space, then $\langle \cdot,\cdot \rangle_{\mathcal{Y}}$ denotes its scalar product. Units of C^*–algebras are always denoted by $\mathbf{1}$.

1.1 Spaces of Antiperiodic Functions on Discrete Tori

We start by defining spaces of antiperiodic functions taking values in a fixed Hilbert space and next give the definition of the antiperiodic discrete delta function:

(i): Fix $\beta \in \mathbb{R}^+$, an *even* integer $n \in 2\mathbb{N}$ and let

$$\mathbb{T}_n \doteq \left\{ -\beta + kn^{-1}\beta : k \in \{1, 2, \ldots, 2n\} \right\} \subset (-\beta, \beta] \tag{1}$$

be the discrete torus of length 2β. This means that $-\beta \equiv \beta$. Pick any Hilbert space $\mathfrak{h}$ and let $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ be the Hilbert space of functions from $\mathbb{T}_n$ to $\mathfrak{h}$ which are *antiperiodic*. That is here, for any $f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$,

$$f(\alpha + \beta) = -f(\alpha) \ , \qquad \alpha \in \mathbb{T}_n \ .$$

The scalar product on $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ is then defined to be

$$\langle f_1, f_2 \rangle_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})} \doteq n^{-1}\beta \sum_{\alpha \in \mathbb{T}_n} \langle f_1(\alpha), f_2(\alpha) \rangle_{\mathfrak{h}} \ , \qquad f_1, f_2 \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h}) \ .$$

The parameter β is interpreted as being the inverse temperature in (fermionic and non–relativistic) quantum field theory, while $\mathfrak{h}$ refers to the so–called *one–particle* Hilbert space in the same context. The use of antiperiodic functions on the torus is related to the KMS property of equilibrium states and the canonical anticommutation relations (CAR). The discretization of the torus, leading to $\mathbb{T}_n$

for $n \in 2\mathbb{N}$, arises from the use of the Trotter–Kato formula in the construction of correlation functions of such KMS states as Berezin–Grassmann integrals.

(ii): We see the Hilbert space $\mathfrak{h}$ as a subset of $\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})$ by using the discrete delta function $\delta_{\mathrm{ap}} \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ defined by

$$\delta_{\mathrm{ap}}(\alpha) \doteq \begin{cases} 0 & \text{if} \quad \alpha \notin \{0,\beta\} \ . \\ \frac{\beta^{-1}n}{2} & \text{if} \quad \alpha = 0 \ . \\ -\frac{\beta^{-1}n}{2} & \text{if} \quad \alpha = \beta \ . \end{cases} \tag{2}$$

Vectors φ of $\mathfrak{h}$ are viewed as antiperiodic functions $\hat{\varphi}$ of $\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})$ via the definition

$$\hat{\varphi}(\alpha) \doteq \delta_{\mathrm{ap}}(\alpha)\,\varphi\ , \qquad \alpha \in \mathbb{T}_n\ . \tag{3}$$

Note that this identification is isometric up to a constant, since

$$\langle \hat{\varphi}_1, \hat{\varphi}_2 \rangle_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})} = \frac{\beta^{-1}n}{2} \langle \varphi_1, \varphi_2 \rangle_{\mathfrak{h}}\ , \qquad \varphi_1, \varphi_2 \in \mathfrak{h}\ . \tag{4}$$

The discrete delta function δ_{ap} is useful here because of the property

$$g * \delta_{\mathrm{ap}} = g\ , \qquad g \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})\ , \tag{5}$$

with the convolution being defined by

$$g*f(\alpha) \doteq n^{-1}\beta \sum_{\vartheta \in \mathbb{T}_n} g(\alpha - \vartheta)\, f(\vartheta)\ ,\ g \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}),\ f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C}),\ \alpha \in \mathbb{T}_n\ . \tag{6}$$

Indeed, δ_{ap} is used below to construct the inverse of some discrete difference operator, see Equation (28).

1.2 Discrete–time Covariance

The discrete–time covariance is an operator defined from (i) a self–adjoint operator acting on the Hilbert space $\mathfrak{h}$ and (ii) the discrete derivative operator acting on the space of antiperiodic functions:

(i): Any (possibly unbounded) operator A acting on $\mathfrak{h}$ with domain $\mathrm{dom}(A)$ is viewed as an operator $\hat{A}$ with domain

$$\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathrm{dom}(A)) \subset \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})$$

by the definition

$$[\hat{A}f]\,(\alpha) \doteq A\,(f\,(\alpha))\ , \qquad f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathrm{dom}(A)),\ \alpha \in \mathbb{T}_n\ . \tag{7}$$

If $A = H = H^*$ then $\hat{H}$ is also self–adjoint on the Hilbert space $\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})$ of antiperiodic functions.

The (possibly unbounded) self–adjoint operator $H = H^*$ acting on the Hilbert space $\mathfrak{h}$ is viewed as the so–called *one–particle Hamiltonian* in (fermionic and non–relativistic) quantum field theory. Indeed, its second quantization refers to the free part of the full interaction of the fermion system.

(ii): The discrete derivative operator $\partial \in \mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}))$ is the bounded operator defined by

$$\partial f\,(\alpha) \doteq \beta^{-1}n\left(f\left(\alpha + n^{-1}\beta\right) - f\,(\alpha)\right)\ , \qquad f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}),\ \alpha \in \mathbb{T}_n\ . \tag{8}$$

It is a normal invertible operator. Combining (7) and (8) we remark that

$$[\hat{A},\partial] \doteq \hat{A}\partial - \partial\hat{A} = 0$$

for any operator A acting on $\mathfrak{h}$. Because the discrete derivative operator ∂ acts on a space of antiperiodic functions,

$$\inf\mathrm{spec}\left(|\mathrm{Im}\partial|\right) > 0\ .$$

Hence, if $H = H^*$ is any self–adjoint operator acting on $\mathfrak{h}$, then $(\partial + \hat{H})$ is a (possibly unbounded) normal operator with bounded inverse. The *discrete–time covariance* is thus defined to be

$$C_H \doteq -2\left(\partial + \hat{H}\right)^{-1} \in \mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}))\ . \tag{9}$$

This type of operator appears as the covariance of Gaussian Berezin–Grassmann integrals used in the construction of correlation functions for systems of interacting fermions, see [S]. The discrete–time derivative is related to the corresponding Trotter–Kato product formula used to define such integrals, as already mentioned in Section 1.1.

1.3 Determinant Bounds in Constructive Quantum Field Theory

Correlation functions of interacting fermions can be constructed by perturbation series in the regime of weak couplings. In this context, the self–adjoint (possibly unbounded) operator $H = H^*$ acting on $\mathfrak{h}$ is the generator of the unperturbed dynamics of the fermion system.

Now, suppose, for simplicity, that $\mathfrak{h}$ is a *separable* Hilbert space with ONB $\{\varphi_{\mathfrak{i}}\}_{\mathfrak{i}\in\mathbb{I}}$, $\mathbb{I}$ being countable, and set

$$\boldsymbol{\omega}_{H,\kappa} \doteq \limsup_{n\to\infty} \sup_{\mathfrak{i}\in\mathbb{I}} \left\{ n^{-1}\beta \sum_{\vartheta\in\mathbb{T}_n} \sum_{q\in\mathbb{I}} \left| \left\langle \varphi_q, \left(C_H \kappa(\hat{H})\hat{\varphi}_{\mathfrak{i}} \right)(\vartheta) \right\rangle_{\mathfrak{h}} \right| \right\} \qquad (10)$$

for any $\beta \in \mathbb{R}^+$, $H = H^*$ and measurable function κ from $\mathbb{R}$ to $\mathbb{R}_0^+$. See (3), (7) and (9). We have in mind cutoff functions $\kappa : \mathbb{R} \to [0,1]$.

Another essential quantity in non–relativistic fermionic constructive quantum field theory is the so–called *determinant bound* of H and κ defined as follows:

Definition 1.2 (Determinant bounds)
The parameter $\gamma_{H,\kappa} \in \mathbb{R}^+$ is a determinant bound of $H = H^*$ and the measurable function $\kappa : \mathbb{R} \to \mathbb{R}_0^+$ if, for any $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m, N \in \mathbb{N}$, $\mathfrak{M} \in \mathrm{Mat}\,(m,\mathbb{R})$ with $\mathfrak{M} \geq 0$, and all parameters

$$\{(\alpha_q, \mathfrak{i}_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0,\beta) \times \mathbb{I} \times \{1,\ldots,m\} ,$$

the following bound holds true:

$$\left| \det \left[\mathfrak{M}_{j_k, j_{N+l}} \left\langle \varphi_{\mathfrak{i}_{N+l}}, \left(C_H \kappa(\hat{H})\hat{\varphi}_{\mathfrak{i}_k} \right)(\alpha_k - \alpha_{N+l}) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \gamma_{H,\kappa}^{2N} \prod_{q=1}^{2N} \mathfrak{M}_{j_q,j_q}^{1/2} . \qquad (11)$$

For $\mathfrak{M}$ we have in mind positive matrices appearing in the so–called *Brydges–Kennedy tree expansions* which have the following structure: For each non–oriented graph $\mathfrak{g}$ with m vertices, all functions $\boldsymbol{\alpha} \in [0,1]^{\mathfrak{g}}$ and any parameter $s \in [0,1]$, we define the subgraph

$$\mathfrak{g}\,(\boldsymbol{\alpha}, s) \doteq \mathfrak{g} \backslash \{\ell \in \mathfrak{g} : \boldsymbol{\alpha}\,(\ell) \geq s\} \subset \mathfrak{g} .$$

In fact, only minimally connected graphs (trees) $\mathfrak{g}$ are relevant for the Brydges–Kennedy tree expansions. Let $\mathcal{R}_{\mathfrak{g}(\boldsymbol{\alpha},s)} \subset \{1,\ldots,m\}^2$ denote the smallest equivalence relation for which one has $(k,l) \in \mathcal{R}_{\mathfrak{g}(\boldsymbol{\alpha},s)}$ for all $k,l \in \{1,\ldots,m\}$ such that the line $\{k,l\}$ belongs to the graph $\mathfrak{g}\,(\boldsymbol{\alpha}, s)$. Then, for any $t \in [0,1]$, $\mathfrak{M} = \mathfrak{M}\,(\mathfrak{g}, \boldsymbol{\alpha}, t)$ is the symmetric positive $m \times m$ real matrix defined by

$$[\mathfrak{M}\,(\mathfrak{g}, \boldsymbol{\alpha}, t)]_{k,l} \doteq \int_0^t \mathbf{1}\left[(k,l) \in \mathcal{R}_{\mathfrak{g}(\boldsymbol{\alpha},s)}\right] \mathrm{d}s , \qquad k,l \in \{1,\ldots,m\} .$$

See for instance [AR, BK, SW].

Assume that the matrix entries, with respect to some fix orthonormal basis, of the interparticle interaction decay sufficiently fast, and let $u \in \mathbb{R}$ be the coupling constant of the considered interacting fermion system, i.e., u quantifies the strength of the interparticle interaction. Then, it can be shown that, if the parameter $\boldsymbol{\omega}_{H,\mathbf{1}_{\mathbb{R}}}\gamma_{H,\mathbf{1}_{\mathbb{R}}}^{2}|u|$ is small enough, the perturbation expansion of *all correlation functions* in terms of powers of u converges absolutely. More precisely, all correlation functions are analytic functions of the coupling u at $u = 0$ with analyticity radius of order $\boldsymbol{\omega}_{H,\mathbf{1}_{\mathbb{R}}}^{-1}\gamma_{H,\mathbf{1}_{\mathbb{R}}}^{-2}$. See for instance [AR, SW].

The use of the cutoff function κ is important in *multiscale analyses* of correlation functions of interacting fermion systems. Indeed, even for couplings $|u|$ much larger than the convergence radius $\boldsymbol{\omega}_{H,\mathbf{1}_{\mathbb{R}}}^{-1}\gamma_{H,\mathbf{1}_{\mathbb{R}}}^{-2}$ correlations functions can still be constructed via multiscale schemes related to the Wilson renormalization group: Take a family $\{\kappa_L\}_{L\in\mathbb{N}}$ of measurable functions from $\mathbb{R}$ to $[0,1]$ such that

$$\sum_{L=1}^{\infty}\kappa_L(x) = 1 , \qquad x \in \mathbb{R} .$$

(I.e., the family is a partition of unity.) If $\boldsymbol{\omega}_{H,\kappa_L}\gamma_{H,\kappa_L}^{2}|u|$ is small enough for all $L \in \mathbb{N}$, then, up to technical details, the perturbation series *at scale* L in terms of powers of u converges absolutely. In general, the smallness of the parameters $\boldsymbol{\omega}_{H,\kappa_L}\gamma_{H,\kappa_L}^{2}|u|$ at all scales is a much weaker condition than the smallness of $\boldsymbol{\omega}_{H,\mathbf{1}_{\mathbb{R}}}\gamma_{H,\mathbf{1}_{\mathbb{R}}}^{2}|u|$. See for instance [dSP].

Note that the form of cutoff function we consider does not depend on the α variables, that is, the dependency on the Matsubara frequency of covariance *does not need* to be regularized, in contrast to other approaches like for instance [GM, BGPS, GMP].

Indeed, coming back to the estimate of the form (11), one easily shows from the *Gram bound* for determinants that

$$\left|\det\left[\mathfrak{M}_{j_k,j_{N+l}}\left\langle \varphi_{\mathfrak{i}_{N+l}},\left(C_H\kappa(\hat{H})\hat{\varphi}_{\mathfrak{i}_k}\right)(\alpha_k-\alpha_{N+l})\right\rangle_{\mathfrak{h}}\right]_{k,l=1}^{N}\right| \\ \leq \|C_H\|_{\mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}))}^{N}\prod_{q=1}^{2N}\left\|\sqrt{\kappa(\hat{H})}\hat{\varphi}_{\mathfrak{i}_q}\right\|_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})}\mathfrak{M}_{j_q,j_q}^{1/2} .$$

This kind of estimate gives no finite determinant bound of H and κ because, in general, the norm of C_H diverges, as $n \to \infty$. This problem appears already for *bounded* $H \in \mathcal{B}(\mathfrak{h})$ when $0 \in \mathrm{spec}(H)$, because in this case

$$\|C_H\|^{1/2}_{\mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h}))} = \mathcal{O}\left(\sqrt{n}\right) \qquad \text{and} \qquad \|\hat{\varphi}_{\mathrm{i}_q}\|_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathfrak{h})} = \mathcal{O}\left(\sqrt{n}\right) ,$$

as $n \to \infty$. See (4). Nevertheless, similar to the multiscale analysis presented above, one can tackle this problem by using the Gram bound as previously for some regularized covariances $C_H \hat{\kappa}_L(\hat{H}, i\partial)$ at every $L \in \mathbb{N}$. Here, for any $L \in \mathbb{N}$, $\hat{\kappa}_L : \mathbb{R}^2 \to [0,1]$ is some measurable function of two variables in such a way that

$$\sum_{L=1}^{\infty} \hat{\kappa}_L(x,y) = \kappa(x) , \qquad x,y \in \mathbb{R} .$$

This decomposition can be chosen such that there are constants $\hat{\gamma}_L \in \mathbb{R}^+$, $L \in \mathbb{N}$, which at least *do not depend* on $n \in 2\mathbb{N}$ and meanwhile satisfy

$$\left| \det \left[\mathfrak{M}_{j_k, j_{N+l}} \left\langle \varphi_{\mathrm{i}_{N+l}}, \left(C_H \hat{\kappa}_L(\hat{H}, i\partial) \hat{\varphi}_{\mathrm{i}_k} \right) (\alpha_k - \alpha_{N+l}) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \hat{\gamma}_L^{2N} \prod_{q=1}^{2N} \mathfrak{M}^{1/2}_{j_q, j_q} .$$

As already mentioned, such a bound follows from the *usual Gram bound* for determinants. This kind of strategy is used for instance in [BGPS, Section 3], [GM, Section 3.2], (more recently) [GMP, Section 5.A.], and in many others works. [dSPS] shows that this multiscale analysis for the so–called the Matsubara UV problem is *not* necessary, by proving a new bound for determinants that generalizes the original Gram bound, see [dSPS, Theorem 1.3]. Note finally that using multiscale analysis to treat the Matsubara UV problem can, moreover, render useful properties of the *full* covariance less transparent. Hence, avoiding this kind of procedure brings various technical benefits.

In the same spirit, we derive direct bounds of the type (11) that do not need the UV regularization of the Matsubara frequency. One technical advantage of the approach we present here is that the given covariance does not need to be decomposed as in [dSPS, Eq. (8)] in order to obtain determinant bounds. Moreover, our estimates are *sharp* (or optimal) and hold true for *all* (possibly unbounded, the latter not being limited to semibounded) one–particle Hamiltonians. Observe that [dSPS] gives sharp estimates *up to a prefactor 2* for the class of *bounded* operators it applies, see [dSPS, Theorem 2.4 and discussions below it].

In this paper we show the (possibly infinite) general bound

$$\begin{aligned} \mathfrak{x} \doteq\ & \sup \Big\{ \inf \left\{ \gamma_{H,\mathbf{1}_{\mathbb{R}}} \in \mathbb{R}^+ : \gamma_{H,\mathbf{1}_{\mathbb{R}}} \text{ determinant bound of } H \text{ and } \mathbf{1}_{\mathbb{R}} \right\} \\ & : \ H = H^* \text{ acting on a separable Hilbert space } \mathfrak{h} \text{ with ONB } \{\varphi_{\mathrm{i}}\}_{\mathrm{i} \in \mathbb{I}} \Big\} , \end{aligned} \tag{12}$$

named here the *universal determinant bound*, is equal to $\mathfrak{x} = 1$. (Even if the class of all separable Hilbert spaces is not a set, the supremum is well–defined because of the separation axiom.) In particular, the convergence of perturbation series at $u = 0$ of any non–relativistic fermionic quantum field theory (possibly in the continuum) is ensured by the smallness of the positive parameter $\boldsymbol{\omega}_{H,\mathbf{1}_{\mathbb{R}}}$, i.e., if the interaction and the covariance are summable, only. To our knowledge, such estimates are *unknown* for unbounded self–adjoint operators H, even for semi-bounded ones. Similar statements can also be derived while taking into account the (cutoff) function κ, see Corollary 2.3. Note that we consider separable Hilbert spaces in (12) to avoid technical issues.

Remark 1.3 (Covariance in the continuum)
In the continuous case, we would like to stress that, in contrast to the lattice case, we do not have in mind covariances of the form

$$\mathbf{c}((x_1,\alpha_1),(x_2,\alpha_2)) = \int_{\mathbb{R}^d} \frac{\mathrm{e}^{(\alpha_1-\alpha_2)E(p)+i\langle p,(x_1-x_2)\rangle_{\mathbb{R}^d}}}{1+\mathrm{e}^{\beta E(p)}} \mathrm{d}^d p \ ,$$

with $(x_1,\alpha_1),(x_2,\alpha_2) \in \mathbb{R}^d \times [0,\beta)$ and $\alpha_1 \geq \alpha_2$, i.e., Fourier transforms of the Fermi–Dirac distribution associated with dispersion relations $E : \mathbb{R}^d \to \mathbb{R}$. Indeed, such functions generally diverge for $x_1 = x_2$ when $\alpha_1 - \alpha_2$ tends to β and, hence, cannot have a finite determinant bound. Formally, such covariances would correspond to use (Dirac) delta functions in (10), instead of the orthonormal vectors φ_{i}.

Remark 1.4 (Determinant bounds in the continuum)
For any fixed $\varphi \in L^2(\mathbb{R}^d)$, its Fourier transform has, of course, to decay at large frequencies. However, we cannot conclude from this that determinant bounds derived here are related to the boundedness of spacial frequencies, because the bounds are uniform with respect to the choice of the unit vectors φ_{i}.

2 Main Results

The proofs are based on two consecutive transformations of the determinant of the left–hand side of Inequality (11):

(a) We first write this determinant as the limit $\nu \to \infty$ of correlation functions associated with quasi–free states ρ_{S_ν}. This is reminiscent of [dSPS, Theorem 3.7], which represents determinants as time–ordered correlation functions of Fock states (a special case of quasi–free state). In contrast to the present work, [dSPS, Theorem 3.7] cannot be applied to the full covariance, but, rather, for each term of the decomposition [dSPS, Eq. (8)].

(b) For any $\nu \in \mathbb{R}^+$, these correlation functions are represented as scalar products involving modular operators in the GNS representation of ρ_{S_ν}. See Equation (76). As compared to [dSPS], the representation of the determinant of (11) obtained from this second transformation has the advantage of avoiding the decomposition [dSPS, Eq. (8)], which can be non–trivial to verify for general Hamiltonians and lead to artificial prefactors in the bounds.

These two transformations allow us to get bounds of the form (11) by using [AM, (A.2)], which can be viewed as Hölder inequalities for general non–commutative L^p–spaces.

Sections 2.1 and 2.2 explain the main lines of (a). The details of this first transformation are postponed to Sections 3.1 and 3.2. In Section 2.3, we give a few key definitions and results on the Tomita–Takesaki modular theory used for the transformation (b), which is described in detail in Section 3.3. In particular, we explain the origin of modular objects appearing in our main theorem, that is, Theorem 2.2. This section is devoted to the readers who may not be acquainted with the Tomita–Takesaki modular theory. The main results of this paper, that is, Theorem 2.2 and Corollaries 2.3–2.4, are found in Section 2.4, while Section 2.5 illustrates the central arguments of the proofs in the finite dimensional case via Hölder inequalities for Schatten norms.

Recall that $\mathfrak{h}$ is an arbitrary *separable* Hilbert space. In all the section, we fix $\beta \in \mathbb{R}^+$, $n \in 2\mathbb{N}$, $m \in \mathbb{N}$, $\mathfrak{M} \in \mathrm{Mat}\,(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, while $H = H^*$ is any self–adjoint operator acting on $\mathfrak{h}$. Note again that H must not be bounded. To avoid triviality of assertions, we assume $\mathfrak{M} \neq 0$.

2.1 Quasi–Free States Associated with the Determinants of the Discrete–time Covariance

The aim of this section is to represent the determinant of (11) in terms of quasi–free states. To this end, we first define CAR C^*–algebras $\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})$ constructed from a fixed $\mathfrak{h}$ and some finite–dimensional Hilbert spaces $\mathbb{M}$, having in mind the positive matrices $\mathfrak{M}$ appearing in the Brydges–Kennedy tree expansions:

(i): The (generic) non–vanishing positive matrix $\mathfrak{M}$ gives rise to a positive sesquilinear form defined on $\mathbb{C}^m$ by

$$\langle (x_1, \ldots, x_m), (y_1, \ldots, y_m) \rangle_{\mathbb{C}^m}^{\mathfrak{M}} \doteq \sum_{p,q=1}^{m} \overline{x_p}\, y_q \mathfrak{M}_{p,q} \ . \tag{13}$$

In general, this sesquilinear form is degenerated. The vector space $\mathbb{M}$ is then defined to be the quotient

$$\mathbb{M} \doteq \mathbb{C}^m / \{x \in \mathbb{C}^m : \langle x, x \rangle_{\mathbb{C}^m}^{\mathfrak{M}} = 0\} \ .$$

Then, as usual, we introduce a scalar product on $\mathbb{M}$ as

$$\langle [x], [y] \rangle_{\mathbb{M}} \doteq \langle x, y \rangle_{\mathbb{C}^m}^{\mathfrak{M}} \ , \qquad x, y \in \mathbb{C}^m \ ,$$

and $\mathbb{M}$ denotes the Hilbert space $(\mathbb{M}, \langle \cdot, \cdot \rangle_{\mathbb{M}})$. Using the notation $\mathfrak{e}_k \doteq [e_k] \in \mathbb{M}$, where $\{e_k\}_{k=1}^m$ is the canonical basis of $\mathbb{C}^m$, note that

$$\mathfrak{M}_{k,l} = \langle \mathfrak{e}_k, \mathfrak{e}_l \rangle_{\mathbb{M}} \ , \qquad k, l \in \{1, \ldots, m\} \ . \tag{14}$$

(ii): The (extended) CAR C^*–algebra associated with $\mathfrak{M}$ is the unital C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ generated by the unit $\mathbf{1}$ and the family $\{a(\Psi)\}_{\Psi \in \mathfrak{h} \otimes \mathbb{M}}$ of elements satisfying the canonical anticommutation relations (CAR), see (43)–(44) with $\mathcal{H} = \mathfrak{h} \otimes \mathbb{M}$. Notice that such a family always exists and two families satisfying these CAR are related to each other by a unique $*$–automorphism on the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$. See, e.g., [BR2, Theorem 5.2.5].

The element $a(\Psi) \in \mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ is, in fermionic quantum field theory, the annihilation operator associated with $\Psi \in \mathfrak{h} \otimes \mathbb{M}$ whereas its adjoint

$$a^+(\Psi) \doteq a(\Psi)^* \ , \qquad \Psi \in \mathfrak{h} \otimes \mathbb{M} \ ,$$

is the corresponding creation operator.

Considering that $\mathfrak{h}$ represents the one–particle Hilbert space, $\mathrm{CAR}(\mathfrak{h})$ is the C^*–algebra that allows to represent the corresponding many–fermion system within the algebraic formulation of quantum mechanics. The extension of this C^*–algebra to $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ is pivotal to control the determinant of (11). Such determinants are naturally expressed through limits of *quasi–free states* on the C^*–algebra

$\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})$: Quasi–free states are positive linear functionals $\rho\in\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})^{*}$ such that $\rho(\mathbf{1})=1$ and, for all $N_1,N_2\in\mathbb{N}$ and $\Psi_1,\ldots,\Psi_{N_1+N_2}\in\mathfrak{h}\otimes\mathbb{M}$,

$$\rho\left(a^{+}(\Psi_1)\cdots a^{+}(\Psi_{N_1})a(\Psi_{N_1+N_2})\cdots a(\Psi_{N_1+1})\right)=0 \tag{15}$$

if $N_1\neq N_2$, while in the case $N_1=N_2\equiv N$,

$$\rho\left(a^{+}(\Psi_1)\cdots a^{+}(\Psi_N)a(\Psi_{2N})\cdots a(\Psi_{N+1})\right)=\det\left[\rho\left(a^{+}(\Psi_k)a(\Psi_{N+l})\right)\right]_{k,l=1}^{N}. \tag{16}$$

Remark 2.1 (Other definitions of quasi–free states in the literature)
Some authors relax Condition (15) in the definition of quasi–free states. Within this more general framework (known as the self–dual formalism) quasi–free states fulfilling (15) are then referred as gauge invariant quasi–free states of the corresponding CAR C^{*}–algebras. For instance, see [A, Definition 3.1]. Note indeed that [A, Definition 3.1, Condition (3.1)] only imposes on the quasi–free state to be even, but not necessarily gauge invariant.

The operator $S^{(\rho)}\in\mathcal{B}(\mathfrak{h}\otimes\mathbb{M})$ defined from

$$\left\langle\Psi_2,S^{(\rho)}\Psi_1\right\rangle_{\mathfrak{h}\otimes\mathbb{M}}=\rho\left(a^{+}(\Psi_1)a(\Psi_2)\right)\ ,\qquad \Psi_1,\Psi_2\in\mathfrak{h}\otimes\mathbb{M}\ , \tag{17}$$

is named the *symbol* (or one–particle density matrix) of the quasi–free state ρ. By the positivity and normalization of states, it follows that symbols are positive (self–adjoint) operators with spectrum lying on the unit interval $[0,1]$. Conversely, any such positive operator $S\leq\mathbf{1}_{\mathfrak{h}\otimes\mathbb{M}}$ on $\mathfrak{h}\otimes\mathbb{M}$ uniquely defines a quasi–free state ρ_S on $\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})$ such that

$$\rho_S\left(a^{+}(\Psi_1)a(\Psi_2)\right)=\langle\Psi_2,S\Psi_1\rangle_{\mathfrak{h}\otimes\mathbb{M}}\ ,\qquad \Psi_1,\Psi_2\in\mathfrak{h}\otimes\mathbb{M}\ . \tag{18}$$

The symbols allowing us to represent the determinant of (11) in terms of quasi–free states are defined as follows: For all $\nu\in\mathbb{R}^{+}$, define the function

$$F_{\nu}(\lambda)\doteq\begin{cases}-\beta^{-1}n\ln|1-n^{-1}\beta\lambda| & \text{if}\quad \lambda\in\mathbb{R}\backslash\{\beta^{-1}n\}\ ,\\ \nu & \text{if}\quad \lambda=\beta^{-1}n\ ,\end{cases} \tag{19}$$

and let

$$H_{\nu}\doteq F_{\nu}(H)\ ,\qquad \nu\in\mathbb{R}^{+}\ . \tag{20}$$

The relevant quasi–free states on the C^{*}–algebra $\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})$ are those with symbol

$$S_{\nu}\doteq\frac{1}{1+\mathrm{e}^{\beta H_{\nu}\otimes\mathbf{1}_{\mathbb{M}}}}=\frac{1}{1+\mathrm{e}^{\beta H_{\nu}}}\otimes\mathbf{1}_{\mathbb{M}}\in\mathcal{B}(\mathfrak{h}\otimes\mathbb{M})\ ,\quad \nu\in\mathbb{R}^{+}, \tag{21}$$

observing that $0 < S_\nu \leq \mathbf{1}_{\mathfrak{h}\otimes\mathbb{M}}$. The precise relationship between the quasi–free states ρ_{S_ν}, $\nu \in \mathbb{R}^+$, and the covariance appearing in the determinant of (11) is described below.

2.2 Discrete–time Covariance and Bernoulli–Euler Approximations

At fixed $\lambda \in \mathbb{R}$ and large $n \gg 1$, note from (19) that

$$\mathrm{e}^{\mp\beta F_\nu(\lambda)} = \left(1 - n^{-1}\beta\lambda\right)^{\pm n} = \mathrm{e}^{\mp\beta\lambda} + o(1) \tag{22}$$

is the well–known Bernoulli–Euler approximation of the exponential function $\mathrm{e}^{\mp\beta\lambda}$. In particular, H_ν, as defined by (20), can be viewed as an approximation of the self–adjoint operator H. The relevance of the function F_ν results from the following observations:

(i): By the spectral theorem, there is a (σ–finite) measure space $(\Omega_H, \mathfrak{A}_H, \mu_H)$, a unitary map U_H from $\mathfrak{h}$ to $L^2(\Omega_H; \mathbb{C})$ and a $\mathfrak{A}_H$–measurable function $\lambda_H : \Omega_H \to \mathbb{R}$ such that

$$U_H H U_H^* = \mathrm{m}_{\lambda_H} , \tag{23}$$

where m_{λ_H} is the multiplication operator on $L^2(\Omega_H; \mathbb{C})$ with the function λ_H. Using the unitary U_H we can identify $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ with $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; L^2(\Omega_H; \mathbb{C}))$, i.e.,

$$\hat{U}_H \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h}) = \ell^2_{\mathrm{ap}}(\mathbb{T}_n; L^2(\Omega_H; \mathbb{C})) .$$

Recall that $\hat{A}$ is the extension to $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ of any operator A acting on $\mathfrak{h}$, as defined by (7). The latter, in turn, is canonically identified with

$$\int_{\Omega_H}^{\oplus} \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathbb{C}) \mu_H(\mathrm{d}\mathfrak{a}) \equiv L^2(\Omega_H; \ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathbb{C})) . \tag{24}$$

In other words, by using U_H, we identify $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathfrak{h})$ with (24). Note that the above direct integral is well–defined because $\ell^2_{\mathrm{ap}}(\mathbb{T}_n; \mathbb{C})$ is finite dimensional and $(\Omega_H, \mathfrak{A}_H, \mu_H)$ is a σ–finite measure space, since $\mathfrak{h}$ is assumed to be separable.

(ii): With this convention,

$$\hat{U}_H \hat{H} \hat{U}_H^* = \int_{\Omega_H}^{\oplus} \lambda_H(\mathfrak{a})\, \mathbf{1}_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})}\, \mu_H(\mathrm{d}\mathfrak{a}) .$$

The discrete derivative ∂ defined by (8) is meanwhile written in the new Hilbert space as

$$\hat{U}_H \, \partial \, \hat{U}_H^* = \int_{\Omega_H}^{\oplus} \mathfrak{d} \, \mu_H \left(\mathrm{d}\mathfrak{a} \right) \ ,$$

where $\mathfrak{d} \in \mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C}))$ is defined by

$$\mathfrak{d} f \left(\alpha \right) \doteq \beta^{-1} n \left(f \left(\alpha + n^{-1} \beta \right) - f \left(\alpha \right) \right) , \qquad f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C}), \ \alpha \in \mathbb{T}_n \ .$$

In particular, the discrete–time covariance C_H, defined by (9), can be represented as

$$\hat{U}_H C_H \hat{U}_H^* = -2 \int_{\Omega_H}^{\oplus} R \left(\mathfrak{d}, \lambda_H \left(\mathfrak{a} \right) \right) \mu_H \left(\mathrm{d}\mathfrak{a} \right) \ , \tag{25}$$

where $R \left(\mathfrak{d}, \lambda \right) \in \mathcal{B}(\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C}))$ is the resolvent

$$R \left(\mathfrak{d}, \lambda \right) \doteq \left(\mathfrak{d} + \lambda \, \mathbf{1}_{\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})} \right)^{-1} \ , \qquad \lambda \in \mathbb{R} \ .$$

<u>(iii):</u> It is convenient to represent the last resolvent as a convolution (6) with an antiperiodic function. To this end, we solve the following equation

$$-2R \left(\mathfrak{d}, \lambda \right) f = g_\lambda * f \ , \qquad f \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C}) \ , \tag{26}$$

in $g_\lambda \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ for any fixed $\lambda \in \mathbb{R}$. (Compare with (25).)

<u>(iii.a):</u> For $\lambda \neq \beta^{-1} n$ and $\nu \in \mathbb{R}^+$, the antiperiodic function $g_\lambda \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ defined by

$$g_\lambda \left(\alpha \right) \doteq \frac{\left(1 - n^{-1} \beta \lambda \right)^{\beta^{-1} n \left(\alpha - n^{-1} \beta \right)}}{1 + \mathrm{e}^{\beta F_\nu \left(\lambda \right)}} \ , \qquad \alpha \in \mathbb{T}_n \cap \left(-\beta, 0 \right] \ , \tag{27}$$

is the unique solution on $\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ of the difference equation

$$\mathfrak{d} f \left(\alpha \right) + \lambda f \left(\alpha \right) = -2 \delta_{\mathrm{ap}} \left(\alpha \right) \ , \qquad \alpha \in \mathbb{T}_n \ , \tag{28}$$

with the discrete delta function $\delta_{\mathrm{ap}} \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ being defined by (2). In particular, $g_\lambda \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ solves (26) for $\lambda \neq \beta^{-1} n$.

Note that we take $n \in 2\mathbb{N}$ to ensure that

$$\left(1 - n^{-1} \beta \lambda \right)^n = \left| 1 - n^{-1} \beta \lambda \right|^n = \mathrm{e}^{-\beta F_\nu \left(\lambda \right)}$$

and observe meanwhile that $\alpha\beta^{-1}n \in \mathbb{Z}$ if $\alpha \in \mathbb{T}_n$. Therefore, for any $\lambda \neq \beta^{-1}n$ and $\nu \in \mathbb{R}^+$,

$$g_\lambda(\alpha) = \left(\operatorname{sgn}\left(1 - n^{-1}\beta\lambda\right)\right)^{\beta^{-1}n(n^{-1}\beta-\alpha)} \frac{\mathrm{e}^{-(\alpha-n^{-1}\beta)F_\nu(\lambda)}}{1+\mathrm{e}^{\beta F_\nu(\lambda)}}\ , \quad \alpha \in \mathbb{T}_n \cap (-\beta, 0]\ . \tag{29}$$

Recall that sgn is the sign function defined here as follows: $\operatorname{sgn}(x) \doteq 1$ for $x \in \mathbb{R}_0^+$ and $\operatorname{sgn}(x) \doteq -1$ otherwise.

(iii.b): For $\lambda = \beta^{-1}n$, the (unique) solution on $\ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ of the difference equation (28) is equal to

$$g_{\beta^{-1}n}(\alpha) \doteq \begin{cases} 0 & \text{if} \quad \alpha \in \mathbb{T}_n \backslash \{n^{-1}\beta, -\beta + n^{-1}\beta\}\ . \\ -1 & \text{if} \quad \alpha = n^{-1}\beta\ . \\ 1 & \text{if} \quad \alpha = -\beta + n^{-1}\beta\ . \end{cases} .$$

We can write this function as the following limit:

$$g_{\beta^{-1}n}(\alpha) = \begin{cases} 0 & \text{if} \quad \alpha \in \mathbb{T}_n \backslash \{n^{-1}\beta, n^{-1}\beta - \beta\}\ . \\ -\lim_{\nu\to\infty} \frac{\mathrm{e}^{(\beta-(\alpha-n^{-1}\beta))F_\nu(\beta^{-1}n)}}{1+\mathrm{e}^{\beta F_\nu(\beta^{-1}n)}} & \text{if} \quad \alpha = n^{-1}\beta\ . \\ \lim_{\nu\to\infty} \frac{\mathrm{e}^{-(\alpha-n^{-1}\beta)F_\nu(\beta^{-1}n)}}{1+\mathrm{e}^{\beta F_\nu(\beta^{-1}n)}} & \text{if} \quad \alpha = n^{-1}\beta - \beta\ . \end{cases} \tag{30}$$

In particular, $g_\lambda \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ solves (26) for $\lambda = \beta^{-1}n$. Compare also (30) with (29).

(iv): The relationship between the function $g_\lambda \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ and the symbols S_ν (21) defining the quasi–free states ρ_{S_ν}, $\nu \in \mathbb{R}^+$, can be heuristically understood by considering the limit case $n = \infty$:

(iv.a): The function $g_\lambda \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C})$ plays the role, in the discrete case ($n < \infty$), of the antiperiodic function $g_\lambda^{(\infty)} : \mathbb{R} \to \mathbb{R}$ defined by

$$g_\lambda^{(\infty)}(\alpha) \doteq \frac{\mathrm{e}^{-\alpha\lambda}}{1+\mathrm{e}^{\beta\lambda}}, \qquad \alpha \in (-\beta, 0]\ , \tag{31}$$

which solves the differential equation

$$y' + \lambda y = \sum_{l=-\infty}^{\infty} (-1)^{l+1} \delta_{\beta l}\ .$$

Here, δ_x is the delta distribution at $x \in \mathbb{R}$. Compare the last equation with (28). Up to the observation (22) and the special case $\lambda = \beta^{-1}n$, the qualitative difference between (31) and (29) concerns the replacement of α in (31) by $\alpha - n^{-1}\beta$ in (29) and the prefactor

$$\left(\operatorname{sgn}\left(1 - n^{-1}\beta\lambda\right)\right)^{\beta^{-1}n\left(\alpha - n^{-1}\beta\right)} .$$

(iv.b): Using the symbol

$$S_H \doteq \frac{1}{1 + \mathrm{e}^{\beta H \otimes \mathbf{1}_{\mathbb{M}}}} = \frac{1}{1 + \mathrm{e}^{\beta H}} \otimes \mathbf{1}_{\mathbb{M}} \in \mathcal{B}\left(\mathfrak{h} \otimes \mathbb{M}\right) ,$$

for any $\alpha_1, \alpha_2 \in \mathbb{T}_\infty \doteq (-\beta, \beta]$ (seen as a torus) with $\alpha_1 \leq \alpha_2$, all entire analytic vectors φ_1, φ_2 of H and every $j_1, j_2 \in \{1, \ldots, m\}$,

$$\rho_{S_H}\left(a^+\left(\left(\mathrm{e}^{-\alpha_1 H}\varphi_1\right) \otimes \mathfrak{e}_{j_1}\right) a\left(\left(\mathrm{e}^{\alpha_2 H}\varphi_2\right) \otimes \mathfrak{e}_{j_2}\right)\right) = \mathfrak{M}_{j_1,j_2}\left\langle \varphi_2, \frac{\mathrm{e}^{(\alpha_2 - \alpha_1)H}}{1 + \mathrm{e}^{\beta H}}\varphi_1 \right\rangle_{\mathfrak{h}}$$

with $\mathfrak{e}_j \doteq [e_j] \in \mathbb{M}$ being the vectors of $\mathbb{M}$ satisfying (14). The symbol S_H is directly related to the the antiperiodic function $g_\lambda^{(\infty)}$ since

$$U_H S_H U_H^* = \int_{\Omega_H}^{\oplus} g_{\lambda_H(\mathfrak{a})}^{(\infty)}(0)\, \mathbf{1}_{\mathbb{C}}\, \mu_H\left(\mathrm{d}\mathfrak{a}\right) .$$

Similar identities hold true in the discrete case for which S_H and $g_\lambda^{(\infty)}$ are replaced with S_ν (21) and g_λ (29)–(30). In particular, the determinant of (11) can be represented in terms of a limit $\nu \to \infty$ (cf. (30)) of quasi-free states ρ_{S_ν} with symbol S_ν (21). See Lemma 3.2 and Corollary 3.3.

2.3 Modular Objects Associated with Discrete–time Covariance

Our estimates are based on non–commutative Hölder inequalities [AM, (A.2)] (see also (75)), which requires the celebrated Tomita–Takesaki (modular) theory. Modular objects associated with discrete–time covariance are constructed, for any fixed $\nu \in \mathbb{R}^+$, from the quasi–free state ρ_{S_ν} with symbol S_ν (21) as follows:

(i): Let $(\mathfrak{H}_\nu, \varkappa_\nu, \eta_\nu)$ be a cyclic representation of ρ_{S_ν}. The weak closure of the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ is the von Neumann algebra

$$\mathcal{X}_\nu \doteq \varkappa_\nu\left(\mathrm{CAR}\left(\mathfrak{h} \otimes \mathbb{M}\right)\right)'' \subset \mathcal{B}\left(\mathfrak{H}_\nu\right) . \tag{32}$$

As is usual, $\mathfrak{M}''$ denotes the bicommutant of any subset $\mathfrak{M}$ of the space of bounded operators acting on a Hilbert space.

(ii): The vector η_ν is, by assumption, a cyclic vector for $\mathcal{X}_\nu$, i.e., $\mathcal{H}$ is the closure of (the linear span of) the set

$$\mathcal{X}_\nu \eta_\nu \doteq \{A\eta_\nu : A \in \mathcal{X}_\nu\} \ .$$

Because the vector η_ν represents a KMS state (see Section 3.3), it is also separating for $\mathcal{X}_\nu$, i.e., for all $A \in \mathcal{X}_\nu$, $A\eta_\nu = 0$ iff $A = 0$.

(iii): We define two anti–linear operators $\mathcal{S}_0$ and $\mathcal{F}_0$ respectively by

$$\mathcal{S}_0 A\eta_\nu = A^*\eta_\nu \qquad \text{and} \qquad \mathcal{F}_0 B\eta_\nu = B^*\eta_\nu$$

for any $A \in \mathcal{X}_\nu$ and $B \in \mathcal{X}_\nu'$. Since a cyclic and separating vector for $\mathcal{X}_\nu$ is also cyclic and separating for its commutant $\mathcal{X}_\nu'$, both operators are well–defined on the dense domains $\mathrm{Dom}(\mathcal{S}_0) = \mathcal{X}_\nu\eta_\nu$ and $\mathrm{Dom}(\mathcal{F}_0) = \mathcal{X}_\nu'\eta_\nu$. By [BR1, Proposition 2.5.9], $\mathcal{S}_0$ and $\mathcal{F}_0$ are closable and their closure are denoted by $\mathcal{S}$ and $\mathcal{F}$, respectively. In fact, $\mathcal{F} = \mathcal{S}^*$ and $\mathcal{S} = \mathcal{F}^*$.

(iv): The modular operator $\mathbf{\Delta}_\nu$ and conjugation J_ν associated with the pair $(\mathcal{X}_\nu, \eta_\nu)$ are respectively the unique, positive, self–adjoint operator and the unique anti–unitary operator occurring in the polar decomposition of $\mathcal{S} = J_\nu \mathbf{\Delta}_\nu^{1/2}$. The main result of the modular Tomita–Takesaki theory is the Tomita–Takesaki theorem [BR1, Theorem 2.5.14], which states in the current context that

$$J_\nu \mathcal{X}_\nu J_\nu = \mathcal{X}_\nu' \qquad \text{and} \qquad \mathbf{\Delta}_\nu^{it} \mathcal{X}_\nu \mathbf{\Delta}_\nu^{-it} = \mathcal{X}_\nu$$

for all $t \in \mathbb{R}$. The second assertion is related with the so–called modular automorphism group, as defined by (70) in its β–rescaled version.

For more details on the theory of von Neumann algebras and modular objects, see for instance [BR1]. To make its key points more transparent, this theory is illustrated in the finite dimensional case in Section 2.5. In the same spirit, the non–commutative Hölder inequalities [AM, (A.2)], corresponding here to (75), are derived in the finite dimensional case from Hölder inequalities for Schatten norms. See (40)–(42).

2.4 Determinant Bounds from Non–commutative Hölder Inequalities

To prove our estimates, we rewrite the determinant of (11) by using cyclic representations of quasi–free states on the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$, as explained in

Section 2.1. This allows us to use the bound [AM, (A.2)], which can be viewed as Hölder inequalities for general non–commutative L^p–spaces. This yields the following assertions on determinants of fermionic covariances:

Theorem 2.2 (Representation of determinants of fermionic covariances) *Let $\mathfrak{h}$ be any separable Hilbert space. Take $\beta \in \mathbb{R}^+$, $m \in \mathbb{N}$, $n \in 2\mathbb{N}$, any self–adjoint operator $H = H^*$ acting on $\mathfrak{h}$, and a non–vanishing $\mathfrak{M} \in \mathrm{Mat}\,(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$. Then there are von Neumann algebras $\mathcal{X}_\nu \subset \mathcal{B}(\mathfrak{H}_\nu)$, cyclic and separating unit vectors $\eta_\nu \in \mathfrak{H}_\nu$ (for $\mathcal{X}_\nu$) and C^*–homomorphisms $\varkappa_\nu$ (from $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ to $\mathcal{X}_\nu$), where $\nu \in \mathbb{R}^+$, such that for each bounded measurable positive function κ from $\mathbb{R}$ to $\mathbb{R}_0^+$, all parameters*

$$\{(\alpha_q, \varphi_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0, \beta) \times \mathfrak{h} \times \{1, \ldots, m\} \ ,$$

and for any permutation π of $2N \in \mathbb{N}$ elements with sign $(-1)^\pi$ so that[1]

$$\vartheta_q \doteq \beta^{-1}(\tilde{\alpha}_{\pi^{-1}(q)} - \tilde{\alpha}_{\pi^{-1}(q-1)}) \geq 0 \ , \quad \alpha_{\pi^{-1}(q)} - \alpha_{\pi^{-1}(q-1)} \geq 0 \ , \quad q \in \{2, \ldots, 2N\} \ ,$$

where $\tilde{\alpha}_q \doteq \alpha_q$ for $q \in \{1, \ldots, N\}$ and $\tilde{\alpha}_q \doteq \alpha_q + n^{-1}\beta$ for $q \in \{N+1, \ldots, 2N\}$, the following assertion holds true:

$$\begin{aligned} &\det\left[\mathfrak{M}_{j_k, j_{N+l}} \left\langle \varphi_{N+l}, \left(C_H \kappa(\hat{H}) \hat{\varphi}_k\right) (\alpha_k - \alpha_{N+l}) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^N \qquad (33) \\ &= (-1)^\pi \lim_{\nu \to \infty} \Big\langle \Delta_\nu^{\frac{1}{2} - \beta^{-1}\tilde{\alpha}_{\pi^{-1}(p-1)}} x_{p-1}^* \Delta_\nu^{\vartheta_{p-1}} \cdots x_2^* \Delta_\nu^{\vartheta_2} x_1^* \eta_\nu \ , \\ &\qquad\qquad \Delta_\nu^{\beta^{-1}\tilde{\alpha}_{\pi^{-1}(p)} - \frac{1}{2}} x_p \Delta_\nu^{\vartheta_{p+1}} x_{p+1} \cdots \Delta_\nu^{\vartheta_{2N}} x_{2N} \eta_\nu \Big\rangle_{\mathfrak{H}_\nu} \ . \end{aligned}$$

The integer p is defined to be the smallest element of $\{1, \ldots, 2N\}$ so that $\tilde{\alpha}_{\pi(p)} \geq \beta/2$. Δ_ν is the modular operator associated with the pair $(\mathcal{X}_\nu, \eta_\nu)$. For $q \in \{1, \ldots, 2N\}$ such that $\pi^{-1}(q) \in \{1, \ldots, N\}$,

$$x_q \doteq \varkappa_\nu \left(a^+ \left(\left(\mathrm{sgn}\left(1 - n^{-1}\beta H\right)^{-\beta^{-1} n \tilde{\alpha}_{\pi^{-1}(q)}} \sqrt{\kappa\,(H)} \varphi_{\pi^{-1}(q)} \right) \otimes \mathfrak{e}_{j_{\pi^{-1}(q)}} \right) \right) \ ,$$

while for $q \in \{1, \ldots, 2N\}$ such that $\pi^{-1}(q) \in \{N+1, \ldots, 2N\}$,

$$x_q \doteq \varkappa_\nu \left(a \left(\left(\mathrm{sgn}\left(1 - n^{-1}\beta H\right)^{\beta^{-1} n \tilde{\alpha}_{\pi^{-1}(q)}} \sqrt{\kappa\,(H)} \varphi_{\pi^{-1}(q)} \right) \otimes \mathfrak{e}_{j_{\pi^{-1}(q)}} \right) \right) \ .$$

[1] The conditions on π impose that it is a permutation of $2N$ elements which orders the numbers α_q, $q \in \{1, \ldots, 2N\}$, in the following way: $\pi(k) < \pi(l)$ whenever $\alpha_k < \alpha_l$ for $k, l \in \{1, \ldots, 2N\}$ while $\pi(k) < \pi(N+l)$ whenever $\alpha_k = \alpha_{N+l}$ for $k, l \in \{1, \ldots, N\}$.

Here, sgn *is the sign function defined as follows:* $\mathrm{sgn}\,(x) \doteq 1$ *for* $x \in \mathbb{R}_0^+$ *and* $\mathrm{sgn}\,(x) \doteq -1$ *otherwise.*

Proof: Combining Lemma 3.1 and Corollary 3.3 with the construction done in Section 3.3, in particular Equation (76), one gets the assertion when all functions $\varphi_1, \ldots, \varphi_N \in \mathfrak{D} \subset \mathfrak{h}$ belong the dense space (59). To extend it to all $\varphi_1, \ldots, \varphi_N \in \mathfrak{h}$, by (75), note that both sides of Equation (33) are continuous with respect to $\varphi_1, \ldots, \varphi_N$.

For an explicit description of $(\mathfrak{H}_\nu, \varkappa_\nu, \eta_\nu)$, which is a cyclic representation of the quasi–free state ρ_{S_ν} for $\nu \in \mathbb{R}^+$, see Sections 2.1 and 2.3. Heuristic arguments can be found in Section 2.2. ■

Corollary 2.3 (Determinant bounds)
Under the assumptions of Theorem 2.2,

$$\left| \det \left[\mathfrak{M}_{j_k, j_{N+l}} \left\langle \varphi_{N+l}, \left(C_H \kappa(\hat{H}) \hat{\varphi}_k \right) (\alpha_k - \alpha_{N+l}) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \leq \prod_{q=1}^{2N} \left\| \sqrt{\kappa\,(H)} \varphi_q \right\|_{\mathfrak{h}} \mathfrak{M}_{j_q, j_q}^{1/2} \,.$$

Compare with Definition 1.2.

Proof: This corollary is a direct consequence of Theorem 2.2 and Inequality (75). In fact, inequalities of the form [AM, (A.2)] (which generalize (75)) are intimately related to Hölder inequalities for non–commutative L^p–spaces. In the finite dimensional case, the non–commutative L^p–spaces correspond to spaces of Schatten class operators, as explained in Section 2.5. ■

Corollary 2.4 (Universal determinant bounds)
The universal determinant bound defined by (12) equals $\mathfrak{x} = 1$.

Proof: Invoking Corollary 2.3, we deduce $\mathfrak{x} \leq 1$, see (12) and Definition 1.2. Now, let $\mathfrak{h} = \ell^2(\mathbb{N}; \mathbb{C})$ with canonical ONB denoted by $\{e_{\mathrm{i}}\}_{\mathrm{i} \in \mathbb{N}}$. Take $\beta \in \mathbb{R}^+$, $\kappa = \mathbf{1}_{\mathbb{R}}$, $H = \lambda \mathbf{1}_{\mathfrak{h}}$ with $\lambda \in \mathbb{R}$ and $\mathfrak{M} \in \mathrm{Mat}\,(1, \mathbb{R})$ with $\mathfrak{M}_{1,1} = 1$. Then, from Corollary 3.3 together with (21) and (16)–(17) , for each $n \in 2\mathbb{N}$ and all $N \in \mathbb{N}$, we directly compute that, for sufficiently large $n \gg 1$,

$$\left| \det \left[\left\langle e_k, \left(C_{\lambda \mathbf{1}_{\mathfrak{h}}} \hat{e}_l \right) (0) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| = \left(1 - n^{-1} \beta \lambda \right)^{-N} \left(1 + \left| 1 - n^{-1} \beta \lambda \right|^{-n} \right)^{-N} .$$

In particular, for every $\varepsilon > 0$ and $\beta \in \mathbb{R}^+$, there are $\lambda_{\varepsilon,\beta} \in \mathbb{R}$ and $n_{\varepsilon,\beta} \in \mathbb{N}$ such that, for all $n \geq n_{\varepsilon,\beta}$ and $N \in \mathbb{N}$,

$$\left| \det \left[\left\langle e_k, \left(C_{\lambda_{\varepsilon,\beta} \mathbf{1}_{\mathfrak{h}}} \hat{e}_l \right) (0) \right\rangle_{\mathfrak{h}} \right]_{k,l=1}^{N} \right| \geq (1-\varepsilon)^{2N} \ .$$

Using this lower bound and Corollary 2.3, we then arrive at the equality $\mathfrak{r} = 1$. ■

2.5 Finite Dimensional Case and Hölder Inequalities for Schatten Norms

As already discussed, we use Hölder inequalities for non–commutative L^p–spaces to derive determinant bounds (Definition 1.2). Here, we illustrate this approach in the finite dimensional case via Hölder inequalities for Schatten norms:

(i): Assume that $\mathfrak{h}$ is a *finite dimensional* Hilbert space. Then, the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ associated with $\mathfrak{h} \otimes \mathbb{M}$ can be identified with the space $\mathcal{B}(\mathbb{F})$ of all linear operators acting on the fermionic Fock space

$$\mathbb{F} \doteq \wedge (\mathfrak{h} \otimes \mathbb{M})$$

constructed from the one–particle Hilbert space $\mathfrak{h} \otimes \mathbb{M}$.

(ii): Take any faithful state ρ on $\mathcal{B}(\mathbb{F})$ with cyclic representation $(\mathfrak{H}, \varkappa, \eta)$. By finite dimensionality, it follows that

$$\varkappa (\mathrm{CAR} (\mathfrak{h} \otimes \mathbb{M}))'' = \varkappa (\mathrm{CAR} (\mathfrak{h} \otimes \mathbb{M})) \ .$$

Because ρ is faithful and $\mathcal{B}(\mathbb{F})$ is a matrix algebra, η is separating for $\varkappa (\mathrm{CAR} (\mathfrak{h} \otimes \mathbb{M}))$ and the (Tomita–Takesaki) modular objects associated with it are well–defined. Denote by $\Delta \in \mathcal{B}(\mathfrak{H})$ the modular operator associated with the pair $(\varkappa (\mathrm{CAR} (\mathfrak{h} \otimes \mathbb{M})), \eta)$. See Section 2.3.

The cyclic representation $(\mathfrak{H}, \varkappa, \eta)$ is uniquely defined, up to a unitary transformation. It is explicitly given, for instance, by the so–called standard (cyclic) representation [DF, Section 5.4]: The space $\mathfrak{H}$ corresponds to the linear space $\mathcal{B}(\mathbb{F})$ endowed with the Hilbert–Schmidt scalar product

$$\langle A, B \rangle_{\mathfrak{H}} \doteq \mathrm{Tr}_{\mathbb{F}}(A^* B) \ , \qquad A, B \in \mathfrak{H} \ . \tag{34}$$

For any $A \in \mathcal{B}(\mathbb{F})$ we define the left and right multiplication operators $\underrightarrow{A}$ and $\underleftarrow{A}$ acting on $\mathcal{B}(\mathbb{F})$ by

$$B \mapsto \underrightarrow{A} B \doteq AB \qquad \text{and} \qquad B \mapsto \underleftarrow{A} B \doteq BA \ ,$$

respectively. The representation $\varkappa$ is the left multiplication, i.e.,

$$\varkappa(A) \doteq \underrightarrow{A} \ , \qquad A \in \mathcal{B}(\mathbb{F}) \ .$$

The cyclic vector η is defined by

$$\eta \doteq \mathrm{D}^{1/2} \in \mathfrak{H}$$

with $\mathrm{D} \in \mathcal{B}(\mathbb{F})$ being the unique positive operator such that

$$\rho(A) \doteq \mathrm{Tr}_{\mathbb{F}}(\mathrm{D}A) \ , \qquad A \in \mathcal{B}(\mathbb{F}) \ . \tag{35}$$

I.e., D is the density matrix of the state ρ. In this representation, the modular operator Δ associated with ρ is equal to

$$\Delta = \underrightarrow{\mathrm{D}} \, \underleftarrow{\mathrm{D}^{-1}} \in \mathcal{B}(\mathfrak{H}) \ . \tag{36}$$

Note that if a state is faithful then its density matrix D is invertible. The (β–rescaled) modular group is the one–parameter group $\sigma \equiv \{\sigma_t\}_{t\in\mathbb{R}}$ defined by

$$\sigma_t(\underrightarrow{A}) \doteq \Delta^{-it\beta^{-1}} \underrightarrow{A} \Delta^{it\beta^{-1}} \qquad A \in \mathcal{B}(\mathbb{F}) \ . \tag{37}$$

<u>(iii):</u> Now, we fix $n \in 2\mathbb{N}$ and apply this last construction to the quasi–free states $\rho = \rho_{S_\nu}$, $\nu \in \mathbb{R}^+$, which are defined from symbols S_ν (21). See Section 2.1. Denote their standard representations by $(\mathfrak{H}_\nu, \varkappa_\nu, \eta_\nu)$, their density matrices by D_ν and the associated modular operators by Δ_ν. We infer from (34), (35), (36), Corollary 3.3, the defining properties of Bogoliubov automorphisms (compare (37) with (69)–(71)), the cyclicity of traces, and the assumptions and definitions

of Theorem 2.2 that

$$
\begin{aligned}
&\det\left[\mathfrak{M}_{j_k,j_{N+l}}\left\langle \varphi_{N+l},\left(C_H\kappa(\hat{H})\hat{\varphi}_k\right)(\alpha_k-\alpha_{N+l})\right\rangle_{\mathfrak{h}}\right]_{k,l=1}^{N}\\
&=\lim_{\nu\to\infty}(-1)^{\pi}\,\mathrm{Tr}_{\mathbb{F}}\left(\mathrm{D}_\nu^{\tilde{\alpha}_{\pi^{-1}(1)}\beta^{-1}}\mathrm{D}_\nu^{\frac{1}{2}}x_1\left(\prod_{j=2}^{p-1}\left(\mathrm{D}_\nu^{\vartheta_j}x_j\right)\right)\mathrm{D}_\nu^{\frac{1}{2}-\beta^{-1}\tilde{\alpha}_{\pi^{-1}(p-1)}}\right.\\
&\qquad\left.\mathrm{D}_\nu^{\beta^{-1}\tilde{\alpha}_{\pi^{-1}(p)}-\frac{1}{2}}x_p\left(\prod_{j=p+1}^{2N}\left(\mathrm{D}_\nu^{\vartheta_j}x_j\right)\right)\mathrm{D}_\nu^{\frac{1}{2}}\mathrm{D}_\nu^{-\beta^{-1}\tilde{\alpha}_{\pi^{-1}(2N)}}\right)\\
&=\lim_{\nu\to\infty}(-1)^{\pi}\left\langle\Delta_\nu^{\frac{1}{2}-\beta^{-1}\tilde{\alpha}_{\pi^{-1}(p-1)}}x_{p-1}^*\Delta_\nu^{\vartheta_{p-1}}\cdots x_2^*\Delta_\nu^{\vartheta_2}x_1^*\eta_\nu\,,\right.\\
&\qquad\left.\Delta_\nu^{\beta^{-1}\tilde{\alpha}_{\pi^{-1}(p)}-\frac{1}{2}}x_p\Delta_\nu^{\vartheta_{p+1}}x_{p+1}\cdots\Delta_\nu^{\vartheta_{2N}}x_{2N}\eta_\nu\right\rangle_{\mathfrak{H}_\nu}\,,
\end{aligned}
\tag{38}
$$

that is, Equation (33).

(iv): Schatten norms on $\mathcal{B}(\mathbb{F})$ are defined by

$$\|A\|_s\doteq\left(\mathrm{Tr}_{\mathbb{F}}\left(|A|^s\right)\right)^{\frac{1}{s}}\ ,\qquad A\in\mathcal{B}(\mathbb{F})\ ,\ s\geq 1\ ,$$

and

$$\|A\|_\infty\doteq\lim_{s\to\infty}\left(\mathrm{Tr}_{\mathbb{F}}\left(|A|^s\right)\right)^{\frac{1}{s}}=\|A\|_{\mathcal{B}(\mathbb{F})}\ ,\qquad A\in\mathcal{B}(\mathbb{F})\ .$$

Remark that the norm on the Hilbert space $\mathfrak{H}$ defined from the scalar product (34) is the Hilbert–Schmidt norm, i.e.,

$$\|A\|_{\mathfrak{H}}=\|A\|_2\ ,\qquad A\in\mathcal{B}(\mathbb{F})\equiv\mathfrak{H}\ . \tag{39}$$

(v): Hölder inequalities for Schatten norms refer to the following bounds: For any $n\in 2\mathbb{N}$, $r,s_1,\ldots,s_n\in[1,\infty]$ such that $\sum_{j=1}^n 1/s_j=1/r$, and all operators $A_1,\ldots,A_n\in\mathcal{B}(\mathbb{F})$,

$$\|A_1\cdots A_n\|_r\leq\prod_{j=1}^n\|A_j\|_{s_j}\ . \tag{40}$$

This type of inequality combined with (38) implies Corollary 2.3 in the finite dimensional case.

(vi): Indeed, for any integer $N \in \mathbb{N}$ and strictly positive parameter $\zeta \in \mathbb{R}^+$, define the tube

$$\mathfrak{T}_N^{(\zeta)} \doteq \left\{ (z_1, \ldots, z_N) \in \mathbb{C}^N : \forall j \in \{1, \ldots, N\},\ \mathrm{Re}(z_j) \geq 0,\ \sum_{j=1}^{N} \mathrm{Re}(z_j) \leq \zeta \right\} . \tag{41}$$

Let ρ be a faithful quasi–free state on $\mathcal{B}(\mathbb{F})$ and denote by $H_\rho = H_\rho^* \in \mathcal{B}(\mathfrak{h} \otimes \mathbb{M})$ the unique self–adjoint operator such that the symbol $S^{(\rho)}$ of ρ equals

$$S^{(\rho)} = \frac{1}{1 + \mathrm{e}^{H_\rho}} .$$

See beginning of Section 2.1 for more explanations on quasi–free states in relation with their symbols.

Choose $\Psi_1, \ldots, \Psi_N \in \mathfrak{h} \otimes \mathbb{M}$ and pick a family$\{a^\# (\Psi_q)\}_{q=1}^N$ of elements of $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$, where the notation "$a^\#$" stands for either "$a^+$" or "$a$". For any complex vector $(z_1, \ldots, z_N) \in \mathfrak{T}_N^{(1/2)}$, we observe from (36) that

$$\begin{aligned} & \Delta^{z_1} \varkappa \left(a^\# (\Psi_1) \right) \Delta^{z_2} \cdots \Delta^{z_N} \varkappa \left(a^\# (\Psi_N) \right) \eta \\ = \ & \mathrm{D}^{\mathrm{Re}(z_1)} a^\# \left(\mathrm{e}^{-i\mathrm{Im}(z_1) H_\rho} \Psi_1 \right) \mathrm{D}^{\mathrm{Re}(z_2)} a^\# \left(\mathrm{e}^{-i(\mathrm{Im}(z_1)+\mathrm{Im}(z_2)) H_\rho} \Psi_2 \right) \\ & \cdots \mathrm{D}^{\mathrm{Re}(z_N)} a^\# \left(\mathrm{e}^{-i(\mathrm{Im}(z_1)+\cdots+\mathrm{Im}(z_N)) H_\rho} \Psi_N \right) \mathrm{D}^{1/2-(\mathrm{Re}(z_1)+\cdots+\mathrm{Re}(z_N))} . \end{aligned}$$

By applying Hölder inequalities (39) and (40), we obtain from the last equality that

$$\begin{aligned} & \left\| \Delta^{z_1} \varkappa \left(a^\# (\Psi_1) \right) \Delta^{z_2} \cdots \Delta^{z_N} \varkappa \left(a^\# (\Psi_N) \right) \eta \right\|_{\mathfrak{H}} \\ & \leq \left\| \mathrm{D}^{1/2-(\mathrm{Re}(z_1)+\cdots+\mathrm{Re}(z_N))} \right\|_{\frac{1}{1/2-(\mathrm{Re}(z_1)+\cdots+\mathrm{Re}(z_N))}} \\ & \qquad \times \prod_{q=1}^{N} \left\| \mathrm{D}^{\mathrm{Re}(z_q)} \right\|_{\frac{1}{\mathrm{Re}(z_q)}} \left\| a^\# \left(\mathrm{e}^{i(\mathrm{Im}(z_1)+\cdots+\mathrm{Im}(z_q)) H_\rho} \Psi_q \right) \right\|_\infty , \end{aligned}$$

which, combined with $\|\mathrm{D}\|_1 = 1$, in turn implies that

$$\left\| \Delta^{z_1} \varkappa \left(a^\# (\Psi_1) \right) \Delta^{z_2} \cdots \Delta^{z_N} \varkappa \left(a^\# (\Psi_N) \right) \eta \right\|_{\mathfrak{H}} \leq \prod_{q=1}^{N} \|\Psi_q\|_{\mathfrak{h} \otimes \mathbb{M}} . \tag{42}$$

This inequality corresponds to (75) in the finite dimensional case. Therefore, Equation (38) combined with Inequality (42) implies Corollary 2.3 when $\mathfrak{h}$ is a finite dimensional Hilbert space.

3 Technical Proofs

3.1 Quasi–Free States on General Monomials

Let $\mathcal{H}$ be some Hilbert space and $\mathrm{CAR}(\mathcal{H})$ the associated CAR C^*–algebra generated by the unit $\mathbf{1}$ and the family $\{a(\varphi)\}_{\varphi\in\mathcal{H}}$ of elements satisfying the canonical commutation relations (CAR): For any $\varphi_1, \varphi_2 \in \mathcal{H}$,

$$a(\varphi_1)a(\varphi_2) + a(\varphi_2)a(\varphi_1) = 0 , \tag{43}$$

$$a(\varphi_1)^*a(\varphi_2) + a(\varphi_2)a(\varphi_1)^* = \langle\varphi_2, \varphi_1\rangle_{\mathcal{H}} \mathbf{1} . \tag{44}$$

Strictly speaking, the above conditions only define $\mathrm{CAR}(\mathcal{H})$ up to an isomorphism of C^*–algebras. See, e.g., [BR2, Theorem 5.2.5]. As explained in Section 2.1 for the special case $\mathcal{H} = \mathfrak{h} \otimes \mathbb{M}$, the generator $a(\varphi) \in \mathrm{CAR}(\mathcal{H})$ is interpreted as the annihilation operator associated with $\varphi \in \mathcal{H}$ whereas its adjoint

$$a^+(\varphi) \doteq a(\varphi)^* , \qquad \varphi \in \mathcal{H} ,$$

is the corresponding creation operator.

A monomial in the annihilation and creation operators is *normally ordered* if the creation operators appearing in the monomial are on the left side of all annihilation operators in the same monomial, like

$$a^+(\varphi_1) \cdots a^+(\varphi_{N_1})a(\varphi_{2N_1}) \cdots a(\varphi_{N_1+1}) .$$

By the above definition, if ρ is a quasi–free state and $\mathcal{M} \in \mathrm{CAR}(\mathcal{H})$ is a normally ordered monomial in the annihilation and creation operators, then $\rho(\mathcal{M})$ is the determinant of a matrix, the entries of which are given by ρ acting on monomials of degree two. We show below that this pivotal property of quasi–free states remains valid even if $\mathcal{M}$ is *not* normally ordered.

This is not surprising. For instance, [A, Definition 3.1, Condition (3.2)] also essentially says that if the state is quasi–free then expectation values (with respect to this state) of any monomial (not necessarily normally ordered) of arbitrary even degree is a determinant of a matrix, the entries of which are expectation values of monomials of degree two. However, beyond this fact, we would like to give the *explicit* behavior of such expectation values with respect to arbitrary permutations of creation and annihilation operators in large monomials. This point is crucial here and is given by Lemma 3.1.

To this end, we introduce some notation. If π is a permutation of $n \in \mathbb{N}$ elements (i.e., a bijective function from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$) with sign $(-1)^\pi$, we define the monomial $\mathbb{O}_\pi(A_1,\ldots,A_n)\in \mathrm{CAR}(\mathcal{H})$ in $A_1,\ldots,A_n\in \mathrm{CAR}(\mathcal{H})$ by the product

$$\mathbb{O}_\pi\left(A_1,\ldots,A_n\right) \doteq (-1)^\pi A_{\pi^{-1}(1)}\cdots A_{\pi^{-1}(n)} \ . \tag{45}$$

In other words, $\mathbb{O}_\pi$ places the operator A_k at the $\pi(k)$th position in the monomial $(-1)^\pi A_{\pi^{-1}(1)}\cdots A_{\pi^{-1}(n)}$. Further, for all $k,l\in\{1,\ldots,n\}$, $k\neq l$,

$$\pi_{k,l}:\{1,2\}\to\{1,2\} \tag{46}$$

is the identity function if $\pi(k)<\pi(l)$, otherwise $\pi_{k,l}$ interchanges 1 and 2. Then, the following property of quasi–free states holds true:

Lemma 3.1 (Quasi–free states on general monomials)
Let ρ be a quasi–free state on the C^*–algebra $\mathrm{CAR}(\mathcal{H})$, as defined by (15)–(16) for $\mathcal{H}=\mathfrak{h}\otimes\mathbb{M}$. For any $N_1,N_2\in\mathbb{N}$, all permutations π of N_1+N_2 elements and $\varphi_1,\ldots,\varphi_{N_1+N_2}\in\mathcal{H}$,

$$\rho\Big(\mathbb{O}_\pi\left(a^+(\varphi_1),\ldots,a^+(\varphi_{N_1}),a(\varphi_{N_1+N_2}),\ldots,a(\varphi_{N_1+1})\right)\Big)=0 \tag{47}$$

if $N_1\neq N_2$, while in the case $N_1=N_2\equiv N$,

$$\begin{aligned}&\rho\Big(\mathbb{O}_\pi\left(a^+(\varphi_1),\ldots,a^+(\varphi_N),a(\varphi_{2N}),\ldots,a(\varphi_{N+1})\right)\Big)\\ =\ &\det\Big[\rho\Big(\mathbb{O}_{\pi_{k,N+l}}\left(a^+(\varphi_k),a(\varphi_{N+l})\right)\Big)\Big]_{k,l=1}^N \ .\end{aligned} \tag{48}$$

Proof: By (43) and (44), if the monomial

$$\mathbb{O}_\pi\left(a^+(\varphi_1),\ldots,a^+(\varphi_{N_1}),a(\varphi_{N_1+N_2}),\ldots,a(\varphi_{N_1+1})\right)$$

contains different numbers of annihilation and creation operators (i.e., $N_1\neq N_2$), then it can be written as a sum of normally ordered monomials with the same property. By (15) and the linearity of states, we thus deduce (47).

We consider the case $N_1=N_2\equiv N\in\mathbb{N}$. Assertion (48) trivially holds if $N=1$ and we can assume from now on that $N\geq 2$.

For convenience, the notation "$a^\#$" stands for either "a^+" or "a". In particular, we write the monomial

$$\mathbb{O}_\pi\left(a^+(\varphi_1),\ldots,a^+(\varphi_n),a(\varphi_{2N}),\ldots,a(\varphi_{N+1})\right)=(-1)^\pi a_1^\#\cdots a_{2N}^\# \ .$$

Let

$$\begin{aligned} k_\pi &\doteq \min\{\pi(N+1),\ldots,\pi(2N)\} \leq N+1 \ , \\ k_\pi^+ &\doteq \max\{\pi(1),\ldots,\pi(N)\} \geq N \ . \end{aligned}$$

The parameter k_π is the position the first annihilation operator appearing in the monomial $a_1^\# \cdots a_{2N}^\#$ while k_π^+ is the position of the last creation operator appearing in $a_1^\# \cdots a_{2N}^\#$. In other words,

$$\begin{aligned} & \mathbb{O}_\pi\left(a^+(\varphi_1),\ldots,a^+(\varphi_N),a(\varphi_{2N}),\ldots,a(\varphi_{N+1})\right) \\ = \ & (-1)^\pi a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi} a_{k_\pi+1}^\# \cdots a_{k_\pi^+-1}^\# a_{k_\pi^+}^+ a_{k_\pi^++1} \cdots a_{2N} \ . \end{aligned}$$

Note that $k_\pi = N+1$ iff the monomial is normally ordered. The same holds true if $k_\pi^+ = N$. In particular, $k_\pi = N+1$ iff $k_\pi^+ = N$. We will prove Assertion (48) by induction in the parameter

$$N_\pi \doteq k_\pi^+ - k_\pi + 1 \geq 0 \ .$$

Observe that $N_\pi = 0$ iff the monomial is normally ordered and Assertion (48) holds in this case because of (43), (16) and the antisymmetry of the determinant under permutations of its lines or rows.

Assume now that $N_\pi \geq 1$. Thus, $k_\pi \leq N$ and $k_\pi^+ \geq N+1$. If $k_\pi > 2$ and $2N - k_\pi > 3$ then

$$\begin{aligned} & (-1)^\pi\, \mathbb{O}_\pi\left(a^+(\varphi_1),\ldots,a^+(\varphi_N),a(\varphi_{2N}),\ldots,a(\varphi_{N+1})\right) \\ = \ & a_1^+ \cdots a_{k_\pi-1}^+ \{a_{k_\pi}, a_{k_\pi+1}^\#\} a_{k_\pi+2}^\# \cdots a_{2N}^\# \qquad (49) \\ & - a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \{a_{k_\pi}, a_{k_\pi+2}^\#\} a_{k_\pi+3}^\# \cdots a_{2N}^\# \\ & + a_1^+ \cdots a_{k_\pi-1}^+ \sum_{l=3}^{2N-k_\pi-2} (-1)^{l-1} a_{k_\pi+1}^\# \cdots a_{k_\pi+l-1}^\# \{a_{k_\pi}, a_{k_\pi+l}^\#\} a_{k_\pi+l+1}^\# \cdots a_{2N}^\# \\ & + (-1)^{2N-k_\pi-2} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N-2}^\# \{a_{k_\pi}, a_{2N-1}^\#\} a_{2N}^\# \\ & + (-1)^{2N-k_\pi-1} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N-1}^\# \{a_{k_\pi}, a_{2N}^\#\} \\ & + (-1)^{2N-k_\pi} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N}^\# a_{k_\pi} \ , \end{aligned}$$

with $\{A,B\} \doteq AB + BA$. Mutatis mutandis if $k_\pi = 1, 2$ or $2N - k_\pi = 2, 3$. It is convenient to use the definition

$$q_\pi \doteq 2N - \pi^{-1}(k_\pi) + 1 \in \{1,\ldots,N\} \ ,$$

which implies $a_{k_\pi} = a(\varphi_{N+q_\pi})$. By combining (49) with the CAR (43) and (44), we deduce the equality

$$\begin{aligned}
& (-1)^\pi \, \mathbb{O}_\pi \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right) \\
= \;& (-1)^{2N-k_\pi} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N}^\# a_{k_\pi} \qquad (50) \\
& + \sum_{k=1}^N \left\langle \varphi_{N+q_\pi}, \varphi_k \right\rangle_{\mathcal{H}} \Big\{ \mathbf{1}\left[k_\pi + 1 = \pi(k)\right] a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+2}^\# \cdots a_{2N}^\# \\
& - \mathbf{1}\left[k_\pi + 2 = \pi(k)\right] a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# a_{k_\pi+3}^\# \cdots a_{2N}^\# \\
& + \mathbf{1}\left[2N - 1 > \pi(k) > k_\pi + 2\right] (-1)^{\pi(k)-k_\pi-1} \\
& \qquad a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{\pi(k)-1}^\# a_{\pi(k)+1}^\# \cdots a_{2N}^\# \\
& + \mathbf{1}\left[2N - 1 = \pi(k)\right] (-1)^{2N-k_\pi-2} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N-2}^\# a_{2N}^\# \\
& + \mathbf{1}\left[2N = \pi(k)\right] (-1)^{2N-k_\pi-1} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N-1}^\# \Big\}
\end{aligned}$$

when $k_\pi > 2$ and $2N - k_\pi > 3$. Mutatis mutandis if $k_\pi = 1, 2$ or $2N - k_\pi = 2, 3$. For any $k \in \{1, \ldots, N\}$, we fix a permutation $\pi^{(k)}$ of $2(N-1)$ elements such that

$$\begin{aligned}
& \mathbb{O}_{\pi^{(k)}} \Big(a^+(\varphi_1), \ldots, a^+(\varphi_{k-1}), a^+(\varphi_{k+1}), \ldots, a^+(\varphi_N), \\
& \qquad a(\varphi_{2N}), \ldots, a(\varphi_{N+q_\pi+1}), a(\varphi_{N+q_\pi-1}), \ldots, a(\varphi_{N+1}) \Big) \\
& \qquad\qquad = (-1)^{\pi^{(k)}} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{\pi(k)-1}^\# a_{\pi(k)+1}^\# \cdots a_{2N}^\# \, .
\end{aligned}$$

(Recall that $N \geq 2$ is assumed without loss of generality.) Similarly, $\tilde{\pi}$ is a permutation of $2N$ elements such that

$$\begin{aligned}
& \mathbb{O}_{\tilde{\pi}} \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right) \\
& \qquad\qquad = (-1)^{\tilde{\pi}} a_1^+ \cdots a_{k_\pi-1}^+ a_{k_\pi+1}^\# \cdots a_{2N}^\# a_{k_\pi} \, .
\end{aligned}$$

By using this notation, we rewrite (50) as

$$\begin{aligned}
& (-1)^\pi \, \mathbb{O}_\pi \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right) \\
= \;& (-1)^{k_\pi} (-1)^{\tilde{\pi}} \, \mathbb{O}_{\tilde{\pi}} \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right) \qquad (51) \\
& + \sum_{k=1}^N \mathbf{1}\left[\pi(k) > k_\pi\right] (-1)^{\pi(k)-k_\pi-1} \left\langle \varphi_{N+q_\pi}, \varphi_k \right\rangle_{\mathcal{H}} (-1)^{\pi^{(k)}} \\
& \times \mathbb{O}_{\pi^{(k)}} \Big(a^+(\varphi_1), \ldots, a^+(\varphi_{k-1}), a^+(\varphi_{k+1}), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \\
& \qquad \ldots, a(\varphi_{N+q_\pi+1}), a(\varphi_{N+q_\pi-1}), \ldots, a(\varphi_{N+1}) \Big) \, .
\end{aligned}$$

For all $k \in \{1, \ldots, N\}$, note that

$$k^+_{\pi^{(k)}} \leq k^+_\pi - 2 \qquad \text{and} \qquad k_{\pi^{(k)}} \geq k_\pi \ .$$

As a consequence, for any $k \in \{1, \ldots, N\}$, the induction parameter $N_{\pi^{(k)}}$ associated with the permutation $\pi^{(k)}$ satisfies:

$$N_{\pi^{(k)}} \doteq k^+_{\pi^{(k)}} - k_{\pi^{(k)}} + 1 \leq k^+_\pi - k_\pi - 1 = N_\pi - 2 \ . \tag{52}$$

Similarly,

$$k^+_{\tilde{\pi}} = k^+_\pi - 1 \qquad \text{and} \qquad k_{\tilde{\pi}} \geq k_\pi \ ,$$

which in turn imply

$$N_{\tilde{\pi}} \doteq k^+_{\tilde{\pi}} - k_{\tilde{\pi}} + 1 \leq k^+_\pi - k_\pi = N_\pi - 1 \ . \tag{53}$$

Observe furthermore that, for any $k \in \{1, \ldots, N\}$ such that $\pi(k) > k_\pi$,

$$(-1)^{\tilde{\pi}} = (-1)^\pi (-1)^{k_\pi} \qquad \text{and} \qquad (-1)^{\pi^{(k)}} = (-1)^\pi (-1)^{q_\pi + k + k_\pi + \pi(k)} \ . \tag{54}$$

Therefore, by using (54) together with (51), we arrive at the equality

$$\begin{aligned} & \mathbb{O}_\pi \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right) \\ = \ & \mathbb{O}_{\tilde{\pi}} \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1})\right) \\ & - \sum_{k=1}^N \mathbf{1}\left[\pi(k) > k_\pi\right] (-1)^{q_\pi + k} \left\langle \varphi_{N+q_\pi}, \varphi_k \right\rangle_{\mathcal{H}} \\ & \qquad \mathbb{O}_{\pi^{(k)}} \Big(a^+(\varphi_1), \ldots, a^+(\varphi_{k-1}), a^+(\varphi_{k+1}), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \\ & \qquad \qquad \ldots, a(\varphi_{N+q_\pi+1}), a(\varphi_{N+q_\pi-1}), \ldots, a(\varphi_{N+1})\Big) \ . \end{aligned} \tag{55}$$

We use now the following definitions: For any $k, l \in \{1, \ldots, N\}$, the coefficients

$$\begin{aligned} \mathrm{M}_{k,l} & \doteq \rho\left(\mathbb{O}_{\pi_{k,N+l}}\left(a^+(\varphi_k), a(\varphi_{N+l})\right)\right) \ , \\ \tilde{\mathrm{M}}_{k,l} & \doteq \rho\left(\mathbb{O}_{\tilde{\pi}_{k,N+l}}\left(a^+(\varphi_k), a(\varphi_{N+l})\right)\right) \ , \end{aligned}$$

$k, l \in \{1, \ldots, N\}$, are the entries of two matrices M and $\tilde{\mathrm{M}}$, respectively. Let

$$\mathrm{M}^{(k,l)} \doteq \det\left(\left[\mathrm{M}_{i,j}\right]_{i,j\in\{1,\ldots,N\},i\neq k,j\neq l}\right)$$

be the $k, l \in \{1, \ldots, N\}$ minor of M, that is, the determinant of the $(N-1) \times (N-1)$ matrix that results from deleting the kth row and the lth column of M. From the Laplace expansion for determinants (sometimes called cofactor expansion),

$$\det \mathrm{M} = \sum_{k=1}^{N} (-1)^{q_\pi + k} \rho \left(\mathbb{O}_{\pi_{k,N+l}} \left(a^+(\varphi_k), a(\varphi_{N+q_\pi}) \right) \right) \mathrm{M}^{(k,q_\pi)} \tag{56}$$

$$\det \tilde{\mathrm{M}} = \sum_{k=1}^{N} (-1)^{q_\pi + k} \rho \left(a^+(\varphi_k) a(\varphi_{N+q_\pi}) \right) \mathrm{M}^{(k,q_\pi)} . \tag{57}$$

To derive the equality (57) we also use that $\tilde{\pi}_{k,N+l} = \pi_{k,N+l}$ whenever $l \neq q_\pi$, whereas it is the identity of the set $\{1, 2\}$ for $l = q_\pi$. On the other hand, using (52)–(53) and the induction hypothesis for all $\tilde{N}_\pi \geq 0$ with $\tilde{N}_\pi < N_\pi$, we deduce that

$$\mathrm{M}^{(k,q_\pi)} = \rho \Big(\mathbb{O}_{\pi^{(k)}} \big(a^+(\varphi_1), \ldots, a^+(\varphi_{k-1}), a^+(\varphi_{k+1}), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \\ \ldots, a(\varphi_{N+q_\pi+1}), a(\varphi_{N+q_\pi-1}), \ldots, a(\varphi_{N+1}) \big) \Big)$$

and

$$\det \tilde{\mathrm{M}} = \rho \Big(\mathbb{O}_{\tilde{\pi}} \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1}) \right) \Big) .$$

Thus, by induction, it follows from (44), (55), (56) and (57) that

$$\begin{aligned} & \rho \Big(\mathbb{O}_{\pi} \left(a^+(\varphi_1), \ldots, a^+(\varphi_N), a(\varphi_{2N}), \ldots, a(\varphi_{N+1}) \right) \Big) \\ = & \sum_{k=1}^{N} (-1)^{q_\pi + k} \left\{ \rho \left(a^+(\varphi_k) a(\varphi_{N+q_\pi}) \right) - \mathbf{1} \left[\pi(k) > k_\pi \right] \langle \varphi_{N+q_\pi}, \varphi_k \rangle_{\mathcal{H}} \right\} \mathrm{M}^{(k,q_\pi)} \\ = & \sum_{k=1}^{N} (-1)^{q_\pi + k} \rho \Big(\mathbb{O}_{\pi_{k,N+q_\pi}} \left(a^+(\varphi_k), a(\varphi_{N+q_\pi}) \right) \Big) \mathrm{M}^{(k,q_\pi)} = \det \mathrm{M} . \end{aligned}$$

■

3.2 Representation of Discrete–time Covariance by Quasi–Free States

(i): We pick a (possibly unbounded) self–adjoint operator $H = H^*$ acting on $\mathfrak{h}$ and fix from now on $n \in 2\mathbb{N}$. Then, because of (25), (26), (29) and (30), for any fixed $\beta, \nu \in \mathbb{R}^+$ we introduce the unitary operator

$$\mathrm{E} \doteq \mathrm{sgn}\left(\mathbf{1}_{\mathfrak{h}} - n^{-1}\beta H\right) \in \mathcal{B}(\mathfrak{h}) \qquad (58)$$

and the (possibly unbounded) operator $H_\nu \doteq F_\nu(H)$, see (19) and (20). For any $\nu \in \mathbb{R}$, the Hamiltonian H_ν gives rise to the symbol S_ν (21), which, as explained in Section 2.1, in turn yields a quasi–free state ρ_{S_ν}, with symbol $S_\nu > 0$, on the CAR C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$.

Let

$$\mathfrak{D} \doteq \bigcup_{D \in \mathbb{R}^+} \mathrm{ran}\left(\mathbf{1}\left[-D \leq H_\nu \leq D\right]\right) , \qquad \nu \in \mathbb{R} . \qquad (59)$$

By the spectral theorem, it is a dense subspace of entire analytic vectors of H_ν. Note additionally that $\mathfrak{D}$ does not depend on $\nu \in \mathbb{R}$.

(ii): Similar to the permutation (46), for all $\alpha_1, \alpha_2 \in \mathbb{T}_n \cap [0, \beta)$, we define the permutation

$$\pi_{\alpha_1,\alpha_2} : \{1, 2\} \to \{1, 2\}$$

as the identity map if $\alpha_1 \leq \alpha_2$, while π_{α_1,α_2} interchanges 1 and 2 when $\alpha_1 > \alpha_2$.

Then, quasi–free states ρ_{S_ν}, $\nu \in \mathbb{R}^+$, give rise to the following representation of the discrete–time covariance:

Lemma 3.2 (Representation of the covariance by a quasi–free state)
Let $\mathfrak{h}$ be any separable Hilbert space. Fix $\beta \in \mathbb{R}^+$, a self–adjoint operator $H = H^*$ acting on $\mathfrak{h}$, and $n \in 2\mathbb{N}$. Then, for each bounded measurable positive function κ from $\mathbb{R}$ to $\mathbb{R}_0^+$, all $m \in \mathbb{N}$, non–vanishing $\mathfrak{M} \in \mathrm{Mat}(m, \mathbb{R})$ with $\mathfrak{M} \geq 0$, $\alpha_1, \alpha_2 \in \mathbb{T}_n \cap [0, \beta)$, $\varphi_1, \varphi_2 \in \mathfrak{D}$ and $j_1, j_2 \in \{1, \ldots, m\}$,

$$\begin{aligned}\mathfrak{M}_{j_1,j_2} &\left\langle \varphi_2, \left(C_H \kappa(\hat{H})\hat{\varphi}_1\right)(\alpha_1 - \alpha_2)\right\rangle_{\mathfrak{h}} \\ &= \lim_{\nu \to \infty} \rho_{S_\nu}\Big(\mathbb{O}_{\pi_{\alpha_1,\alpha_2}}\Big(a^+\left((\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\kappa(H)^{1/2}\varphi_1) \otimes \mathfrak{e}_{j_1}\right), \\ &\qquad a\left((\mathrm{e}^{(\alpha_2 + n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2 + 1}\kappa(H)^{1/2}\varphi_2) \otimes \mathfrak{e}_{j_2}\right)\Big)\Big)\end{aligned}$$

with $\mathfrak{e}_j \doteq [e_j] \in \mathbb{M}$ being the vectors of $\mathbb{M}$ satisfying (14) and where $\mathbb{O}_{\pi_{\alpha_1,\alpha_2}}$ is defined by (45) for $\pi = \pi_{\alpha_1,\alpha_2}$.

Proof: Fix all the parameters of the lemma. Note that

$$\left\langle \varphi_2, \left(C_H \kappa(\hat{H})\hat{\varphi}_1\right)(\alpha_1 - \alpha_2)\right\rangle_{\mathfrak{h}} = \left\langle \kappa(H)^{1/2}\varphi_2, \left(C_H \kappa(\hat{H})^{1/2}\hat{\varphi}_1\right)(\alpha_1 - \alpha_2)\right\rangle_{\mathfrak{h}} .$$

Therefore, we can assume without loss of generality that $\kappa = \mathbf{1}_{\mathbb{R}}$. We deduce from Equations (25) and (26) that, for any

$$\begin{aligned} &\langle \varphi_2, (C_H \hat{\varphi}_1)(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} \qquad (60) \\ = & \left\langle \psi_2, \left(\int_{\Omega_H}^{\oplus} g_{\lambda_H(\mathfrak{a})} * \hat{\psi}_1(\mathfrak{a}) \ \mu_H(\mathrm{d}\mathfrak{a}) \right) (\alpha_1 - \alpha_2) \right\rangle_{L^2(\Omega_H;\mathbb{C})} \end{aligned}$$

with

$$\psi_{1,2} \doteq U_H \varphi_{1,2} \qquad \text{and} \qquad \hat{\psi}_{1,2} \doteq \hat{U}_H \hat{\varphi}_{1,2} \in \ell^2_{\mathrm{ap}}(\mathbb{T}_n; L^2(\Omega_H)) \ .$$

In the right–hand side of (60) observe that $\hat{\psi}_1$ is seen as an element of $L^2(\Omega_H; \ell^2_{\mathrm{ap}}(\mathbb{T}_n;\mathbb{C}))$, see (24). By (3), observe that

$$[\hat{\psi}_1(\mathfrak{a})](\alpha) = \delta_{\mathrm{ap}}(\alpha) \cdot (U_H \varphi_1)(\mathfrak{a}) \ , \qquad \mathfrak{a} \in \Omega_H, \ \alpha \in \mathbb{T}_n \ ,$$

which, combined with Equations (5) and (60), yields

$$\begin{aligned} &\langle \varphi_2, (C_H \hat{\varphi}_1)(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} \qquad (61) \\ = & \left\langle \psi_2, \left(\int_{\Omega_H}^{\oplus} g_{\lambda_H(\mathfrak{a})} (\alpha_1 - \alpha_2) \, \psi_1(\mathfrak{a}) \, \mu_H(\mathrm{d}\mathfrak{a}) \right) \right\rangle_{L^2(\Omega_H;\mathbb{C})} . \end{aligned}$$

Therefore, by using the explicit expressions (20), (29)–(30) and (58), we deduce from (23) and (61) the equality

$$\langle \varphi_2, (C_H \hat{\varphi}_1)(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} = \lim_{\nu \to \infty} \left\langle \mathrm{E}^{\beta^{-1} n \alpha_2 + 1} \varphi_2, \frac{\mathrm{e}^{(\alpha_2 + n^{-1}\beta - \alpha_1) H_\nu}}{1 + \mathrm{e}^{\beta H_\nu}} \mathrm{E}^{-\beta^{-1} n \alpha_1} \varphi_1 \right\rangle_{\mathfrak{h}} \qquad (62)$$

for any $\alpha_1 \leq \alpha_2$ while, for any $\alpha_1 > \alpha_2$,

$$\begin{aligned} &\langle \varphi_2, (C_H \hat{\varphi}_1)(\alpha_1 - \alpha_2) \rangle_{\mathfrak{h}} \qquad (63) \\ = & -\lim_{\nu \to \infty} \left\langle \mathrm{E}^{\beta^{-1} n \alpha_2 + 1} \varphi_2, \frac{\mathrm{e}^{(\beta - (\alpha_1 - \alpha_2 - n^{-1}\beta)) H_\nu}}{1 + \mathrm{e}^{\beta H_\nu}} \mathrm{E}^{-\beta^{-1} n \alpha_1} \varphi_1 \right\rangle_{\mathfrak{h}} , \end{aligned}$$

using $n \in 2\mathbb{N}$. On the other hand, if $\alpha_1 \leq \alpha_2$ then $\pi_{\alpha_1,\alpha_2} = \mathbf{1}_{\{1,2\}}$ and we infer from Equations (13), (14), (18), (21) and (45) that

$$\begin{aligned} &\rho_{S_\nu} \Big(\mathbb{O}_{\pi_{\alpha_1,\alpha_2}} \Big(a^+ \left((\mathrm{e}^{-\alpha_1 H_\nu} \mathrm{E}^{-\beta^{-1} n \alpha_1} \varphi_1) \otimes \mathfrak{e}_{j_1} \right), \\ &\qquad a \left((\mathrm{e}^{(\alpha_2 + n^{-1}\beta) H_\nu} \mathrm{E}^{\beta^{-1} n \alpha_2 + 1} \varphi_2) \otimes \mathfrak{e}_{j_2} \right) \Big) \Big) \\ = & \ \mathfrak{M}_{j_1,j_2} \left\langle \mathrm{E}^{\beta^{-1} n \alpha_2 + 1} \varphi_2, \frac{\mathrm{e}^{(\alpha_2 + n^{-1}\beta - \alpha_1) H_\nu}}{1 + \mathrm{e}^{\beta H_\nu}} \mathrm{E}^{-\beta^{-1} n \alpha_1} \varphi_1 \right\rangle_{\mathfrak{h}} \qquad (64) \end{aligned}$$

and the assertion holds true when $\alpha_1 \leq \alpha_2$. If $\alpha_1 > \alpha_2$ then

$$\rho_{S_\nu}\Big(\mathbb{O}_{\pi_{\alpha_1,\alpha_2}}\Big(a^+\left((\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1)\otimes \mathfrak{e}_{j_1}\right), a\left((\mathrm{e}^{(\alpha_2+n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2)\otimes \mathfrak{e}_{j_2}\right)\Big)\Big)$$
$$= -\rho_{S_\nu}\Big(a\left((\mathrm{e}^{(\alpha_2+n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2)\otimes \mathfrak{e}_{j_2}\right) a^+\left((\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1)\otimes \mathfrak{e}_{j_1}\right)\Big) ,$$

which, combined with (44), implies that

$$\rho_{S_\nu}\Big(\mathbb{O}_{\pi_{\alpha_1,\alpha_2}}\Big(a^+\left((\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1)\otimes \mathfrak{e}_{j_1}\right), a\left((\mathrm{e}^{(\alpha_2+n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2)\otimes \mathfrak{e}_{j_2}\right)\Big)\Big)$$
$$= \rho_{S_\nu}\Big(a^+\left((\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1)\otimes \mathfrak{e}_{j_1}\right) a\left((\mathrm{e}^{(\alpha_2+n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2)\otimes \mathfrak{e}_{j_2}\right)\Big)$$
$$-\left\langle (\mathrm{e}^{(\alpha_2+n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2)\otimes \mathfrak{e}_{j_2}, (\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1)\otimes \mathfrak{e}_{j_1}\right\rangle_{\mathfrak{h}\otimes\mathbb{M}} .$$

Using again (13), (14) and (21), we thus arrive from the last equality at

$$\rho_{S_\nu}\Big(\mathbb{O}_{\pi_{\alpha_1,\alpha_2}}\Big(a^+\left((\mathrm{e}^{-\alpha_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1)\otimes \mathfrak{e}_{j_1}\right), a\left((\mathrm{e}^{(\alpha_2+n^{-1}\beta)H_\nu}\mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2)\otimes \mathfrak{e}_{j_2}\right)\Big)\Big)$$
$$= \quad -\mathfrak{M}_{j_1,j_2}\left\langle \mathrm{E}^{\beta^{-1}n\alpha_2+1}\varphi_2, \frac{\mathrm{e}^{(\beta-(\alpha_1-\alpha_2-n^{-1}\beta))H_\nu}}{1+\mathrm{e}^{\beta H_\nu}}\mathrm{E}^{-\beta^{-1}n\alpha_1}\varphi_1\right\rangle_{\mathfrak{h}} . \quad (65)$$

By combining (62) and (63) with (64) and (65), we arrive at the assertion with $\kappa = \mathbf{1}_{\mathbb{R}}$. ■

Corollary 3.3 (Determinants of the covariance and quasi–free states)
Let $\mathfrak{h}$ be any separable Hilbert space. Fix $\beta \in \mathbb{R}^+$, a self–adjoint operator $H = H^*$ acting on $\mathfrak{h}$, and $n \in 2\mathbb{N}$. Then, for each bounded measurable positive function κ from $\mathbb{R}$ to $\mathbb{R}_0^+$, all $m, N \in \mathbb{N}$, non–vanishing $\mathfrak{M} \in \mathrm{Mat}\,(m,\mathbb{R})$ with $\mathfrak{M} \geq 0$ and

$$\{(\alpha_q, \varphi_q, j_q)\}_{q=1}^{2N} \subset \mathbb{T}_n \cap [0,\beta) \times \mathfrak{D} \times \{1,\ldots,m\} ,$$

the following identity holds true:

$$\det\left[\mathfrak{M}_{j_k,j_{N+l}}\left\langle\varphi_{N+l},\left(C_H\kappa(\hat{H})\hat{\varphi}_k\right)(\alpha_k-\alpha_{N+l})\right\rangle_{\mathfrak{h}}\right]_{k,l=1}^{N}$$

$$\begin{aligned}= \lim_{\nu\to\infty}\rho_{S_\nu}\Big(\mathbb{O}_\pi\Big(& a^+\left((\mathrm{e}^{-\tilde{\alpha}_1 H_\nu}\mathrm{E}^{-\beta^{-1}n\tilde{\alpha}_1}\kappa(H)^{1/2}\varphi_1)\otimes\mathfrak{e}_{j_1}\right),\ldots, \\ & a^+\left((\mathrm{e}^{-\tilde{\alpha}_N H_\nu}\mathrm{E}^{-\beta^{-1}n\tilde{\alpha}_N}\kappa(H)^{1/2}\varphi_N)\otimes\mathfrak{e}_{j_N}\right), \\ & a\left((\mathrm{e}^{\tilde{\alpha}_{2N} H_\nu}\mathrm{E}^{\beta^{-1}n\tilde{\alpha}_{2N}}\kappa(H)^{1/2}\varphi_{2N})\otimes\mathfrak{e}_{j_{2N}}\right), \\ & \ldots, a\left((\mathrm{e}^{\tilde{\alpha}_{N+1} H_\nu}\mathrm{E}^{\beta^{-1}n\tilde{\alpha}_{N+1}}\kappa(H)^{1/2}\varphi_{N+1})\otimes\mathfrak{e}_{j_{N+1}}\right)\Big)\Big)\end{aligned} \tag{66}$$

for any permutation π of $2N$ elements such that

$$\tilde{\alpha}_{\pi^{-1}(q)}-\tilde{\alpha}_{\pi^{-1}(q-1)}\geq 0\ ,\ \ \alpha_{\pi^{-1}(q)}-\alpha_{\pi^{-1}(q-1)}\geq 0\ ,\quad q\in\{2,\ldots,2N\}\ , \tag{67}$$

where $\tilde{\alpha}_q \doteq \alpha_q$ for $q\in\{1,\ldots,N\}$ and $\tilde{\alpha}_q\doteq\alpha_q+n^{-1}\beta$ for $q\in\{N+1,\ldots,2N\}$.

Proof: Fix all the parameters of the corollary. Take any permutation π of $2N$ elements such that

$$\pi_{\alpha_k,\alpha_{N+l}}=\pi_{k,N+l}\ ,\qquad k,l\in\{1,\ldots,N\}\ . \tag{68}$$

See, respectively, (iv) before Lemma 3.2 and Equation (46) for the definitions of the permutations $\pi_{\alpha_k,\alpha_{N+l}}$ and $\pi_{k,N+l}$ of two elements. Then, (66) follows from Lemmata 3.1 and 3.2. To conclude the proof observe that a permutation π of $2N$ elements satisfying (67) exists and also satisfies (68), keeping in mind Equation (1). ∎

3.3 Correlation Functions and Tomita–Takesaki Modular Theory

(i): As above, fix $\beta\in\mathbb{R}^+$, a self–adjoint operator $H=H^*$ acting on $\mathfrak{h}$, $n\in 2\mathbb{N}$, $\nu\in\mathbb{R}^+$, and a non–vanishing positive real matrix $\mathfrak{M}\in\mathrm{Mat}\,(m,\mathbb{R})$ with $m\in\mathbb{N}$. Let $\tau\equiv\{\tau_t\}_{t\in\mathbb{R}}$ be the unique C_0–group (that is, strongly continuous group) of automorphisms on the C^*–algebra $\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})$ satisfying

$$\tau_t\left(a\left(\varphi\otimes g\right)\right)=a(\mathrm{e}^{itH_\nu\otimes\mathbf{1}_{\mathfrak{h}\otimes\mathbb{M}}}\varphi\otimes g)=a\left((\mathrm{e}^{itH_\nu}\varphi)\otimes g\right)\ ,\qquad \varphi\in\mathfrak{h},\ g\in\mathbb{M}\ . \tag{69}$$

See (20). It is well—known that the quasi–free state ρ_{S_ν}, which is defined from the symbol S_ν (21), is the unique (τ, β)–KMS state on $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$.

(ii): Recall that $(\mathfrak{H}_\nu, \varkappa_\nu, \eta_\nu)$ is a cyclic representation of ρ_{S_ν} (Section 2.3). The weak closure of the C^*–algebra $\mathrm{CAR}(\mathfrak{h}\otimes\mathbb{M})$ is the von Neumann algebra $\mathcal{X}_\nu$ (32). The state $\rho_{S_\nu} \circ \varkappa_\nu$ on $\varkappa_\nu(\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M}))$ extends uniquely to a normal state on the von Neumann algebra $\mathcal{X}_\nu$ and the C_0–group $\{\tau_t \circ \varkappa_\nu\}_{t\in\mathbb{R}}$ also uniquely extends to a σ–weakly continuous $*$–automorphism group on $\mathcal{X}_\nu$. Both extensions are again denoted by ρ_{S_ν} and $\{\tau_t\}_{t\in\mathbb{R}}$, respectively. By [BR2, Corollary 5.3.4], ρ_{S_ν} is again a (τ, β)–KMS state on $\mathcal{X}_\nu$.

(iii): By [BR2, Corollary 5.3.9], the cyclic vector η_ν is separating for $\mathcal{X}_\nu$, i.e., $A\eta_\nu = 0$ implies $A = 0$ for all $A \in \mathcal{X}_\nu$. Denote by Δ_ν the (possibly unbounded) Tomita–Takesaki modular operator of the pair $(\mathcal{X}_\nu, \eta_\nu)$. The ($\beta$–rescaled) modular group is the σ–weakly continuous one-parameter group $\sigma \equiv \{\sigma_t\}_{t\in\mathbb{R}}$ defined by

$$\sigma_t (A) \doteq \Delta_\nu^{-it\beta^{-1}} A \Delta_\nu^{it\beta^{-1}} \qquad A \in \mathcal{X}_\nu \ . \tag{70}$$

(If $\beta = -1$ then σ is the well–known modular automorphism group associated with the pair $(\mathcal{X}_\nu, \eta_\nu)$, see [BR1, Definition 2.5.15].) By Takesaki's theorem [BR2, Theorem 5.3.10], we deduce that $\sigma = \tau$. In particular, using (69) we arrive at the equality

$$\varkappa_\nu \left(a\left((\mathrm{e}^{itH_\nu}\varphi) \otimes g\right)\right) = \Delta_\nu^{-it\beta^{-1}} \varkappa_\nu \left(a\left(\varphi \otimes g\right)\right) \Delta_\nu^{it\beta^{-1}} \ , \qquad \varphi \in \mathfrak{h},\ g \in \mathbb{M} \ . \tag{71}$$

(iv): Recall that $\mathfrak{D} \subseteq \mathfrak{h}$ (59) is a dense subspace of entire analytic vectors for H_ν, while for any $N \in \mathbb{N}$ and $\zeta \in \mathbb{R}^+$, $\mathfrak{T}_N^{(\zeta)}$ is the tube defined by (41). For any $\varphi \in \mathfrak{D}$ and $g \in \mathbb{M}$, the maps

$$z \mapsto a^+ \left((\mathrm{e}^{-zH_\nu}\varphi) \otimes g\right) \qquad \text{and} \qquad z \mapsto a\left((\mathrm{e}^{\bar{z}H_\nu}\varphi) \otimes g\right) \tag{72}$$

from $\mathbb{C}$ to the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$ are entire analytic functions. Fix $N \in \mathbb{N}$, $\varphi_1, \ldots, \varphi_N \in \mathfrak{D}$, $g_1, \ldots, g_N \in \mathbb{M}$, and pick a family

$$\left\{a^\# \left(\varphi_q \otimes g_q\right)\right\}_{q=1}^N \subset \mathrm{CAR}\left(\mathfrak{h} \otimes \mathbb{M}\right) \ , \tag{73}$$

where the notation "$a^\#$" stands for either "a^+" or "a". For any $\varphi \in \mathfrak{D}$, $g \in \mathbb{M}$ and $z \in \mathbb{C}$, we also use the convention

$$a^\# \left((\mathrm{e}^{z^\# H_\nu}\varphi) \otimes g\right) = \begin{cases} a^+ \left((\mathrm{e}^{-zH_\nu}\varphi) \otimes g\right) & \text{when} \quad a^\# = a^+, \\ a\left((\mathrm{e}^{\bar{z}H_\nu}\varphi) \otimes g\right) & \text{when} \quad a^\# = a \ , \end{cases}$$

with

$$(z_1 + z_2)^\# \doteq z_1^\# + z_2^\# \ , \qquad z_1, z_2 \in \mathbb{C} \ .$$

Then, for any fixed integer $p \in \{1, \ldots, N\}$, the map Υ from $\mathbb{C}^{N+1}$ to $\mathbb{C}$ defined by

$$\begin{aligned}
&\Upsilon(z_1, \ldots, z_p, \tilde{z}_p, z_{p+1}, \ldots, z_N) \qquad (74)\\
&\doteq \rho\Big(a^\#\big((\mathrm{e}^{z_1^\# H_\nu}\varphi_1) \otimes g_1\big) \cdots a^\#\Big((\mathrm{e}^{(z_1^\# + \cdots + z_{p-1}^\#)H_\nu}\varphi_{p-1}) \otimes g_{p-1}\Big)\\
&\qquad a^\#\big((\mathrm{e}^{(z_1^\# + \cdots + z_p^\# + \tilde{z}_p^\#)H_\nu}\varphi_p) \otimes g_p\big) a^\#\big((\mathrm{e}^{(z_1^\# + \cdots + z_p^\# + \tilde{z}_p^\# + z_{p+1}^\#)H_\nu}\varphi_{p+1}) \otimes g_{p+1}\big)\\
&\qquad\qquad \cdots a^\#\big((\mathrm{e}^{(z_1^\# + \cdots + z_p^\# + \tilde{z}_p^\# + z_{p+1}^\# + \cdots + z_N^\#)H_\nu}\varphi_N) \otimes g_N\big)\Big)
\end{aligned}$$

is an entire analytic function.

(v): For $\Psi_1, \ldots, \Psi_N \in \mathfrak{h} \otimes \mathbb{M}$, define a family $\{a^\#(\Psi_q)\}_{q=1}^N$ of elements of the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$, where, as before, "$a^\# = a^+$" or "$a^\# = a$". Then, by applying [AM, Lemma A.1], we obtain the following assertions:

(A) The (cyclic and separating) vector η_ν belongs to the domain of definition of the possibly unbounded operator

$$\Delta_\nu^{z_1\beta^{-1}} \varkappa_\nu\left(a^\#(\Psi_1)\right) \cdots \Delta_\nu^{z_N\beta^{-1}} \varkappa_\nu\left(a^\#(\Psi_N)\right)$$

for any $(z_1, \ldots, z_N) \in \mathfrak{T}_N^{(\beta/2)}$ with

$$\left\|\Delta_\nu^{z_1\beta^{-1}} \varkappa_\nu\left(a^\#(\Psi_1)\right) \cdots \Delta_\nu^{z_N\beta^{-1}} \varkappa_\nu\left(a^\#(\Psi_N)\right)\eta_\nu\right\|_{\mathfrak{H}_\nu} \leq \prod_{q=1}^N \|\Psi_q\|_{\mathfrak{h}\otimes\mathbb{M}} \ . \qquad (75)$$

This inequality is a special case of [AM, (A.2)], which is intimately related to Hölder inequalities for non–commutative L^p–spaces.

(B) The map from $\mathfrak{T}_N^{(\beta/2)}$ to $\mathfrak{H}_\nu$ defined by

$$(z_1, \ldots, z_N) \mapsto \Delta_\nu^{z_1\beta^{-1}} \varkappa_\nu\left(a^\#(\Psi_1)\right) \cdots \Delta_\nu^{z_N\beta^{-1}} \varkappa_\nu\left(a^\#(\Psi_N)\right)\eta_\nu$$

is norm continuous on the whole tube $\mathfrak{T}_N^{(\beta/2)}$ and analytic on its interior.

Using the notation

$$x_q \doteq \varkappa_\nu\left(a^\#(\varphi_q \otimes g_q)\right) \ , \qquad q \in \{1, \ldots, N\} \ ,$$

we consider now the map Θ from $\mathfrak{T}_p^{(\beta/2)} \times \mathfrak{T}_{N-p+1}^{(\beta/2)}$ to $\mathbb{R}$ defined by

$$\begin{aligned}\Theta\big((z_1,\ldots,z_p),(\tilde{z}_p,z_{p+1},\ldots,z_N)\big) \\ \doteq \Big\langle \Delta_\nu^{\bar{\tilde{z}}_p\beta^{-1}} x_{p-1}^* \Delta_\nu^{\bar{z}_{p-1}\beta^{-1}} \cdots x_2^* \Delta_\nu^{\bar{z}_2\beta^{-1}} x_1^* \eta_\nu \ , \\ \Delta_\nu^{\tilde{z}_p\beta^{-1}} x_p \Delta_\nu^{z_{p+1}\beta^{-1}} x_{p+1} \cdots \Delta_\nu^{z_N\beta^{-1}} x_N \eta_\nu \Big\rangle_{\mathfrak{H}_\nu} .\end{aligned}$$

Compare for instance with [AM, Lemma A], which explains the properties of Θ. (Notice that [AM] uses a different convention for sesquilinear forms.) By (75), this function is uniformly bounded for all $n \in 2\mathbb{N}$. The same is trivially true for the map Υ (74) on

$$\mathfrak{T}_p^{(\beta/2)} \times \mathfrak{T}_{N-p+1}^{(\beta/2)} \subset \mathbb{C}^{N+1} .$$

Moreover, by using (71) we deduce that Υ and Θ are equal to each other on $i\mathbb{R}^p \times i\mathbb{R}^{N-p+1}$. For each fixed imaginary vector $(z_1,\ldots,z_p) \in i\mathbb{R}^p$, the maps Υ and Θ are both continuous as functions of $(\tilde{z}_p,z_{p+1},\ldots,z_N) \in \mathfrak{T}_{N-p+1}^{(\beta/2)}$ and analytic in the interior of $\mathfrak{T}_{N-p+1}^{(\beta/2)}$, by (B) [AM, Lemma A.1]. Hence, from Hadamard's three line theorem (see, e.g., [RS2, Appendix to IX.4]), Υ and Θ are equal to each other for any fixed imaginary vector $(z_1,\ldots,z_p) \in i\mathbb{R}^p$ and all complex vectors $(\tilde{z}_p,z_{p+1},\ldots,z_N) \in \mathfrak{T}_{N-p+1}^{(\beta/2)}$. Applying this argument again at fixed $(\tilde{z}_p,z_{p+1},\ldots,z_N) \in \mathfrak{T}_{N-p+1}^{(\beta/2)}$ for Υ and Θ viewed as functions of $(z_1,\ldots,z_p) \in \mathfrak{T}_p^{(\beta/2)}$, we conclude that $\Upsilon = \Theta$ on $\mathfrak{T}_p^{(\beta/2)} \times \mathfrak{T}_{N-p+1}^{(\beta/2)}$.

In particular, for $N \in \mathbb{N}$, any family (73) of elements of the C^*–algebra $\mathrm{CAR}(\mathfrak{h} \otimes \mathbb{M})$, and all $\alpha_1,\ldots,\alpha_N \in [0,\beta]$ such that

$$\vartheta_q \doteq \beta^{-1}(\alpha_q - \alpha_{q-1}) \geq 0 , \qquad q \in \{2,\ldots,N\} ,$$

the following equality holds true:

$$\begin{aligned}\rho\Big(a^{\#}\big((\mathrm{e}^{\alpha_1^{\#}H_\nu}\varphi_1)\otimes g_1\big)\ldots a^{\#}\big((\mathrm{e}^{\alpha_N^{\#}H_\nu}\varphi_N)\otimes g_N\big)\Big) \\ = \Big\langle \Delta_\nu^{\frac{1}{2}-\beta^{-1}\alpha_{p-1}} x_{p-1}^* \Delta_\nu^{\vartheta_{p-1}} \cdots x_2^* \Delta_\nu^{\vartheta_2} x_1^* \eta_\nu \ , \\ \Delta_\nu^{\beta^{-1}\alpha_p - \frac{1}{2}} x_p \Delta_\nu^{\vartheta_{p+1}} x_{p+1} \cdots \Delta_\nu^{\vartheta_N} x_N \eta_\nu \Big\rangle_{\mathfrak{H}_\nu}\end{aligned} \tag{76}$$

with p defined to be the smallest element of $\{1,\ldots,N\}$ such that $\alpha_p \geq \beta/2$.

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