IMPROVED $A_1 - A_\infty$ AND RELATED ESTIMATES FOR COMMUTATORS OF ROUGH SINGULAR INTEGRALS

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ABSTRACT. An $A_1 - A_\infty$ estimate improving a previous result in [22] for $[b, T_\Omega]$ with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and $b \in \text{BMO}$ is obtained. Also a new result in terms of the A_∞ constant and the one supremum $A_q - A_\infty^{\exp}$ constant is proved, providing a counterpart for commutators of the result obained in [19]. Both of the preceding results rely upon a sparse domination result in terms of bilinear forms which is established using techniques from [13].

1. INTRODUCTION

We recall that a weight w, namely a non negative locally integrable function, belongs to A_p if

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{\frac{1}{1-p}}\right)^{p-1} < \infty \quad 1 < p < \infty$$

or in the case $p = 1$ if

$$[w]_{A_1} = \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty.$$

Given $\Omega \in L(\mathbb{S}^{n-1})$ with $\int_{\mathbb{S}^{n-1}} \Omega = 0$ we define the rough singular integral T_Ω by

$$T_\Omega f(x) = pv \int_{\mathbb{R}^n} \frac{\Omega(y')}{|y|^n} f(x-y)dy$$

where $y' = \frac{y}{|y|}$.

During the last years an increasing interest in the study of the sharp dependence on the A_p constants of rough singular integrals has appeared. In particular it was established in [10] that

$$\|T_\Omega\|_{L^2(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_2}^2.$$

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Recently the following sparse domination (very recently reproved in [13] for the case $\Omega \in L^\infty(\mathbb{S}^{n-1})$) was established in [3].

Theorem. *For all $1<p<\infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$, we have that*

$$\Big|\int_{\mathbb{R}^n} T_\Omega(f) g dx\Big| \leq c_n C_T s' \sup_{\mathcal{S}} \sum_{Q\in\mathcal{S}} \Big(\int_Q |f|\Big)\Big(\frac{1}{|Q|}\int_Q |g|^s\Big)^{1/s}, \tag{1.1}$$

where each $\mathcal{S}$ is a sparse family of a dyadic lattice $\mathcal{D}$,

$$\begin{cases} 1<s<\infty & \text{if } \Omega \in L^\infty(\mathbb{S}^{n-1}) \\ q' \leq s < \infty & \text{if } \Omega \in L^{q,1}\log L(\mathbb{S}^{n-1}) \end{cases}$$

and

$$C_T = \begin{cases} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1}),} & \text{if } \Omega \in L^\infty(\mathbb{S}^{n-1}) \\ \|\Omega\|_{L^{q,1}\log L(\mathbb{S}^{n-1})} & \text{if } \Omega \in L^{q,1}\log L(\mathbb{S}^{n-1}). \end{cases}$$

The preceding sparse domination was widely exploited in [20]. Among other estimates, the following $A_1 - A_\infty$ estimate was established in that paper (see Lemma 2.2 in Section 2 for the definition of the A_∞ constant)

$$\|T_\Omega\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{\frac{1}{p'}}.$$

The preceding inequality is an improvement of the following estimate established earlier in [22]

$$\|T_\Omega\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{1+\frac{1}{p'}}.$$

Now we recall that the commutator of an operator T and a symbol b is defined as

$$[b,T]f(x) = T(bf)(x) - b(x)Tf(x).$$

In the case of T being a Calderón-Zygmund operator this operator was introduced by R.R. Coifman, R. Rochberg and G. Weiss in [2]. They established that $b \in \mathrm{BMO}$ is a sufficient condition for $[b,T]$ to be bounded on L^p for every $1<p<\infty$ and also a converse result in terms of the Riesz transforms, namely that the boundedness of $[b,R_j]$ on L^p for some $1<p<\infty$ and for every Riesz transform implies that $b \in \mathrm{BMO}$.

In [22] the following estimate for commutators of rough singular integrals and a symbol $b \in \mathrm{BMO}$ was obtained.

$$\|[b,T_\Omega]\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{2+\frac{1}{p}}. \tag{1.2}$$

One of the main goals of this paper is to improve the dependence on the $[w]_{A_\infty}$ constant in (1.2). Our result is the following.

Theorem 1.1. *Let T_Ω be a rough homogeneous singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$ and let $b \in \mathrm{BMO}$. For every weight w we have that*

$$\|[b, T_\Omega]\|_{L^p(M_r(w)) \to L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}} (p')^3 p^2 (r')^{1+\frac{1}{p'}} \tag{1.3}$$

where $r > 1$. Assuming additionally that $w \in A_\infty$

$$\|[b, T_\Omega]\|_{L^p(M(w)) \to L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}} (p')^3 p^2 [w]_{A_\infty}^{1+\frac{1}{p'}}$$

and, furthermore, if $w \in A_1$, then

$$\|[b, T_\Omega]\|_{L^p(w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}} (p')^3 p^2 [w]_{A_1}^{\frac{1}{p}} [w]_{A_\infty}^{1+\frac{1}{p'}}.$$

Very recently a conjecture left open by K. Moen and A. Lerner in [18] was solved by K. Li in [19]. Actually he obtained a more general result.

Theorem. *Let T be a Calderón-Zygmund operator or a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then for every $1 < q < p < \infty$*

$$\|T\|_{L^p(w)} \leq c_{n,p,q} c_T [w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}}$$

where

$$[w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}} = \sup_Q \langle w \rangle_Q \langle w^{\frac{1}{1-q}} \rangle_Q^{\frac{q-1}{p}} \exp\left(\langle \log w^{-1} \rangle_Q\right)^{\frac{1}{p'}}$$

and

$$c_T = \begin{cases} \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} & \text{if } T = T_\Omega \text{ with } \Omega \in L^\infty(\mathbb{S}^{n-1}), \\ c_K + \|T\|_{L^2} + \|\omega\|_{Dini} & \text{if } T \text{ is an } \omega\text{-Calderón-Zygmund operator.} \end{cases}$$

This result can be regarded as an improvement of the linear dependence on the A_q constant established in [20], and that, as it was stated there, follows from the linear dependence on the A_1 constant by [5, Corollary 4.3]. Such an improvement stems from the fact that

$$[w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}} \leq c_n [w]_{A_q}.$$

In the next Theorem we provide a counterpart of the preceding result for commutators.

Theorem 1.2. *Let T be a Calderón-Zygmund operator or a rough singular integral with $\Omega \in L^\infty(\mathbb{S}^{n-1})$. Then for every $1 < q < p < \infty$*

$$\|[b, T]\|_{L^p(w)} \leq c_{n,p,q} c_T [w]_{A_\infty} [w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}} \tag{1.4}$$

We would like to recall the following known estimates.

$$\|[b, T]\|_{L^p(w)} \leq c [w]_{A_q}^2,$$
$$\|[b, T_\Omega]\|_{L^p(w)} \leq c [w]_{A_q}^3.$$

The first of them can be derived as a consequence of the quadratic dependence on the A_1 constant of $[b,T]$ obtained in [24] combined with [5, Corollary 4.3], while the second one was established in [22]. In both cases we improve the dependence on the A_q constant since we are able to prove a mixed $A_\infty - A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}$ bound and

$$\max\{[w]_{A_\infty},[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\}\leq c_n[w]_{A_q}.$$

In order to establish Theorems 1.2 and 1.1 we will rely upon a suitable sparse domination result for $[b,T_\Omega]$. This result will be a natural bilinear counterpart of the result obtained in [17] for $[b,T]$ with T a Calderón-Zygmund operator and also of (1.1). The precise statement is the following.

Theorem 1.3. *Let T_Ω be a rough homogeneous singular integral with $\Omega\in L^\infty(\mathbb{S}^{n-1})$. Then, for every compactly supported $f,g\in\mathcal{C}^\infty(\mathbb{R}^n)$ every $b\in\mathrm{BMO}$ and $1<p<\infty$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j\subset\mathcal{D}_j$ such that*

$$|\langle[b,T_\Omega]f,g\rangle|\leq C_np'\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\sum_{j=1}^{\infty}\Big(\mathcal{T}_{\mathcal{S}_j,1,p}(b,f,g)+\mathcal{T}^*_{\mathcal{S}_j,1,p}(b,f,g)\Big)$$

where

$$\mathcal{T}_{\mathcal{S}_j,r,s}(b,f,g)=\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{r,Q}\langle(b-b_Q)g\rangle_{s,Q}|Q|$$

$$\mathcal{T}^*_{\mathcal{S}_j,r,s}(b,f,g)=\sum_{Q\in\mathcal{S}_j}\langle(b-b_Q)f\rangle_{r,Q}\langle g\rangle_{s,Q}|Q|$$

Remark 1.4. In the preceding Theorem and throughout the rest of this work $\langle h\rangle^w_{\alpha,Q}=\left(\frac{1}{w(Q)}\int_Q|h|^\alpha wdx\right)^{\frac{1}{\alpha}}$. We may drop α in the case $\alpha=1$ and w when we consider the Lebesgue measure.

The rest of the paper is organized as follows. We devote Section 2 to gather some results and definitions that will be needed to prove the main theorems. Section 3 is devoted to the proof of Theorem 1.3. In Section 4 we prove Theorem 1.1. We end this work providing a proof of Theorem 1.2 in Section 5.

2. Preliminaries

In this section we gather some definitions and results that will be necessary for the proofs of the main theorems.

We start borrowing some definitions and a basic lemma from [14]. Given a cube $Q_0\subset\mathbb{R}^n$, we denote by $\mathcal{D}(Q_0)$ the family of all dyadic cubes with respect to Q_0, namely, the cubes obtained subdividing repeatedly Q_0 and each of its descendants into 2^n subcubes of the same sidelength.

We say that $\mathcal{D}$ is a dyadic lattice if it is a collection of cubes of $\mathbb{R}^n$ such that:

(1) If $Q \in \mathcal{D}$, then $\mathcal{D}(Q_0) \subset \mathcal{D}$.
(2) For every pair of cubes $Q', Q'' \in \mathcal{D}$ there exists a common ancestor, namely, we can find $Q \in \mathcal{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$.
(3) For every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathcal{D}$ such that $K \subset Q$.

Lemma 2.1 (3^n dyadic lattices lemma)**.** *Given a dyadic lattice $\mathcal{D}$, there exist 3^n dyadic lattices $\mathcal{D}_1, \dots, \mathcal{D}_{3^n}$ such that*

$$\{3Q : Q \in \mathcal{D}\} = \cup_{j=1}^{3^n} \mathcal{D}_j$$

and for each cube $Q \in \mathcal{D}$ and $j = 1, \dots, 3^n$, there exists a unique cube $R \in \mathcal{D}_j$ with sidelength $l(R) = 3l(Q)$ containing Q.

Now we gather some results that will be needed to prove Theorem 1.1. The first of them is the so called Reverse Hölder inequality that was proved in [8] (see also [9]).

Lemma 2.2. *For every $w \in A_\infty$, namely for every weight such that*

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty,$$

the following estimate holds

$$\left(\frac{1}{|Q|}\int_Q w^{r_w}\right)^{\frac{1}{r_w}} \leq 2\left(\frac{1}{|Q|}\int_Q w\right)$$

where $r_w = 1 + \frac{1}{\tau_n [w]_{A_\infty}}$ and $\tau_n > 0$ is a constant independent of w.

At this point we would like to recall that if $w \in A_p \subsetneq A_\infty$ then $[w]_{A_\infty} \leq c_n [w]_{A_p}$. This fact makes mixed $A_\infty - A_p$ bounds interesting, since they provide a sharper dependence than A_p bounds. We also need to borrow the following lemma from [22].

Lemma 2.3. *Let $w \in A_\infty$. Let $\mathcal{D}$ be a dyadic lattice and $\mathcal{S} \subset \mathcal{D}$ be an η-sparse family. Let Ψ be a Young function. Given a measurable function f on $\mathbb{R}^n$ define*

$$\mathcal{B}_{\mathcal{S}} f(x) := \sum_{Q \in \mathcal{S}} \|f\|_{\Psi(L),Q} \chi_Q(x).$$

Then we have

$$\|\mathcal{B}_{\mathcal{S}} f\|_{L^1(w)} \leq \frac{4}{\eta} [w]_{A_\infty} \|M_{\Psi(L)} f\|_{L^1(w)}.$$

We recall that $\Psi : [0,\infty) \to [0,\infty)$ is a Young function if it is a convex, increasing function such that $\Psi(0) = 0$. We define the local Orlicz norm associated to a Young function Ψ as

$$\|f\|_{\Psi(L)(\mu),E} = \inf\left\{\lambda > 0 : \frac{1}{\mu(E)}\int_E \Psi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1\right\}$$

where E is a set of finite measure. We note that in the case $\Psi(t) = t^r$ we recover the standard L^r local norm. We shall drop μ from the notation in the case of the Lebesgue measure and write w instead of wdx for measures that are absolutely continuous with respect to the Lebesgue measure.

Using the preceding definition of local norm, we can define the maximal function associated to a Young function Ψ in the natural way,

$$M_{\Psi(L)}f(x) = \sup_{x\in Q} \|f\|_{\Psi(L)(\mu),Q}.$$

We end this section recalling two basic estimates that work for doubling measures. The first of them is a particular case of the generalized Hölder inequality and the second can be derived, for example, from [1, Lemma 4.1].

$$\begin{aligned}\frac{1}{\mu(Q)}\int_Q |f - f_Q||g|d\mu &\leq \|f - f_Q\|_{\exp L(\mu),Q}\|g\|_{L\log L(\mu),Q} \\ &\leq c_n\|f\|_{\mathrm{BMO}(\mu)}\|g\|_{L\log L(\mu),Q} \quad \text{if } \mu = wdx \text{ with } w \in A_\infty.\end{aligned} \tag{2.1}$$

$$\|f\|_{L\log L(\mu),Q} \leq c_n r'\left(\frac{1}{\mu(Q)}\int_Q w^r d\mu\right)^{\frac{1}{r}} \quad r > 1 \tag{2.2}$$

For a detailed account of local Orlicz norms and maximal functions associated to Young functions we encourage the reader to consult references such as [25], [23], [21] or [4].

3. Proof of Theorem 1.3

The proof of Theorem 1.3 relies upon techniques recently developed by A. K. Lerner in [13]. Given an operator T we define the bilinear operator $\mathcal{M}_T$ by

$$\mathcal{M}_T(f,g)(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |T(f\chi_{\mathbb{R}^n\setminus 3Q})||g|dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. Our first result provides a sparse domination principle based on that bilinear operator.

Theorem 3.1. *Let $1 \le q \le r$ and $s \ge 1$. Assume that T is a sublinear operator of weak type (q,q), and $\mathcal{M}_T$ maps $L^r \times L^s$ into $L^{\nu,\infty}$, where $\frac{1}{\nu} = \frac{1}{r} + \frac{1}{s}$. Then, for every compactly supported $f, g \in \mathcal{C}^\infty(\mathbb{R}^n)$ and every $b \in \mathrm{BMO}$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that*

$$|\langle [b,T]f, g\rangle| \le K \sum_{j=1}^{\infty} \Big(\mathcal{T}_{\mathcal{S}_j,r,s}(b,f,g) + \mathcal{T}^*_{\mathcal{S}_j,r,s}(b,f,g) \Big) \tag{3.1}$$

where

$$\mathcal{T}_{\mathcal{S}_j,r,s}(b,f,g) = \sum_{Q\in\mathcal{S}_j} \langle f\rangle_{r,Q} \langle (b-b_Q)g\rangle_{s,Q} |Q|$$

$$\mathcal{T}^*_{\mathcal{S}_j,r,s}(b,f,g) = \sum_{Q\in\mathcal{S}_j} \langle (b-b_Q)f\rangle_{r,Q} \langle g\rangle_{s,Q} |Q|$$

and

$$K = C_n \big(\|T\|_{L^q \to L^{q,\infty}} + \|\mathcal{M}_T\|_{L^r \times L^s \to L^{\nu,\infty}} \big).$$

It is possible to relax the condition imposed on b for this result and the subsequent ones, but we restrict ourselves to this choice for the sake of clarity.

Proof of Theorem 3.1. By Lemma 2.1, there exist 3^n dyadic lattices $\mathcal{D}_j$ such that for every $Q \subset \mathbb{R}^n$, there is a cube $R = R_Q \in \mathcal{D}_j$ for some j, for which $3Q \subset R_Q$ and $|R_Q| \le 9^n |Q|$.

Let us fix a cube $Q_0 \subset \mathbb{R}^n$. Now we can define a local analogue of $\mathcal{M}_T$ by

$$\mathcal{M}_{T,Q_0}(f,g)(x) = \sup_{Q \ni x, Q\subset Q_0} \frac{1}{|Q|} \int_Q |T(f\chi_{3Q_0 \setminus 3Q})||g| dy.$$

We define the sets E_i $i = 1, \dots 4$ as follows

$E_1 = \{x \in Q_0 : |T(f\chi_{3Q_0})(x)| > A_1 \langle f\rangle_{q,3Q_0}\}$,

$E_2 = \{x \in Q_0 : \mathcal{M}_{T,Q_0}(f, g(b - b_{R_{Q_0}}))(x) > A_2 \langle f\rangle_{r,3Q_0} \langle g(b-b_{R_{Q_0}})\rangle_{s,Q_0}\}$,

$E_3 = \{x \in Q_0 : |T(f\chi_{3Q_0}(b - b_{R_{Q_0}}))(x)| > A_3 \langle f(b-b_{R_{Q_0}})\rangle_{q,3Q_0}\}$,

$E_4 = \{x \in Q_0 : \mathcal{M}_{T,Q_0}(f(b - b_{R_{Q_0}}), g)(x) > A_4 \langle (b-b_{R_{Q_0}})f\rangle_{r,3Q_0} \langle g\rangle_{s,Q_0}\}$.

We can choose A_i in such a way that

$$\max(|E_1|, |E_2|, |E_3|, |E_4|) \le \frac{1}{2^{n+5}} |Q_0|.$$

Actually it suffices to take

$$A_1, A_3 = (c_n)^{1/q} \|T\|_{L^q \to L^{q,\infty}} \quad \text{and} \quad A_2, A_4 = c_{n,r,\nu} \|\mathcal{M}_T\|_{L^r \times L^s \to L^{\nu,\infty}}$$

with $c_n, c_{n,r,\nu}$ large enough. For this choice of E_i the set $\Omega = \cup_i E_i$ satisfies $|\Omega| \le \frac{1}{2^{n+2}} |Q_0|$.

Now applying Calderón-Zygmund decomposition to the function χ_Ω on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$ we obtain pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$ such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| \leq \frac{1}{2}|P_j|$$

and also $|\Omega \setminus \cup_j P_j| = 0$. From the properties of the cubes it readily follows that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and $P_j \cap \Omega^c \neq \emptyset$.

Now, since $|\Omega \setminus \cup_j P_j| = 0$, we have that

$$\int_{Q_0\setminus\cup_j P_j} |T(f\chi_{3Q_0})||(b-b_{R_{Q_0}})g| \leq A_1\langle f\rangle_{q,3Q_0}\int_{Q_0}|g(b-b_{R_{Q_0}})|$$
$$\int_{Q_0\setminus\cup_j P_j} |T((b-b_{R_{Q_0}})f\chi_{3Q_0})||g| \leq A_3\langle (b-b_{R_{Q_0}})f\rangle_{q,3Q_0}\int_{Q_0}|g|.$$

Also, since $P_j \cap \Omega^c \neq \emptyset$, we obtain

$$\int_{P_j} |T((b-b_{R_{Q_0}})f\chi_{3Q_0\setminus 3P_j})||g| \leq A_2\langle (b-b_{R_{Q_0}})f\rangle_{r,3Q_0}\langle g\rangle_{s,Q_0}|Q_0|$$
$$\int_{P_j} |T(f\chi_{3Q_0\setminus 3P_j})||(b-b_{R_{Q_0}})g| \leq A_4\langle f\rangle_{r,3Q_0}\langle (b-b_{R_{Q_0}})g\rangle_{s,Q_0}|Q_0|.$$

Our next step is to observe that for any arbitrary pairwise disjoint cubes $P_j \in \mathcal{D}(Q_0)$,

$$\begin{aligned}
&\int_{Q_0} |[b,T](f\chi_{3Q_0})||g| \\
&= \int_{Q_0\setminus\cup_j P_j} |[b,T](f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |[b,T](f\chi_{3Q_0})||g| \\
&\leq \int_{Q_0\setminus\cup_j P_j} |[b,T](f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |[b,T](f\chi_{3Q_0\setminus 3P_j})||g| \\
&+ \sum_j \int_{P_j} |[b,T](f\chi_{3P_j})||g|.
\end{aligned}$$

For the first two terms, using that $[b,T]f = [b-c,T]f$ for any $c \in \mathbb{R}$, we obtain

$$\begin{aligned}
&\int_{Q_0\setminus\cup_j P_j} |[b,T](f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |[b,T](f\chi_{3Q_0\setminus 3P_j})||g| \\
&\leq \int_{Q_0\setminus\cup_j P_j} |b-b_{R_{Q_0}}||T(f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |b-b_{R_{Q_0}}||T(f\chi_{3Q_0\setminus 3P_j})||g| \\
&+ \int_{Q_0\setminus\cup_j P_j} |T((b-b_{R_{Q_0}})f\chi_{3Q_0})||g| + \sum_j \int_{P_j} |T((b-b_{R_{Q_0}})f\chi_{3Q_0\setminus 3P_j})||g|.
\end{aligned}$$

Therefore, combining all the preceding estimates with Hölder's inequality (here we take into account $q \le r$ and $s \ge 1$) and calling $A = \sum_i A_i$ we have that

$$\int_{Q_0} |[b,T](f\chi_{3Q_0})||g| \le \sum_j \int_{P_j} |[b,T](f\chi_{3P_j})||g|$$
$$+ A\left(\langle f\rangle_{r,3Q_0}\langle (b-b_{R_{Q_0}})g\rangle_{s,Q_0}|Q_0| + \langle (b-b_{R_{Q_0}})f\rangle_{r,3Q_0}\langle g\rangle_{s,Q_0}|Q_0|\right).$$

Since $\sum_j |P_j| \le \frac{1}{2}|Q_0|$, iterating the above estimate, we obtain that there is a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that

$$\begin{aligned}\int_{Q_0} |[b,T](f\chi_{3Q_0})||g| &\le A\sum_{Q\in\mathcal{F}} \langle (b-b_{R_Q})f\rangle_{r,3Q}\langle g\rangle_{s,Q}|Q| \\ &\quad + A\sum_{Q\in\mathcal{F}} \langle f\rangle_{r,3Q}\langle g(b-b_{R_Q})\rangle_{s,Q}|Q| \end{aligned} \tag{3.2}$$

To end the proof, take now a partition of $\mathbb{R}^n$ by cubes R_j such that $\operatorname{supp}(f) \subset 3R_j$ for each j. One way to do that is the following. We take a cube Q_0 such that $\operatorname{supp}(f) \subset Q_0$ and cover $3Q_0 \setminus Q_0$ by $3^n - 1$ congruent cubes R_j. Each of them satisfies $Q_0 \subset 3R_j$. We continue covering in the same way $9Q_0 \setminus 3Q_0$, and so on. The family of the resulting cubes of this process, including Q_0, satisfies the desired property.

Having such a partition, apply (3.2) to each R_j. We obtain a $\frac{1}{2}$-sparse family $\mathcal{F}_j \subset \mathcal{D}(R_j)$ such that

$$\begin{aligned}\int_{R_j} |[b,T](f)||g| &\le A\sum_{Q\in\mathcal{F}_j} \langle (b-b_{R_Q})f\rangle_{r,3Q}\langle g\rangle_{s,Q}|Q| \\ &\quad + A\sum_{Q\in\mathcal{F}_j} \langle f\rangle_{r,3Q}\langle g(b-b_{R_Q})\rangle_{s,Q}|Q|\end{aligned}$$

Therefore, setting $\mathcal{F} = \cup_j \mathcal{F}_j$

$$\begin{aligned}\int_{\mathbb{R}^n} |[b,T](f)||g| &\le A\sum_{Q\in\mathcal{F}} \langle (b-b_{R_Q})f\rangle_{r,3Q}\langle g\rangle_{s,Q}|Q| \\ &\quad + A\sum_{Q\in\mathcal{F}} \langle f\rangle_{r,3Q}\langle g(b-b_{R_Q})\rangle_{s,Q}|Q|.\end{aligned}$$

Now since $3Q \subset R_Q$ and $|R_Q| \le 3^n|3Q|$, clearly $\langle h\rangle_{\alpha,3Q} \le c_n\langle h\rangle_{\alpha,R_Q}$. Further, setting $\mathcal{S}_j = \{R_Q \in \mathcal{D}_j : Q \in \mathcal{F}\}$, and using that $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Hence

$$\int_{\mathbb{R}^n} |[b,T](f)||g| \leq c_n A \sum_{j=1}^{3^n} \sum_{R\in\mathcal{S}_j} \langle (b-b_R)f\rangle_{r,R}\langle g\rangle_{s,R}|R|$$
$$+ c_n A \sum_{j=1}^{3^n} \sum_{R\in\mathcal{S}_j} \langle f\rangle_{r,R}\langle g(b-b_R)\rangle_{s,R}|R|$$

and (3.1) holds. □

Given $1\leq p\leq\infty$, we define the maximal operator $\mathcal{M}_{p,T}$ by

$$\mathcal{M}_{p,T}f(x) = \sup_{Q\ni x}\left(\frac{1}{|Q|}\int_Q |T(f\chi_{\mathbb{R}^n\setminus 3Q})|^p dy\right)^{1/p}$$

(in the case $p=\infty$ we call $\mathcal{M}_{p,T}f(x) = M_T f(x)$).

Our next step is to provide a suitable version of [13, Corollary 3.2] for the commutator. The result is the following.

Corollary 3.2. *Let $1\leq q\leq r$ and $s\geq 1$. Assume that T is a sublinear operator of weak type (q,q), and $\mathcal{M}_{s',T}$ is of weak type (r,r). Then, for every compactly supported $f,g\in\mathcal{C}^\infty(\mathbb{R}^n)$ and every $b\in$ BMO, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j\subset\mathcal{D}_j$ such that*

$$|\langle [b,T]f,g\rangle| \leq K\sum_{j=1}^{\infty}\left(\mathcal{T}_{\mathcal{S}_j,r,s}(b,f,g)+\mathcal{T}^*_{\mathcal{S}_j,r,s}(b,f,g)\right)$$

where

$$\mathcal{T}_{\mathcal{S}_j,r,s}(b,f,g) = \sum_{Q\in\mathcal{S}_j}\langle f\rangle_{r,Q}\langle (b-b_Q)g\rangle_{s,Q}|Q|$$
$$\mathcal{T}^*_{\mathcal{S}_j,r,s}(b,f,g) = \sum_{Q\in\mathcal{S}_j}\langle (b-b_Q)f\rangle_{r,Q}\langle g\rangle_{s,Q}|Q|$$

and

$$K = C_n\left(\|T\|_{L^q\to L^{q,\infty}} + \|\mathcal{M}_{s',T}\|_{L^r\to L^{r,\infty}}\right).$$

Proof. The proof is the same as [13, Corollary 3.2]. It suffices to observe that

$$\|\mathcal{M}_T\|_{L^r\times L^s\to L^{\nu,\infty}} \leq C_n\|\mathcal{M}_{s',T}\|_{L^r\to L^{r,\infty}} \quad (1/\nu = 1/r+1/s),$$

and to apply Theorem 3.1. □

Remark 3.3. At this point we would like to note that if T is an ω-Calderón-Zygmund operator, with ω satisfying a Dini condition, since M_T is of weak-type $(1,1)$ with

$$\|M_T\|_{L^1\to L^{1,\infty}} \leq c_n\left(C_K + \|T\|_{L^2} + \|\omega\|_{\text{Dini}}\right)$$

(see [12], also for the notation) and we have that

$$\|T\|_{L^1\to L^{1,\infty}} \le c_n \left(\|T\|_{L^2} + \|\omega\|_{\text{Dini}}\right),$$

then from the preceding Corollary we recover a bilinear version of the sparse domination established in [17].

In order to use Corollary 3.2 to obtain Theorem 1.3, we need to borrow some results from [13]. Given an operator T, we define the maximal operator $M_{\lambda,T}$ by

$$M_{\lambda,T}f(x) = \sup_{Q\ni x} (T(f\chi_{\mathbb{R}^n\setminus 3Q})\chi_Q)^*(\lambda|Q|) \quad 0<\lambda<1.$$

That operator was proved to be of weak type $(1,1)$ in [13] where the following estimate was established.

Theorem 3.4. *If $\Omega \in L^\infty(\mathbb{S}^{n-1})$, then*

$$\|M_{\lambda,T_\Omega}\|_{L^1\to L^{1,\infty}} \le C_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\left(1+\log\frac{1}{\lambda}\right) \quad 0<\lambda<1.$$

Also in [13] the following result showing the relationship between the $L^1 \to L^{1,\infty}$ norms of the operators $M_{\lambda,T}$ and $\mathcal{M}_{p,T}$ was provided.

Lemma 3.5. *Let $0<\gamma\le 1$ and let T be a sublinear operator. The following statements are equivalent:*

(1) *there exists $C>0$ such that for all $p\ge 1$,*
$$\|\mathcal{M}_{p,T}f\|_{L^1\to L^{1,\infty}} \le Cp^\gamma;$$
(2) *there exists $C>0$ such that for all $0<\lambda<1$,*
$$\|M_{\lambda,T}f\|_{L^1\to L^{1,\infty}} \le C\left(1+\log\frac{1}{\lambda}\right)^\gamma.$$

At this point we are in the position to prove that Theorem 1.3 follows as a corollary from the previous results.

Proof of Theorem 1.3. Theorem 3.4 combined with Lemma 3.5 with $\gamma=1$ yields

$$\|\mathcal{M}_{p,T_\Omega}\|_{L^1\to L^{1,\infty}} \le c_n p\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}$$

with $p\ge 1$. Also, by [26], we have that

$$\|T_\Omega\|_{L^1\to L^{1,\infty}} \le C_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}.$$

Hence, by Corollary 3.2 with $q=r=1$ and $s=p>1$, there exist 3^n dyadic lattices $\mathcal{D}_j$ and 3^n sparse families $\mathcal{S}_j\subset\mathcal{D}_j$ such that

$$|\langle [b,T_\Omega]f,g\rangle| \le C_n p'\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\sum_{j=1}^{3^n}\left(\mathcal{T}_{\mathcal{S}_j,1,p}(b,f,g)+\mathcal{T}_{\mathcal{S}_j,1,p}(b,f,g)\right).$$

□

4. Proof of Theorem 1.1

We start providing a proof for (1.3). We follow some of the key ideas from [15, 16] (see also [22]). By duality, it suffices to prove (1.3) it suffices to show that

$$\left\|\frac{[b,T_\Omega]f}{M_rw}\right\|_{L^{p'}(M_rw)}\leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\|b\|_{\mathrm{BMO}}(p')^3p^2(r')^{1+\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}.$$

We can calculate the norm by duality. Then,

$$\left\|\frac{[b,T_\Omega]f}{M_rw}\right\|_{L^{p'}(M_rw)}=\sup_{\|h\|_{L^p(M_rw)}=1}\left|\int_{\mathbb{R}^n}[b,T_\Omega]f(x)h(x)dx\right|.$$

Let us define now a Rubio de Francia algorithm suited for this situation (see [6, Chapter IV.5] and [4] for plenty of applications of the Rubio de Francia algorithm). First we consider the operator

$$S(f)=\frac{M\left(f(M_rw)^{\frac{1}{p}}\right)}{(M_rw)^{\frac{1}{p}}}$$

and we observe that S is bounded on $L^p(M_rw)$ with norm bounded by a dimensional multiple of p'. Relying upon S we define

$$R(h)=\sum_{k=0}^\infty\frac{1}{2^k}\frac{S^kh}{\|S\|^k_{L^p(M_rw)}}.$$

This operator has the following properties:

(a) $0\leq h\leq R(h)$,
(b) $\|Rh\|_{L^p(M_rw)}\leq 2\|h\|_{L^p(M_rw)}$,
(c) $R(h)(M_rw)^{\frac{1}{p}}\in A_1$ with $[R(h)(M_rw)^{\frac{1}{p}}]_{A_1}\leq cp'$. We also note that $[Rh]_{A_\infty}\leq[Rh]_{A_3}\leq c_np'$.

Using Theorem 1.3 and taking into account (a) we have that,

$$\begin{aligned}&\left|\int_{\mathbb{R}^n}[b,T_\Omega]f(x)h(x)dx\right|\\&\leq C_ns'\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\sum_{j=1}^\infty\left(\mathcal{T}_{\mathcal{S}_j,1,s}(b,f,h)+\mathcal{T}^*_{\mathcal{S}_j,1,s}(b,f,h)\right)\\&\leq C_ns'\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\sum_{j=1}^\infty\left(\mathcal{T}_{\mathcal{S}_j,1,s}(b,f,Rh)+\mathcal{T}^*_{\mathcal{S}_j,1,s}(b,f,Rh)\right)\end{aligned}$$

and it suffices to obtain estimates for

$$I:=\mathcal{T}_{\mathcal{S}_j,1,s}(b,f,Rh)\qquad\text{and}\qquad II:=\mathcal{T}^*_{\mathcal{S}_j,1,s}(b,f,Rh).$$

First we focus on I. Now we choose $r, s > 1$ such that $sr = 1 + \frac{1}{\tau_n[Rh]_{A_\infty}}$. For instance, choosing $r = 1 + \frac{1}{2\tau_n[Rh]_{A_\infty}}$ we have that $s = 2\frac{1+\tau_n[Rh]_{A_\infty}}{1+2\tau_n[Rh]_{A_\infty}}$ and also that $sr' = 2(1+\tau_n[Rh]_{A_\infty}) \simeq [Rh]_{A_\infty}$. Now we recall that for every $0 < t < \infty$ it was established in [7, Corollary 3.1.8] that

$$\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^t dx\right)^{\frac{1}{t}} \leq (t\Gamma(t))^{\frac{1}{t}}\, e^{\frac{1}{t}+1}2^n\|b\|_{\mathrm{BMO}}$$

For $t > 1$ we have that $(t\Gamma(t))^{\frac{1}{t}}\, e^{\frac{1}{t}+1}2^n \leq c_n t$. Taking into account the preceding estimate, the choices for r and s, the reverse Hölder inequality (Lemma 2.2), and the property (c) above, we have that

$$\begin{aligned}
I &\leq \sum_{Q\in\mathcal{S}_j}\left(\frac{1}{|Q|}\int_Q |b(x)-b_Q|^s|Rh(x)|^s dx\right)^{\frac{1}{s}}\int_Q |f|dy\\
&\leq \sum_{Q\in\mathcal{S}_j}\langle b-b_Q\rangle_{sr',Q}\langle Rh\rangle_{sr,Q}\int_Q |f|dy\\
&\leq c_n(sr')\|b\|_{\mathrm{BMO}}\sum_{Q\in\mathcal{S}_j}\left(\frac{1}{|Q|}\int_Q Rh\right)\int_Q |f|dy\\
&\leq c_n[Rh]_{A_\infty}\|b\|_{\mathrm{BMO}}\sum_{Q\in\mathcal{S}_j}Rh(Q)\frac{1}{|Q|}\int_Q |f|dy\\
&\leq c_n p'\|b\|_{\mathrm{BMO}}\sum_{Q\in\mathcal{S}_j}Rh(Q)\frac{1}{|Q|}\int_Q |f|dy.
\end{aligned}$$

An application of Lemma 2.3 with $\Psi(t) = t$ yields

$$\sum_{Q\in\mathcal{S}_j}Rh(Q)\frac{1}{|Q|}\int_Q |f|dy \leq 8[Rh]_{A_\infty}\|Mf\|_{L^1(Rh)} \leq c_n p'\|Mf\|_{L^1(Rh)}.$$

From here

$$\begin{aligned}
\|Mf\|_{L^1(Rh)} &\leq \Big(\int_{\mathbb{R}^n}|Mf|^{p'}(M_r w)^{1-p'}\Big)^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^n}(Rh)^p M_r w\Big)^{\frac{1}{p}}\\
&\leq 2\Big\|\frac{Mf}{M_r w}\Big\|_{L^{p'}(M_r w)}.
\end{aligned}$$

Now by [15, Lemma 3.4] (see also [24, Lemma 2.9])

$$\Big\|\frac{Mf}{M_r w}\Big\|_{L^{p'}(M_r w)} \leq cp(r')^{\frac{1}{p'}}\Big\|\frac{f}{w}\Big\|_{L^{p'}(w)}.$$

Gathering all the preceding estimates we have that

$$I \leq c_n\|b\|_{\mathrm{BMO}}p(p')^3(r')^{\frac{1}{p'}}\left\|\frac{f}{w}\right\|_{L^{p'}(w)}.$$

Now we turn our attention to II. Recalling that we have chosen $rs = 1 + \frac{1}{\tau_n [Rh]_{A_\infty}}$, taking into account the Reverse Hölder inequality and applying also (2.1) we have that

$$\begin{aligned} II &\leq \sum_{Q\in\mathcal{S}_j} \Big(\frac{1}{|Q|}\int_Q |b(y)-b_Q| f(y)dy\Big)\langle Rh\rangle_{s,Q}|Q| \\ &\leq \sum_{Q\in\mathcal{S}_j} \Big(\frac{1}{|Q|}\int_Q |b(y)-b_Q| f(y)dy\Big)\langle Rh\rangle_{rs,Q}|Q| \\ &\leq c_n\|b\|_{\mathrm{BMO}} \sum_{Q\in\mathcal{S}_j} \|f\|_{L\log L,Q} Rh(Q). \end{aligned}$$

Then a direct application of Lemma 2.3 with $\Psi(t) = t\log(e+t)$ yields the following estimate

$$\sum_{Q\in\mathcal{S}_j} \|f\|_{L\log L,Q} Rh(Q) \leq 8[Rh]_{A_\infty} \|M_{L\log L} f\|_{L^1(Rh)}.$$

Arguing as in the estimate of I,

$$\|M_{L\log L} f\|_{L^1(Rh)} \leq 2\Big\|\frac{M_{L\log L} f}{M_r w}\Big\|_{L^{p'}(M_r w)}.$$

Now [24, Proposition 3.2] gives

$$\Big\|\frac{M_{L\log L} f}{M_r w}\Big\|_{L^{p'}(M_r w)} \leq c_n p^2 (r')^{1+\frac{1}{p'}} \Big\|\frac{f}{w}\Big\|_{L^{p'}(w)}.$$

Combining all the estimates we have that

$$II \leq c_n \|b\|_{\mathrm{BMO}} (p')^2 p^2 (r')^{1+\frac{1}{p'}} \Big\|\frac{f}{w}\Big\|_{L^{p'}(w)}.$$

Finally, collecting the estimates we have obtained for I and II, we arrive at the desired bound, namely

$$\Big\|\frac{[b,T_\Omega]f}{M_r w}\Big\|_{L^{p'}(M_r w)} \leq c_n \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \|b\|_{\mathrm{BMO}} (p')^3 p^2 (r')^{1+\frac{1}{p'}} \Big\|\frac{f}{w}\Big\|_{L^{p'}(w)}.$$

We end the proof observing that the A_∞ and the $A_1 - A_\infty$ results are a direct consequence of the estimate we have just established and of the Reverse-Hölder inequality (see [15, 16, 8] for this kind of argument). □

5. Proof of Theorem 1.2

Let us consider first the case in which T is a Calderón-Zygmund operator. Calculating the norm by duality we have that

$$\|[b,T]f\|_{L^p(w)} = \sup_{\|g\|_{L^{p'}(w)}=1} \Big|\int [b,T](f) g w\Big|.$$

Now taking into account Remark 3.3 (or [17]) we have that

$$\left|\int [b,T](f)gw\right| \leq c_n c_T \sum_{j=1}^{3^n} \Big(\mathcal{T}_{\mathcal{S}_j,1,1}(b,f,gw) + \mathcal{T}^*_{\mathcal{S}_j,1,1}(b,f,gw)\Big)$$

so it suffices to provide estimates for

$$\mathcal{T}_{\mathcal{S},1,1}(b,f,gw) \quad \text{and} \quad \mathcal{T}^*_{\mathcal{S},1,1}(b,f,gw).$$

First we work on $\mathcal{T}_{\mathcal{S}_j,1,1}(b,f,gw)$. Following ideas in [19] we have that

$$\langle w\rangle_Q \langle w^{\frac{1}{1-q}}\rangle^{q-1} = \langle w\rangle_Q \langle \sigma^{\frac{1}{p'}}\rangle^p_{\overline{A},Q}$$

where $\overline{A}(t) = t^{\frac{p}{q-1}}$ and $\sigma = w^{1-p'}$. Then, choosing $s < p'$ and taking into account [11, Lemma 6], (2.1) and (2.2),

$$\begin{aligned}
&\mathcal{T}_{\mathcal{S},1,1}(b,f,gw) = \sum_{Q\in\mathcal{S}} \langle f\rangle_Q \langle g(b-b_Q)w\rangle_Q |Q| \\
&\leq c \sum_{Q\in\mathcal{S}} \langle f w^{\frac{1}{p}}\rangle_{A,Q} \langle w^{-\frac{1}{p}}\rangle_{\overline{A},Q} \|g\|_{L\log L(w),Q} \|b-b_Q\|_{\exp L(w),Q} \\
&\leq cs' \|b\|_{\mathrm{BMO}} [w]_{A_\infty} \sum_{Q\in\mathcal{S}} \langle f w^{\frac{1}{p}}\rangle_{A,Q} \langle w^{-\frac{1}{p}}\rangle_{\overline{A},Q} \langle g\rangle^w_{s,Q} \\
&\times \exp\big(\langle \log w^{-1}\rangle_Q\big)^{\frac{1}{p'}} \exp\left(\langle \log w\rangle_Q\right)^{\frac{1}{p'}} \\
&\leq cs' \|b\|_{\mathrm{BMO}} [w]_{A_\infty} [w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}} \Big(\sum_{Q\in\mathcal{S}} \langle f w^{\frac{1}{p}}\rangle^p_{A,Q} |Q|\Big)^{\frac{1}{p}} \\
&\times \Big(\sum_{Q\in\mathcal{S}} \big(\langle g\rangle^w_{s,Q}\big)^{p'} \exp\big(\langle \log w^{-1}\rangle_Q\big)^{\frac{1}{p'}} |Q|\Big)^{\frac{1}{p'}} \\
&\leq c_n \gamma^{-1} \|M_A\|_{L^p} \|b\|_{\mathrm{BMO}} [w]_{A_\infty} [w]_{A_q^{\frac{1}{p}} (A_\infty^{\exp})^{\frac{1}{p'}}} \|f\|_{L^p(w)} \|g\|_{L^{p'}(w)},
\end{aligned}$$

where in the last step we use the Carleson embedding Theorem [8, Theorem 4.5] and the sparsity of $\mathcal{S}$.

Now we turn our attention to $\mathcal{T}^*_{\mathcal{S},1,1}(b,f,gw)$. We observe that for any $r > 1$

$$\begin{aligned}
\mathcal{T}^*_{\mathcal{S},1,1}(b,f,gw) &= \sum_{Q\in\mathcal{S}} \langle f(b-b_Q)\rangle_Q \langle gw\rangle_Q |Q| \\
&\leq \sum_{Q\in\mathcal{S}} \langle f\rangle_{r,Q} \langle b-b_Q\rangle_{r',Q} \langle gw\rangle_Q |Q| \\
&\leq c\|b\|_{\mathrm{BMO}} \sum_{Q\in\mathcal{S}} \langle f\rangle_{r,Q} \langle gw\rangle_Q |Q|
\end{aligned}$$

and from this point it suffices to follow the proof of [19, Theorem 3.1] to obtain the following estimate

$$\mathcal{T}^*_{\mathcal{S},1,1}(b,f,gw)\leq c[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

Combining the estimates for $\mathcal{T}_{\mathcal{S},1,1}(b,f,gw)$ and $\mathcal{T}^*_{\mathcal{S},1,1}(b,f,gw)$ we obtain (1.4) in the case of T being a Calderón-Zygmund operator.

Let us consider now the remaining case. Assume that T is a rough singular integral with $\Omega\in L^\infty(\mathbb{S}^{n-1})$. Calculating the norm by duality and denoting by $[b,T]^t$ the adjoint of $[b,T]$ we have that

$$\|[b,T]f\|_{L^p(w)}=\sup_{\|g\|_{L^{p'}(w)}=1}\left|\int[b,T](f)gw\right|=\sup_{\|g\|_{L^{p'}(w)}=1}\left|\int[b,T]^t(gw)f\right|.$$

Taking into account that $[b,T]^t$ is also a commutator we can use the sparse domination obtained in Theorem 1.3 so we have that

$$\left|\int[b,T]^t(gw)f\right|\leq c_n u'\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}\sum_{j=1}^{3^n}\Big(\mathcal{T}_{\mathcal{S}_j,u,1}(b,f,gw)+\mathcal{T}^*_{\mathcal{S}_j,u,1}(b,f,gw)\Big)$$

and then the question reduces to control both

$$\mathcal{T}_{\mathcal{S}_j,u,1}(b,f,gw)\quad\text{and}\quad\mathcal{T}^*_{\mathcal{S}_j,u,1}(b,f,gw).$$

We begin observing that, arguing as before, choosing $1<s<p'$

$$\begin{aligned}\mathcal{T}_{\mathcal{S}_j,u,1}(b,f,gw)&=\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{u,Q}\langle(b-b_Q)gw\rangle_{1,Q}|Q|\\&\leq cs'[w]_{A_\infty}\|b\|_{\mathrm{BMO}}\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{u,Q}\langle g\rangle^w_{s,Q}w(Q)=c[w]_{A_\infty}\|b\|_{\mathrm{BMO}}B_1.\end{aligned}$$

On the other hand we have that for $s_1>1$ to be chosen later

$$\begin{aligned}\mathcal{T}^*_{\mathcal{S},u,1}(b,f,gw)&=\sum_{Q\in\mathcal{S}}\langle(b-b_Q)f\rangle_{u,Q}\langle gw\rangle_{1,Q}|Q|\\&\leq\sum_{Q\in\mathcal{S}}\langle f\rangle_{us_1,Q}\langle b-b_Q\rangle_{us_1',Q}\langle gw\rangle_{1,Q}|Q|\\&\leq c\|b\|_{\mathrm{BMO}}\sum_{Q\in\mathcal{S}}\langle f\rangle_{us_1,Q}\langle gw\rangle_{1,Q}|Q|=c\|b\|_{\mathrm{BMO}}B_2.\end{aligned}$$

By Hölder inequality, we have that both B_1 and B_2 are controlled by

$$\sum_{Q\in\mathcal{S}}\langle f\rangle_{us_1,Q}\langle g\rangle^w_{s,Q}w(Q).$$

We note that we can choose us_1 as close to 1 as we want so let us rename $us_1=r$. Now denoting $\overline{B}(t)=t^{\frac{p}{r(q-1)}}$ and arguing as in [19, Theorem 3.1] we have that

$$\begin{aligned}\sum_{Q\in\mathcal{S}_j}\langle f\rangle_{r,Q}\langle g\rangle_{s,Q}^{w}w(Q) &\leq [w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\Big(\sum_{Q\in\mathcal{S}}\langle f^r w^{\frac{r}{p}}\rangle_{B,Q}^{\frac{p}{r}}|Q|\Big)^{\frac{1}{p}}\\ &\times\Big(\sum_{Q\in\mathcal{S}}(\langle g\rangle_{s,Q}^{w})^{p'}\exp(\langle\log w\rangle_Q)|Q|\Big)^{\frac{1}{p'}}\\ &\leq c_n\gamma^{-1}p\|M_B\|_{L^{p/r}}^{\frac{1}{r}}[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}\end{aligned}$$

where in the last step we have used again the sparsity of $\mathcal{S}$ and the Carleson embedding theorem ([8, Theorem 4.5]). Collecting all the estimates

$$\left|\int[b,T]^t(gw)f\right|\leq c_n\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}[w]_{A_\infty}[w]_{A_q^{\frac{1}{p}}(A_\infty^{\exp})^{\frac{1}{p'}}}\|f\|_{L^p(w)}\|g\|_{L^{p'}(w)}.$$

This ends the proof of Theorem 1.2. □

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References

[1] M. Caldarelli, A. K. Lerner, S. Ombrosi, *On a counterexample related to weighted weak type estimates for singular integrals.* Proc. Amer. Math. Soc. 145 (2017), no. 7. 3005–3012

[2] R.R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., 103 (1976), no. 3, 611–635

[3] J.M. Conde-Alonso, A. Culiuc, F. Di Plinio and Y. Ou, *A sparse domination principle for rough singular integrals*, Anal. PDE 10 (2017), no. 5, 1255–1284.

[4] D. Cruz-Uribe, J. M. Martell, and C. Pérez, *Weights, Extrapolation and the Theory of Rubio de Francia,* Operator Theory: Advances and Applications, 215. Birkhäuser/Springer Basel AG, Basel, 2011.

[5] J. Duoandikoetxea, *Extrapolation of weights revisited: New proofs and sharp bounds*, J. Funct. Anal. 260 (2015), 1886–1901.

[6] J. García-Cuerva, J. L. Rubio de Francia *Weighted Norm Inequalities and Related Topics,* North-Holland Mathematics Studies, 116. Notas de Matemática (Mathematical Notes), 104. North-Holland Publishing Co., Amsterdam, 1985.

[7] L. Grafakos, *Modern Fourier analysis.* Third edition. Graduate Texts in Mathematics, 250. Springer, New York, 2014.

[8] T.P. Hytönen, C. Pérez, *Sharp weighted bounds involving A_∞.* Anal. PDE 6 (2013), no. 4., 777–818.

[9] T.P. Hytönen, C. Pérez, E. Rela, *Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type.* J. Funct. Anal. 263 (2012), no. 12. 3883–3899.

[10] T.P. Hytönen, L. Roncal and O. Tapiola, *Quantitative weighted estimates for rough homogeneous singular integrals*, Israel J. Math., **218** (2017), no. 1, 133–164.

[11] G.H. Ibañez-Firnkorn, I.P. Rivera-Ríos, *Sparse and weighted estimates for generalized Hörmander operators and commutators*, preprint. Available at https://arxiv.org/abs/1704.01018
[12] A.K. Lerner, *On pointwise estimates involving sparse operators*, New York J. Math., 22 (2016), 341–349.
[13] A.K. Lerner *A weak type estimate for rough singular integrals*, preprint. Available at https://arxiv.org/abs/1705.07397
[14] A.K. Lerner and F. Nazarov, *Intuitive dyadic calculus: the basics*, preprint. Available at http://arxiv.org/abs/1508.05639
[15] A. K. Lerner, S. Ombrosi, and C. Pérez, *Sharp A_1 bounds for Calderón-Zygmund operators and the relationship with a problem of Muckenhoupt and Wheeden,* Int. Math. Res. Not. IMRN 14 (2008), Art. ID rnm161, 11 pp.
[16] A. K. Lerner, S. Ombrosi, and C. Pérez, *A_1 bounds for Calderón–Zygmund operators related to a problem of Muckenhoupt and Wheeden*, Math. Res. Lett. 16 (2009), no. 1, 149–156.
[17] A. K. Lerner, S. Ombrosi, and I. P. Rivera-Ríos, *On pointwise and weighted estimates for commutators of Calderón-Zygmund operators*, Adv. Math. 319 (2017), 153–181.
[18] A.K. Lerner, K. Moen, Mixed $A_p - A_\infty$ estimates with one supremum. Studia Math. 219 (2013), no. 3, 247–267.
[19] K. Li, *Sharp weighted estimates involving one supremum*, C. R. Math. Acad. Sci. Paris 355 (2017), no. 8, 906–909.
[20] K. Li, C. Pérez, I. P. Rivera-Ríos, L. Roncal, *Weighted norm inequalities for rough singular integral operators*, preprint. Available at https://arxiv.org/abs/1701.05170
[21] C. Pérez, On sufficient conditions for the boundedness of the Hardy-Littlewood maximal operator between weighted L^p-spaces with different weights. Proc. London Math. Soc. (3) 71 (1995), no. 1, 135–157.
[22] C. Pérez, I. P. Rivera-Ríos, L. Roncal, *A_1 theory of weights for rough homogeneous singular integrals and commutators*, preprint. Available at https://arxiv.org/abs/1607.06432 , to appear in Ann. Sci. Scuola Norm. Sup. (Scienze)
[23] L. Pick, A. Kufner, O. John, and S. Fučik, *Function spaces. Vol. 1.* Second revised and extended edition. De Gruyter Series in Nonlinear Analysis and Applications, 14. Walter de Gruyter and Co., Berlin, 2013.
[24] C. Ortiz-Caraballo, *Quadratic A_1 bounds for commutators of singular integrals with* BMO *functions*, Indiana Univ. Math. J. 60 (2011), no. 6, 2107–2130.
[25] M.M. Rao and Z. D. Ren, *Theory of Orlicz Spaces.* Monographs and Textbooks in Pure and Applied Mathematics,146. Marcel Dekker, Inc., New York, 1991.
[26] A. Seeger, *Singular integral operators with rough convolution kernels*, J. Amer. Math. Soc. 9 (1996), no. 1, 95–105.

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