KLEIN'S PARADOX AND THE RELATIVISTIC δ-SHELL INTERACTION IN $\mathbb{R}^3$

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ABSTRACT. Under certain hypothesis of smallness of the regular potential $\mathbf{V}$, we prove that the Dirac operator in $\mathbb{R}^3$ coupled with a suitable rescaling of $\mathbf{V}$ converges in the strong resolvent sense to the Hamiltonian coupled with a δ-shell potential supported on Σ, a bounded C^2 surface. Nevertheless, the coupling constant depends non-linearly on the potential $\mathbf{V}$: the Klein's Paradox comes into play.

1. INTRODUCTION

The "Klein's Paradox" is a counter-intuitive relativistic phenomenon related to scattering theory for high-barrier (or equivalently low-well) potentials for the Dirac equation. When an electron is approaching to a barrier, its wave function can be split in two parts: the reflected one and the transmitted one. In a non-relativistic situation, it is well known that the transmitted wave-function decays exponentially depending on the high of the potential, see [24] and the references therein. In the case of the Dirac equation it has been observed, in [13] for the first time, that the transmitted wave-function depends weakly on the power of the barrier, and it becomes almost transparent for very high barriers. This means that outside the barrier the wave-function behaves like an electronic solution and inside the barrier it behaves like a positronic one, violating the principle of the conservation of the charge. This incongruence comes from the fact that, in the Dirac equation, the behavior of electrons and positrons is described by different components of the same spinor wave-function, see [12]. Roughly speaking, this contradiction derives from the fact that even if a very high barrier is reflective for electrons, it is attractive for the positrons.

From a mathematical perspective, the problem appears when approximating the Dirac operator coupled with a δ-shell potential by the corresponding operator using local potentials with shrinking support. The idea of coupling Hamiltonians with singular potentials supported on subsets of lower dimension with respect to the ambient space (commonly called *singular perturbations*) is quite classic in quantum mechanics. One important example is the model of a particle in a one-dimensional lattice that analyses the evolution of an electron on a straight line perturbed by a potential caused by ions in the periodic structure of the crystal that create an electromagnetic field. In 1931, Kronig and Penney [15] idealized this system: in their model the electron is free to move in regions of the whole space separated by some periodical barriers which are zero everywhere except at a single point, where they take infinite value. In a modern language, this corresponds to a δ-point potential. For the Schrödinger operator, this problem is described in the manuscript [1] for finite and infinite δ-point interactions and in [10] for singular

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potentials supported on hypersurfaces. The reader may look at [8, 7, 3, 4, 18] and the references therein for the case of the Dirac operator, and to [19] for a much more general scenario.

Nevertheless, one has to keep in mind that, even if this kind of model is easier to be mathematically understood, since the analysis can be reduced to an algebraic problem, it is and ideal model that cannot be physically reproduced. This is the reason why it is interesting to approximate this kind of operators by more regular ones. For instance, in one dimension, if $V \in C_c^\infty(\mathbb{R})$ then

$$V_\epsilon(t) := \tfrac{1}{\epsilon}\, V\big(\tfrac{t}{\epsilon}\big) \to (\textstyle\int V)\delta_0 \quad \text{when } \epsilon \to 0$$

in the sense of distributions, where δ_0 denotes the Dirac measure at the origin. In [1] it is proved that $\Delta + V_\epsilon \to \Delta + (\int V)\delta_0$ in the norm resolvent sense when $\epsilon \to 0$, and in [5] this result is generalized to higher dimensions for singular perturbations on general smooth hypersurfaces.

These kind of results do not hold for the Dirac operator. In fact, in [22] it is proved that, in the 1-dimensional case, the convergence holds in the norm resolvent sense but the coupling constant does depend non-linearly on the potential V, unlike in the case of Schrödinger operators. This non-linear phenomenon, which may also occur in higher dimensions, is a consequence of the fact that, in a sense, the free Dirac operator is critical with respect to the set where the δ-shell interaction is performed, unlike the Laplacian (the Dirac/Laplace operator is a first/second order differential operator, respectively, and the set where the interaction is performed has codimension 1 with respect to the ambient space). The present paper is devoted to the study of the 3-dimensional case, where we investigate if it is possible obtain the same results as in one dimension. We advance that, for δ-shell interactions on bounded smooth hypersurfaces, we get the same non-linear phenomenon on the coupling constant but we are only able to show convergence in the strong resolvent sense.

Given $m \geq 0$, the free Dirac operator in $\mathbb{R}^3$ is defined by

$$H := -i\alpha \cdot \nabla + m\beta,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{for } j = 1,2,3, \quad \beta = \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \mathbb{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.1}$$

is the family of *Pauli's matrices*. It is well known that H is self-adjoint on the Sobolev space $H^1(\mathbb{R}^3)^4 =: D(H)$, see [23, Theorem 1.1]. Throughout this article we assume that $m > 0$.

In the sequel $\Omega \subset \mathbb{R}^3$ denotes a bounded C^2 domain and $\Sigma := \partial\Omega$ denotes its boundary. By a C^2 domain we mean the following: for each point $Q \in \Sigma$ there exist a ball $B \subset \mathbb{R}^3$ centered at Q, a C^2 function $\psi : \mathbb{R}^2 \to \mathbb{R}$ and a coordinate system $\{(x, x_3) : x \in \mathbb{R}^2,\, x_3 \in \mathbb{R}\}$ so that, with respect to this coordinate system, $Q = (0,0)$ and

$$\begin{aligned} B \cap \Omega &= B \cap \{(x, x_3) : x_3 > \psi(x)\}, \\ B \cap \Sigma &= B \cap \{(x, x_3) : x_3 = \psi(x)\}. \end{aligned}$$

By compactness, one can find a finite covering of Σ made of such coordinate systems, thus the Lipschitz constant of those ψ can be taken uniformly bounded on Σ.

Set $\Omega_\epsilon := \{x \in \mathbb{R}^3 : d(x,\Sigma) < \epsilon\}$ for $\epsilon > 0$. Following [5, Appendix B], there exists $\eta > 0$ small enough depending on Σ so that for every $0 < \epsilon \leq \eta$ one can parametrize Ω_ϵ as

$$\Omega_\epsilon = \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma,\, t \in (-\epsilon,\epsilon)\}, \tag{1.2}$$

where $\nu(x_\Sigma)$ denotes the outward (with respect to Ω) unit normal vector field on Σ evaluated at x_Σ. This parametrization is a bijective correspondence between Ω_ϵ and $\Sigma \times (-\epsilon,\epsilon)$, it can be understood as *tangential* and *normal coordinates*. For $t \in [-\eta,\eta]$, we set

$$\Sigma_t := \{x_\Sigma + t\nu(x_\Sigma) : x_\Sigma \in \Sigma\}. \tag{1.3}$$

In particular, $\Sigma_t = \partial\Omega_t \setminus \Omega$ if $t > 0$, $\Sigma_t = \partial\Omega_{|t|} \cap \Omega$ if $t < 0$ and $\Sigma_0 = \Sigma$. Let σ_t denote the surface measure on Σ_t and, for simplicity of notation, we set $\sigma := \sigma_0$, the surface measure on Σ.

Given $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta,\eta]$ and $0 < \epsilon \leq \eta$ define

$$V_\epsilon(t) := \frac{\eta}{\epsilon} V\Big(\frac{\eta t}{\epsilon}\Big)$$

and, for $x \in \mathbb{R}^3$,

$$\mathbf{V}_\epsilon(x) := \begin{cases} V_\epsilon(t) & \text{if } x \in \Omega_\epsilon, \text{ where } x = x_\Sigma + t\nu(x_\Sigma) \text{ for a unique } (x_\Sigma,t) \in \Sigma \times (-\epsilon,\epsilon), \\ 0 & \text{if } x \notin \Omega_\epsilon. \end{cases} \tag{1.4}$$

Finally, set

$$\begin{gathered} \mathbf{u}_\epsilon := |\mathbf{V}_\epsilon|^{1/2}, \quad \mathbf{v}_\epsilon := \operatorname{sign}(\mathbf{V}_\epsilon)|\mathbf{V}_\epsilon|^{1/2}, \\ u(t) := |\eta V(\eta t)|^{1/2}, \quad v(t) := \operatorname{sign}(V(\eta t))u(t). \end{gathered} \tag{1.5}$$

Note that $\mathbf{u}_\epsilon, \mathbf{v}_\epsilon \in L^\infty(\mathbb{R}^3)$ are supported in $\overline{\Omega_\epsilon}$ and $u, v \in L^\infty(\mathbb{R})$ are supported in $[-1,1]$.

Definition 1.1. Given η, $\delta > 0$, we say that $V \in L^\infty(\mathbb{R})$ is (δ,η)-small if

$$\operatorname{supp} V \subset [-\eta,\eta] \quad \text{and} \quad \|V\|_{L^\infty(\mathbb{R})} \leq \frac{\delta}{\eta}.$$

Observe that if V is (δ,η)-small then $\|V\|_{L^1(\mathbb{R})} \leq 2\delta$, this is the reason why we call it a "small" potential.

In this article we study the asymptotic behavior, in a strong resolvent sense, of the couplings of the free Dirac operator with electrostatic and Lorentz scalar short-range potentials of the form

$$H + \mathbf{V}_\epsilon \quad \text{and} \quad H + \beta\mathbf{V}_\epsilon, \tag{1.6}$$

respectively, where V_ϵ is given by (1.4) for some (δ,η)-small V with δ and η small enough only depending on Σ. By [23, Theorem 4.2], both couplings in (1.6) are self-adjoint operators on $H^1(\mathbb{R}^3)^4$. Given $\eta > 0$ small enough so that (1.2) holds, and given u and v as in (1.5) for some $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta,\eta]$, set

$$\mathcal{K}_V f(t) := \frac{i}{2}\int_{\mathbb{R}} u(t)\operatorname{sign}(t-s)v(s)f(s)\,ds \quad \text{for } f \in L^1_{loc}(\mathbb{R}). \tag{1.7}$$

The main result in this article reads as follows.

Theorem 1.2. *There exist η_0, $\delta > 0$ small enough only depending on Σ such that, for any $0 < \eta \leq \eta_0$ and (δ,η)-small V,*

$$H + \mathbf{V}_\epsilon \to H + \lambda_e\delta_\Sigma \quad \textit{in the strong resolvent sense when } \epsilon \to 0, \tag{1.8}$$

$$H + \beta\mathbf{V}_\epsilon \to H + \lambda_s\beta\,\delta_\Sigma \quad \textit{in the strong resolvent sense when } \epsilon \to 0, \tag{1.9}$$

where

$$\lambda_e := \textstyle\int_{\mathbb{R}} v(t)\,((1-\mathcal{K}_V^2)^{-1}u)(t)\,dt \in \mathbb{R}, \tag{1.10}$$

$$\lambda_s := \textstyle\int_{\mathbb{R}} v(t)\,((1+\mathcal{K}_V^2)^{-1}u)(t)\,dt \in \mathbb{R} \tag{1.11}$$

and $H+\lambda_e\delta_\Sigma$ *and* $H+\lambda_s\beta\,\delta_\Sigma$ *are the electrostatic and Lorentz scalar shell interactions given by* (2.9) *and* (2.11)*, respectively.*

To define λ_e in (1.10) and λ_s in (1.11), the invertibility of $1\pm\mathcal{K}_V^2$ is required. However, since $\mathcal{K}_V$ is a Hilbert-Schmidt operator, we know that $\|\mathcal{K}_V\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}$ is controlled by the norm of its kernel in $L^2(\mathbb{R}\times\mathbb{R})$, which is exactly $\|u\|_{L^2(\mathbb{R})}\|v\|_{L^2(\mathbb{R})} = \|V\|_{L^1(\mathbb{R})} \le 2\delta < 1$, assuming that $\delta < 1/2$ and that V is (δ,η)-small with $\eta\le\eta_0$. We must stress that the way to construct λ_e and λ_s is the same as in the one dimensional case, see [22, Theorem 1].

From Theorem 1.2 we deduce that if $a\in\sigma(H+\lambda_e\delta_\Sigma)$, where $\sigma(\cdot)$ denotes the spectrum, then there exists a sequence $\{a_\epsilon\}$ such that $a_\epsilon\in\sigma(H+\mathbf{V}_\epsilon)$ and $a_\epsilon\to a$ when $\epsilon\to 0$. The kind of instruments we used to prove Theorem 1.2 suggest us that the norm resolvent convergence may not hold in general, thus we cannot ensure that the vice-versa spectral implication also holds. Nevertheless, if Σ is a sphere, one has more information than in the general scenario, see [16]. The Lorentz scalar case is analogous.

The non-linear behavior of the limiting coupling constant with respect to the approximating potentials mentioned in the first paragraphs of the introduction is depicted by (1.10) and (1.11); the reader may compare this to the analogous result [5, Theorem 1.1] in the non-relativistic scenario. However, unlike in [5, Theorem 1.1], in Theorem 1.2 we demand an smallness assumption on the potential, the (δ,η)-smallness from Definition 1.1. We use this assumption in Corollary 3.3 below, where the strong convergence of some inverse operators $(1+B_\epsilon(a))^{-1}$ when $\epsilon\to 0$ is shown. The proof of Theorem 1.2 follows the strategy of [5, Theorem 1.1], but dealing with the Dirac operator instead of the Laplacian makes a big difference at this point. In the non-relativistic scenario, the fundamental solution of $-\Delta+a^2$ in $\mathbb{R}^3$ for $a>0$ has exponential decay at infinity and behaves like $1/|x|$ near the origin, which is locally integrable in $\mathbb{R}^2$ and thus its integral tends to zero as we integrate on shrinking balls in $\mathbb{R}^2$ centered at the origin. This facts are used in [5] to show that their corresponding $(1+B_\epsilon(a))^{-1}$ can be uniformly bounded in ϵ just by taking a big enough. In our situation, the fundamental solution of $H-a$ in $\mathbb{R}^3$ can still be taken with exponential decay at infinity for $a\in\mathbb{C}\setminus\mathbb{R}$, but it is not locally absolutely integrable in $\mathbb{R}^2$. Actually, its most singular part behaves like $x/|x|^3$ near the origin, and thus it yields a singular integral operator in $\mathbb{R}^2$. This means that the contribution near the origin can not be disesteemed as in [5] just by shrinking the domain of integration and taking $a\in\mathbb{C}\setminus\mathbb{R}$ big enough, something else is required. We impose smallness on V to obtain smallness on $B_\epsilon(a)$ and ensure the uniform invertibility of $1+B_\epsilon(a)$ with respect to ϵ; this is the only point where the (δ,η)-smallness is used.

Let η_0, $\delta>0$ be as in Theorem 1.2. Take $0<\eta\le\eta_0$ and $V=\frac{\tau}{2}\chi_{(-\eta,\eta)}$ for some $\tau\in\mathbb{R}$ such that $0<|\tau|\eta\le 2\delta$. Then, arguing as in [22, Remark 1], one gets that

$$\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u = \sum_{n=0}^{\infty}\int_{\mathbb{R}} v\,\mathcal{K}_V^{2n}u = 2\tan\Big(\frac{\tau\eta}{2}\Big).$$

Since V is $(\delta,\eta)-$small, using (1.10) and (1.8) we obtain that

$$H+\mathbf{V}_\epsilon \to H+2\tan(\tfrac{\tau\eta}{2})\delta_\Sigma \quad \text{in the strong resolvent sense when } \epsilon\to 0,$$

analogously to [22, Remark 1]. Similarly, one can check that $\int v\,(1+\mathcal{K}_V^2)^{-1}u = 2\tanh(\frac{\tau\eta}{2})$. Then, (1.11) and (1.9) yield

$$H+\beta\,\mathbf{V}_\epsilon \to H+2\tanh(\tfrac{\tau\eta}{2})\beta\delta_\Sigma \quad \text{in the strong resolvent sense when } \epsilon\to 0.$$

Regarding the structure of the paper, Section 2 is devoted to the preliminaries, which refer to basic rudiments with a geometric measure theory flavor and spectral properties of the short range and shell interactions appearing in Theorem 1.2. In Section 3 we present the first main step to prove Theorem 1.2, a decomposition of the resolvent of the approximating interaction into three concrete operators. This type of decomposition, which is made through a scaling operator, already appears in [5, 22]. Section 3 also contains some auxiliary results concerning these three operators, whose proofs are carried out later on, and the proof of Theorem 1.2, see Section 3.1. Sections 4, 5, 6 and 7 are devoted to prove all those auxiliary results presented in Section 3.

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2. Preliminaries

As usual, in the sequel the letter 'C' (or 'c') stands for some constant which may change its value at different occurrences. We will also make use of constants with subscripts, both to highlight the dependence on some other parameters and to stress that they retain their value from one equation to another. The precise meaning of the subscripts will be clear from the context in each situation.

2.1. Geometric and measure theoretic considerations.

In this section we recall some geometric and measure theoretic properties of Σ and the domains presented in (1.2). At the end, we provide some growth estimates of the measures associated to the layers introduced in (1.3).

The following definition and propositions correspond to Definition 2.2 and Propositions 2.4 and 2.6 in [5], respectively. The reader should look at [5] for the details.

Definition 2.1 (Weingarten map)**.** Let Σ be parametrized by the family $\{\varphi_i, U_i, V_i\}_{i\in I}$, that is, I is a finite set, $U_i\subset\mathbb{R}^2$, $V_i\subset\mathbb{R}^3$, $\Sigma\subset\cup_{i\in I}V_i$ and $\varphi_i(U_i)=V_i\cap\Sigma$ for all $i\in I$. For

$$x=\varphi_i(u)\in\Sigma\cap V_i$$

with $u\in U_i$, $i\in I$, one defines the Weingarten map $W(x): T_x\to T_x$, where T_x denotes the tangent space of Σ on x, as the linear operator acting on the basis vector $\{\partial_j\varphi_i(u)\}_{j=1,2}$ of T_x as

$$W(x)\partial_j\varphi_i(u) := -\partial_j\nu(\varphi_i(u)).$$

Proposition 2.2. *The Weingarten map $W(x)$ is symmetric with respect to the inner product induced by the first fundamental form and its eigenvalues are uniformly bounded for all $x \in \Sigma$.*

Given $0 < \epsilon \leq \eta$ and Ω_ϵ as in (1.2), let $i_\epsilon : \Sigma \times (-\epsilon, \epsilon) \to \Omega_\epsilon$ be the bijection defined by

$$i_\epsilon(x_\Sigma, t) := x_\Sigma + t\nu(x_\Sigma).$$

For future purposes, we also introduce the projection $P_\Sigma : \Omega_\epsilon \to \Sigma$ given by

$$P_\Sigma(x_\Sigma + t\nu(x_\Sigma)) := x_\Sigma. \tag{2.1}$$

For $1 \leq p < +\infty$, let $L^p(\Omega_\epsilon)$ and $L^p(\Sigma \times (-1, 1))$ be the Banach spaces endowed with the norms

$$\|f\|^p_{L^p(\Omega_\epsilon)} := \int_{\Omega_\epsilon} |f|^p \, d\mathcal{L}, \qquad \|f\|^p_{L^p(\Sigma\times(-1,1))} := \int_{-1}^{1} \int_\Sigma |f|^p \, d\sigma \, dt, \tag{2.2}$$

respectively, where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$. The Banach spaces corresponding to the endpoint case $p = +\infty$ are defined, as usual, in terms of essential suprema with respect to the measures associated to Ω_ϵ and $\Sigma \times (-1, 1)$ in (2.2), respectively.

Proposition 2.3. *If $\eta > 0$ is small enough, there exist $0 < c_1, c_2 < +\infty$ such that*

$$c_1\|f\|_{L^1(\Omega_\epsilon)} \leq \|f \circ i_\epsilon\|_{L^1(\Sigma\times(-\epsilon,\epsilon))} \leq c_2\|f\|_{L^1(\Omega_\epsilon)} \quad \text{for all } f \in L^1(\Omega_\epsilon),\ 0 < \epsilon \leq \eta.$$

Moreover, if W denotes the Weingarten map associated to Σ from Definition 2.1*,*

$$\int_{\Omega_\epsilon} f(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_\Sigma f(x_\Sigma + t\nu(x_\Sigma)) \det(1 - tW(x_\Sigma))\, d\sigma(x_\Sigma)\, dt \quad \text{for all } f \in L^1(\Omega_\epsilon). \tag{2.3}$$

The eigenvalues of the Weingarten map $W(x)$ are the principal curvatures of Σ on $x \in \Sigma$, and they are independent of the parametrization of Σ. Therefore, the term $\det(1 - tW(x_\Sigma))$ in (2.3) is also independent of the parametrization of Σ.

Remark 2.4. Let $h : \Omega_\epsilon \to (-\epsilon, \epsilon)$ be defined by $h(x_\Sigma + t\nu(x_\Sigma)) := t$. Then $|\nabla h| = 1$ in Ω_ϵ, so the coarea formula (see [2, Remark 2.94], for example) gives

$$\int_{\Omega_\epsilon} f(x)\, dx = \int_{-\epsilon}^{\epsilon} \int_{\Sigma_t} f(x)\, d\sigma_t(x)\, dt \quad \text{for all } f \in L^1(\Omega_\epsilon).$$

In view of (2.3), one deduces that

$$\int_{\Sigma_t} f \, d\sigma_t = \int_\Sigma f(x_\Sigma + t\nu(x_\Sigma)) \det(1 - tW(x_\Sigma))\, d\sigma(x_\Sigma) \tag{2.4}$$

for all $t \in (-\epsilon, \epsilon)$ and all $f \in L^1(\Sigma_t)$.

In the following lemma we give uniform growth estimates on the measures σ_t, for $t \in [-\eta, \eta]$, that exhibit their 2-dimensional nature. These estimates will be used many times in the sequel, mostly for the case of σ.

Lemma 2.5. *If $\eta > 0$ is small enough, there exist $c_1, c_2 > 0$ such that*

$$\sigma_t(B_r(x)) \leq c_1 r^2 \quad \text{for all } x \in \mathbb{R}^3,\ r > 0,\ t \in [-\eta, \eta], \tag{2.5}$$

$$\sigma_t(B_r(x)) \geq c_2 r^2 \quad \text{for all } x \in \Sigma_t,\ 0 < r < 2\operatorname{diam}(\Omega_\eta),\ t \in [-\eta, \eta], \tag{2.6}$$

being $B_r(x)$ the ball of radius r centered at x.

Proof. We first prove (2.5). Let $r_0 > 0$ be a constant small enough to be fixed later on. If $r \geq r_0$, then

$$\sigma_t(B_r(x)) \leq \max_{t\in[-\eta,\eta]} \sigma_t(\mathbb{R}^3) \leq C = \frac{C}{r_0^2} r_0^2 \leq C_0 r^2,$$

where $C_0 := C/r_0^2 > 0$ only depends on r_0 and η. Therefore, we can assume that $r < r_0$. Let us see that we can also suppose that $x \in \Sigma_t$. In fact, if η and r_0 are small enough and $0 < r < r_0$, given $x \in \mathbb{R}^3$ one can always find $\tilde{x} \in \Sigma_t$ such that $\sigma_t(B_r(x)) \leq 2\sigma_t(B_r(\tilde{x}))$ (if $x \in \Omega_\eta$ just take $\tilde{x} = P_\Sigma x + t\nu(P_\Sigma x)$). Then if (2.5) holds for $\tilde{x}$, one gets $\sigma_t(B_r(x)) \leq 2\sigma_t(B_r(\tilde{x})) \leq Cr^2$, as desired.

Thus, it is enough to prove (2.5) for $x \in \Sigma_t$ and $r < r_0$. If r_0 and η are small enough, covering Σ_t by local chards we can find an open and bounded set $V_{t,r} \subset \mathbb{R}^2$ and a C^1 diffeomorphism $\varphi_t : \mathbb{R}^2 \to \varphi_t(\mathbb{R}^2) \subset \mathbb{R}^3$ such that $\varphi_t(V_{t,r}) = \Sigma_t \cap B_r(x)$. By means of a rotation if necessary, we can further assume that φ_t is of the form $\varphi_t(y') = (y', T_t(y'))$, i.e. φ_t is the graph of a C^1 function $T_t : \mathbb{R}^2 \to \mathbb{R}$, and that $\max_{t\in[-\eta,\eta]} \|\nabla T_t\|_\infty \leq C$ (this follows from the regularity of Σ). Then, if $x' \in V_{t,r}$ is such that $\varphi_t(x') = x$, for any $y' \in V_{t,r}$ we get

$$r^2 \geq |\varphi_t(y') - \varphi_t(x')|^2 \geq |y' - x'|^2,$$

which means that $V_{t,r} \subset \{y' \in \mathbb{R}^2 : |x' - y'| < r\} =: B' \subset \mathbb{R}^2$. Denoting by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure, from [17, Theorem 7.5] we get

$$\sigma_t(B_r(x)) = \mathcal{H}^2(\varphi_t(V_{t,r})) \leq \mathcal{H}^2(\varphi_t(B')) \leq \|\nabla\varphi_t\|_\infty^2 \mathcal{H}^2(B') \leq Cr^2$$

for all $t \in [-\eta, \eta]$, so (2.5) is finally proved.

Let us now deal with (2.6). Given $r_0 > 0$, by the regularity and boundedness of Σ it is clear that $\inf_{t\in[-\eta,\eta],\, x\in\Sigma_t} \sigma_t(B_{r_0}(x)) \geq C > 0$. As before, for any $r_0 \leq r < 2\mathrm{diam}(\Omega_\eta)$ we easily see that

$$\sigma_t(B_r(x)) \geq \sigma_t(B_{r_0}(x)) \geq C = \frac{C}{4\mathrm{diam}(\Omega_\eta)^2} 4\mathrm{diam}(\Omega_\eta)^2 \geq C_1 r^2,$$

where $C_1 := C/4\mathrm{diam}(\Omega_\eta)^2 > 0$ only depends on r_0 and η. Hence (2.6) is proved for all $r_0 \leq r < 2\mathrm{diam}(\Omega_\eta)$.

The case $0 < r < r_0$ is treated, as before, using the local parametrization of Σ_t around x by the graph of a function. Taking η and r_0 small enough, we may assume the existence of $V_{t,r}$ and φ_t as above, so let us set $\varphi_t(x') = x$ for some $x' \in V_{t,r}$. The fact that φ_t is of the form $\varphi_t(y') = (y', T_t(y'))$ and that $\varphi_t(V_{t,r}) = \Sigma_t \cap B_r(x)$ implies that $B'' := \{y' \in \mathbb{R}^2 : |x' - y'| < C_2 r\} \subset V_{t,r}$ for some $C_2 > 0$ small enough only depending on $\max_{t\in[-\eta,\eta]} \|\nabla T_t\|_\infty$, which is finite by assumption. Then, we easily see that

$$\sigma_t(B_r(x)) = \sigma_t(\varphi_t(V_{t,r})) \geq \sigma_t(\varphi_t(B'')) = \int_{B''} \sqrt{1 + |\nabla T_t(y')|^2}\, dy' \geq \int_{B''} dy' = Cr^2,$$

where $C > 0$ only depends on C_2. The lemma is finally proved. $\square$

2.2. **Shell interactions for Dirac operators.**

In this section we briefly recall some useful instruments regarding the δ-shell interactions studied in [3, 4]. The reader should look at [4, Section 2 and Section 5] for the details.

Let $a \in \mathbb{C}$. A fundamental solution of $H - a$ is given by

$$\phi^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\Big(a + m\beta + \Big(1 + \sqrt{m^2 - a^2}|x|\Big)\, i\alpha \cdot \frac{x}{|x|^2}\Big) \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

where $\sqrt{m^2 - a^2}$ is chosen with positive real part whenever $a \in (\mathbb{C} \setminus \mathbb{R}) \cup \big((-m, m) \times \{0\}\big)$. To guarantee the exponential decay of ϕ^a at ∞, from now on we assume that $a \in (\mathbb{C} \setminus \mathbb{R}) \cup \big((-m, m) \times \{0\}\big)$. Given $G \in L^2(\mathbb{R}^3)^4$ and $g \in L^2(\sigma)^4$ we define

$$\Phi^a(G, g)(x) := \int_{\mathbb{R}^3} \phi^a(x - y)\, G(y)\, dy + \int_{\Sigma} \phi^a(x - y) g(y)\, d\sigma(y) \quad \text{for } x \in \mathbb{R}^3 \setminus \Sigma. \tag{2.7}$$

Then, $\Phi^a : L^2(\mathbb{R}^3)^4 \times L^2(\sigma)^4 \to L^2(\mathbb{R}^3)^4$ is linear and bounded and $\Phi^a(G, 0) \in H^1(\mathbb{R}^3)^4$. We also set

$$\Phi^a_\sigma G := \mathrm{tr}_\sigma(\Phi^a(G, 0)) \in L^2(\sigma)^4,$$

being tr_σ the trace operator on Σ. Finally, given $x \in \Sigma$ we define

$$C^a_\sigma g(x) := \lim_{\epsilon \searrow 0} \int_{\Sigma \cap \{|x-y|>\epsilon\}} \phi^a(x - y) g(y)\, d\sigma(y) \quad \text{and} \quad C^a_\pm g(x) := \lim_{\Omega_\pm \ni y \overset{nt}{\to} x} \Phi^a(0, g)(y),$$

where $\Omega_\pm \ni y \overset{nt}{\to} x$ means that y tends to x non-tangentially from the interior/exterior of Ω, respectively, i.e. $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^3 \setminus \overline{\Omega}$. The operators C^a_σ and $C^a_\pm$ are linear and bounded in $L^2(\sigma)^4$. Moreover, the following Plemelj-Sokhotski jump formulae holds:

$$C^a_\pm = \mp \frac{i}{2}(\alpha \cdot \nu) + C^a_\sigma. \tag{2.8}$$

Let $\lambda_e \in \mathbb{R}$. Using Φ^a, we define the electrostatic δ-shell interaction appearing in Theorem 1.2 as follows:

$$\begin{aligned} &D(H + \lambda_e \delta_\Sigma) := \{\Phi^0(G, g) : G \in L^2(\mathbb{R}^3)^4,\ g \in L^2(\sigma)^4,\ \lambda_e \Phi^0_\sigma G = -(1 + \lambda_e C^0_\sigma) g\},\\ &(H + \lambda_e \delta_\Sigma)\varphi := H\varphi + \lambda_e \frac{\varphi_+ + \varphi_-}{2}\,\sigma \quad \text{for } \varphi \in D(H + \lambda_e \delta_\Sigma), \end{aligned} \tag{2.9}$$

where $H\varphi$ in the right hand side of the second statement in (2.9) is understood in the sense of distributions and $\varphi_\pm$ denotes the boundary traces of φ when one approaches to Σ from $\Omega_\pm$. In particular, one has $(H + \lambda_e \delta_\Sigma)\varphi = G \in L^2(\mathbb{R}^3)^4$ for all $\varphi = \Phi^0(G, g) \in D(H + \lambda_e \delta_\Sigma)$. We should mention that one recovers the free Dirac operator in $H^1(\mathbb{R}^3)^4$ when $\lambda_e = 0$.

From [4, Section 3.1] we know that $H + \lambda_e \delta_\Sigma$ is self-adjoint for all $\lambda_e \neq \pm 2$. Besides, if $\lambda_e \neq 0$, given $a \in (-m, m)$ and $\varphi = \Phi^0(G, g) \in D(H + \lambda_e \delta_\Sigma)$,

$$(H + \lambda_e \delta_\Sigma - a)\varphi = 0 \quad \text{if and only if} \quad (\tfrac{1}{\lambda_e} + C^a_\sigma) g = 0. \tag{2.10}$$

This corresponds to the Birman-Swinger principle in the electrostatic δ-shell interaction setting. Since the case $\lambda_e = 0$ corresponds to the free Dirac operator, it can be excluded from this consideration because it is well known that the free Dirac operator doesn't have pure point spectrum. Moreover, the relation (2.10) can be easily extended to the case of $a \in (\mathbb{C} \setminus \mathbb{R}) \cup \big((-m, m) \times \{0\}\big)$ (one still has exponential decay of a fundamental solution of $H - a$).

In the same vein, given $\lambda_s \in \mathbb{R}$, we define the Lorentz scalar δ-shell interaction as follows:

$$\begin{aligned} &D(H + \lambda_s \beta\, \delta_\Sigma) := \{\Phi^0(G, g) : G \in L^2(\mathbb{R}^3)^4,\ g \in L^2(\sigma)^4,\ \lambda_s \Phi^0_\sigma G = -(\beta + \lambda_s C^0_\sigma) g\},\\ &(H + \lambda_s \beta\, \delta_\Sigma)\varphi := H\varphi + \lambda_s \beta\, \frac{\varphi_+ + \varphi_-}{2}\,\sigma \quad \text{for } \varphi \in D(H + \lambda_s \beta\, \delta_\Sigma). \end{aligned} \tag{2.11}$$

From [4, Section 5.1] we know that $H+\lambda_s\beta\,\delta_\Sigma$ is self-adjoint for all $\lambda_s\in\mathbb{R}$. Besides, given $\lambda_s\neq 0$, $a\in(\mathbb{C}\setminus\mathbb{R})\cup\big((-m,m)\times\{0\}\big)$ and $\varphi=\Phi^0(G,g)\in D(H+\lambda_s\beta\,\delta_\Sigma)$, arguing as in (2.10) one gets

$$(H+\lambda_s\beta\,\delta_\Sigma-a)\varphi=0\quad\text{if and only if}\quad(\tfrac{\beta}{\lambda_s}+C^a_\sigma)g=0. \tag{2.12}$$

The following lemma describes the resolvent operator of the δ-shell interactions presented in (2.9) and (2.11).

Lemma 2.6. *Given $\lambda_e,\lambda_s\in\mathbb{R}$ with $\lambda_e\neq\pm2$, $a\in\mathbb{C}\setminus\mathbb{R}$ and $F\in L^2(\mathbb{R}^3)^4$, the following identities hold:*

$$(H+\lambda_e\delta_\Sigma-a)^{-1}F=(H-a)^{-1}F-\lambda_e\Phi^a\big(0,(1+\lambda_eC^a_\sigma)^{-1}\,\Phi^a_\sigma F\big), \tag{2.13}$$

$$(H+\lambda_s\beta\,\delta_\Sigma-a)^{-1}F=(H-a)^{-1}F-\lambda_s\Phi^a\big(0,(\beta+\lambda_sC^a_\sigma)^{-1}\,\Phi^a_\sigma F\big). \tag{2.14}$$

Proof. We will only show (2.13), the proof of (2.14) is analogous. Since $H+\lambda_e\delta_\Sigma$ is self-adjoint for $\lambda_e\neq\pm2$, $(H+\lambda_e\delta_\Sigma-a)^{-1}$ is well-defined and bounded in $L^2(\mathbb{R}^3)^4$. For $\lambda_e=0$ there is nothing to prove, so we assume $\lambda_e\neq0$.

Let $\varphi=\Phi^0(G,g)\in D(H+\lambda_e\delta_\Sigma)$ as in (2.9) and $F=(H+\lambda_e\delta_\Sigma-a)\varphi\in L^2(\mathbb{R}^3)^4$. Then,

$$F=(H+\lambda_e\delta_\Sigma-a)\Phi^0(G,g)=G-a\Phi^0(G,g). \tag{2.15}$$

If we apply H on both sides of (2.15) and we use that $H\Phi^0(G,g)=G+g\sigma$ in the sense of distributions, we get $HF=HG-a(G+g\sigma)$, that is, $(H-a)G=(H-a)F+aF+ag\sigma$. Convolving with ϕ^a the left and right hand sides of this last equation, we obtain $G=F+a\Phi^a(F,0)+a\Phi^a(0,g)$, thus $G-F=a\Phi^a(F,g)$. This, combined with (2.15), yields

$$\Phi^0(G,g)=\Phi^a(F,g). \tag{2.16}$$

Therefore, taking non-tangential boundary values on Σ from inside/outside of Ω in (2.16) we obtain

$$\Phi^0_\sigma G+C^0_\pm g=\Phi^a_\sigma F+C^a_\pm g.$$

Since $\Phi^0(G,g)\in D(H+\lambda_e\delta_\Sigma)$, thanks to (2.9) and (2.8) we conclude that

$$\Phi^a_\sigma F=-\Big(\frac{1}{\lambda_e}+C^a_\sigma\Big)g. \tag{2.17}$$

Since $a\in\mathbb{C}\setminus\mathbb{R}$ and $H+\lambda_e\delta_\Sigma$ is self-adjoint for $\lambda_e\neq\pm2$, by (2.10) we see that $\operatorname{Kernel}(\frac{1}{\lambda_e}+C^a_\sigma)=\{0\}$. Moreover, using the ideas of the proof of [3, Lemma 3.7] and that $\lambda_e\neq\pm2$, one can show that $\frac{1}{\lambda_e}+C^a_\sigma$ has closed range. Finally, since we are taking the square root so that

$$\overline{\sqrt{m^2-a^2}}=\sqrt{m^2-\bar a^2},$$

following [3, Lemma 3.1] we see that $\overline{(\phi^a)^t}(x)=\phi^{\bar a}(-x)$. Here, $(\phi^a)^t$ denotes the transpose matrix of ϕ^a. Thus we conclude that $(\operatorname{Range}(\frac{1}{\lambda_e}+C^a_\sigma))^\perp=\operatorname{Kernel}(\frac{1}{\lambda_e}+C^{\bar a}_\sigma)=\{0\}$, and so $\frac{1}{\lambda_e}+C^a_\sigma$ is invertible. Then, by (2.17), we obtain

$$g=-\Big(\frac{1}{\lambda_e}+C^a_\sigma\Big)^{-1}\Phi^a_\sigma F. \tag{2.18}$$

Thanks to (2.16) and (2.18), we finally get

$$(H+\lambda_e\delta_\Sigma - a)^{-1}F = \varphi = \Phi^0(G,g) = \Phi^a(F,g) = \Phi^a\Big(F, -\Big(\frac{1}{\lambda_e}+C_\sigma^a\Big)^{-1}\Phi_\sigma^a F\Big)$$
$$= \Phi^a(F,0) - \lambda_e\Phi^a\big(0,(1+\lambda_e C_\sigma^a)^{-1}\,\Phi_\sigma^a F\big),$$

and the lemma follows because $\Phi^a(\cdot,0) = (H-a)^{-1}$ as a bounded operator in $L^2(\mathbb{R}^3)^4$. □

2.3. **Coupling the free Dirac operator with short range potentials as in (1.6).**

Given $\mathbf{V}_\epsilon$ as in (1.4), set

$$H_\epsilon^e := H + \mathbf{V}_\epsilon \qquad \text{and} \qquad H_\epsilon^s := H + \beta\mathbf{V}_\epsilon.$$

Recall that these operators are self-adjoint on $H^1(\mathbb{R}^3)^4$. In the following, we give the resolvent formulae for H_ϵ^e and H_ϵ^s.

Throughout this section we make an abuse of notation. Remember that, given $G \in L^2(\mathbb{R}^3)^4$ and $g \in L^2(\sigma)^4$, in (2.7) we already defined $\Phi^a(G,g)$. However, now we make the identification $\Phi^a(\cdot) \equiv \Phi^a(\cdot,0)$, that is, in this section we identify Φ^a with an operator acting on $L^2(\mathbb{R}^3)^4$ by always assuming that the second entrance in Φ^a vanishes. Besides, in this section we use the symbol $\sigma(\cdot)$ to denote the spectrum of an operator, the reader should not confuse it with the symbol σ for the surface measure on Σ.

Proposition 2.7. *Let $\mathbf{u}_\epsilon$ and $\mathbf{v}_\epsilon$ be as in (1.5). Then,*

(i) *$a \in \rho(H_\epsilon^e)$ if and only if $-1 \in \rho(\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon)$, where $\rho(\cdot)$ denotes the resolvent set,*
(ii) *$a \in \sigma_{pp}(H_\epsilon^e)$ if and only if $-1 \in \sigma_{pp}(\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon)$, where $\sigma_{pp}(\cdot)$ denotes the pure point spectrum. Moreover, the multiplicity of a as eigenvalue of H_ϵ^e coincides with the multiplicity of -1 as eigenvalue of $\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon$.*

Furthermore, the following resolvent formula holds:

$$(H_\epsilon^e - a)^{-1} = \Phi^a - \Phi^a\mathbf{v}_\epsilon\,(1+\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon)^{-1}\,\mathbf{u}_\epsilon\Phi^a. \tag{2.19}$$

Proof. To prove (i) and (ii) it is enough to verify that the assumptions of [14, Lemma 1] are satisfied. That is, we just need to show that $a \in \sigma_{pp}(H_\epsilon^e)$ if and only if $-1 \in \sigma_{pp}(\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon)$ and that there exists $a \in \rho(H_\epsilon^e)$ such that $-1 \in \rho(\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon)$.

Assume that $a \in \sigma_{pp}(H_\epsilon^e)$. Then $(H+\mathbf{V}_\epsilon - a)F = 0$ for some $F \in L^2(\mathbb{R}^3)^4$ with $F \not\equiv 0$, so $(H-a)F = -\mathbf{V}_\epsilon F$. Using that $\sigma(H) = \sigma_{ess}(H)$, where $\sigma_{ess}(\cdot)$ denotes the essential spectrum, it is not hard to show that indeed $\mathbf{V}_\epsilon F \not\equiv 0$. Since $\mathbf{V}_\epsilon = \mathbf{v}_\epsilon\mathbf{u}_\epsilon$, by setting $G = \mathbf{u}_\epsilon F \in L^2(\mathbb{R}^3)^4$ we get that $G \not\equiv 0$ and

$$(H-a)F = -\mathbf{v}_\epsilon G. \tag{2.20}$$

From [23, Theorem 4.7] we know that $\sigma_{ess}(H+\mathbf{V}_\epsilon) = \sigma_{ess}(H) = \sigma(H)$. Since $\sigma(H_\epsilon^e)$ is the disjoint union of the pure point spectrum and the essential spectrum, we resume that $\sigma_{pp}(H_\epsilon^e) \subset \rho(H)$, which means that $(H-a)^{-1} = \Phi^a$ is a bounded operator on $L^2(\mathbb{R}^3)^4$. By (2.20), $F = -\Phi^a\mathbf{v}_\epsilon G$. If we multiply both sides of this last equation by $\mathbf{u}_\epsilon$ we obtain $G = \mathbf{u}_\epsilon F = -\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon G$, so $-1 \in \sigma_{pp}(\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon)$ as desired.

On the contrary, assume now that there exists a nontrivial $G \in L^2(\mathbb{R}^3)^4$ such that $\mathbf{u}_\epsilon\Phi^a\mathbf{v}_\epsilon G = -G$. If we take $F = \Phi^a\mathbf{v}_\epsilon G \in L^2(\mathbb{R}^3)$, we easily see that $F \not\equiv 0$ and $\mathbf{V}_\epsilon F = -(H-a)F$, which means that a is an eigenvalue of H_ϵ^e.

To conclude the first part of the proof, it remains to show that there exists $a \in \rho(H_\epsilon^e)$ such that $-1 \in \rho(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$. By [23, Theorem 4.23] we know that $\sigma_{pp}(H_\epsilon^e)$ is a finite sequence contained in $(-m, m)$, so we can chose $a \in (-m, m) \cap \rho(H_\epsilon^e)$. Moreover, by [21, Lemma 2], $\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon$ is a compact operator. Then, by Fredholm's alternative, either $-1 \in \sigma_{pp}(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$ or $-1 \in \rho(\mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$. But we can discard the first option, otherwise $a \in \sigma_{pp}(H_\epsilon^e)$, in contradiction with $a \in \rho(H_\epsilon^e)$.

Let us now prove (2.19). Writing $\mathbf{V}_\epsilon = \mathbf{v}_\epsilon \mathbf{u}_\epsilon$ and using that $(H - a)^{-1} = \Phi^a$, we have

$$\begin{aligned}
&(H_\epsilon^e - a)\big(\Phi^a - \Phi^a \mathbf{v}_\epsilon (1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)^{-1} \mathbf{u}_\epsilon \Phi^a\big) \\
&\quad = 1 - \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a + \mathbf{v}_\epsilon \mathbf{u}_\epsilon \Phi^a - \mathbf{v}_\epsilon (-1 + 1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon) \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a \\
&\quad = 1 - \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a + \mathbf{v}_\epsilon \mathbf{u}_\epsilon \Phi^a + \mathbf{v}_\epsilon \left(1 + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a - \mathbf{v}_\epsilon \mathbf{u}_\epsilon \Phi^a = 1,
\end{aligned}$$

as desired. This completes the proof of the proposition. □

The following result can be proved in the same way, we leave the details for the reader.

Proposition 2.8. *Let $\mathbf{u}_\epsilon$ and $\mathbf{v}_\epsilon$ be as in (1.5). Then,*

(i) $a \in \rho(H_\epsilon^s)$ if and only if $-1 \in \rho(\beta \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$,
(ii) $a \in \sigma_{pp}(H_\epsilon^s)$ if and only if $-1 \in \sigma_{pp}(\beta \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon)$. Moreover, the multiplicity of a as eigenvalue of H_ϵ^s coincides with the multiplicity of -1 as eigenvalue of $\beta \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon$.

Furthermore, the following resolvent formula holds:

$$(H_\epsilon^s - a)^{-1} = \Phi^a - \Phi^a \mathbf{v}_\epsilon \left(\beta + \mathbf{u}_\epsilon \Phi^a \mathbf{v}_\epsilon\right)^{-1} \mathbf{u}_\epsilon \Phi^a. \tag{2.21}$$

3. The main decomposition and the proof of Theorem 1.2

Following the ideas in [22, 5], the first key step to prove Theorem 1.2 is to decompose $(H_\epsilon^e - a)^{-1}$ and $(H_\epsilon^s - a)^{-1}$, using a scaling operator, in terms of the operators $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ introduced below (see Lemma 3.1).

Let $\eta_0 > 0$ be some constant small enough to be fixed later on. In particular, we take η_0 so that (1.2) holds for all $0 < \epsilon \leq \eta_0$. Given $0 < \epsilon \leq \eta_0$, define

$$\begin{aligned}
&\mathcal{I}_\epsilon : L^2(\Sigma \times (-\epsilon, \epsilon))^4 \to L^2(\Omega_\epsilon)^4 \quad \text{by} \quad (\mathcal{I}_\epsilon f)(x_\Sigma + t\nu(x_\Sigma)) := f(x_\Sigma, t), \\
&\mathcal{S}_\epsilon : L^2(\Sigma \times (-1, 1))^4 \to L^2(\Sigma \times (-\epsilon, \epsilon))^4 \quad \text{by} \quad (\mathcal{S}_\epsilon g)(x_\Sigma, t) := \frac{1}{\sqrt{\epsilon}}\, g\Big(x_\Sigma, \frac{t}{\epsilon}\Big).
\end{aligned}$$

Thanks to the regularity of Σ, $\mathcal{I}_\epsilon$ is well-defined, bounded and invertible for all $0 < \epsilon \leq \eta_0$ if η_0 is small enough. Note also that $\mathcal{S}_\epsilon$ is a unitary and invertible operator.

Let $0 < \eta \leq \eta_0$, $V \in L^\infty(\mathbb{R})$ with $\operatorname{supp} V \subset [-\eta, \eta]$ and $u, v \in L^\infty(\mathbb{R})$ be the functions with support in $[-1, 1]$ introduced in (1.5), that is,

$$u(t) := |\eta V(\eta t)|^{1/2} \quad \text{and} \quad v(t) := \operatorname{sign}(V(\eta t)) u(t). \tag{3.1}$$

Using the notation related to (2.3), for $0 < \epsilon \leq \eta_0$ we consider the integral operators

$$\begin{aligned}
&A_\epsilon(a) : L^2(\Sigma \times (-1, 1))^4 \to L^2(\mathbb{R}^3)^4, \\
&B_\epsilon(a) : L^2(\Sigma \times (-1, 1))^4 \to L^2(\Sigma \times (-1, 1))^4, \\
&C_\epsilon(a) : L^2(\mathbb{R}^3)^4 \to L^2(\Sigma \times (-1, 1))^4
\end{aligned} \tag{3.2}$$

defined by

$$
\begin{aligned}
&(A_\epsilon(a)g)(x) := \int_{-1}^{1}\int_\Sigma \phi^a(x-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
&(B_\epsilon(a)g)(x_\Sigma,t) := u(t)\int_{-1}^{1}\int_\Sigma \phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\times\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
&(C_\epsilon(a)g)(x_\Sigma,t) := u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y)g(y)\,dy.
\end{aligned}
\tag{3.3}
$$

Recall that, given $F\in L^2(\mathbb{R}^3)^4$ and $f\in L^2(\sigma)^4$, in (2.7) we defined $\Phi^a(F,f)$. However, in Section 2.3 we made the identification $\Phi^a(\cdot)\equiv\Phi^a(\cdot,0)$, which enabled us to write $(H-a)^{-1}=\Phi^a$. Here, and in the sequel, we recover the initial definition for Φ^a given in (2.7) and we assume that $a\in\mathbb{C}\setminus\mathbb{R}$; now we must write $(H-a)^{-1}=\Phi^a(\cdot,0)$, which is a bounded operator in $L^2(\mathbb{R}^3)^4$.

Proceeding as in the proof of [5, Lemma 3.2], one can show the following result.

Lemma 3.1. *The following operator identities hold for all $0<\epsilon\le\eta$:*

$$
\begin{aligned}
A_\epsilon(a) &= \Phi^a(\cdot,0)\mathbf{v}_\epsilon\mathcal{I}_\epsilon\mathcal{S}_\epsilon,\\
B_\epsilon(a) &= \mathcal{S}_\epsilon^{-1}\mathcal{I}_\epsilon^{-1}\mathbf{u}_\epsilon\,\Phi^a(\cdot,0)\mathbf{v}_\epsilon\mathcal{I}_\epsilon\mathcal{S}_\epsilon,\\
C_\epsilon(a) &= \mathcal{S}_\epsilon^{-1}\mathcal{I}_\epsilon^{-1}\mathbf{u}_\epsilon\,\Phi^a(\cdot,0).
\end{aligned}
\tag{3.4}
$$

Moreover, the following resolvent formulae hold:

$$(H_\epsilon^e-a)^{-1}=(H-a)^{-1}+A_\epsilon(a)\big(1+B_\epsilon(a)\big)^{-1}C_\epsilon(a),\tag{3.5}$$

$$(H_\epsilon^s-a)^{-1}=(H-a)^{-1}+A_\epsilon(a)\big(\beta+B_\epsilon(a)\big)^{-1}C_\epsilon(a).\tag{3.6}$$

In (3.4), $A_\epsilon(a)=\Phi^a(\cdot,0)\mathbf{v}_\epsilon\mathcal{I}_\epsilon\mathcal{S}_\epsilon$ means that $A_\epsilon(a)g=\Phi^a(\mathbf{v}_\epsilon\mathcal{I}_\epsilon\mathcal{S}_\epsilon g,0)$ for all $g\in L^2(\Sigma\times(-1,1))^4$, and similarly for $B_\epsilon(a)$ and $C_\epsilon(a)$. Since both $\mathcal{I}_\epsilon$ and $\mathcal{S}_\epsilon$ are an isometry, $V\in L^\infty(\mathbb{R})$ is supported in $[-\eta,\eta]$ and $\Phi^a(\cdot,0)$ is bounded by assumption, from (3.4) we deduce that $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ are well-defined and bounded, so (3.2) is fully justified. Once (3.4) is proved, the resolvent formulae (3.5) and (3.6) follow from (2.19) and (2.21), respectively. We stress that, in (2.19) and (2.21), there is the abuse of notation in the definition of Φ^a commented before.

Lemma 3.1 connects $(H_\epsilon^e-a)^{-1}$ and $(H_\epsilon^s-a)^{-1}$ to $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$. When $\epsilon\to0$, the limit of the former ones is also connected to the limit of the latter ones. We now introduce those limit operators for $A_\epsilon(a)$, $B_\epsilon(a)$ and $C_\epsilon(a)$ when $\epsilon\to0$. Let

$$
\begin{aligned}
&A_0(a): L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4,\\
&B_0(a): L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4,\\
&B': L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4,\\
&C_0(a): L^2(\mathbb{R}^3)^4\to L^2(\Sigma\times(-1,1))^4
\end{aligned}
\tag{3.7}
$$

be the operators given by

$$
\begin{aligned}
&(A_0(a)g)(x) := \int_{-1}^{1}\int_{\Sigma} \phi^a(x-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
&(B_0(a)g)(x_\Sigma,t) := \lim_{\epsilon\to 0} u(t)\int_{-1}^{1}\int_{|x_\Sigma-y_\Sigma|>\epsilon} \phi^a(x_\Sigma-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\
&(B'g)(x_\Sigma,t) := (\alpha\cdot\nu(x_\Sigma))\,\frac{i}{2}\,u(t)\int_{-1}^{1}\operatorname{sign}(t-s)v(s)g(x_\Sigma,s)\,ds,\\
&(C_0(a)g)(x_\Sigma,t) := u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma-y)g(y)\,dy.
\end{aligned}
\tag{3.8}
$$

The next theorem corresponds to the core of this article. Its proof is quite technical and is carried out in Sections 4, 5 and 6. We also postpone the proof of (3.7) to those sections, where each operator is studied in detail. Anyway, the boundedness of B' is trivial.

Theorem 3.2. *The following convergences of operators hold in the strong sense:*

$$A_\epsilon(a)\to A_0(a) \quad \textit{when } \epsilon\to 0, \tag{3.9}$$

$$B_\epsilon(a)\to B_0(a)+B' \quad \textit{when } \epsilon\to 0, \tag{3.10}$$

$$C_\epsilon(a)\to C_0(a) \quad \textit{when } \epsilon\to 0. \tag{3.11}$$

The proof of the following corollary is also postponed to Section 7. It combines Theorem 3.2, (3.5) and (3.6), but it requires some fine estimates developed in Sections 4, 5 and 6.

Corollary 3.3. *There exist $\eta_0,\delta>0$ small enough only depending on Σ such that, for any $a\in\mathbb{C}\setminus\mathbb{R}$ with $|a|\le 1$, $0<\eta\le\eta_0$ and (δ,η)-small V (see* Definition 1.1*), the following convergences of operators hold in the strong sense:*

$$
\begin{aligned}
&(H+\mathbf{V}_\epsilon-a)^{-1}\to (H-a)^{-1}+A_0(a)\big(1+B_0(a)+B'\big)^{-1}C_0(a) \quad \textit{when } \epsilon\to 0,\\
&(H+\beta\mathbf{V}_\epsilon-a)^{-1}\to (H-a)^{-1}+A_0(a)\big(\beta+B_0(a)+B'\big)^{-1}C_0(a) \quad \textit{when } \epsilon\to 0.
\end{aligned}
$$

In particular, $\big(1+B_0(a)+B'\big)^{-1}$ and $\big(\beta+B_0(a)+B'\big)^{-1}$ are well-defined bounded operators in $L^2(\Sigma\times(-1,1))^4$.

3.1. **Proof of Theorem 1.2.**

Thanks to [20, Theorem VIII.19], to prove the theorem it is enough to show that, for some $a\in\mathbb{C}\setminus\mathbb{R}$, the following convergences of operators hold in the strong sense:

$$(H+\mathbf{V}_\epsilon-a)^{-1}\to (H+\lambda_e\delta_\Sigma-a)^{-1} \quad \text{when } \epsilon\to 0, \tag{3.12}$$

$$(H+\beta\mathbf{V}_\epsilon-a)^{-1}\to (H+\lambda_s\beta\delta_\Sigma-a)^{-1} \quad \text{when } \epsilon\to 0. \tag{3.13}$$

Thus, from now on, we fix $a\in\mathbb{C}\setminus\mathbb{R}$ with $|a|\le 1$.

We introduce the operators

$$\widehat{V}: L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma)^4 \quad \text{and} \quad \widehat{U}: L^2(\Sigma)^4\to L^2(\Sigma\times(-1,1))^4$$

given by

$$\widehat{V}f(x_\Sigma) := \int_{-1}^{1} v(s)\,f(x_\Sigma,s)\,ds \quad \text{and} \quad \widehat{U}f(x_\Sigma,t) := u(t)\,f(x_\Sigma).$$

Observe that, by Fubini's theorem,

$$A_0(a) = \Phi^a(0,\cdot)\widehat{V}, \qquad B_0(a) = \widehat{U}C^a_\sigma\widehat{V}, \qquad C_0(a) = \widehat{U}\Phi^a_\sigma. \tag{3.14}$$

Hence, from Corollary 3.3 and (3.14) we deduce that, in the strong sense,

$$(H + \mathbf{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + \Phi^a(0,\cdot)\widehat{V}\big(1 + \widehat{U}C^a_\sigma\widehat{V} + B'\big)^{-1}\widehat{U}\Phi^a_\sigma \quad \text{when } \epsilon \to 0, \tag{3.15}$$

$$(H + \beta\mathbf{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + \Phi^a(0,\cdot)\widehat{V}\big(\beta + \widehat{U}C^a_\sigma\widehat{V} + B'\big)^{-1}\widehat{U}\Phi^a_\sigma \quad \text{when } \epsilon \to 0. \tag{3.16}$$

For convenience of notation, set

$$\widetilde{\mathcal{K}}g(x_\Sigma, t) := \mathcal{K}_V(g(x_\Sigma,\cdot))(t) \quad \text{for } g \in L^2(\Sigma \times (-1,1)),$$

where $\mathcal{K}_V$ is as in (1.7). Then, we get

$$1 + B' = \mathbb{I}_4 + (\alpha\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_4 = \begin{pmatrix} \mathbb{I}_2 & (\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 \\ (\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix}.$$

Here, $\sigma := (\sigma_1, \sigma_2, \sigma_3)$ (see (1.1)), $\mathbb{I}_4$ denotes the 4×4 identity matrix and $\widetilde{\mathcal{K}}\mathbb{I}_4$ denotes the diagonal 4×4 operator matrix whose nontrivial entries are $\widetilde{\mathcal{K}}$, and analogously for $\widetilde{\mathcal{K}}\mathbb{I}_2$. Since the operators that compose the matrix $1 + B'$ commute, if we set $\mathcal{K} := \widetilde{\mathcal{K}}\mathbb{I}_4$, we get

$$\begin{aligned}(1 + B')^{-1} &= (1 - \widetilde{\mathcal{K}}^2)^{-1} \otimes \begin{pmatrix} \mathbb{I}_2 & -(\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 \\ -(\sigma\cdot\nu)\widetilde{\mathcal{K}}\mathbb{I}_2 & \mathbb{I}_2 \end{pmatrix} \\ &= (1 - \mathcal{K}^2)^{-1} - (\alpha\cdot\nu)(1 - \mathcal{K}^2)^{-1}\mathcal{K}.\end{aligned} \tag{3.17}$$

With this at hand, we can compute

$$\begin{aligned}(1 + \widehat{U}C^a_\sigma\widehat{V} + B')^{-1} &= \Big(1 + (1 + B')^{-1}\widehat{U}C^a_\sigma\widehat{V}\Big)^{-1}(1 + B')^{-1} \\ &= \Big(1 + (1 - \mathcal{K}^2)^{-1}\widehat{U}C^a_\sigma\widehat{V} - (\alpha\cdot\nu)(1 - \mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C^a_\sigma\widehat{V}\Big)^{-1} \\ &\qquad \circ \Big((1 - \mathcal{K}^2)^{-1} - (\alpha\cdot\nu)(1 - \mathcal{K}^2)^{-1}\mathcal{K}\Big).\end{aligned} \tag{3.18}$$

Notice that

$$\begin{aligned}&\widehat{V}\Big(1 + (1 - \mathcal{K}^2)^{-1}\widehat{U}C^a_\sigma\widehat{V} - (\alpha\cdot\nu)(1 - \mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C^a_\sigma\widehat{V}\Big) \\ &\qquad = \Big(1 + \widehat{V}(1 - \mathcal{K}^2)^{-1}\widehat{U}C^a_\sigma - (\alpha\cdot\nu)\widehat{V}(1 - \mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C^a_\sigma\Big)\widehat{V},\end{aligned}$$

which obviously yields

$$\begin{aligned}&\widehat{V}\Big(1 + (1 - \mathcal{K}^2)^{-1}\widehat{U}C^a_\sigma\widehat{V} - (\alpha\cdot\nu)(1 - \mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C^a_\sigma\widehat{V}\Big)^{-1} \\ &\qquad = \Big(1 + \widehat{V}(1 - \mathcal{K}^2)^{-1}\widehat{U}C^a_\sigma - (\alpha\cdot\nu)\widehat{V}(1 - \mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C^a_\sigma\Big)^{-1}\widehat{V}.\end{aligned} \tag{3.19}$$

Besides, by the definition of $\mathcal{K}_V$ in (1.7), we see that

$$\begin{aligned}\widehat{V}(1 - \mathcal{K}^2)^{-1}\widehat{U} &= \Big(\int_{\mathbb{R}} v\,(1 - \mathcal{K}_V^2)^{-1}u\Big)\mathbb{I}_4 = \lambda_e\mathbb{I}_4, \\ \widehat{V}(1 - \mathcal{K}^2)^{-1}\mathcal{K}\widehat{U} &= \Big(\int_{\mathbb{R}} v\,(1 - \mathcal{K}_V^2)^{-1}\mathcal{K}_V u\Big)\mathbb{I}_4 = 0.\end{aligned} \tag{3.20}$$

Indeed, from (1.10) in Theorem 1.2, $\lambda_e = \int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}u$. Let us focus on $\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u$. Note that, for any $n \geq 0$,

$$\int_{\mathbb{R}} v\mathcal{K}_V^{2n+1}u = \Big(-\frac{i}{2}\Big)^{2n+1}\int_{(-\eta,\eta)^{2n+2}} V(t_0)V(t_1)\cdots V(t_{2n+1})\times \\ \operatorname{sign}(t_0-t_1)\cdots\operatorname{sign}(t_{2n}-t_{2n+1})\,dt_0dt_1\dots dt_{2n+1}.$$

Set $s_j := t_{2n+1-j}$ for $j \in \{0,\dots,2n+1\}$. Then,

$$\operatorname{sign}(t_0-t_1)\cdots\operatorname{sign}(t_{2n}-t_{2n+1}) = (-1)^{2n+1}\operatorname{sign}(s_0-s_1)\cdots\operatorname{sign}(s_{2n}-s_{2n+1}),$$

thus, by Fubini's theorem, $\int_{\mathbb{R}} v\mathcal{K}_V^{2n+1}u = 0$. This implies that $\int_{\mathbb{R}} v\,(1-\mathcal{K}_V^2)^{-1}\mathcal{K}_V u = 0$ by a Neumann series argument, and therefore $\widehat{V}(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U} = 0$.

Hence, combining (3.19) and (3.20) we have that

$$\widehat{V}\Big(1+(1-\mathcal{K}^2)^{-1}\widehat{U}C_\sigma^a\widehat{V}-(\alpha\cdot\nu)(1-\mathcal{K}^2)^{-1}\mathcal{K}\widehat{U}C_\sigma^a\widehat{V}\Big)^{-1} = (1+\lambda_e C_\sigma^a)^{-1}\widehat{V}. \tag{3.21}$$

Then, from (3.18), (3.21) and (3.20), we finally get

$$\Phi^a(0,\cdot)\widehat{V}(1+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1}\widehat{U}\Phi_\sigma^a = \Phi^a(0,\cdot)(1+\lambda_e C_\sigma^a)^{-1}\lambda_e\Phi_\sigma^a.$$

This last identity combined with (3.15) and (2.13) yields (3.12).

The proof of (3.13) follows the same lines. Similarly to (3.17),

$$(\beta+B')^{-1} = (1+\mathcal{K}^2)^{-1}\beta-(\alpha\cdot\nu)(1+\mathcal{K}^2)^{-1}.$$

One can then make the computations analogous to (3.18), (3.19), (3.20) and (3.21). Since $\lambda_s = \int_{\mathbb{R}} v\,(1+\mathcal{K}_V^2)^{-1}u$, we now get

$$\Phi^a(0,\cdot)\widehat{V}(\beta+\widehat{U}C_\sigma^a\widehat{V}+B')^{-1}\widehat{U}\Phi_\sigma^a = \Phi^a(0,\cdot)(\beta+\lambda_s C_\sigma^a)^{-1}\lambda_s\Phi_\sigma^a.$$

From this, (3.16) and (2.14) we obtain (3.13). This finishes the proof of Theorem 1.2, except for the boundedness stated in (3.7), the proof of Corollary 3.3 in Section 7, and Theorem 3.2, whose proof is fragmented as follows: (3.9) in Section 6, (3.10) in Section 5 and (3.11) in Section 4.

4. Proof of (3.11): $C_\epsilon(a) \to C_0(a)$ in the strong sense when $\epsilon \to 0$

Recall from (3.3) and (3.8) that $C_\epsilon(a)$ with $0 < \epsilon \leq \eta_0$ and $C_0(a)$ are defined by

$$(C_\epsilon(a)g)(x_\Sigma,t) = u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma+\epsilon t\nu(x_\Sigma)-y)g(y)\,dy,$$
$$(C_0(a)g)(x_\Sigma,t) = u(t)\int_{\mathbb{R}^3}\phi^a(x_\Sigma-y)g(y)\,dy.$$

Let us first show that $C_\epsilon(a)$ is bounded from $L^2(\mathbb{R}^3)^4$ to $L^2(\Sigma\times(-1,1))^4$ with a norm uniformly bounded on $0 \leq \epsilon \leq \eta_0$. For this purpose, we write

$$(C_\epsilon(a)g)(x_\Sigma,t) = u(t)(\phi^a * g)(x_\Sigma+\epsilon t\nu(x_\Sigma)), \tag{4.1}$$

where $\phi^a * g$ denotes the convolution of the matrix-valued function ϕ^a with the vector-valued function $g \in L^2(\mathbb{R}^3)^4$. Since we are assuming that $a \in \mathbb{C}\setminus\mathbb{R}$ and, in the definition of ϕ^a, we are taking $\sqrt{m^2-a^2}$ with positive real part, the same arguments as the ones in the proof of [3, Lemma 2.8] (essentially Plancherel's theorem) show that

$$\|\phi^a * g\|_{H^1(\mathbb{R}^3)^4} \leq C\|g\|_{L^2(\mathbb{R}^3)^4} \quad \text{for all } g \in L^2(\mathbb{R}^3)^4,$$

where $C>0$ only depends on a. Besides, thanks to the C^2 regularity of Σ, if η_0 is small enough it is not hard to show that the Sobolev trace inequality from $H^1(\mathbb{R}^3)^4$ to $L^2(\Sigma_{\epsilon t})^4$ holds for all $0\leq\epsilon\leq\eta_0$ and $t\in[-1,1]$ with a constant only depending on η_0 (and Σ, of course). Combining these two facts, we obtain that

$$\|\phi^a * g\|_{L^2(\Sigma_{\epsilon t})^4}\leq C\|g\|_{L^2(\mathbb{R}^3)^4}\quad\text{for all } g\in L^2(\mathbb{R}^3)^4,\ 0\leq\epsilon\leq\eta_0 \text{ and } t\in[-1,1]. \tag{4.2}$$

By Proposition 2.2, if η_0 is small enough there exists $C>0$ such that

$$C^{-1}\leq \det(1-\epsilon tW(P_\Sigma x))\leq C\quad\text{for all } 0<\epsilon\leq\eta_0,\ t\in(-1,1) \text{ and } x\in\Sigma_{\epsilon t}. \tag{4.3}$$

Therefore, an application of (4.1), (2.4), (4.3) and (4.2) finally yields

$$\begin{aligned}\|C_\epsilon(a)g\|^2_{L^2(\Sigma\times(-1,1))^4}&=\int_{-1}^1\int_\Sigma\big|u(t)(\phi^a*g)(x_\Sigma+\epsilon t\nu(x_\Sigma))\big|^2d\sigma(x_\Sigma)\,dt\\&\leq\|u\|^2_{L^\infty(\mathbb{R})}\int_{-1}^1\int_{\Sigma_{\epsilon t}}\big|\det(1-\epsilon tW(P_\Sigma x))^{-1/2}(\phi^a*g)(x)\big|^2d\sigma_{\epsilon t}(x)\,dt\\&\leq C\|u\|^2_{L^\infty(\mathbb{R})}\int_{-1}^1\|\phi^a*g\|^2_{L^2(\Sigma_{\epsilon t})^4}\,dt\leq C\|u\|^2_{L^\infty(\mathbb{R})}\|g\|^2_{L^2(\mathbb{R}^3)^4}.\end{aligned}$$

That is, if η_0 is small enough there exists $C_1>0$ only depending on η_0 and a such that

$$\|C_\epsilon(a)\|_{L^2(\mathbb{R}^3)^4\to L^2(\Sigma\times(-1,1))^4}\leq C_1\|u\|_{L^\infty(\mathbb{R})}\quad\text{for all } 0\leq\epsilon\leq\eta_0. \tag{4.4}$$

In particular, the boundedness stated in (3.7) holds for $C_0(a)$.

In order to prove the strong convergence of $C_\epsilon(a)$ to $C_0(a)$ when $\epsilon\to0$, fix $g\in L^2(\mathbb{R}^3)^4$. We must show that, given $\delta>0$, there exists $\epsilon_0>0$ such that

$$\|C_\epsilon(a)g-C_0(a)g\|_{L^2(\Sigma\times(-1,1))^4}\leq\delta\quad\text{for all } 0\leq\epsilon\leq\epsilon_0. \tag{4.5}$$

For every $0<d\leq\eta_0$, using (4.4) we can estimate

$$\begin{aligned}\|C_\epsilon(a)g-&C_0(a)g\|_{L^2(\Sigma\times(-1,1))^4}\\&\leq\|C_\epsilon(a)(\chi_{\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}+\|C_0(a)(\chi_{\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}\\&\quad+\|(C_\epsilon(a)-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}\\&\leq 2C_1\|u\|_{L^\infty(\mathbb{R})}\|\chi_{\Omega_d}g\|_{L^2(\mathbb{R}^3)^4}+\|(C_\epsilon(a)-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d}g)\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{4.6}$$

On one hand, since $g\in L^2(\mathbb{R}^3)^4$ and $\mathcal{L}(\Sigma)=0$ ($\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$), we can take $d>0$ small enough so that

$$\|\chi_{\Omega_d}g\|_{L^2(\mathbb{R}^3)^4}\leq\frac{\delta}{4C_1\|u\|_{L^\infty(\mathbb{R})}}. \tag{4.7}$$

On the other hand, note that

$$|(x_\Sigma+\epsilon t\nu(x_\Sigma))-x_\Sigma|=\epsilon|t||\nu(x_\Sigma)|\leq\epsilon\leq\frac{d}{2}=\frac12\operatorname{dist}(\Sigma,\mathbb{R}^3\setminus\Omega_d)\leq\frac12|x_\Sigma-y| \tag{4.8}$$

for all $0\leq\epsilon\leq\frac{d}{2}$, $t\in(-1,1)$, $x_\Sigma\in\Sigma$ and $y\in\mathbb{R}^3\setminus\Omega_d$.

As we said before, we are assuming that $a\in\mathbb{C}\setminus\mathbb{R}$ and, in the definition of ϕ^a, we are taking $\sqrt{m^2-a^2}$ with positive real part, so the components of $\phi^a(x)$ decay exponentially as $|x|\to\infty$. In particular, there exist $C,r>0$ only depending on a such that

$$\begin{aligned}|\partial\phi^a(x)|&\leq Ce^{-r|x|}\quad\text{for all } |x|\geq1,\\|\partial\phi^a(x)|&\leq C|x|^{-3}\quad\text{for all } 0<|x|<1,\end{aligned} \tag{4.9}$$

where by the left hand side in (4.9) we mean the absolute value of any derivative of any component of the matrix $\phi^a(x)$. Therefore, using the mean value theorem, (4.9) and (4.8), we see that there exists $C_{a,d} > 0$ only depending on a and d such that

$$|\phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y) - \phi^a(x_\Sigma - y)| \leq C_{a,d}\,\frac{\epsilon}{|x_\Sigma - y|^3}$$

for all $0 \leq \epsilon \leq \frac{d}{2}$, $t \in (-1,1)$, $x_\Sigma \in \Sigma$ and $y \in \mathbb{R}^3 \setminus \Omega_d$. Hence, we can easily estimate

$$\begin{aligned}
|(C_\epsilon(a)&-C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d} g)(x_\Sigma, t)| \\
&\leq \|u\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^3\setminus\Omega_d} |\phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y) - \phi^a(x_\Sigma - y)||g(y)|\,dy \\
&\leq C_{a,d}\|u\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}^3\setminus\Omega_d} \frac{\epsilon|g(y)|}{|x_\Sigma - y|^3}\,dy \\
&\leq C_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\Big(\int_{\mathbb{R}^3\setminus B_d(x_\Sigma)} \frac{dy}{|x_\Sigma - y|^6}\Big)^{1/2}\|g\|_{L^2(\mathbb{R}^3)^4} \leq C'_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4},
\end{aligned}$$

where $C'_{a,d} > 0$ only depends on a and d. Then,

$$\|(C_\epsilon(a) - C_0(a))(\chi_{\mathbb{R}^3\setminus\Omega_d} g)\|_{L^2(\Sigma\times(-1,1))^4} \leq C'_{a,d}\,\epsilon\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4} \tag{4.10}$$

for a possibly bigger constant $C'_{a,d} > 0$.

With these ingredients, the proof of (4.5) is straightforward. Given $\delta > 0$, take $d > 0$ small enough so that (4.7) holds. For this fixed d, take

$$\epsilon_0 = \min\Big\{\frac{\delta}{2C'_{a,d}\|u\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\mathbb{R}^3)^4}}, \frac{d}{2}\Big\}.$$

Then, (4.5) follows from (4.6), (4.7) and (4.10). In conclusion, we have shown that

$$\lim_{\epsilon\to 0}\|(C_\epsilon(a) - C_0(a))g\|_{L^2(\Sigma\times(-1,1))^4} = 0 \quad \text{for all } g \in L^2(\mathbb{R}^3)^4, \tag{4.11}$$

which is (3.11).

5. Proof of (3.10): $B_\epsilon(a) \to B_0(a) + B'$ in the strong sense when $\epsilon \to 0$

Recall from (3.3) and (3.8) that $B_\epsilon(a)$ with $0 < \epsilon \leq \eta_0$, $B_0(a)$ and B' are defined by

$$\begin{aligned}
(B_\epsilon(a)g)(x_\Sigma, t) &= u(t)\int_{-1}^1\int_\Sigma \phi^a(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))v(s) \\
&\qquad\qquad\qquad\times \det(1 - \epsilon sW(y_\Sigma))g(y_\Sigma, s)\,d\sigma(y_\Sigma)\,ds, \\
(B_0(a)g)(x_\Sigma, t) &= \lim_{\epsilon\to 0} u(t)\int_{-1}^1\int_{|x_\Sigma - y_\Sigma|>\epsilon} \phi^a(x_\Sigma - y_\Sigma)v(s)g(y_\Sigma, s)\,ds\,d\sigma(y_\Sigma), \\
(B'g)(x_\Sigma, t) &= (\alpha\cdot\nu(x_\Sigma))\,\frac{i}{2}\,u(t)\int_{-1}^1 \operatorname{sign}(t - s)v(s)g(x_\Sigma, s)\,ds.
\end{aligned}$$

We already know that $B_\epsilon(a)$ and B' are bounded in $L^2(\Sigma\times(-1,1))^4$. Let us postpone to Section 5.2 the proof of the boundedness of $B_0(a)$ stated in (3.7). The first step to prove (3.10) is to decompose ϕ^a as in [4, Lemma 3.2], that is,

$$\begin{aligned}\phi^a(x) &= \frac{e^{-\sqrt{m^2-a^2}|x|}}{4\pi|x|}\Big(a + m\beta + \sqrt{m^2-a^2}\, i\alpha\cdot\frac{x}{|x|}\Big)\\ &\quad + \frac{e^{-\sqrt{m^2-a^2}|x|}-1}{4\pi}\, i\alpha\cdot\frac{x}{|x|^3} + \frac{i}{4\pi}\,\alpha\cdot\frac{x}{|x|^3} =: \omega_1^a(x)+\omega_2^a(x)+\omega_3(x).\end{aligned} \tag{5.1}$$

Then we can write

$$\begin{aligned}B_\epsilon(a) &= B_{\epsilon,\omega_1^a} + B_{\epsilon,\omega_2^a} + B_{\epsilon,\omega_3},\\ B_0(a) &= B_{0,\omega_1^a} + B_{0,\omega_2^a} + B_{0,\omega_3},\end{aligned} \tag{5.2}$$

where B_{ϵ,ω_1^a}, B_{ϵ,ω_2^a} and B_{ϵ,ω_3} are defined as $B_\epsilon(a)$ but replacing ϕ^a by ω_1^a, ω_2^a and ω_3, respectively, and analogously for the case of $B_0(a)$.

For $j=1,2$, we see that $|\omega_j^a(x)| = O(|x|^{-1})$ and $|\partial\omega_j^a(x)| = O(|x|^{-2})|$ for $|x|\to 0$, with the understanding that $|\omega_j^a(x)|$ means the absolute value of any component of the matrix $\omega_j^a(x)$ and $|\partial\omega_j^a(x)|$ means the absolute value of any first order derivative of any component of $\omega_j^a(x)$. Therefore, the integrals defining B_{ϵ,ω_j^a} and B_{0,ω_j^a} are of fractional type for $j=1,2$ (recall Lemma 2.5) and they are taken over bounded sets, so the strong convergence follows by standard methods. However, one can also follow the arguments in the proof of [5, Lemma 3.4] to show, for $j=1,2$, the convergence of B_{ϵ,ω_j^a} to B_{0,ω_j^a} in the norm sense when $\epsilon\to 0$, that is,

$$\lim_{\epsilon\to 0}\|B_{\epsilon,\omega_j^a} - B_{0,\omega_j^a}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} = 0 \quad \text{for } j=1,2. \tag{5.3}$$

A comment is in order. Since the integrals involved in (5.3) are taken over $\Sigma\times(-1,1)$, which is bounded, the exponential decay at infinity from [5, Proposition A.1] is not necessary in the setting of (3.10), hence the local estimate of $|\omega_j^a(x)|$ and $|\partial\omega_j^a(x)|$ near the origin is enough to adapt the proof of [5, Lemma 3.4] to get (5.3).

Thanks to (5.2) and (5.3), to prove (3.10) we only need to show that $B_{\epsilon,\omega_3}\to B_{0,\omega_3}+B'$ in the strong sense when $\epsilon\to 0$. This will be done in two main steps. First, we will show that

$$\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma,t) = B_{0,\omega_3} g(x_\Sigma,t) + B' g(x_\Sigma,t) \quad \text{for allmost all } (x_\Sigma,t)\in\Sigma\times(-1,1) \tag{5.4}$$

and all $g\in L^\infty(\Sigma\times(-1,1))^4$ such that $\sup_{|t|<1}|g(x_\Sigma,t)-g(y_\Sigma,t)|\le C|x_\Sigma-y_\Sigma|$ for all x_Σ, $y_\Sigma\in\Sigma$ and some $C>0$ which may depend on g. This is done in Section 5.1. Then, for a general $g\in L^2(\Sigma\times(-1,1))^4$, we will estimate $|B_{\epsilon,\omega_3}g(x_\Sigma,t)|$ in terms of some bounded maximal operators that will allow us to prove the pointwise limit (5.4) for almost every $(x_\Sigma,t)\in\Sigma\times(-1,1)$ and the desired strong convergence of B_{ϵ,ω_3} to $B_{0,\omega_3}+B'$, see Section 5.2.

5.1. The pointwise limit of $B_{\epsilon,\omega_3}g(x_\Sigma,t)$ when $\epsilon\to 0$ for g in a dense subspace of $L^2(\Sigma\times(-1,1))^4$.

Observe that the function u in front of the definitions of B_{ϵ,ω_3}, B_{0,ω_3} and B' does not affect to the validity of the limit in (5.4), so we can assume without loss of generality that $u\equiv 1$ in $(-1,1)$.

We are going to prove (5.4) by showing the pointwise limit component by component, that is, we are going to work in $L^\infty(\Sigma\times(-1,1))$ instead of $L^\infty(\Sigma\times(-1,1))^4$. In order to do so, we need to introduce some definitions. Set

$$k(x) := \frac{x}{4\pi|x|^3} \quad \text{for } x\in\mathbb{R}^3\setminus\{0\}. \tag{5.5}$$

Given $t \in (-1,1)$ and $0 < \epsilon \leq \eta_0$ with η_0 small enough and $f \in L^\infty(\Sigma \times (-1,1))$ such that $\sup_{|t|<1} |f(x_\Sigma,t) - f(y_\Sigma,t)| \leq C|x_\Sigma - y_\Sigma|$ for all x_Σ, $y_\Sigma \in \Sigma$ and some $C > 0$, we define

$$T_t^\epsilon f(x_\Sigma) := \int_{-1}^1 \int_\Sigma k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma)) f(y_\Sigma, s) \det(1 - \epsilon s W(y_\Sigma))\, d\sigma(y_\Sigma)\, ds.$$

By (2.4),

$$T_t^\epsilon f(x_\Sigma) = \int_{-1}^1 \int_{\Sigma_{\epsilon s}} k(x_{\epsilon t} - y_{\epsilon s}) f(P_\Sigma y_{\epsilon s}, s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds, \tag{5.6}$$

where $x_{\epsilon t} := x_\Sigma + \epsilon t\nu(x_\Sigma)$, $y_{\epsilon s} := y_\Sigma + \epsilon s\nu(y_\Sigma)$ and P_Σ is given by (2.1). We also set

$$T_t f(x_\Sigma) := \lim_{\delta \to 0} \int_{-1}^1 \int_{|x_\Sigma - y_\Sigma| > \delta} k(x_\Sigma - y_\Sigma) f(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds + \frac{\nu(x_\Sigma)}{2} \int_{-1}^1 \operatorname{sign}(t-s) f(x_\Sigma, s)\, ds.$$

We are going to prove that

$$\lim_{\epsilon \to 0} T_t^\epsilon f(x_\Sigma) = T_t f(x_\Sigma) \tag{5.7}$$

for almost all $(x_\Sigma, t) \in \Sigma \times (-1,1)$. Once this is proved, it is not hard to get (5.4). Indeed, note that $k = (k_1, k_2, k_3)$ with $k_j(x) := \frac{x_j}{4\pi|x|^3}$ being the scalar components of the vector kernel $k(x)$. Thus, we can write

$$T_t^\epsilon f(x_\Sigma) = \big((T_t^\epsilon f(x_\Sigma))_1, (T_t^\epsilon f(x_\Sigma))_2, (T_t^\epsilon f(x_\Sigma))_3\big),$$

where each $(T_t^\epsilon f(x_\Sigma))_j$ is defined as in (5.6) but replacing k by k_j. Then, (5.7) holds if and only if $(T_t^\epsilon f(x_\Sigma))_j \to (T_t f(x_\Sigma))_j$ when $\epsilon \to 0$ for $j = 1, 2, 3$. From this limits, if we let $f(y_\Sigma, s)$ in the definitions of $T_t^\epsilon f$ and $T_t f$ be the different components of $v(s)g(y_\Sigma, s)$, we easily deduce (5.4). Thus, we are reduced to prove (5.7).

The proof of (5.7) follows the strategy of the proof of [11, Proposition 3.30]. Set

$$E(x) := -\frac{1}{4\pi|x|} \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\},$$

the fundamental solution of the Laplace operator in $\mathbb{R}^3$. Note that $\nabla E = k = (k_1, k_2, k_3)$. In particular, if we set $\nu = (\nu_1, \nu_2, \nu_3)$ and $x = (x_1, x_2, x_3)$, for $x \in \mathbb{R}^3$ and $y \in \Sigma$ with $x \neq y$ we can decompose

$$\begin{aligned}
k_j(x-y) &= \partial_{x_j} E(x-y) = |\nu(y)|^2\, \partial_{x_j} E(x-y)\\
&= \sum_n \nu_n(y)^2 \partial_{x_j} E(x-y) + \sum_n \nu_j(y)\nu_n(y)\partial_{x_n} E(x-y) - \sum_n \nu_j(y)\nu_n(y)\partial_{x_n} E(x-y)\\
&= \nu_j(y) \sum_n \partial_{x_n} E(x-y)\nu_n(y) + \sum_n \Big(\nu_n(y)\partial_{x_j} E(x-y) - \nu_j(y)\partial_{x_n} E(x-y)\Big)\nu_n(y)\\
&= \nu_j(y)\nabla_{\nu(y)} E(x-y) + \sum_n \nabla_{\nu(y)}^{j,n} E(x-y)\nu_n(y),
\end{aligned} \tag{5.8}$$

where we have taken

$$\begin{aligned}
\nabla_{\nu(y)} E(x-y) &:= \sum_n \nu_n(y)\partial_{x_n} E(x-y) = \nabla_x E(x-y) \cdot \nu(y),\\
\nabla_{\nu(y)}^{j,n} E(x-y) &:= \nu_n(y)\partial_{x_j} E(x-y) - \nu_j(y)\partial_{x_n} E(x-y).
\end{aligned} \tag{5.9}$$

For $j,\, n \in \{1,2,3\}$ we define

$$
\begin{aligned}
T^\epsilon_\nu f(x_\Sigma,t) &:= \int_{-1}^1 \int_{\Sigma_{\epsilon s}} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t}-y_{\epsilon s}) f(P_\Sigma y_{\epsilon s},s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds,\\
T^\epsilon_{j,n} f(x_\Sigma,t) &:= \int_{-1}^1 \int_{\Sigma_{\epsilon s}} \nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t}-y_{\epsilon s}) f(P_\Sigma y_{\epsilon s},s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds,
\end{aligned} \tag{5.10}
$$

being $\nu_{\epsilon s}(y_{\epsilon s}) := \nu(y_\Sigma)$ a normal vector field to $\Sigma_{\epsilon s}$. Besides, the terms $\nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t}-y_{\epsilon s})$ and $\nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t}-y_{\epsilon s})$ in (5.10) are defined as in (5.9) with the obvious replacements.

Given $f\in L^\infty(\Sigma\times(-1,1))$ such that $\sup_{|t|<1} |f(x_\Sigma,t)-f(y_\Sigma,t)| \le C|x_\Sigma-y_\Sigma|$ for all $x_\Sigma,\, y_\Sigma \in \Sigma$ and some $C>0$, by (5.8) we see that

$$
(T^\epsilon_t f(x_\Sigma))_j = T^\epsilon_\nu h_j(x_\Sigma,t) + \sum_n T^\epsilon_{j,n} h_n(x_\Sigma,t), \tag{5.11}
$$

where $h_n(P_\Sigma y_{\epsilon s},s) := (\nu_{\epsilon s}(y_{\epsilon s}))_n\, f(P_\Sigma y_{\epsilon s},s)$ for $n=1,2,3$. We are going to prove that

$$
\begin{aligned}
\lim_{\epsilon\to 0} T^\epsilon_\nu h_j(x_\Sigma,t) &= \lim_{\delta\to 0}\int_{-1}^1 \int_{|x_\Sigma-y_\Sigma|>\delta} \nabla_{\nu(y_\Sigma)} E(x_\Sigma-y_\Sigma) h_j(y_\Sigma,s)\, d\sigma(y_\Sigma)\, ds\\
&\quad + \frac12 \int_{-1}^1 \operatorname{sign}(t-s) h_j(x_\Sigma,s)\, ds,
\end{aligned} \tag{5.12}
$$

$$
\lim_{\epsilon\to 0} T^\epsilon_{j,n} h_n(x_\Sigma,t) = \lim_{\delta\to 0}\int_{-1}^1 \int_{|x_\Sigma-y_\Sigma|>\delta} \nabla^{j,n}_{\nu(y_\Sigma)} E(x_\Sigma-y_\Sigma) h_n(y_\Sigma,s)\, d\sigma(y_\Sigma)\, ds \tag{5.13}
$$

for $n=1,2,3$. Then, combining (5.11), (5.12) and (5.13), we obtain (5.7). Therefore, it is enough to show (5.12) and (5.13).

We first deal with (5.12). Remember that $\nabla E = k$ so, given $\delta>0$, from (5.9) and (5.10) we can split

$$
\begin{aligned}
T^\epsilon_\nu h_j(x_\Sigma,t) = &\int_{-1}^1 \int_{|x_{\epsilon s}-y_{\epsilon s}|>\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s})\, h_j(P_\Sigma y_{\epsilon s},s)\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds\\
&+ \int_{-1}^1 \int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s}) \Big(h_j(P_\Sigma y_{\epsilon s},s) - h_j(P_\Sigma x_{\epsilon s},s)\Big) d\sigma_{\epsilon s}(y_{\epsilon s})\, ds\\
&+ \int_{-1}^1 h_j(P_\Sigma x_{\epsilon s},s) \int_{|x_{\epsilon s}-y_{\epsilon s}|\le\delta} k(x_{\epsilon t}-y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s})\, ds\\
=: &\,\mathscr{A}_{\epsilon,\delta} + \mathscr{B}_{\epsilon,\delta} + \mathscr{C}_{\epsilon,\delta},
\end{aligned}
$$

and we easily see that

$$
\lim_{\epsilon\to 0} T^\epsilon_\nu h_j(x_\Sigma,t) = \lim_{\delta\to 0}\lim_{\epsilon\to 0} \big(\mathscr{A}_{\epsilon,\delta} + \mathscr{B}_{\epsilon,\delta} + \mathscr{C}_{\epsilon,\delta}\big). \tag{5.14}
$$

We study the three terms on the right hand side of (5.14) separately.

For the case of $\mathscr{A}_{\epsilon,\delta}$, note that $k\in C^\infty(\mathbb{R}^3\setminus B_\delta(0))^3$ and it has polynomial decay at ∞, so

$$
|k(x)| + |\partial k(x)| \le C < +\infty \quad \text{for all } x\in \mathbb{R}^3\setminus B_\delta(0),
$$

where $C>0$ only depends on δ, and ∂k denotes any first order derivative of any component of k. Moreover, h_j is bounded on $\Sigma\times(-1,1)$ and Σ is bounded and of class C^2. Therefore, fixed

$\delta > 0$, the uniform boundedness of the integrand combined with the regularity of k and Σ and the dominated convergence theorem yields

$$\lim_{\epsilon\to 0} \mathscr{A}_{\epsilon,\delta} = \int_{-1}^{1} \int_{|x_\Sigma - y_\Sigma|>\delta} k(x_\Sigma - y_\Sigma)\cdot \nu(y_\Sigma)\, h_j(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds. \tag{5.15}$$

Then, if we let $\delta \to 0$, from (5.15) we get the first term on the right hand side of (5.12).

Recall that the function h_j appearing in $\mathscr{B}_{\epsilon,\delta}$ is constructed from the one in (5.4) using v (see below (5.7)) and $\nu_{\epsilon s}$ (see below (5.11)). Hence $h_j \in L^\infty(\Sigma\times(-1,1))$ and $\sup_{|t|<1} |h_j(x_\Sigma, t) - h_j(y_\Sigma, t)| \leq C|x_\Sigma - y_\Sigma|$ for all $x_\Sigma, y_\Sigma \in \Sigma$ and some $C > 0$. Thus, if η_0 and δ are small enough, by the mean value theorem there exists $C > 0$ such that

$$\big|k(x_{\epsilon t} - y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s})(h_j(P_\Sigma y_{\epsilon s}, s) - h_j(P_\Sigma x_{\epsilon s}, s))\big| \leq C\frac{|P_\Sigma y_{\epsilon s} - P_\Sigma x_{\epsilon s}|}{|x_{\epsilon t} - y_{\epsilon s}|^2} \leq \frac{C}{|y_{\epsilon s} - x_{\epsilon s}|} \tag{5.16}$$

for all $0 \leq \epsilon \leq \eta_0$ and $|x_{\epsilon s} - y_{\epsilon s}| \leq \delta$. In the last inequality in (5.16) we used that P_Σ is Lipschitz on Ω_{η_0} and that $|x_{\epsilon s} - y_{\epsilon s}| \leq C|x_{\epsilon t} - y_{\epsilon s}|$ if $|x_{\epsilon s} - y_{\epsilon s}| \leq \delta$ and δ is small enough (due to the regularity of Σ). From the local integrability of the right hand side of (5.16) with respect to $\sigma_{\epsilon s}$ (see Lemma 2.5) and standard arguments, we easily deduce the existence of $C_\delta > 0$ such that $\sup_{0\leq\epsilon\leq\eta_0} |\mathscr{B}_{\epsilon,\delta}| \leq C_\delta$ and $C_\delta \to 0$ when $\delta \to 0$, see [5, equation (A.7)] for a similar argument. Then, we can resume

$$\Big|\lim_{\delta\to 0}\lim_{\epsilon\to 0} \mathscr{B}_{\epsilon,\delta}\Big| \leq \lim_{\delta\to 0} \sup_{0\leq\epsilon\leq\eta_0} |\mathscr{B}_{\epsilon,\delta}| \leq \lim_{\delta\to 0} C_\delta = 0. \tag{5.17}$$

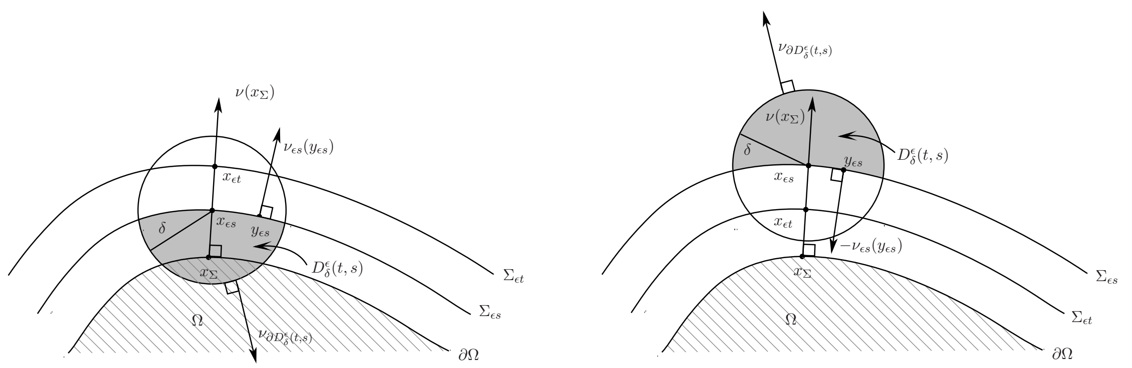
Let us finally focus on $\mathscr{C}_{\epsilon,\delta}$. Since $k = \nabla E$, from (5.9) we get

$$\int_{|x_{\epsilon s} - y_{\epsilon s}|\leq\delta} k(x_{\epsilon t} - y_{\epsilon s})\cdot \nu_{\epsilon s}(y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s}) = \int_{|x_{\epsilon s} - y_{\epsilon s}|\leq\delta} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t} - y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s}).$$

Consider the set

$$D_\delta^\epsilon(t,s) := \begin{cases} B_\delta(x_{\epsilon s}) \setminus \overline{\Omega(\epsilon, s)} & \text{if } t \leq s, \\ B_\delta(x_{\epsilon s}) \cap \Omega(\epsilon, s) & \text{if } t > s, \end{cases}$$

where $\Omega(\epsilon, s)$ denotes the bounded connected component of $\mathbb{R}^3 \setminus \Sigma_{\epsilon s}$ that contains Ω if $s \geq 0$ and that is included in Ω if $s < 0$.



$D_\delta^\epsilon(t,s)$ in the case $t > s > 0$, $D_\delta^\epsilon(t,s)$ in the case $s > t > 0$.

FIG. 1. The set $D_\delta^\epsilon(t,s)$.

Set $E_x(y) := E(x-y)$ for $x,\, y \in \mathbb{R}^3$ with $x \neq y$. Then $\Delta E_{x_{\epsilon t}} = 0$ in $D^\epsilon_\delta(t,s)$ and $\nabla E_{x_{\epsilon t}}(y) = -\nabla E(x_{\epsilon t} - y)$. If $\nu_{\partial D^\epsilon_\delta(t,s)}$ denotes the normal vector field on $\partial D^\epsilon_\delta(t,s)$ pointing outside $D^\epsilon_\delta(t,s)$, by the divergence theorem,

$$\begin{aligned}0 &= \int_{D^\epsilon_\delta(t,s)} \Delta E_{x_{\epsilon t}}(y)\, dy = -\int_{\partial D^\epsilon_\delta(t,s)} \nabla E(x_{\epsilon t} - y) \cdot \nu_{\partial D^\epsilon_\delta(t,s)}(y)\, d\mathcal{H}^2(y)\\ &= -\operatorname{sign}(t-s) \int_{|x_{\epsilon s} - y_{\epsilon s}| \leq \delta} \nabla_{\nu_{\epsilon s}(y_{\epsilon s})} E(x_{\epsilon t} - y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s})\\ &\quad - \int_{\{y \in \mathbb{R}^3 :\, |x_{\epsilon s} - y| = \delta\} \cap A^\epsilon_{t,s}} \nabla E(x_{\epsilon t} - y) \cdot \frac{y - x_{\epsilon s}}{|y - x_{\epsilon s}|}\, d\mathcal{H}^2(y),\end{aligned} \tag{5.18}$$

where

$$A^\epsilon_{t,s} := \mathbb{R}^3 \setminus \overline{\Omega(\epsilon,s)} \text{ if } t \leq s \qquad \text{and} \qquad A^\epsilon_{t,s} := \Omega(\epsilon,s) \text{ if } t > s.$$

Remember also that $\mathcal{H}^2$ denotes the 2-dimensional Hausdorff measure. Since $\nabla E = k$, from (5.18) and (5.9) we deduce that

$$\begin{aligned}&\int_{|x_{\epsilon s} - y_{\epsilon s}| \leq \delta} k(x_{\epsilon t} - y_{\epsilon s}) \cdot \nu_{\epsilon s}(y_{\epsilon s})\, d\sigma_{\epsilon s}(y_{\epsilon s})\\ &\qquad\qquad = \operatorname{sign}(t-s) \int_{\partial B_\delta(x_{\epsilon s}) \cap A^\epsilon_{t,s}} k(x_{\epsilon t} - y) \cdot \frac{x_{\epsilon s} - y}{|x_{\epsilon s} - y|}\, d\mathcal{H}^2(y).\end{aligned} \tag{5.19}$$

Note that $x_{\epsilon t} \notin D^\epsilon_\delta(t,s)$ by construction, see Figure 1. Moreover, by the regularity of Σ, given $\delta > 0$ small enough we can find $\epsilon_0 > 0$ so that $|x_{\epsilon t} - y| \geq \delta/2$ for all $0 < \epsilon \leq \epsilon_0$, $s,t \in [-1,1]$ and $y \in \partial B_\delta(x_{\epsilon s}) \cap A^\epsilon_{t,s}$. In particular,

$$|k(x_{\epsilon t} - y)| \leq C < +\infty \qquad \text{for all } y \in \partial B_\delta(x_{\epsilon s}) \cap A^\epsilon_{t,s}, \tag{5.20}$$

where C only depends on δ and ϵ_0. Then,

$$\begin{aligned}&\chi_{\partial B_\delta(x_{\epsilon s}) \cap A^\epsilon_{t,s}}(y)\, k(x_{\epsilon t} - y) \cdot \frac{x_{\epsilon s} - y}{|x_{\epsilon s} - y|}\, d\mathcal{H}^2(y)\\ &\qquad\qquad = \chi_{\partial B_\delta(x_{\epsilon s}) \cap A^\epsilon_{t,s}}(y)\, \frac{x_{\epsilon t} - y}{4\pi |x_{\epsilon t} - y|^3} \cdot \frac{x_{\epsilon s} - y}{|x_{\epsilon s} - y|}\, d\mathcal{H}^2(y)\\ &\qquad\qquad \to \frac{\chi_{\partial B_\delta(x_\Sigma) \cap D(t,s)}(y)}{4\pi |x_\Sigma - y|^2}\, d\mathcal{H}^2(y) \quad \text{when } \epsilon \to 0,\end{aligned} \tag{5.21}$$

where

$$D(t,s) := \mathbb{R}^3 \setminus \overline{\Omega} \text{ if } t \leq s \qquad \text{and} \qquad D(t,s) := \Omega \text{ if } t > s.$$

The limit in (5.21) refers to weak-$*$ convergence of finite Borel measures in $\mathbb{R}^3$ (acting on the variable y). Using (5.21), the uniform estimate (5.20), the boundedness of h_j and the dominated convergence theorem, we see that

$$\begin{aligned}&\lim_{\epsilon \to 0} \int_{-1}^1 \operatorname{sign}(t-s) h_j(x_\Sigma, s) \int_{\partial B_\delta(x_{\epsilon s}) \cap A^\epsilon_{t,s}} k(x_{\epsilon t} - y) \cdot \frac{x_{\epsilon s} - y}{|x_{\epsilon s} - y|}\, d\mathcal{H}^2(y)\, ds\\ &\qquad = \int_{-1}^1 \operatorname{sign}(t-s) h_j(x_\Sigma, s) \int_{\partial B_\delta(x_\Sigma) \cap D(t,s)} \frac{1}{4\pi |x_\Sigma - y|^2}\, d\mathcal{H}^2(y)\, ds\\ &\qquad = \int_{-1}^1 \operatorname{sign}(t-s) h_j(x_\Sigma, s) \frac{\mathcal{H}^2\big(\partial B_\delta(x_\Sigma) \cap D(t,s)\big)}{\mathcal{H}^2(\partial B_\delta(x_\Sigma))}\, ds.\end{aligned}$$

Then, using the regularity of Σ and the dominated convergence theorem once again, we get

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}} k(x_{\epsilon t}-y)\cdot\frac{x_{\epsilon s}-y}{|x_{\epsilon s}-y|}\,d\mathcal{H}^2(y)\,ds\\ =\frac{1}{2}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\,ds.\end{aligned}\tag{5.22}$$

By (5.19), (5.22) and the definition of $\mathscr{C}_{\epsilon,\delta}$ before (5.14), we get

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\mathscr{C}_{\epsilon,\delta}=\frac{1}{2}\int_{-1}^{1}\operatorname{sign}(t-s)h_j(x_\Sigma,s)\,ds.\tag{5.23}$$

The proof of (5.12) is a straightforward combination of (5.14), (5.15), (5.17) and (5.23).

To prove (5.13) we use the same approach as in (5.12), that is, we split

$$T^\epsilon_{j,n}h_n(x_\Sigma,t)=:\mathscr{A}_{\epsilon,\delta}+\mathscr{B}_{\epsilon,\delta}+\mathscr{C}_{\epsilon,\delta}$$

like above (5.14). The first two terms can be treated analogously and one gets the desired result, the details are left for the reader. To estimate $\mathscr{C}_{\epsilon,\delta}$ we use the notation introduced before. Recall that $E_{x_{\epsilon t}}$ is smooth in $\overline{D^\epsilon_\delta(t,s)}$ (assuming $t\neq s$) and $k(x_{\epsilon t}-y)=\nabla E(x_{\epsilon t}-y)=-\nabla E_{x_{\epsilon t}}(y)$. So, by the divergence theorem (see also (5.9)),

$$\begin{aligned}&\int_{\partial D^\epsilon_\delta(t,s)}\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\\ &=\int_{\partial D^\epsilon_\delta(t,s)}\Big((\nu_{\partial D^\epsilon_\delta(t,s)}(y))_n\partial_{x_j}E(x_{\epsilon t}-y)-(\nu_{\partial D^\epsilon_\delta(t,s)}(y))_j\partial_{x_n}E(x_{\epsilon t}-y)\Big)d\mathcal{H}^2(y)\\ &=\int_{D^\epsilon_\delta(t,s)}\big(\partial_{y_j}\partial_{y_n}E_{x_{\epsilon t}}-\partial_{y_n}\partial_{y_j}E_{x_{\epsilon t}}\big)(y)\,dy=0.\end{aligned}\tag{5.24}$$

Since $\partial D^\epsilon_\delta(t,s)=(B_\delta(x_{\epsilon s})\cap\Sigma_{\epsilon s})\cup(\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s})$, from (5.24) we have

$$\Big|\int_{|x_{\epsilon s}-y_{\epsilon s}|\leq\delta}\nabla^{j,n}_{\nu_{\epsilon s}(y_{\epsilon s})}E(x_{\epsilon st}-y_{\epsilon s})\,d\sigma_{\epsilon s}(y_{\epsilon s})\Big|=\Big|\int_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\Big|.$$

Observe that

$$\begin{aligned}&\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\,\nabla^{j,n}_{\nu_{\partial D^\epsilon_\delta(t,s)}(y)}E(x_{\epsilon t}-y)\,d\mathcal{H}^2(y)\\ &=\chi_{\partial B_\delta(x_{\epsilon s})\cap A^\epsilon_{t,s}}(y)\Big((\nu_{\partial D^\epsilon_\delta(t,s)}(y))_j\partial_{y_n}E_{x_{\epsilon t}}(y)-(\nu_{\partial D^\epsilon_\delta(t,s)}(y))_n\partial_{y_j}E_{x_{\epsilon t}}(y)\Big)\,d\mathcal{H}^2(y)\\ &\to\chi_{\partial B_\delta(x_\Sigma)\cap D(t,s)}(y)\Big(\frac{(y-x_\Sigma)_j}{|y-x_\Sigma|}\partial_{y_n}E_{x_\Sigma}(y)-\frac{(y-x_\Sigma)_n}{|y-x_\Sigma|}\partial_{y_j}E_{x_\Sigma}(y)\Big)\,d\mathcal{H}^2(y)=0\end{aligned}\tag{5.25}$$

when $\epsilon\to 0$. The limit measure in (5.25) vanishes because its density function corresponds to a tangential derivative of E_{x_Σ} on $\partial B_\delta(x_\Sigma)$, which is a constant function on $\partial B_\delta(x_\Sigma)$. Therefore, arguing as in the proof of (5.12) but replacing (5.21) by (5.25), we can resume that, now,

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\mathscr{C}_{\epsilon,\delta}=0.$$

This yields (5.13) and concludes the proof of (5.4).

5.2. **A pointwise estimate of $|B_{\epsilon,\omega_3} g(x_\Sigma, t)|$ by maximal operators.**

We begin this section by setting

$$k(x) := \frac{x_j}{4\pi|x|^3} \quad \text{for } j = 1, 2, 3,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{0\}. \tag{5.26}$$

In (5.5) we already introduced a kernel k which, in fact, corresponds to the vectorial version of the ones introduced in (5.26). So, by an abuse of notation, throughout this section we mean by $k(x)$ any of the components of the kernel given in (5.5).

Note that $k(-x) = -k(x)$ for all $x \in \mathbb{R}^3 \setminus \{0\}$ and, besides, there exists $C > 0$ such that

$$\begin{aligned} &|k(x-y)| \leq \frac{C}{|x-y|^2} \quad \text{for all } x, y \in \mathbb{R}^3 \text{ such that } |x-y| > 0,\\ &|k(z-y) - k(x-y)| \leq C\frac{|z-x|}{|x-y|^3} \quad \text{for all } x, y, z \in \mathbb{R}^3 \text{ with } 0 < |z-x| \leq \frac{1}{2}|x-y|. \end{aligned} \tag{5.27}$$

As in Section 5.1, we are going to work componentwise. More precisely, in order to deal with the different components of $B_{\epsilon,\omega_3} g(x_\Sigma, t)$ for $g \in L^2(\Sigma \times (-1,1))^4$, we are going to study the following scalar version. Given $0 < \epsilon \leq \eta_0$, $g \in L^2(\Sigma \times (-1,1))$ and $(x_\Sigma, t) \in \Sigma \times (-1,1)$, define

$$\begin{aligned} \widetilde{B}_\epsilon g(x_\Sigma, t) := u(t) \int_{-1}^{1} \int_\Sigma & k(x_\Sigma + \epsilon t\nu(x_\Sigma) - y_\Sigma - \epsilon s\nu(y_\Sigma))\\ & \times v(s) \det(1 - \epsilon s W(y_\Sigma)) g(y_\Sigma, s)\, d\sigma(y_\Sigma)\, ds, \end{aligned} \tag{5.28}$$

where u and v are as in (3.1) for some $0 < \eta \leq \eta_0$. It is clear that pointwise estimates of $|\widetilde{B}_\epsilon g(x_\Sigma, t)|$ for a given $g \in L^2(\Sigma \times (-1,1))$ directly transfer to pointwise estimates of $|B_{\epsilon,\omega_3} h(x_\Sigma, t)|$ for a given $h \in L^2(\Sigma \times (-1,1))^4$, so we are reduced to estimate $|\widetilde{B}_\epsilon g(x_\Sigma, t)|$ for $g \in L^2(\Sigma \times (-1,1))$.

A key ingredient to find those suitable pointwise estimates is to relate $\widetilde{B}_\epsilon$ to the Hardy-Littlewood maximal operator and some maximal singular integral operators from Calderón-Zygmund theory. The Hardy-Littlewood maximal operator is given by

$$M_* f(x_\Sigma) := \sup_{\delta > 0} \frac{1}{\sigma(B_\delta(x_\Sigma))} \int_{B_\delta(x_\Sigma)} |f|\, d\sigma, \quad M_* : L^2(\Sigma) \to L^2(\Sigma) \text{ bounded}, \tag{5.29}$$

see [17, 2.19 Theorem] for a proof of the boundedness. The above mentioned maximal singular integral operators are

$$T_* f(x_\Sigma) := \sup_{\delta > 0} \Big| \int_{|x_\Sigma - y_\Sigma| > \delta} k(x_\Sigma - y_\Sigma) f(y_\Sigma)\, d\sigma(y_\Sigma) \Big|, \quad T_* : L^2(\Sigma) \to L^2(\Sigma) \text{ bounded}, \tag{5.30}$$

see [6, Proposition 4 bis] for a proof of the boundedness. We also introduce some integral versions of these maximal operators to connect them to the space $L^2(\Sigma \times (-1,1))$. Set

$$\begin{aligned} &\widetilde{M}_* g(x_\Sigma) := \Big(\int_{-1}^{1} M_*(g(\cdot, s))(x_\Sigma)^2\, ds \Big)^{1/2}, \quad \widetilde{M}_* : L^2(\Sigma \times (-1,1)) \to L^2(\Sigma) \text{ bounded},\\ &\widetilde{T}_* g(x_\Sigma) := \int_{-1}^{1} T_*(g(\cdot, s))(x_\Sigma)\, ds, \quad \widetilde{T}_* : L^2(\Sigma \times (-1,1)) \to L^2(\Sigma) \text{ bounded}. \end{aligned} \tag{5.31}$$

Indeed, by Fubini's theorem and (5.29),

$$\begin{aligned}\|\widetilde{M}_*g\|^2_{L^2(\Sigma)} &= \int_\Sigma\int_{-1}^1 M_*(g(\cdot,s))(x_\Sigma)^2\,ds\,d\sigma(x_\Sigma) = \int_{-1}^1 \|M_*(g(\cdot,s))\|^2_{L^2(\Sigma)}\,ds\\ &\leq C\int_{-1}^1 \|g(\cdot,s)\|^2_{L^2(\Sigma)}\,ds = C\|g\|^2_{L^2(\Sigma\times(-1,1))}.\end{aligned}$$

By Cauchy-Schwarz inequality, Fubini's theorem and (5.30), we also see that $\widetilde{T}_*$ is bounded, so (5.31) is fully justified.

Let us focus for a moment on the boundedness of $B_0(a)$ stated in (3.7). The fact that, for $g\in L^2(\Sigma\times(-1,1))^4$, the limit in the definition of $(B_0(a)g)(x_\Sigma,t)$ exists for almost every $(x_\Sigma,t)\in\Sigma\times(-1,1)$ is a consequence of the decomposition (see (5.1))

$$\phi^a = \omega_1^a + \omega_2^a + \omega_3,$$

the integrals of fractional type on bounded sets in the case of ω_1^a and ω_2^a and, for ω_3, that

$$\lim_{\epsilon\to 0}\int_{|x_\Sigma-y_\Sigma|>\epsilon} k(x_\Sigma-y_\Sigma)f(y_\Sigma)\,d\sigma(y_\Sigma)\quad\text{exists for }\sigma\text{-almost every }x_\Sigma\in\Sigma \tag{5.32}$$

if $f\in L^2(\Sigma)$ (see [17, 20.27 Theorem] for a proof) and that

$$\int_{-1}^1 v(s)g(\cdot,s)\,ds\in L^2(\Sigma)^4.$$

Of course, (5.32) directly applies to B_{0,ω_3} (see (5.2) for the definition). From the boundedness of $\widetilde{T}_*$ and working component by component, we easily see that B_{0,ω_3} is bounded in $L^2(\Sigma\times(-1,1))^4$. By the comments regarding B_{0,ω_1^a} and B_{0,ω_2^a} from the paragraph which contains (5.3), we also get that $B_0(a)$ is bounded in $L^2(\Sigma\times(-1,1))^4$, which gives (3.7) in this case.

With the maximal operators at hand, we proceed to pointwise estimate $|\widetilde{B}_\epsilon g(x_\Sigma,t)|$ for $g\in L^2(\Sigma\times(-1,1))$. Set

$$g_\epsilon(y_\Sigma,s) := v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s). \tag{5.33}$$

Then, since the eigenvalues of W are uniformly bounded by Proposition 2.2, there exists $C>0$ only depending on η_0 such that

$$|g_\epsilon(y_\Sigma,s)|\leq C\|v\|_{L^\infty(\mathbb{R})}|g(y_\Sigma,s)|\quad\text{for all } 0<\epsilon\leq\eta_0,\ (y_\Sigma,s)\in\Sigma\times(-1,1). \tag{5.34}$$

Besides, the regularity and boundedness of Σ implies the existence of $L>0$ such that

$$|\nu(x_\Sigma)-\nu(y_\Sigma)|\leq L|x_\Sigma-y_\Sigma|\quad\text{for all } x_\Sigma,y_\Sigma\in\Sigma. \tag{5.35}$$

We make the following splitting of $\widetilde{B}_\epsilon g(x_\Sigma,t)$ (see (5.28) for the definition):

$$
\begin{aligned}
\widetilde{B}_\epsilon g(x_\Sigma,t) = {}& u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|\le 4\epsilon|t-s|} k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
&+u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\epsilon|t-s|}\Big(k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\\
&\qquad -k(x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\Big)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
&+u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\epsilon|t-s|}\Big(k(x_\Sigma+\epsilon s(\nu(x_\Sigma)-\nu(y_\Sigma))-y_\Sigma)-k(x_\Sigma-y_\Sigma)\Big)\\
&\qquad\times g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
&+u(t)\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\epsilon|t-s|} k(x_\Sigma-y_\Sigma)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds\\
=:{}&\widetilde{B}_{\epsilon,1}g(x_\Sigma,t)+\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)+\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)+\widetilde{B}_{\epsilon,4}g(x_\Sigma,t).
\end{aligned}
\tag{5.36}
$$

We are going to estimate the four terms on the right hand side of (5.36) separately.

Concerning $\widetilde{B}_{\epsilon,1}g(x_\Sigma,t)$, note that

$$\epsilon|t-s| = \mathrm{dist}(x_\Sigma+\epsilon t\nu(x_\Sigma),\Sigma_{\epsilon s}) \le |x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))|$$

for all $(y_\Sigma,s)\in\Sigma\times(-1,1)$, thus $|k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))|\le\frac{1}{\epsilon^2|t-s|^2}$ by (5.27), and then

$$
\begin{aligned}
|\widetilde{B}_{\epsilon,1}g(x_\Sigma,t)| &\le \|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\frac{1}{\epsilon^2|t-s|^2}\int_{|x_\Sigma-y_\Sigma|\le 4\epsilon|t-s|}|g_\epsilon(y_\Sigma,s)|\,d\sigma(y_\Sigma)\,ds\\
&\le C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1 M_*(g_\epsilon(\cdot,s))(x_\Sigma)\,ds \le C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),
\end{aligned}
\tag{5.37}
$$

where we used the Cauchy-Schwarz inequality and (5.34) in the last inequality above.

For the case of $\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)$, we split the integral over Σ on dyadic annuli as follows. Set

$$N := \Big[\Big|\log_2\Big(\frac{\mathrm{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\Big)\Big|\Big]+1 \tag{5.38}$$

for $t\ne s$, where $[\,\cdot\,]$ denotes the integer part. Then, $2^N\epsilon|t-s|>\mathrm{diam}(\Omega_{\eta_0})$ and

$$|\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)|\le\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\sum_{n=2}^N\int_{2^{n+1}\epsilon|t-s|\ge|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\cdots\,d\sigma(y_\Sigma)\,ds, \tag{5.39}$$

where "$\cdots$" means $\big|k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\big||g_\epsilon(y_\Sigma,s)|$. By (5.35),

$$
\begin{aligned}
(1-\eta_0L)|x_\Sigma-y_\Sigma| &\le |x_\Sigma-y_\Sigma|-\eta_0|\nu(x_\Sigma)-\nu(y_\Sigma)|\\
&\le |x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)|\\
&\le |x_\Sigma-y_\Sigma|+\eta_0|\nu(x_\Sigma)-\nu(y_\Sigma)|\le(1+\eta_0L)|x_\Sigma-y_\Sigma|,
\end{aligned}
$$

thus if we take $\eta_0\le\frac{1}{2L}$ we get

$$\frac12|x_\Sigma-y_\Sigma|\le|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)|\le 2|x_\Sigma-y_\Sigma|. \tag{5.40}$$

Besides, for $2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|$, using (5.40) we see that

$$\begin{aligned}|x_\Sigma+\epsilon t\nu(x_\Sigma)-(x_\Sigma+\epsilon s\nu(x_\Sigma))|&=\epsilon|t-s|<2^{-n}|x_\Sigma-y_\Sigma|\\&\leq 2^{-n+1}|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)|\\&\leq\frac{1}{2}|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)|\end{aligned}\tag{5.41}$$

for all $n=2,\dots,N$. Therefore, combining (5.41), (5.27) and (5.40) we finally get

$$\begin{aligned}&\big|k(x_\Sigma+\epsilon t\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma))\big|\\&\qquad\leq C\frac{|x_\Sigma+\epsilon t\nu(x_\Sigma)-(x_\Sigma+\epsilon s\nu(x_\Sigma))|}{|x_\Sigma+\epsilon s\nu(x_\Sigma)-y_\Sigma-\epsilon s\nu(y_\Sigma)|^3}\leq\frac{C\epsilon|t-s|}{|x_\Sigma-y_\Sigma|^3}<\frac{C}{2^{3n}\epsilon^2|t-s|^2}\end{aligned}$$

for all $s,t\in(-1,1)$, $0<\epsilon\leq\eta_0$, $n=2,\dots,N$ and $2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|$. Plugging this estimate into (5.39) we obtain

$$\begin{aligned}|\widetilde{B}_{\epsilon,2}g(x_\Sigma,t)|&\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\sum_{n=2}^N\int_{2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{2^{3n}\epsilon^2|t-s|^2}\,d\sigma(y_\Sigma)\,ds\\&\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\sum_{n=2}^N\frac{1}{2^n}\int_{|x_\Sigma-y_\Sigma|\leq2^{n+1}\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{(2^{n+1}\epsilon|t-s|)^2}\,d\sigma(y_\Sigma)\,ds\\&\leq C\|u\|_{L^\infty(\mathbb{R})}\sum_{n=2}^\infty\frac{1}{2^n}\int_{-1}^1M_*(g_\epsilon(\cdot,s))(x_\Sigma)\,ds\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),\end{aligned}\tag{5.42}$$

where we used the Cauchy-Schwarz inequality and (5.34) in the last inequality above.

Let us deal now with $\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)$. Since $0<\epsilon\leq\eta_0$ and $s\in(-1,1)$, if we take $\eta_0\leq\frac{1}{2L}$ as before, from (5.35) we see that

$$\big|(x_\Sigma+\epsilon s(\nu(x_\Sigma)-\nu(y_\Sigma)))-x_\Sigma\big|=\epsilon|s||\nu(x_\Sigma)-\nu(y_\Sigma)|\leq\frac{1}{2}|x_\Sigma-y_\Sigma|,$$

and then, by (5.27),

$$\big|k(x_\Sigma+\epsilon s(\nu(x_\Sigma)-\nu(y_\Sigma))-y_\Sigma)-k(x_\Sigma-y_\Sigma)\big|\leq C\frac{\epsilon|s||\nu(x_\Sigma)-\nu(y_\Sigma)|}{|x_\Sigma-y_\Sigma|^3}\leq\frac{C\epsilon}{|x_\Sigma-y_\Sigma|^2}.\tag{5.43}$$

Splitting the integral which defines $\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)$ into dyadic annuli as in (5.39), and using (5.43), (5.34) and (5.38), we get

$$\begin{aligned}|\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)|&\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\sum_{n=2}^N\epsilon\int_{2^{n+1}\epsilon|t-s|\geq|x_\Sigma-y_\Sigma|>2^n\epsilon|t-s|}\frac{|g_\epsilon(y_\Sigma,s)|}{|x_\Sigma-y_\Sigma|^2}\,d\sigma(y_\Sigma)\,ds\\&\leq C\|u\|_{L^\infty(\mathbb{R})}\int_{-1}^1\epsilon\sum_{n=2}^NM_*(g_\epsilon(\cdot,s))(x_\Sigma)\,ds\\&\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^1\epsilon\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right|M_*(g(\cdot,s))(x_\Sigma)\,ds.\end{aligned}\tag{5.44}$$

Note that

$$\epsilon\left|\log_2\left(\frac{\operatorname{diam}(\Omega_{\eta_0})}{\epsilon|t-s|}\right)\right|\leq\epsilon\big(C+|\log_2\epsilon|+|\log_2|t-s||\big)\leq C\big(1+|\log_2|t-s||\big)$$

for all $0 < \epsilon \leq \eta_0$, where $C > 0$ only depends on η_0. Hence, from (5.44) and Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|\widetilde{B}_{\epsilon,3}g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\big(1+|\log_2|t-s||\big)M_*(g(\cdot,s))(x_\Sigma)\,ds\\
&\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\Big(\int_{-1}^{1}\big(1+|\log_2|t-s||\big)^2\,ds\Big)^{1/2}\widetilde{M}_*g(x_\Sigma)\\
&\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma),
\end{aligned}
\tag{5.45}
$$

where we also used that $t\in(-1,1)$, so $\int_{-1}^{1}\big(1+|\log_2|t-s||\big)^2\,ds \leq C\big(1+\int_0^2|\log_2 r|^2\,dr\big) < +\infty$, in the last inequality above.

The term $|\widetilde{B}_{\epsilon,4}g(x_\Sigma,t)|$ can be estimated using the maximal operator $\widetilde{T}_*$ as follows. Let $\lambda_1(y_\Sigma)$ and $\lambda_2(y_\Sigma)$ denote the eigenvalues of the Weingarten map $W(y_\Sigma)$. By definition,

$$
\begin{aligned}
g_\epsilon(y_\Sigma,s) &= v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\\
&= v(s)\big(1+\epsilon^2s^2\lambda_1(y_\Sigma)\lambda_2(y_\Sigma)-\epsilon s\lambda_1(y_\Sigma)-\epsilon s\lambda_2(y_\Sigma)\big)g(y_\Sigma,s).
\end{aligned}
$$

Therefore, the triangle inequality yields

$$
\begin{aligned}
|\widetilde{B}_{\epsilon,4}g(x_\Sigma,t)| &\leq \|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\int_{-1}^{1}\Big(T_*(g(\cdot,s))(x_\Sigma)+\eta_0^2T_*(\lambda_1\lambda_2g(\cdot,s))(x_\Sigma)\\
&\qquad +\eta_0T_*(\lambda_1g(\cdot,s))(x_\Sigma)+\eta_0T_*(\lambda_2g(\cdot,s))(x_\Sigma)\Big)\,ds\\
&\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{T}_*g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2g)(x_\Sigma)+\widetilde{T}_*(\lambda_1g)(x_\Sigma)+\widetilde{T}_*(\lambda_2g)(x_\Sigma)\big).
\end{aligned}
\tag{5.46}
$$

Combining (5.36), (5.37), (5.42), (5.45) and (5.46) and taking the supremum on ϵ we finally get that

$$
\begin{aligned}
\sup_{0<\epsilon\leq\eta_0}|\widetilde{B}_\epsilon g(x_\Sigma,t)| &\leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{M}_*g(x_\Sigma)+\widetilde{T}_*g(x_\Sigma)\\
&\qquad +\widetilde{T}_*(\lambda_1\lambda_2g)(x_\Sigma)+\widetilde{T}_*(\lambda_1g)(x_\Sigma)+\widetilde{T}_*(\lambda_2g)(x_\Sigma)\big),
\end{aligned}
\tag{5.47}
$$

where $C > 0$ only depends on η_0. Define

$$\widetilde{B}_*g(x_\Sigma,t) := \sup_{0<\epsilon\leq\eta_0}|\widetilde{B}_\epsilon g(x_\Sigma,t)| \quad \text{for } (x_\Sigma,t)\in\Sigma\times(-1,1).$$

Then, from (5.47), the boundedness of $\widetilde{M}_*$ and $\widetilde{T}_*$ from $L^2(\Sigma\times(-1,1))$ to $L^2(\Sigma)$ (see (5.31)) and the fact that $\|\lambda_1\|_{L^\infty(\Sigma)}$ and $\|\lambda_2\|_{L^\infty(\Sigma)}$ are finite by Proposition 2.2, we easily conclude that there exists $C > 0$ only depending on η_0 such that

$$\|\widetilde{B}_*g\|_{L^2(\Sigma\times(-1,1))} \leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}. \tag{5.48}$$

5.3. $B_{\epsilon,\omega_3}\to B_{0,\omega_3}+B'$ in the strong sense when $\epsilon\to 0$ and conclusion of the proof of (3.10).

To begin this section, we present a standard result in harmonic analysis about the existence of limit almost everywhere for a sequence of operators acting on a fixed function and its convergence in strong sense. General statements can be found in [9, Theorem 2.2 and the remark below it] and [25, Proposition 6.2], for example. For the sake of completeness, here we present a concrete version with its proof.

Lemma 5.1. *Let $b \in \mathbb{N}$ and (X, μ_X) and (Y, μ_Y) be two Borel measure spaces. Let $\{W_\epsilon\}_{0<\epsilon\leq\eta_0}$ be a family of bounded linear operators from $L^2(\mu_X)^b$ to $L^2(\mu_Y)^b$ such that, if*

$$W_*g(y) := \sup_{0<\epsilon\leq\eta_0} |W_\epsilon g(y)| \quad \text{for } g \in L^2(\mu_X)^b \text{ and } y \in Y, \qquad \text{then } W_* : L^2(\mu_X)^b \to L^2(\mu_Y)$$

is a bounded sublinear operator. Suppose that for any $g \in S$, where $S \subset L^2(\mu_X)^b$ is a dense subspace, $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$. Then, for any $g \in L^2(\mu_X)^b$, $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$ and

$$\lim_{\epsilon\to 0} \big\|W_\epsilon g - \lim_{\delta\to 0} W_\delta g\big\|_{L^2(\mu_Y)^b} = 0. \tag{5.49}$$

In particular, $\lim_{\epsilon\to 0} W_\epsilon$ defines a bounded operator from $L^2(\mu_X)^b$ to $L^2(\mu_Y)^b$.

Proof. We start by proving that, for any $g \in L^2(\mu_X)^b$, $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists for μ_Y-a.e. $y \in Y$. Take $g_k \in S$ such that $\|g_k - g\|_{L^2(\mu_X)^b} \to 0$ for $k \to \infty$, and fix $\lambda > 0$. Since $\lim_{\epsilon\to 0} W_\epsilon g_k(y)$ exists for μ_Y-a.e. $y \in Y$, Chebyshev inequality yields

$$\begin{aligned}
&\mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon g(y) - \liminf_{\epsilon\to 0} W_\epsilon g(y)\Big| > \lambda\Big\}\Big)\\
&\qquad \leq \mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon (g - g_k)(y)\Big| + \Big|\liminf_{\epsilon\to 0} W_\epsilon (g_k - g)(y)\Big| > \lambda\Big\}\Big)\\
&\qquad \leq \mu_Y(\{y \in Y : 2W_*(g - g_k)(y) > \lambda\})\\
&\qquad \leq \frac{4}{\lambda^2}\|W_*(g - g_k)\|^2_{L^2(\mu_Y)} \leq \frac{C}{\lambda^2}\|g - g_k\|^2_{L^2(\mu_X)^b}.
\end{aligned}$$

Letting $k \to \infty$ we deduce that

$$\mu_Y\Big(\Big\{y \in Y : \Big|\limsup_{\epsilon\to 0} W_\epsilon g(y) - \liminf_{\epsilon\to 0} W_\epsilon g(y)\Big| > \lambda\Big\}\Big) = 0.$$

Since this holds for all $\lambda > 0$, we finally get that $\lim_{\epsilon\to 0} W_\epsilon g(y)$ exists μ_Y-a.e.

Note that $|W_\epsilon g(y) - \lim_{\delta\to 0} W_\delta g(y)| \leq 2W_*g(y)$ and $W_*g \in L^2(\mu_Y)$. Thus, (5.49) follows by the dominated convergence theorem. The last statement in the lemma is also a consequence of the boundedness of W_*. □

Thanks to Lemma 5.1 and the results in Sections 5.1 and 5.2, we are ready to conclude the proof of (3.10). As we said before (5.4), to obtain (3.10) we only need to show that $B_{\epsilon,\omega_3} \to B_{0,\omega_3} + B'$ in the strong sense when $\epsilon \to 0$. From (5.4), we know that

$$\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t) = B_{0,\omega_3} g(x_\Sigma, t) + B' g(x_\Sigma, t) \quad \text{for almost all } (x_\Sigma, t) \in \Sigma \times (-1, 1)$$

and all $g \in L^\infty(\Sigma\times(-1,1))^4$ such that $\sup_{|t|<1} |g(x_\Sigma, t) - g(y_\Sigma, t)| \leq C_g|x_\Sigma - y_\Sigma|$ for all x_Σ, $y_\Sigma \in \Sigma$ and some $C_g > 0$ (it may depend on g). Note also that this set of functions g is dense in $L^2(\Sigma \times (-1, 1))^4$. Besides, thanks to (5.48) we see that, if $\eta_0 > 0$ is small enough and we set

$$B_{*,\omega_3} g(x_\Sigma, t) := \sup_{0<\epsilon\leq\eta_0} |B_{\epsilon,\omega_3} g(x_\Sigma, t)| \quad \text{for } (x_\Sigma, t) \in \Sigma \times (-1, 1),$$

then there exists $C > 0$ only depending on η_0 such that

$$\|B_{*,\omega_3} g\|_{L^2(\Sigma\times(-1,1))} \leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{5.50}$$

Therefore, from Lemma 5.1 we get that, for any $g \in L^2(\Sigma \times (-1, 1))^4$, the pointwise limit $\lim_{\epsilon\to 0} B_{\epsilon,\omega_3} g(x_\Sigma, t)$ exists for almost every $(x_\Sigma, t) \in \Sigma \times (-1, 1)$. Recall also that $B_{0,\omega_3} + B'$ is bounded in $L^2(\Sigma\times(-1,1))^4$ (see the comment before (5.33) for B_{0,ω_3}, the case of B' is trivial), so one can easily adapt the proof of Lemma 5.1 to also show that, for any $g\in L^2(\Sigma\times(-1,1))^4$,

$$\lim_{\epsilon\to0} B_{\epsilon,\omega_3}g(x_\Sigma,t)=B_{0,\omega_3}g(x_\Sigma,t)+B'g(x_\Sigma,t)\quad\text{for almost all } (x_\Sigma,t)\in\Sigma\times(-1,1).$$

Finally, (5.49) in Lemma 5.1 yields

$$\lim_{\epsilon\to0}\|(B_{\epsilon,\omega_3}-B_{0,\omega_3}-B')g\|_{L^2(\Sigma\times(-1,1))^4}=0\quad\text{for all } g\in L^2(\Sigma\times(-1,1))^4,$$

which is the required strong convergence of B_{ϵ,ω_3} to $B_{0,\omega_3}+B'$. This finishes the proof of (3.10).

6. Proof of (3.9): $A_\epsilon(a)\to A_0(a)$ in the strong sense when $\epsilon\to0$

Recall from (3.3) and (3.8) that $A_\epsilon(a)$ with $0<\epsilon\le\eta_0$ and $A_0(a)$ are defined by

$$\begin{aligned}(A_\epsilon(a)g)(x)&=\int_{-1}^1\int_\Sigma\phi^a(x-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\(A_0(a)g)(x)&=\int_{-1}^1\int_\Sigma\phi^a(x-y_\Sigma)v(s)g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds.\end{aligned}$$

We already know that $A_\epsilon(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$. To show the boundedness of $A_0(a)$ (and conclude the proof of (3.7)) just note that, by Fubini's theorem, for every $x\in\mathbb{R}^3\setminus\Sigma$ we have

$$(A_0(a)g)(x)=\int_\Sigma\phi^a(x-y_\Sigma)\Big(\int_{-1}^1v(s)g(y_\Sigma,s)\,ds\Big)d\sigma(y_\Sigma),$$

and $\int_{-1}^1v(s)g(\cdot,s)\,ds\in L^2(\Sigma)^4$ if $g\in L^2(\Sigma\times(-1,1))^4$. Since $a\in\mathbb{C}\setminus\mathbb{R}$, [3, Lemma 2.1] shows that $A_0(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$.

We begin the proof of (3.9) by splitting

$$A_\epsilon(a)g=\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_\epsilon(a)g+\chi_{\Omega_{\eta_0}}A_\epsilon(a)g.\tag{6.1}$$

Let us treat first the case of $\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_\epsilon(a)$. As we said before, since $a\in\mathbb{C}\setminus\mathbb{R}$, the components of $\phi^a(x)$ decay exponentially when $|x|\to\infty$. In particular, there exist $C,r>0$ only depending on a and η_0 such that

$$|\phi^a(x)|,|\partial\phi^a(x)|\le Ce^{-r|x|}\quad\text{for all } |x|\ge\frac{\eta_0}{2},\tag{6.2}$$

where the left hand side of (6.2) means the absolute value of any component of the matrix $\phi^a(x)$ and of any first order derivative of it, respectively.

Note that $\eta_0=\text{dist}(\mathbb{R}^3\setminus\Omega_{\eta_0},\Sigma)$. Hence, if $x\in\mathbb{R}^3\setminus\Omega_{\eta_0}$, $y_\Sigma\in\Sigma$, $0\le\epsilon\le\frac{\eta_0}{2}$ and $s\in(-1,1)$ then, for any $0\le q\le1$,

$$\begin{aligned}|q(x-y_\Sigma-\epsilon s\nu(y_\Sigma))+(1-q)(x-y_\Sigma)|&=|x-y_\Sigma-q\epsilon s\nu(y_\Sigma)|\\&\ge|x-y_\Sigma|-q\epsilon|s|\ge|x-y_\Sigma|-\frac{\eta_0}{2}\ge\frac{|x-y_\Sigma|}{2}\ge\frac{\eta_0}{2}.\end{aligned}\tag{6.3}$$

Thus (6.2) applies to $[x,y_\Sigma]_q:=q(x-y_\Sigma-\epsilon s\nu(y_\Sigma))+(1-q)(x-y_\Sigma)$, and a combination of the mean value theorem and (6.3) gives

$$|\phi^a(x-y_\Sigma-\epsilon s\nu(y_\Sigma))-\phi^a(x-y_\Sigma)|\le\epsilon\max_{0\le q\le1}|\partial\phi^a([x,y_\Sigma]_q)|\le C\epsilon e^{-\frac{r}{2}|x-y_\Sigma|}.\tag{6.4}$$

Set $\widetilde{g}_\epsilon(y_\Sigma,s) := \det(1-\epsilon s W(y_\Sigma))g(y_\Sigma,s)$. On one hand, from (6.4), Proposition 2.2 and Cauchy-Schwarz inequality, we get that

$$\begin{aligned}
&\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)|(A_\epsilon(a)g)(x)-(A_0(a)g_\epsilon)(x)|\\
&\qquad\leq C\|v\|_{L^\infty(\mathbb{R})}\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\int_{-1}^1\int_\Sigma \epsilon e^{-\frac{r}{2}|x-y_\Sigma|}|\widetilde{g}_\epsilon(y_\Sigma,s)|\,d\sigma(y_\Sigma)\,ds\\
&\qquad\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|\widetilde{g}_\epsilon\|_{L^2(\Sigma\times(-1,1))^4}\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\Big(\int_\Sigma e^{-r|x-y_\Sigma|}\,d\sigma(y_\Sigma)\Big)^{1/2}\\
&\qquad\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}\xi(x),
\end{aligned}$$

where

$$\xi(x) := \chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(x)\Big(\int_\Sigma e^{-r|x-y_\Sigma|}\,d\sigma(y_\Sigma)\Big)^{1/2}.$$

Since $\xi\in L^2(\mathbb{R}^3)$ because $\sigma(\Sigma)<+\infty$, we deduce that

$$\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)g-A_0(a)\widetilde{g}_\epsilon)\|_{L^2(\mathbb{R}^3)^4}\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{6.5}$$

On the other hand, by Proposition 2.2 we have that

$$|\widetilde{g}_\epsilon(y_\Sigma,s)-g(y_\Sigma,s)| = \big|\det(1-\epsilon s W(y_\Sigma))-1\big||g(y_\Sigma,s)|\leq C\epsilon|g(y_\Sigma,s)|.$$

This, together with the fact that $A_0(a)$ is bounded from $L^2(\Sigma\times(-1,1))^4$ to $L^2(\mathbb{R}^3)^4$ (see above (6.1)), implies that

$$\begin{aligned}
\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}A_0(a)(\widetilde{g}_\epsilon-g)\|_{L^2(\mathbb{R}^3)^4}&\leq C\|v\|_{L^\infty(\mathbb{R})}\|\widetilde{g}_\epsilon-g\|_{L^2(\Sigma\times(-1,1))^4}\\
&\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}.
\end{aligned}\tag{6.6}$$

Using the triangle inequality, (6.5) and (6.6), we finally get that

$$\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)-A_0(a))g\|_{L^2(\mathbb{R}^3)^4}\leq C\epsilon\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}\tag{6.7}$$

for all $0\leq\epsilon\leq\frac{\eta_0}{2}$, where $C>0$ only depends on a and η_0. In particular, this implies that

$$\lim_{\epsilon\to0}\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)-A_0(a))\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4}=0.\tag{6.8}$$

Let us deal now with $\chi_{\Omega_{\eta_0}}A_\epsilon(a)$. Consider the decomposition of ϕ^a given by (5.1). Then, as in (5.2), we write

$$\begin{aligned}
A_\epsilon(a)&=A_{\epsilon,\omega_1^a}+A_{\epsilon,\omega_2^a}+A_{\epsilon,\omega_3},\\
A_0(a)&=A_{0,\omega_1^a}+A_{0,\omega_2^a}+A_{0,\omega_3},
\end{aligned}\tag{6.9}$$

where A_{ϵ,ω_1^a}, A_{ϵ,ω_2^a} and A_{ϵ,ω_3} are defined as $A_\epsilon(a)$ but replacing ϕ^a by ω_1^a, ω_2^a and ω_3, respectively, and analogously for the case of $A_0(a)$. For $j=1,2$, the arguments used to show (5.3) in the case of B_{ϵ,ω_j^a} also apply to $\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_j^a}$, thus we now get

$$\lim_{\epsilon\to0}\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_j^a}-A_{0,\omega_j^a})\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4}=0\quad\text{for } j=1,2.\tag{6.10}$$

It only remains to show the strong convergence of $\chi_{\Omega_{\eta_0}}A_{\epsilon,\omega_3}$. This case is treated similarly to what we did in Sections 5.1, 5.2 and 5.3, as follows.

6.1. **The pointwise limit of $A_{\epsilon,\omega_3}g(x)$ when $\epsilon\to 0$ for $g\in L^2(\Sigma\times(-1,1))^4$.**

This case is much more easy than the one in Section 5.1. Fixed $x\in\mathbb{R}^3\setminus\Sigma$, we can always find $\delta_x, C_x>0$ small enough such that

$$|x-y_\Sigma-\epsilon s\nu(y_\Sigma)|\geq C_x\quad\text{for all } y_\Sigma\in\Sigma,\ s\in(-1,1)\text{ and } 0\leq\epsilon\leq\delta_x.$$

In particular, fixed $x\in\mathbb{R}^3\setminus\Sigma$, $|\omega_3(x-y_\Sigma-\epsilon s\nu(y_\Sigma))|\leq C$ uniformly on $y_\Sigma\in\Sigma$, $s\in(-1,1)$ and $0\leq\epsilon\leq\delta_x$, where $C>0$ depends on x. By Proposition 2.2 and the dominated convergence theorem, given $g\in L^2(\Sigma\times(-1,1))^4$, we have

$$\lim_{\epsilon\to 0}A_{\epsilon,\omega_3}g(x)=A_{0,\omega_3}g(x)\quad\text{for }\mathcal{L}\text{-a.e. } x\in\mathbb{R}^3,\tag{6.11}$$

where $\mathcal{L}$ denotes the Lebesgue measure in $\mathbb{R}^3$.

6.2. **A pointwise estimate of $\chi_{\Omega_{\eta_0}}(x)|A_{\epsilon,\omega_3}g(x)|$ by maximal operators.**

Given $0\leq\epsilon\leq\frac{\eta_0}{4}$, we divide the study of $\chi_{\Omega_{\eta_0}}(x)A_{\epsilon,\omega_3}g(x)$ into two different cases, i.e. $x\in\Omega_{\eta_0}\setminus\Omega_{4\epsilon}$ and $x\in\Omega_{4\epsilon}$. As we did in Section 5.2, we are going to work componentwise, that is, we consider $\mathbb{C}$-valued functions instead of $\mathbb{C}^4$-valued functions. With this in mind, for $g\in L^2(\Sigma\times(-1,1))$ we set

$$\widetilde{A}_\epsilon g(x):=\int_{-1}^{1}\int_\Sigma k(x-y_\Sigma-\epsilon s\nu(y_\Sigma))v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,$$

where k is given by (5.26).

In what follows, we can always assume that $x\in\mathbb{R}^3\setminus\Sigma$ because $\mathcal{L}(\Sigma)=0$. In case that $x\in\Omega_{4\epsilon}$, we can write $x=x_\Sigma+\epsilon t\nu(x_\Sigma)$ for some $t\in(-4,4)$, and then $\widetilde{A}_\epsilon g(x)$ coincides with $\widetilde{B}_\epsilon g(x_\Sigma,t)$ (see (5.28)) except for the term $u(t)$. Therefore, one can carry out all the arguments involved in the estimate of $\widetilde{B}_\epsilon g(x_\Sigma,t)$ (that is, from (5.28) to (5.48)) with minor modifications to get the following result: define

$$\widetilde{A}_* g(x_\Sigma,t):=\sup_{0<\epsilon\leq\eta_0/4}|\widetilde{A}_\epsilon g(x_\Sigma+\epsilon t\nu(x_\Sigma))|\quad\text{for }(x_\Sigma,t)\in\Sigma\times(-4,4).\tag{6.12}$$

Then, if η_0 is small enough, there exists $C>0$ only depending on η_0 such that

$$\|\sup_{|t|<4}\widetilde{A}_* g(\cdot,t)\|_{L^2(\Sigma)}\leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}\quad\text{for all } g\in L^2(\Sigma\times(-1,1)).\tag{6.13}$$

For the proof of (6.13), a remark is in order. The fact that in the present situation $t\in(-4,4)$ instead of $t\in(-1,1)$ (as in the definition of $\widetilde{B}_\epsilon g(x_\Sigma,t)$ in (5.28)) only affects the arguments used to get (5.47) at the comment just below (5.45). Now one should use that $\int_0^5|\log_2 r|^2\,dr<+\infty$ to prove the estimate analogous to (5.45) and to derive the counterpart of (5.47), that is,

$$\widetilde{A}_* g(x_\Sigma,t)\leq C\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{M}_* g(x_\Sigma)+\widetilde{T}_* g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma)+\widetilde{T}_*(\lambda_1 g)(x_\Sigma)+\widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big)$$

for all $(x_\Sigma,t)\in\Sigma\times(-4,4)$, being λ_1 and λ_2 the eigenvalues of the Weingarten map. Combining this estimate (whose right hand side is independent of $t\in(-4,4)$), the boundedness of $\widetilde{M}_*$ and $\widetilde{T}_*$ from $L^2(\Sigma\times(-1,1))$ to $L^2(\Sigma)$ (see (5.31)) and Proposition 2.2, we get (6.13).

Finally, thanks to (6.12), (2.3), Proposition 2.2 and (6.13), for η_0 small enough we conclude

$$\begin{aligned}\Big\|\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{4\epsilon}}|\widetilde{A}_\epsilon g|\Big\|_{L^2(\mathbb{R}^3)}&\leq\Big\|\sup_{|t|<4}\widetilde{A}_*g(P_\Sigma\cdot,t)\Big\|_{L^2(\Omega_{\eta_0})}\\&\leq C\Big\|\sup_{|t|<4}\widetilde{A}_*g(\cdot,t)\Big\|_{L^2(\Sigma)}\leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}.\end{aligned}\tag{6.14}$$

We now focus on $\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}\widetilde{A}_\epsilon$ for $0\leq\epsilon\leq\frac{\eta_0}{4}$. Similarly to what we did in (5.36), we set

$$g_\epsilon(y_\Sigma,s):=v(s)\det(1-\epsilon sW(y_\Sigma))g(y_\Sigma,s)\qquad\text{(see (5.33))}$$

and we split $\widetilde{A}_\epsilon g(x)=\widetilde{A}_{\epsilon,1}g(x)+\widetilde{A}_{\epsilon,2}g(x)+\widetilde{A}_{\epsilon,3}g(x)+\widetilde{A}_{\epsilon,4}g(x)$, where

$$\begin{aligned}\widetilde{A}_{\epsilon,1}g(x)&:=\int_{-1}^1\int_\Sigma\big(k(x-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x-y_\Sigma)\big)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\\widetilde{A}_{\epsilon,2}g(x)&:=\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|\leq4\mathrm{dist}(x,\Sigma)}k(x-y_\Sigma)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\\widetilde{A}_{\epsilon,3}g(x)&:=\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\mathrm{dist}(x,\Sigma)}\big(k(x-y_\Sigma)-k(x_\Sigma-y_\Sigma)\big)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds,\\\widetilde{A}_{\epsilon,4}g(x)&:=\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|>4\mathrm{dist}(x,\Sigma)}k(x_\Sigma-y_\Sigma)g_\epsilon(y_\Sigma,s)\,d\sigma(y_\Sigma)\,ds.\end{aligned}$$

From now on we assume $x\in\Omega_{\eta_0}\setminus\Omega_{4\epsilon}$ and, as always, $y_\Sigma\in\Sigma$. Note that

$$|(y_\Sigma-\epsilon s\nu(y_\Sigma))-y_\Sigma|\leq\epsilon\leq\frac{1}{4}\,\mathrm{dist}(x,\Sigma)\leq\frac{1}{4}\,|x-y_\Sigma|,$$

so (5.27) gives $|k(x-y_\Sigma-\epsilon s\nu(y_\Sigma))-k(x-y_\Sigma)|\leq C\epsilon|x-y_\Sigma|^{-3}$. Furthermore, we have that $|x-y_\Sigma|\geq C|x_\Sigma-y_\Sigma|$ for all $y_\Sigma\in\Sigma$ and some $C>0$ only depending on η_0. We can split the integral on Σ which defines $\widetilde{A}_{\epsilon,1}g(x)$ in dyadic annuli as we did in (5.39) (see also (5.42)) to obtain

$$\begin{aligned}|\widetilde{A}_{\epsilon,1}g(x)|&\leq C\int_{-1}^1\int_{|x_\Sigma-y_\Sigma|<\mathrm{dist}(x,\Sigma)}\frac{\epsilon|g_\epsilon(y_\Sigma,s)|}{\mathrm{dist}(x,\Sigma)^3}\,d\sigma(y_\Sigma)\,ds\\&\qquad+C\int_{-1}^1\sum_{n=0}^\infty\int_{2^n\mathrm{dist}(x,\Sigma)<|x_\Sigma-y_\Sigma|\leq2^{n+1}\mathrm{dist}(x,\Sigma)}\frac{\epsilon|g_\epsilon(y_\Sigma,s)|}{|x-y_\Sigma|^3}\,d\sigma(y_\Sigma)\,ds\\&\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma)+C\int_{-1}^1\sum_{n=0}^\infty\frac{1}{2^n}\int_{|x_\Sigma-y_\Sigma|\leq2^{n+1}\mathrm{dist}(x,\Sigma)}\frac{|g_\epsilon(y_\Sigma,s)|}{(2^n\mathrm{dist}(x,\Sigma))^2}\,d\sigma(y_\Sigma)\,ds\\&\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma)+C\sum_{n=0}^\infty\frac{1}{2^n}\int_{-1}^1M_*(g_\epsilon(\cdot,s))(x_\Sigma)\,ds\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma).\end{aligned}\tag{6.15}$$

Using that $|k(x-y_\Sigma)|\leq C|x-y_\Sigma|^{-2}\leq C\mathrm{dist}(x,\Sigma)^{-2}$ by (5.27), it is easy to show that

$$|\widetilde{A}_{\epsilon,2}g(x)|\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma).\tag{6.16}$$

Since $\mathrm{dist}(x,\Sigma)=|x-x_\Sigma|$, the same arguments as in (6.15) yield

$$|\widetilde{A}_{\epsilon,3}g(x)|\leq C\|v\|_{L^\infty(\mathbb{R})}\widetilde{M}_*g(x_\Sigma).\tag{6.17}$$

Finally, the same arguments as in (5.46) show that

$$|\widetilde{A}_{\epsilon,4}g(x)| \leq C\|v\|_{L^\infty(\mathbb{R})}\big(\widetilde{T}_*g(x_\Sigma)+\widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma)+\widetilde{T}_*(\lambda_1 g)(x_\Sigma)+\widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big). \tag{6.18}$$

Therefore, thanks to (6.15), (6.16), (6.17) and (6.18) we conclude that

$$\begin{aligned}\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}(x)|\widetilde{A}_\epsilon g(x)| \leq C\|v\|_{L^\infty(\mathbb{R})}\big(&\widetilde{M}_*g(x_\Sigma)+\widetilde{T}_*g(x_\Sigma)\\ &+\widetilde{T}_*(\lambda_1\lambda_2 g)(x_\Sigma)+\widetilde{T}_*(\lambda_1 g)(x_\Sigma)+\widetilde{T}_*(\lambda_2 g)(x_\Sigma)\big),\end{aligned}$$

and then, similarly to what we did in (6.14), a combination of (5.31) and Proposition 2.2 gives

$$\Big\|\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{\eta_0}\setminus\Omega_{4\epsilon}}|\widetilde{A}_\epsilon g|\Big\|_{L^2(\mathbb{R}^3)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}. \tag{6.19}$$

Finally, combining (6.14) and (6.19) we get that, if $\eta_0>0$ is small enough, then

$$\Big\|\sup_{0\leq\epsilon\leq\eta_0/4}\chi_{\Omega_{\eta_0}}|\widetilde{A}_\epsilon g|\Big\|_{L^2(\mathbb{R}^3)} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))}, \tag{6.20}$$

where $C>0$ only depends on η_0.

6.3. $A_{\epsilon,\omega_3}\to A_{0,\omega_3}$ in the strong sense when $\epsilon\to 0$ and conclusion of the proof of (3.9).

It only remains to put all the pieces together. Despite that the proof follows more or less the same lines as the one in Section 5.3, this case is easier. Namely, now we don't need to appeal to Lemma 5.1 because the dominated convergence theorem suffices (the developments in Section 6.1 hold for all $g\in L^2(\Sigma\times(-1,1))^4$, not only for a dense subspace like in Section 5.1).

Working component by component and using (6.20) we see that, if we set

$$A_{*,\omega_3}g(x):=\sup_{0\leq\epsilon\leq\eta_0/4}|A_{\epsilon,\omega_3}g(x)| \quad \text{for } x\in\mathbb{R}^3\setminus\Sigma,$$

then there exists $C>0$ only depending on $\eta_0>0$ (being η_0 small enough) such that

$$\|\chi_{\Omega_{\eta_0}}A_{*,\omega_3}g\|_{L^2(\mathbb{R}^3)^4} \leq C\|v\|_{L^\infty(\mathbb{R})}\|g\|_{L^2(\Sigma\times(-1,1))^4}. \tag{6.21}$$

Moreover, given $g\in L^2(\Sigma\times(-1,1))^4$, in (6.11) we showed that $\lim_{\epsilon\to0}A_{\epsilon,\omega_3}g(x)=A_{0,\omega_3}g(x)$ for $\mathcal{L}$-a.e. $x\in\mathbb{R}^3$. Thus (6.21) and the dominated convergence theorem show that

$$\lim_{\epsilon\to0}\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_3}-A_{0,\omega_3})g\|_{L^2(\mathbb{R}^3)^4}=0. \tag{6.22}$$

Then, combining (6.1), (6.9), (6.8), (6.10) and (6.22), we conclude that

$$\begin{aligned}\lim_{\epsilon\to0}\|(A_\epsilon(a)-A_0(a))g\|^2_{L^2(\mathbb{R}^3)^4} \leq \lim_{\epsilon\to0}\Big(&\|\chi_{\mathbb{R}^3\setminus\Omega_{\eta_0}}(A_\epsilon(a)-A_0(a))g\|^2_{L^2(\mathbb{R}^3)^4}\\ &+\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_1^a}-A_{0,\omega_1^a})g\|^2_{L^2(\mathbb{R}^3)^4}\\ &+\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_2^a}-A_{0,\omega_2^a})g\|^2_{L^2(\mathbb{R}^3)^4}\\ &+\|\chi_{\Omega_{\eta_0}}(A_{\epsilon,\omega_3}-A_{0,\omega_3})g\|^2_{L^2(\mathbb{R}^3)^4}\Big)=0\end{aligned}$$

for all $g\in L^2(\Sigma\times(-1,1))^4$. This is precisely (3.9).

7. Proof of Corollary 3.3

We first prove an auxiliary result.

Lemma 7.1. *Let $a \in \mathbb{C}\setminus\mathbb{R}$ and $\eta_0 > 0$ be such that (1.2) holds for all $0 < \epsilon \leq \eta_0$. If η_0 is small enough, then for any $0 < \eta \leq \eta_0$ and $V \in L^\infty(\mathbb{R})$ with $\mathrm{supp} V \subset [-\eta, \eta]$ we have that*

$$\|A_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\mathbb{R}^3)^4},$$
$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4},$$
$$\|C_\epsilon(a)\|_{L^2(\mathbb{R}^3)^4\to L^2(\Sigma\times(-1,1))^4}$$

are uniformly bounded for all $0 \leq \epsilon \leq \eta_0$, with bounds that only depend on a, η_0 and V. Furthermore, if η_0 is small enough there exists $\delta > 0$ only depending on η_0 such that

$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq \frac{1}{3} \tag{7.1}$$

for all $|a| \leq 1$, $0 \leq \epsilon \leq \eta_0$, $0 < \eta \leq \eta_0$ and all (δ, η)-small V.

Proof. The first statement in the lemma comes as a byproduct of the developments carried out in Sections 4, 5 and 6; see (4.4) for the case of $C_\epsilon(a)$, (5.50) and the paragraph which contains (5.3) for $B_\epsilon(a)$, and (6.7), (6.10) and (6.21) for $A_\epsilon(a)$. We should stress that these developments are valid for any $V \in L^\infty(\mathbb{R})$ with $\mathrm{supp} V \subset [-\eta, \eta]$, where $0 < \eta \leq \eta_0$, hence the (δ, η)-small assumption on V in Theorem 1.2 is only required to prove the explicit bound in the second part of the lemma, which will yield the strong convergence of $(1 + B_\epsilon(a))^{-1}$ and $(\beta + B_\epsilon(a))^{-1}$ to $(1 + B_0(a) + B')^{-1}$ and $(\beta + B_0(a) + B')^{-1}$, respectively, in Corollary 3.3.

Recall the decomposition

$$B_\epsilon(a) = B_{\epsilon,\omega_1^a} + B_{\epsilon,\omega_2^a} + B_{\epsilon,\omega_3} \tag{7.2}$$

given by (5.2). Thanks to (5.50), there exists $C_0 > 0$ only depending on η_0 such that

$$\|B_{\epsilon,\omega_3}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq C_0\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \leq \eta_0. \tag{7.3}$$

The comments in the paragraph which contains (5.3) and an inspection of the proof of [5, Lemma 3.4] show that there also exists $C_1 > 0$ only depending on η_0 such that, for any $|a| \leq 1$ and $j = 1, 2$,

$$\|B_{\epsilon,\omega_j^a}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq C_1\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \leq \eta_0. \tag{7.4}$$

Note that the kernel defining B_{ϵ,ω_2^a} is given by

$$\omega_2^a(x) = \frac{e^{-\sqrt{m^2-a^2}|x|} - 1}{4\pi}\, i\alpha \cdot \frac{x}{|x|^3}, \quad \text{so } |\omega_2^a(x)| = O\Big(\frac{\sqrt{|m^2 - a^2|}}{|x|}\Big) \text{ for } |x| \to 0.$$

Therefore, the kernel is of fractional type with respect to σ, but the estimate blows up as $|a| \to \infty$. This is the reason why we restrict ourselves to $|a| \leq 1$ in (7.4), where we have a uniform bound with respect to a. However, for proving Theorem 1.2, one fixed $a \in \mathbb{C}\setminus\mathbb{R}$ suffices, say $a = i$ (see (3.12) and (3.13)).

From (7.2), (7.3) and (7.4), we derive that

$$\|B_\epsilon(a)\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq (C_0 + 2C_1)\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} \quad \text{for all } 0 < \epsilon \leq \eta_0. \tag{7.5}$$

If V is (δ, η)-small (see Definition 1.1) then $\|V\|_{L^\infty(\mathbb{R})} \leq \frac{\delta}{\eta}$, so (1.5) yields

$$\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})} = \eta\|V\|_{L^\infty(\mathbb{R})} \leq \delta.$$

Taking $\delta > 0$ small enough so that $(C_0 + 2C_1)\delta \leq \frac{1}{3}$, from (7.5) we finally get (7.1) for all $0 < \epsilon \leq \eta_0$. The case of $B_0(a)$ follows similarly, just recall the paragraph previous to (5.33) taking into account that the dependence of the norm of $B_0(a)$ with respect to $\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}$ is the same as in the case of $0 < \epsilon \leq \eta_0$. □

7.1. **Proof of Corollary 3.3.**

We are going to prove the corollary for $(H + \mathbf{V}_\epsilon - a)^{-1}$, the case of $(H + \beta\mathbf{V}_\epsilon - a)^{-1}$ follows by the same arguments. Let η_0, $\delta > 0$ be as in Lemma 7.1 and take $a \in \mathbb{C} \setminus \mathbb{R}$ with $|a| \leq 1$. It is trivial to show that

$$\|B'\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq C\|u\|_{L^\infty(\mathbb{R})}\|v\|_{L^\infty(\mathbb{R})}$$

for some $C > 0$ only depending on Σ. Using (1.5), we can take a smaller $\delta > 0$ so that, for any (δ, η)-small V with $0 < \eta \leq \eta_0$,

$$\|B'\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq C\delta \leq \frac{1}{3}.$$

Then, from this and (7.1) in Lemma 7.1 (with $\epsilon = 0$) we deduce that

$$\begin{aligned}\|(1 + B_0(a) + B')g\|_{L^2(\Sigma\times(-1,1))^4} &\geq \|g\|_{L^2(\Sigma\times(-1,1))^4} - \|(B_0(a) + B')g\|_{L^2(\Sigma\times(-1,1))^4}\\ &\geq \frac{1}{3}\|g\|_{L^2(\Sigma\times(-1,1))^4}\end{aligned}$$

for all $g \in L^2(\Sigma \times (-1, 1))^4$. Therefore, $1 + B_0(a) + B'$ is invertible and

$$\|(1 + B_0(a) + B')^{-1}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq 3.$$

This justifies the last comment in the corollary. Similar considerations also apply to $1 + B_\epsilon(a)$, so in this case we deduce that

$$\|(1 + B_\epsilon(a))^{-1}\|_{L^2(\Sigma\times(-1,1))^4\to L^2(\Sigma\times(-1,1))^4} \leq \frac{3}{2} \tag{7.6}$$

for all $0 < \epsilon \leq \eta_0$. Note also that

$$\begin{aligned}&(1 + B_\epsilon(a))^{-1} - (1 + B_0(a) + B')^{-1}\\ &\qquad = (1 + B_\epsilon(a))^{-1}(B_0(a) + B' - B_\epsilon(a))(1 + B_0(a) + B')^{-1}.\end{aligned} \tag{7.7}$$

Given $g \in L^2(\Sigma \times (-1, 1))^4$, set $f = (1 + B_0(a) + B')^{-1}g \in L^2(\Sigma \times (-1, 1))^4$. Then, by (7.7) and (7.6), we see that

$$\begin{aligned}&\big\|\big((1 + B_\epsilon(a))^{-1} - (1+B_0(a) + B')^{-1}\big)g\big\|_{L^2(\Sigma\times(-1,1))^4}\\ &\qquad = \|(1 + B_\epsilon(a))^{-1}(B_0(a) + B' - B_\epsilon(a))f\|_{L^2(\Sigma\times(-1,1))^4}\\ &\qquad \leq \frac{3}{2}\,\|(B_0(a) + B' - B_\epsilon(a))f\|_{L^2(\Sigma\times(-1,1))^4}.\end{aligned} \tag{7.8}$$

By (3.10) in Theorem 3.2, the right hand side of (7.8) converges to zero when $\epsilon \to 0$. Therefore, we deduce that $(1 + B_\epsilon(a))^{-1}$ converges strongly to $(1 + B_0(a) + B')^{-1}$ when $\epsilon \to 0$. Since the composition of strongly convergent operators is strongly convergent, using (3.5) and Theorem 3.2, we finally obtain the desired strong convergence

$$(H + \mathbf{V}_\epsilon - a)^{-1} \to (H - a)^{-1} + A_0(a)\big(1 + B_0(a) + B'\big)^{-1}C_0(a) \quad \text{when } \epsilon \to 0.$$

Corollary 3.3 is finally proved.

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