A PROOF OF THE INTEGRAL IDENTITY CONJECTURE, II

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Abstract. In this note, using Cluckers-Loeser's theory of motivic integration, we prove the integral identity conjecture with framework a localized Grothendieck ring of varieties over an arbitrary base field of characteristic zero.

1. Statement of conjecture and main theorem

1.1. **Equivariant Grothendieck ring of varieties.** Let k be a field of characteristic zero, S an algebraic k-variety, and Var_S the category of S-varieties. Let $K_0(\mathrm{Var}_S)$ be the Grothendieck ring of Var_S, which is the quotient of the free abelian group generated by the S-isomorphism classes $[X \to S]$ in Var_S such that $[X \to S] = [Y \to S] + [X \setminus Y \to S]$ for any Zariski closed subvariety Y of X. It is a commutative ring with respect to fiber product.

We consider the projective system of $\mu_n = \mathrm{Spec}k[t]/(t^n - 1)$ with transitions $\mu_{mn} \to \mu_n$ given by $\lambda \mapsto \lambda^m$, and define $\hat{\mu} = \varprojlim \mu_n$. A good μ_n-action on an S-variety X is a group action each of whose orbits is contained in an affine k-subvariety of X, a good $\hat{\mu}$-action on S-variety X is a good μ_n-action for some n. The $\hat{\mu}$-equivariant Grothendieck group $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ of S-varieties endowed with good $\hat{\mu}$-action is the quotient of the free abelian group generated by the $\hat{\mu}$-equivariant isomorphism classes $[X \to S, \sigma]$, σ being a good $\hat{\mu}$-action on S-variety X, modulo the conditions $[X \to S, \sigma] = [Y \to S, \sigma|_Y] + [X \setminus Y \to S, \sigma|_{X\setminus Y}]$, for Y σ-stable Zariski closed in X, and $[X \times_k \mathbb{A}_k^n \to S, \sigma] = [X \times_k \mathbb{A}_k^n \to S, \sigma']$, whenever σ and σ' lift the same $\hat{\mu}$-action on $X \to S$ to an affine action on $X \times \mathbb{A}_k^n \to S$. The structure of a commutative ring with unity on $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ is given by fiber product. Let $\mathbb{L}$ be the class of the trivial line bundle over S, with trivial $\hat{\mu}$-action. We write $\mathcal{M}_S^{\hat{\mu}}$ for the localization of $K_0^{\hat{\mu}}(\mathrm{Var}_S)$ inverting $\mathbb{L}$, and $\mathcal{M}_{S,\mathrm{loc}}^{\hat{\mu}}$ for the localization of $\mathcal{M}_S^{\hat{\mu}}$ inverting the elements $1 - \mathbb{L}^{-n}$, for every n in $\mathbb{N}_{>0}$. The ring $\mathcal{M}_{\mathrm{Spec}k,\mathrm{loc}}^{\hat{\mu}}$ is rewritten simply by $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

Any morphism of k-varieties $g : S \to S'$ induces a ring morphism $g^* : \mathcal{M}_{S'}^{\hat{\mu}} \to \mathcal{M}_S^{\hat{\mu}}$ by fiber product, and induces a group morphism $g_! : \mathcal{M}_S^{\hat{\mu}} \to \mathcal{M}_{S'}^{\hat{\mu}}$ by composition. When S' is $\mathrm{Spec}k$ we replace the symbol $g_!$ by the symbol $\int_S$. Let loc denote the natural morphism $\mathcal{M}_k^{\hat{\mu}} \to \mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

Consider the ring $\mathcal{M}_S^{\hat{\mu}}[[T]]$, and its subset $\mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}}$ of rational series, which consists of $\mathcal{M}_S^{\hat{\mu}}$-polynomials in variables $\frac{\mathbb{L}^p T^q}{(1-\mathbb{L}^p T^q)}$, with (p,q) in $\mathbb{Z} \times \mathbb{N}_{>0}$. There exists by [2] a unique $\mathcal{M}_S^{\hat{\mu}}$-linear morphism $\lim_{T\to\infty} : \mathcal{M}_S^{\hat{\mu}}[[T]]_{\mathrm{sr}} \to \mathcal{M}_S^{\hat{\mu}}$ such that $\lim_{T\to\infty} \frac{\mathbb{L}^p T^q}{(1-\mathbb{L}^p T^q)} = -1$, for every (p,q) in $\mathbb{Z} \times \mathbb{N}_{>0}$.

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1.2. **Motivic nearby cycles.** Let X be a smooth algebraic k-variety of pure dimension d. For $n \geq 1$, let $\mathcal{L}_n(X)$ be the k-scheme of n-jets on X, which represents the functor sending a k-algebra A to the set of morphisms of k-schemes $\mathrm{Spec}(A[t]/(t^{n+1})) \to X$. These schemes together with morphisms $\mathcal{L}_m(X) \to \mathcal{L}_n(X)$ $(m \geq n)$ induced by truncation form a projective system, and we denote its limit by $\mathcal{L}(X)$. Let f be a regular function on X with nonempty zero locus X_0. For $n \geq 1$, let $\mathcal{X}_n(f)$ be the k-variety of n-jets φ in $\mathcal{L}_n(X)$ with $f(\varphi) = t^n \mod t^{n+1}$, which admits an obvious morphism to X_0 and the natural μ_n-action $(\lambda, \varphi(t)) \mapsto \varphi(\lambda t)$. Write $[\mathcal{X}_n(f)]$ for the class of $\mathcal{X}_n(f) \to X_0$ in $\mathcal{M}_{X_0}^{\hat{\mu}}$. By [2], the series $Z_f(T) := \sum_{n\geq 1}[\mathcal{X}_n(f)]\mathbb{L}^{-nd}T^n$ in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ is rational, and the limit $\mathcal{S}_f := -\lim_{T\to\infty} Z_f(T)$ in $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ is called the motivic nearby cycles of f. If x is a closed point in X_0, we may consider the motivic Milnor fiber of f at x, $\mathcal{S}_{f,x} = i_x^*\mathcal{S}_f$ in $\mathcal{M}_k^{\hat{\mu}}[[T]]$, where i_x is the inclusion of $\{x\}$ in X_0.

1.3. **Conjecture and main theorem.** The integral identity conjecture plays a crucial role in Kontsevich-Soibelman's theory of motivic Donaldson-Thomas invariants for noncommutative Calabi-Yau threefolds [4]. We now state the version for regular functions of the conjecture (for the full version, see [4, Conjecture 4.4]).

Conjecture 1.1 ([4])**.** *Let (x, y, z) be coordinates of the k-vector space $k^d = k^{d_1} \times k^{d_2} \times k^{d_3}$. Let f be in $k[x, y, z]$ such that $f(0, 0, 0) = 0$ and $f(\lambda x, \lambda^{-1}y, z) = f(x, y, z)$ for λ in $\mathbb{G}_{m,k}$. Then the identity $\int_{\mathbb{A}_k^{d_1}} i^*\mathcal{S}_f = \mathbb{L}^{d_1}\mathcal{S}_{\tilde{f},0}$ holds in $\mathcal{M}_k^{\hat{\mu}}$, with $\tilde{f}$ the restriction of f to $\mathbb{A}_k^{d_3}$, and i the inclusion of $\mathbb{A}_k^{d_1}$ in $f^{-1}(0)$.*

Here, we identify $\mathbb{A}_k^{d_1}$ with $\mathbb{A}_k^{d_1} \times \{0\} \times \{0\}$, hence by the homogeneity of f, we consider it as a subvariety of $f^{-1}(0)$. We also identify $\mathbb{A}_k^{d_3}$ with $\{0\} \times \{0\} \times \mathbb{A}_k^{d_3}$, thus by definition, $\tilde{f}(z) = f(0, 0, z)$.

The conjecture was first proved in [5] in the case where f is either a function of Steenbrink type or the composition of a pair of regular functions with a polynomial in two variables. In [6, Theorem 1.2], we show that, if the field k is algebraically closed, Conjecture 1.1 holds in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. Recently, under the weaker assumption that the base field k contains all roots of unity, Nicaise and Payne [7] prove the conjecture with the full context $\mathcal{M}_k^{\hat{\mu}}$.

The main result of this note is the following theorem.

Theorem 1.2. *Conjecture 1.1 is true in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, namely,* $\mathrm{loc}\left(\int_{\mathbb{A}_k^{d_1}} i^*\mathcal{S}_f\right) = \mathrm{loc}\left(\mathbb{L}^{d_1}\mathcal{S}_{\tilde{f},0}\right)$.

Note that our proof for the theorem does not use the assumption that k is algebraically closed, that is, k may be any field of characteristic zero. The materials for the proof are in Cluckers-Loeser's motivic integration of constructible motivic functions [1].

2. Measurable subassignments

2.1. **Definable subassignments.** We consider the formalism of Cluckers and Loeser [1] with a concrete Denef-Pas language $\mathcal{L}_{\mathrm{DP,P}}$ consisting of the ring language $\{+, -, \cdot, 0, 1\}$ for valued fields, also the ring language for residue fields, and the Presburger language $\{+, -, 0, 1, \leq\} \cup \{\equiv_n \mid n \in \mathbb{N}_{>0}\}$ for value groups, where $\equiv_n$ is the equivalence relation modulo n. Let Field_k be the category of algebraically closed fields K containing k in the $\mathcal{L}_{\mathrm{DP,P}}$-language, where sentences take coefficients in k and $k((t))$, and morphisms of Field_k are field morphisms. The theory corresponding to Field_k is the theory of algebraically closed fields containing k, each model of this theory is a triple $(K((t)), K, \mathbb{Z})$ with K in Field_k. The valued fields $K((t))$

are endowed with a natural valuation map $\mathrm{ord}_t : K((t))^\times \to \mathbb{Z}$ augmented by $\mathrm{ord}_t(0) = +\infty$, and with a natural angular component map $\overline{\mathrm{ac}} : K((t)) \to K$, with convention $\overline{\mathrm{ac}}(0) = 0$.

A basic affine definable subassignment has the form $h[m,n,r]$, where $h[m,n,r](K) = K((t))^m \times K^n \times \mathbb{Z}^r$. More generally, if $W = \mathcal{X} \times X \times \mathbb{Z}^r$ with $\mathcal{X}$ a $k((t))$-variety and X a k-variety, we define $h_W(K) := \mathcal{X}(K((t))) \times X(K) \times \mathbb{Z}^r$. An arbitrary definable subassignment is a set of points in $h[m,n,r]$, or in h_W, satisfying a given formula φ.

Among a broad collection of definable subassignments, we now only consider the category Def_k of affine definable subassignments where objects are pairs $(\mathsf{Z}, h[m,n,r])$ with Z being a definable subassignment of $h[m,n,r]$, and a morphism $(\mathsf{Z}, h[m,n,r]) \to (\mathsf{Z}', h[m',n',r'])$ is a definable morphism $\mathsf{Z} \to \mathsf{Z}'$. Due to [1], by a definable morphism $\mathsf{Z} \to \mathsf{Z}'$ one means a morphism of subassignments $\mathsf{Z} \to \mathsf{Z}'$ such that its graph is a definable subassignment of $h[m+m', n+n', r+r']$. Let RDef_k be the full subcategory of Def_k whose objects are definable subassignments of $h_{\mathbb{A}^n_k}$ for n in $\mathbb{N}$.

Let X be an affine algebraic k-variety. A *(good) μ_n-action* on h_X is a definable morphism of definable subassignments $h_{\mu_n \times X} \to h_X$ such that the corresponding morphism of k-varieties $\mu_n \times_k X \to X$ is a (good) μ_n-action. A *good $\hat{\mu}$-action* on h_X is a good μ_n-action on h_X for some integer $n \geq 1$. For an algebraic $k((t))$-variety $\mathcal{X}$, the definable subassignment $h_{\mathcal{X}}$ admits a natural μ_n-action $h_{\mu_n} \times h_{\mathcal{X}} \to h_{\mathcal{X}}$ induced by $(\lambda, t) \mapsto \lambda t$, for all $n \in \mathbb{N}_{>0}$. The profinite group scheme $\hat{\mu}$ acts naturally on $h_{\mathcal{X}}$ via μ_n for some integer $n \geq 1$.

The Grothendieck semiring and ring of the category RDef_k are defined in [1, Section 5.1.2], however, in this note we only want to work with its $\hat{\mu}$-equivariant version. By definition, the *$\hat{\mu}$-equivariant Grothendieck group* $K_0^{\hat{\mu}}(\mathrm{RDef}_k)$ is the quotient of the free abelian group generated by definable $\hat{\mu}$-equivariant isomorphism classes $[\mathsf{X}, \sigma]$, with X in RDef_k endowed with a good $\hat{\mu}$-action σ, modulo the relations: $[\mathsf{X}, \sigma] = [\mathsf{Y}, \sigma|_{\mathsf{Y}}] + [\mathsf{X} \setminus \mathsf{Y}, \sigma|_{\mathsf{X}\setminus\mathsf{Y}}]$, for Y σ-stable definable subassignment of X, and $[\mathsf{X} \times h_{\mathbb{A}^m_k}, \sigma] = [\mathsf{X} \times h_{\mathbb{A}^m_k}, \sigma']$, whenever σ and σ' lift the same $\hat{\mu}$-action on X to an affine action on $\mathsf{X} \times h_{\mathbb{A}^m_k}$, for any integer $m \geq 0$. The cartesian product of subassignments induces a commutative with unity ring structure on $K_0^{\hat{\mu}}(\mathrm{RDef}_k)$.

Put $\mathbb{A} := \mathbb{Z}\left[\mathbb{L}, \mathbb{L}^{-1}, \frac{1}{1-\mathbb{L}^{-n}} \mid n \in \mathbb{N}_{>0}\right]$ where, by abuse of notation, $\mathbb{L}$ also stands for the class of $h_{\mathbb{A}^1_k}$ in $K_0^{\hat{\mu}}(\mathrm{RDef}_k)$. By quantifier elimination for the theory of algebraically closed fields containing k (i.e., the Chevalley constructibility), definable subassignments of $h_{\mathbb{A}^n_k}$, for n in $\mathbb{N}$, are defined by formulas without quantifiers, thus objects in RDef_k may be viewed as constructible sets. This correspondence is compatible with the $\hat{\mu}$-actions mentioned above. Hence, there are canonical isomorphisms of rings $K_0^{\hat{\mu}}(\mathrm{RDef}_k) \cong K_0^{\hat{\mu}}(\mathrm{Var}_k)$ and $K_0^{\hat{\mu}}(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A} \cong \mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

2.2. **Motivic measure.** In view of Theorem 10.1.1 in the paper [1], there is a unique functor from Def_k to the category of abelian semigroups, $\mathsf{X} \mapsto IC_+(\mathsf{X})$, which assigns to the projection $\mathsf{X} \to h_{\mathrm{Spec}k}$ a morphism of semigroups $\mu : \mathrm{IC}_+(\mathsf{X}) \to \mathrm{IC}_+(h_{\mathrm{Spec}k})$, such that the eight axioms (A1) to (A8) in that theorem, characterizing an integration theory, are satisfied. By [1, Proposition 12.2.2], if X is a definable subassignment of $h[m,n,0]$ which is bounded, i.e., there exists an s in $\mathbb{N}$ such that X is contained in the subassignment of $h[m,n,0]$ defined by $\mathrm{ord}_t x_i \geq -s$ for $1 \leq i \leq m$, then the characteristic function $\mathbf{1}_{\mathsf{X}}$ is in $\mathrm{IC}_+(\mathsf{X})$. In this case we call X *motivically measurable* and its motivic measure $\mu(\mathsf{X}) := \mu(\mathbf{1}_{\mathsf{X}}) \in \mathrm{IC}_+(h_{\mathrm{Spec}k})$. Also by [1], there is a canonical morphism from $\mathrm{IC}_+(h_{\mathrm{Spec}k})$ to $K_0(\mathrm{Var}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$, thus by composition we can consider $\mu(\mathsf{X})$ as an element of $K_0(\mathrm{Var}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$.

In the previous definition of boundedness, if we can take $s = 0$, X is called *small* (see [1, Section 16.3] for a more general definition of *small definable subassignments*). There is a canonical action of $\hat{\mu}$ on $h[m,0,0]$ induced by $(\lambda,t)\mapsto \lambda t$. We say that the definable subassignment X is stable under this action if there exists a natural number $n\geq 1$ such that, for every $x=(x_1(t),\ldots,x_m(t))$ in X and λ in μ_n, the point $\lambda\cdot x=(x_1(\lambda t),\ldots,x_m(\lambda t))$ is in X. Since formulas defining X are in the language $\mathcal{L}_{\mathrm{DP,P}}$, by quantifier elimination for algebraically closed fields, they also define a semi-algebraic subset X of the arc space $\mathcal{L}(\mathbb{A}_k^m)$ of $\mathbb{A}_k^m$. The assignment $\mathsf{X}\mapsto X$ carries the canonical $\hat{\mu}$-action on $h[m,0,0]$ to the canonical $\hat{\mu}$-action on $\mathcal{L}(\mathbb{A}_k^m)$, and in that way, X is also stable for the action on $\mathcal{L}(\mathbb{A}_k^m)$. As in [1, Theorem 16.3.1, Remark 16.3.2] we can see that X is measurable as X is measurable, and that since (with the above action) $\mu'(X)$ is in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, the measure $\mu(\mathsf{X})$ of X is also in $K_0^{\hat{\mu}}(\mathrm{RDef}_k)\otimes_{\mathbb{Z}[\mathbb{L}]}\mathbb{A}\cong\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. Here, as explained in [1, Theorem 16.3.1], μ' stands for Denef-Loeser's motivic measure [3], and further by [1, Remark 16.3.2], we can consider that this measure takes value in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$.

3. Sketch of proof of Theorem 1.2

Let us consider the motivic zeta function $Z_f(T)$ of the polynomial f in the theorem. Write the n-th coefficient of $\int_{\mathbb{A}_k^{d_1}} i^*Z_f(T)$ as follows:

$$\int_{\mathbb{A}_k^{d_1}} i^*[\mathcal{X}_n(f)]\mathbb{L}^{-nd}=[\mathcal{U}_n]\,\mathbb{L}^{-nd}+[\mathcal{W}_n]\,\mathbb{L}^{-nd},$$

where $\mathcal{U}_n$ is the set of jets in $\mathcal{X}_n(f)\cap\big(\mathcal{L}_n(\{0\}\times\mathbb{A}_k^{d_2+d_3})\cup\mathcal{L}_n(\mathbb{A}_k^{d_1}\times\{0\}\times\mathbb{A}_k^{d_3})\big)$ originated in $\mathbb{A}_k^{d_1}$, $\mathcal{W}_n$ is the set of jets in $\mathcal{X}_n(f)$ that are originated in $\mathbb{A}_k^{d_1}$ and not contained in $\mathcal{U}_n$. The elements $\mathcal{U}_n$ and $\mathcal{W}_n$ are μ_n-stable, and they give rise to rational series with coefficients in $\mathcal{M}_k^{\hat{\mu}}$. Because of the hypothesis on f, taking $\lim_{T\to\infty}$ for the decomposition of the $\int_{\mathbb{A}_k^{d_1}} i^*Z_f(T)$ reduces the proof of Theorem 1.2 to checking that $\mathrm{loc}\big(\lim_{T\to\infty}\sum_{n\geq 1}[\mathcal{W}_n]\,\mathbb{L}^{-nd}T^n\big)$ vanishes in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$. The latter has been a challenging problem, and the previous attempts [5], [6] and [7] for solving it had to use certain additional assumptions.

Now we write $\mathcal{W}_{n,m}$ for the set of $(\varphi_1,\varphi_2,\varphi_3)$ in $\mathcal{W}_n$ with $\mathrm{ord}_t\varphi_1+\mathrm{ord}_t\varphi_2=m$, and let us observe that it is still stable under the canonical μ_n-action. In what follows we only consider the set $\mathcal{W}_{n,m}$ when it is nonempty. Suggested ideally from [6], with the hypothesis of Theorem 1.2, our approach is to construct a constructible set $\widetilde{\mathcal{W}}_{n,m}$ endowed with a good $\hat{\mu}$-action and a $\hat{\mu}$-equivariant constructible surjective morphism $\mathcal{W}_{n,m}\to\widetilde{\mathcal{W}}_{n,m}$ such that its fiber over a point of residue field k' is isomorphic to $\mathbb{A}_{k'}^{n+1}\setminus\mathbb{A}_{k'}^{n+1-m}$. Once we can do this, it follows (not obvious) that $[\mathcal{W}_{n,m}]=[\widetilde{\mathcal{W}}_{n,m}]\mathbb{L}^{n+1}(1-\mathbb{L}^{-m})$ in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}}$, and then, Theorem 1.2 will be proved completely, because the rest of the proof is elementary.

Our idea is that we use Cluckers-Loeser's motivic integration [1], together to a slight development to $\hat{\mu}$-action context, as seen in Section **??**. Clearly, $\mathbb{G}_{m,k((t))}$ is an algebraic group and the action of $\mathbb{G}_{m,k((t))}$ on the $k((t))$-variety

$$\mathscr{X}:=\big(\mathbb{A}_{k((t))}^{d_1}\setminus\{0\}\big)\times_{k((t))}\big(\mathbb{A}_{k((t))}^{d_2}\setminus\{0\}\big)\times_{k((t))}\mathbb{A}_{k((t))}^{d_3}$$

given by

$$\tau\cdot(\varphi_1,\varphi_2,\varphi_3):=(\tau\varphi_1,\tau^{-1}\varphi_2,\varphi_3)$$

is free. It follows that the space of its orbits is an algebraic $k((t))$-variety and the canonical projection $\phi:\mathscr{X}\to\mathscr{Y}:=\mathscr{X}/\mathbb{G}_{m,k((t))}$ is a surjective morphism of algebraic $k((t))$-varieties. This morphism ϕ induces a definable morphism $h_\phi:h_{\mathscr{X}}\to h_{\mathscr{Y}}$ of definable subassignments in the theory of Cluckers and Loeser. Take the preimage $\mathcal{W}_{n,m}^{\infty}$ of $\mathcal{W}_{n,m}$ under the canonical morphism $\mathcal{L}(\mathbb{A}_k^d) \to \mathcal{L}_n(\mathbb{A}_k^d)$. By [1, Section 16.3], $\mathcal{W}_{n,m}^{\infty}$ corresponds to a small definable subassignment $\mathsf{V}_{n,m}$ of the basic definable subassignment $h[d,0,0]$, both have the same measure $[\mathcal{W}_{n,m}]\mathbb{L}^{-nd}$ in $\mathcal{M}_{\mathrm{loc}}^{\hat{\mu}} \cong K_0^{\hat{\mu}}(\mathrm{RDef}_k) \otimes_{\mathbb{Z}[\mathbb{L}]} \mathbb{A}$ in Denef-Loeser's motivic measure and Cluckers-Loeser's motivic measure, endowed with $\hat{\mu}$-action, respectively. By Denef-Pas quantifier elimination theorem, $h_\phi(\mathsf{V}_{n,m})$ is a definable subassignment and the restriction $h_\phi|_{\mathsf{V}_{n,m}}$ is a definable morphism. Moreover, we can prove that $h_\phi(\mathsf{V}_{n,m})$ and $h_\phi|_{\mathsf{V}_{n,m}}$ are small, and that the fiber of $h_\phi|_{\mathsf{V}_{n,m}}$ over $[(\varphi_1,\varphi_2,\varphi_3)] \in h_\phi(\mathsf{V}_{n,m})$ (of residue field k') equals

$$\left\{\tau \in h_{\mathbb{G}_{m,k'((t))}} \mid -\mathrm{ord}_t\varphi_1 \leq \mathrm{ord}_t\tau < \mathrm{ord}_t\varphi_2\right\} \cong \left\{\tau \in h_{\mathbb{G}_{m,k'((t))}} \mid 0 \leq \mathrm{ord}_t\tau < m\right\}.$$

By [1, Section 16.3], $h_\phi|_{\mathsf{V}_{n,m}}$ gives rise to a μ_n-equivariant semi-algebraic morphism of semi-algebraic sets in Denef-Loeser's framework $p : \mathcal{W}_{n,m}^{\infty} \to \widetilde{\mathcal{W}}_{n,m}^{\infty}$ with fiber over a point of residue field k' isomorphic to $\{\tau \in \mathcal{L}(\mathbb{A}_{k'}^1) \mid 0 \leq \mathrm{ord}_t\tau < m\}$. Finally, we can show that p induces a μ_n-equivariant constructible morphism of constructible sets $p_n : \mathcal{W}_{n,m} \to \widetilde{\mathcal{W}}_{n,m}$ with fiber

$$\{\tau \in \mathcal{L}_n(\mathbb{A}_{k'}^1) \mid 0 \leq \mathrm{ord}_t\tau < m\} \cong \mathbb{A}_{k'}^{n+1} \setminus \mathbb{A}_{k'}^{n+1-m},$$

as desired.

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