

Weghted Lorentz and Lorentz-Morrey estimates to viscosity solutions of fully nonlinear elliptic equations *

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Abstract

We prove a global weighted Lorentz and Lorentz-Morrey estimates of the viscosity solutions to the Dirichlet problem for fully nonlinear elliptic equation $F(D^2u, x) = f(x)$ defined in a bounded $C^{1,1}$ domain. The oscillation of nonlinearity F with respect to x is assumed to be small in the L^n -sense. Here, we employ the Lorentz boundedness of the Hardy-Littlewood maximal operators and an equivalent representation of weighted Lorentz norm.

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Key words: fully nonlinear elliptic equations; weighted Lorentz estimates; viscosity solutions; weight functions; Lorentz-Morrey spaces

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^{n \geq 2}$ with $\partial\Omega \in C^{1,1}$. This paper presents an avenue to understanding a possibility of Calderón-Zygmund type theory in weighted Lorentz spaces to viscosity solutions of the following Dirichlet problem for fully nonlinear elliptic equations:

$$\begin{cases} F(D^2u, x) = f(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $F(M, x) : \mathcal{S}(n) \times \Omega \rightarrow \mathbb{R}$ and $\mathcal{S}(n)$ is the set consisting of symmetric $(n \times n)$ -matrices.

It is well-known that the Calderón-Zygmund estimate is an extremely popular research to various fully nonlinear elliptic and parabolic equations in recent decades. Caffarelli in [?] first proved the interior $W^{2,p}$ estimates

$$\|u\|_{W^{2,q}(B_{\frac{1}{2}})} \leq C (\|u\|_{L^\infty(B_1)} + \|f\|_{L^q(B_1)})$$

with $q > n$ for the viscosity solution of

$$F(D^2u, x) = f(x), \quad \text{in } B_1 = B_1(0)$$

under a small oscillation of F in the variable x , and established the $C^{1,1}$ estimates for the homogeneous equations with constant coefficients $F(D^2w, x_0) = 0$, where the Aleksandrov-Bakel'man-Pucci a priori estimate,

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a covering argument and the Harnack inequality from Krylov-Safonov are employed, see also [?]. Adapting this technique, L. Wang [?] developed the similar interior a priori estimate for fully nonlinear parabolic operator. Furthermore, Winter [?] used this technique to establish the $W^{2,q}$ -solvability of the associated boundary-value problem

$$\begin{cases} F(D^2u, x) = f(x), & \text{in } B_1^+, \\ u = 0, & \text{on } B_1 \cap \{x_n = 0\}. \end{cases}$$

and also proved the global $W^{2,q}$ estimates for $q > n$ when the boundary of the domain is additionally smooth enough. It is also worth noting that the the range of exponents $p > n$ in [?, ?] can be relaxed to $p > n - \varepsilon$ with a small $\varepsilon > 0$ depending on the ellipticity constants, which has been proved by Escauriaza in [?] using a weak reverse Hölder inequality.

It is worth noting that there exists a common point in these papers, that is the nonlinearity F supports linear growths and satisfies a small bounded mean oscillation assumption. Indeed, this assumption turns out to be equivalent to a small oscillation condition in the L^∞ sense, which, particularly in the linear case, is the same as what is required in the classical L^q -theory based on the Calderón-Zygmund theorem when one investigates, using perturbation method and the partitions of unity, the case that the coefficients are uniformly sufficiently close to continuous ones. Krylov et al have some important work about this kind of nonlinearity. For example, Krylov in [?] showed the $W^{2,q}$ -solvability for $q > n$ in the whole space to Bellman's equations with VMO "coefficients" whose local oscillations are measured in a certain average sense allowing rather rough discontinuity. Indeed, this $W^{2,p}$ estimate for nondivergence linear elliptic equations with VMO coefficients should to be trace back to the beginning of Chiarenza-Frasca-Longo [?]. Recently, Dong-Krylov-Li [?] demonstrated the same results for more generalized fully nonlinear elliptic and parabolic equations with VMO "coefficients": there exists $R_0 \in (0, 1]$ such that for any $r \in (0, R_0]$, $x \in \Omega$ and $M \in \mathcal{S}(n)$ with $\|M\| = 1$

$$\int_{B_r(x)} |F(M, x) - \bar{F}(M)| dx \leq \delta, \quad \bar{F}(M) = \int_{B_r(x)} |F(M, x)| dx$$

and gave global Calderón-Zygmund estimates in bounded domain Ω . Krylov [?, ?] further investigate the existence and uniqueness for fully nonlinear elliptic and parabolic operator F under some relaxed convexity assumptions instead of requiring F to be convex or concave with respect to D^2u . Later, Byun et al [?] used rather different geometrical approaches to achieve a global weighted $W^{2,q}$ estimates for fully nonlinear elliptic equations with small BMO "coefficients" in a bounded $C^{1,1}$ domain on the basis of the well-known interior and boundary $W^{2,q}$ -estimates in [?, ?]. This suggests a question of what happens in the limiting case $q \rightarrow \infty$. Caffarelli-Huang's work [?] answered this question: if f belongs to the generalized Campanato-John-Nirenberg spaces, $F(M, x)$ is a small multiplier of BMO in x , and Evans-Krylov estimates hold [?], then the Hessian D^2u of their strong solutions must be correspondingly the generalized Campanato-John-Nirenberg spaces.

On the other hand, there is a great deal of literature on the topic of Lorentz regularity concerning PDEs. For example, Baroni in [?, ?] obtained the Lorentz estimates for evolutionary p -Laplacian systems and obstacle parabolic p -Laplacian by using *the Large-M-inequality principle* introduced by Acerbi-Mingione in [?]. Meanwhile, Mengesha-Phuc in [?] established the gradient estimates in weighted Lorentz spaces for quasilinear p -Laplacian based on a rather different geometrical approach from Byun-Wang's ideas in [?]. In [?], we proved interior unweighted Lorentz estimates of strong solutions to fully nonlinear elliptic and parabolic equations using the Large-M-inequality principle mentioned above. Here, in the present paper, we are mainly devoted to employing a different approach to show a global weighted estimate in the scale of Lorentz spaces for the *viscosity solutions* of Dirichlet problem (??) under certain regular assumptions imposed on the nonlinearity F over a bounded $C^{1,1}$ domain.

For readers' convenience, let us first recall the definitions of weight, Lorentz-Sobolev spaces and Lorentz-Morrey spaces as follows. Let $s \in (1, \infty)$, we say ω is an A_s weight if $\omega \in A_s$, where A_s is the class of all

nonnegative locally integrable function $\omega(x)$ defined on \mathbb{R}^n satisfying

$$[\omega]_s := \sup_{B \subset \mathbb{R}^n} \left(\int_B \omega(x) dx \right) \left(\int_B \omega(x)^{\frac{-1}{s-1}} dx \right)^{s-1} < +\infty.$$

A direct consequence show that A_s is invariant under translations, dilations and multiplication by a positive scalar. For each measurable set $U \subset \mathbb{R}^n$ and a weight ω , we set

$$\omega(U) = \int_U \omega(x) dx.$$

Let $U \subset \mathbb{R}^n$ be an bounded set and $(q, t, \theta) \in (0, \infty) \times (0, \infty] \times (0, n]$. The *weighted Lorentz spaces* $L_\omega^{q,t}(U)$ and $L_\omega^{q,\infty}(U)$ are defined to be the sets of measurable functions $g(x) : U \rightarrow \mathbb{R}$ satisfying

$$\begin{cases} \|g\|_{L_\omega^{q,t}(U)} := \left(q \int_0^\infty (\lambda^q \omega(\{x \in U : |g(x)| > \lambda\}))^{\frac{t}{q}} \frac{d\lambda}{\lambda} \right)^{\frac{1}{t}} < +\infty; \\ \|g\|_{L_\omega^{q,\infty}(U)} := \sup_{\lambda > 0} \lambda \omega(\{x \in U : |g(x)| > \lambda\})^{\frac{1}{q}} < \infty, \end{cases}$$

respectively. Furthermore, the *weighted Lorentz-Sobolev space* $W^2 L_\omega^{q,t}(U)$ is defined by the set of functions $g(x)$ whose distributional derivatives $D^k g(x)$ with integer $0 \leq k \leq 2$ also belong to $L_\omega^{q,t}(U)$, and its norm is naturally given by

$$\|g\|_{W^2 L_\omega^{q,t}(U)} := \sum_{k=0}^2 \|D^k g\|_{L_\omega^{q,t}(U)}.$$

We say a measurable function $g(x)$ belongs to *Lorentz-Morrey space* $L^{q,t;\theta}(U)$ if and only if

$$\|g\|_{L^{q,t;\theta}(U)} := \sup_{(x,r) \in U \times (0, d_0)} r^{\frac{\theta-n}{q}} \|g\|_{L^{q,t}(B_r(x) \cap U)} < +\infty, \quad (1.2)$$

where $d_0 = \text{diam}(\Omega)$ and $B_r(x)$ stands for the ball in \mathbb{R}^n with center x and radius r . If $\omega = 1$ then $L_\omega^{q,t}(U) = L^{q,t}(U)$, so our result is an extension of [?]. If $t = q$ then $L_\omega^{q,q}(U) = L_\omega^q(U)$, $L^{q,t;\theta}(\Omega) = L^{q;\theta}(\Omega)$, so our result is also an extension of [?].

Next, to our ends we need to add the following structure conditions imposed on F .

- H1. $F(M, x)$ is convex in M and $F(0, x) = 0$;
- H2. $F(M, x)$ is uniformly elliptic with some positive constants ϑ_1, ϑ_2 satisfying $\vartheta_1 \leq \vartheta_2$ such that

$$\mathcal{M}^-(M - N, \vartheta_1, \vartheta_2) \leq F(M, x) - F(N, x) \leq \mathcal{M}^+(M - N, \vartheta_1, \vartheta_2), \quad \forall (N, x) \in \mathcal{S}(n) \times \Omega, \quad (1.3)$$

where \mathcal{M}^- and \mathcal{M}^+ stand for the Pucci extremal operators associated with ϑ_1, ϑ_2 :

$$\begin{aligned} \mathcal{M}^-(M, \vartheta_1, \vartheta_2) &:= \vartheta_1 \sum_{\lambda_i > 0} + \vartheta_2 \sum_{\lambda_i < 0}, \\ \mathcal{M}^+(M, \vartheta_1, \vartheta_2) &:= \vartheta_1 \sum_{\lambda_i < 0} + \vartheta_2 \sum_{\lambda_i > 0}, \end{aligned}$$

and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of M .

- H3. $F(M, x)$ is “ (δ, R) -vanishing” with respect to x , that is, there exists a small $\delta > 0$ and R such that

$$\sup_{x_0 \in \Omega} \sup_{0 < r \leq R} \left(\int_{B_r(x_0) \cap \Omega} \Upsilon(x, x_0)^n dx \right)^{\frac{1}{n}} \leq \delta, \quad (1.4)$$

where $\Upsilon(x, x_0)$ stands for the oscillation of F in the variable x , i.e.,

$$\Upsilon(x, x_0) = \sup_{M \in \mathcal{S} \setminus \{0\}} \frac{|F(M, x) - F(M, x_0)|}{\|M\|}.$$

Now we are in a position to present the main result concerning the regularity for the Dirichlet problem .

Theorem 1.1 *Let $(q, t, \theta, p) \in (n, \infty) \times (0, \infty] \times (0, n] \times [n, q)$ and ω be an $A_{\frac{q}{n}}$ weight. Assume that $\partial\Omega \in C^{1,1}$ and there exist a small $\delta > 0$ and R such that F satisfies structure conditions H1-H3.*

(i) *if $f \in L_{\omega}^{q,t}(\Omega)$, then the Dirichlet problem (??) has a unique L^p viscosity solution $u \in W^2 L^{q,t}(\Omega)$ satisfying*

$$\|u\|_{W^2 L_{\omega}^{q,t}(\Omega)} \leq C \|f\|_{L_{\omega}^{q,t}(\Omega)}, \quad (1.5)$$

where δ, R, C depend only on $n, \vartheta_1, \vartheta_2, q, t, [\omega]_{\frac{q}{n}}, d_0$ (except in the case $t = \infty$, where δ, R, C are independent of t);

(ii) *if $f \in L^{q,t;\theta}(\Omega)$, then $D^2 u \in L^{q,t;\theta}(\Omega)$ and*

$$\|D^2 u\|_{L^{q,t;\theta}(\Omega)} \leq C \|f\|_{L^{q,t;\theta}(\Omega)}, \quad (1.6)$$

where δ, R, C depend only on $n, \vartheta_1, \vartheta_2, q, t, \theta, [\omega]_{\frac{q}{n}}, d_0$ (except in the case $t = \infty$, where δ, R, C are independent of t).

Remark 1.2 *Six comments on the hypotheses H1-H3 and Theorem ?? are in order.*

1. *The convexity (or concavity) of F in H1 ensures that the viscosity solutions of the corresponding homogeneous equations with constant coefficients have $C^{1,1}$ interior and boundary estimates (cf. [?, Theorem 6.6], [?, Remark 4.4]).*

2. *H2 amounts to for any nonnegative definite matrix $N \in \mathcal{S}(n)$ and almost all $x \in \Omega$*

$$\vartheta_1 \|N\| \leq F(M + N, x) - F(M, x) \leq \vartheta_2 \|N\|,$$

where $\|N\| := \sup_{|x|=1} |Nx|$ = the maximum eigenvalue of N .

3. *The condition (??) is equivalent to*

$$\sup_{x_0 \in \Omega} \sup_{0 < r \leq R} \int_{B_r(x_0) \cap \Omega} \Upsilon(x, x_0) dx \leq \delta.$$

Also, (??) amounts to a small oscillation of F in the L^{∞} -norm.

4. *A typical example when it is relatively easy to verify our hypotheses H1-H3 is given by the following Bellman’s equation:*

$$\sup_{\kappa \in \mathcal{A}} [a_{ij}^{\kappa}(x) D_{ij} u(x)] = f(x),$$

where the set \mathcal{A} is a countable set, $a^{\kappa}(x) = (a_{ij}^{\kappa}(x))_{i,j=1}^n$ and f are given functions measurable in x for each $\kappa \in \mathcal{A}$. We introduce

$$F(D^2 u, x) = \sup_{\kappa \in \mathcal{A}} [a_{ij}^{\kappa}(x) D_{ij} u(x)],$$

then several conditions in terms of $a_{ij}^\kappa(x)$ can be given that are sufficient for the hypotheses H1-H3 to hold. For instance, the conditions H1-H2 are satisfied if for any $\kappa \in \mathcal{A}$ each function $a_{ij}^\kappa(x)D_{ij}u(x)$ satisfy H1-H2. And, H3 is satisfied if

$$\sup_{(y,r) \in \Omega \times (0,R]} \left(\int_{B_r(y) \cap \Omega} \sup_{\kappa \in \mathcal{A}} |a_{ij}^\kappa(x) - (a_{ij}^\kappa)_{B_r(y) \cap \Omega}|^n dx dt \right)^{\frac{1}{n}} \leq \delta,$$

where $(a_{ij}^\kappa)_{B_r(y) \cap \Omega}$ stands for the mean value of $a_{ij}^\kappa(x)$ on $B_r(y) \cap \Omega$, that is,

$$(a_{ij}^\kappa)_{B_r(y) \cap \Omega} = \frac{1}{|B_r(y) \cap \Omega|} \int_{B_r(y) \cap \Omega} a_{ij}^\kappa(x) dx.$$

5. Since Caffarell's interior $W^{2,q}$ -estimate and Winter's boundary $W^{2,q}$ -estimate can be generalised by Escauriaza's method (weak reverse Hölder inequality) to the range of $n - \check{\epsilon} < q < \infty$ where $\check{\epsilon} = \check{\epsilon}(\frac{\theta_2}{\theta_1}, n)$ (cf, [?, ?]), our result also valid for $(q, t, \theta, p) \in (n - \check{\epsilon}, \infty) \times (0, \infty) \times (0, n) \times [n - \check{\epsilon}, q]$ for some $\check{\epsilon} = \check{\epsilon}(\frac{\theta_2}{\theta_1}, n, d_0)$.
6. Lorentz-Morrey spaces $L^{q,t;\theta}(\Omega)$ are neither rearrangement invariant spaces, nor interpolation spaces. They often show up in the analysis of Schrödinger operators and quasilinear Riccati type equations (cf. [?]).

We would like to mention that a uniformly elliptic condition on the nonlinearity F is not enough to ensure our main result, that is the second derivatives of solutions to equations corresponding to highly oscillatory coefficients cannot be expected to have higher integrability under the assumptions of the Lorentz regularity of the data $f(x)$, see Meyers's work [?] for an counterexample. Therefore, requiring the coefficients to satisfy structure condition H3 is necessary. In addition, by Jerison-Kenig's counterexample in [?], with details given in Mengesha-Phuc's paper [?], the $C^{1,1}$ assumption on the boundary is necessary if we want to obtain the global higher integrability of D^2u .

The rest of the paper is organized as follows. The second section presents some basic lemmas employed in the proof of Theorem ???. The third section is used to prove Theorem ?? (i). The forth section is devoted to proving Theorem ?? (ii).

Notations 1.3 Nevertheless, the coming-up-next symbolic conventions are necessary:

(i) For $r > 0$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we set

$$B_r(x) = \{y \in \mathbb{R}^n : |x - y| < r\}, \quad B_r^+(x) = B_r(x) \cap \{x_1 > 0\}, \quad \Gamma_r(x) = B_r(x) \cap \{x_1 = 0\}.$$

and write $B_r = B_r(0)$, $B_r^+ = B_r^+(0)$, $\Gamma_r = \Gamma_r(0)$ for simplicity.

(ii) $Du = (\partial_{x_i}u, \dots, \partial_{x_n}u)$ and $D^2u = (D_{ij}u) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j=1}^n$ stand for the gradient and the Hessian of $u(x) : \mathbb{R}^n \rightarrow \mathbb{R}$, respectively.

(iii) $a \lesssim b$ means $a \leq Cb$ for a positive constant C ; $a \approx b$ means both $a \lesssim b$ and $b \lesssim a$.

(iv) For a measurable subset $U \subset \mathbb{R}^n$, $|U|$ denotes n -dimensional Lebesgue measure of U .

2 Preliminaries

We start this section with the definition of viscosity solution to fully nonlinear elliptic equation

$$F(D^2u, x) = f(x) \quad \text{in } \Omega. \tag{2.1}$$

Definition 2.1 Let F be continuous in X and measurable in x . Suppose $p > \frac{n}{2}$ and $f \in L^p_{loc}(\Omega)$. A function $u \in C(\Omega)$ is called an L^p -viscosity subsolution (respectively, supersolution) of (??), if for all $\varphi \in W^{2,p}_{loc}(\Omega)$ whenever $\varepsilon > 0$, $U \subset \Omega$ is open and

$$F(D^2\varphi(x), x) - f(x) \leq -\varepsilon \quad \text{a.e. in } U$$

$$\left(\text{respectively, } F(D^2\varphi(x), x) - f(x) \geq \varepsilon \quad \text{a.e. in } U \right),$$

$u - \varphi$ cannot attain a local maximum (respectively, minimum) in U . Moreover, u is L^p -viscosity solution of (??), if it is both an L^p -viscosity subsolution and an L^p -viscosity supersolution.

A function $Y(x)$ is called a paraboloid with opening K if

$$Y(x) = a_0 + b(x) \pm \frac{K}{2}|x|^2, \quad a_0, K \in \mathbb{R}^+, \quad (2.2)$$

where $b(x)$ is a linear function. $Y(x)$ is convex when we choose “+” in (??) and concave when we choose “-” in (??). Let u be a continuous function in bounded domain Ω . For any open set $U \subset \Omega$ and $K > 0$, we define $\underline{G}_K(u, U) := U \setminus \underline{F}_K(u, U)$ and

$$\underline{F}_K(u, U) := \{x_0 \in U : \exists \text{ concave } Y(x) \text{ such that } Y(x) \leq u(x) \text{ and } Y(x_0) = u(x_0) \text{ for any } x \in U\}.$$

We analogously define $\overline{F}_K(u, U)$ and $\overline{G}_K(u, U)$ by using convex paraboloid. We also define $F_K(u, U) := \underline{F}_K(u, U) \cap \overline{F}_K(u, U)$ and $G_K(u, U) := \underline{G}_K(u, U) \cap \overline{G}_K(u, U)$. For any $x \in U$, set

$$\underline{\Theta}(u, U)(x) := \inf\{K > 0 : x \in \underline{F}_K(u, U)\};$$

$$\overline{\Theta}(u, U)(x) := \inf\{K > 0 : x \in \overline{F}_K(u, U)\};$$

$$\Theta(u, U)(x) := \sup\{\underline{\Theta}(u, U)(x), \overline{\Theta}(u, U)(x)\}.$$

Lemma 2.2 Let $(q, t, s) \in (1, \infty) \times (0, \infty] \times (1, \infty)$ and ω be an A_s weight, and u be a continuous function in a bounded domain $U \subset \mathbb{R}^n$. If $\Theta(u, r)(x) \in L^{q,t}_\omega(U)$, then $D^2u \in L^{q,t}_\omega(U)$ and

$$\|D^2u\|_{L^{q,t}_\omega(U)} \lesssim \|\Theta(u, r)\|_{L^{q,t}_\omega(U)}, \quad (2.3)$$

where

$$\Theta(u, r)(x) := \Theta(u, U \cap B_r(x))(x) \quad \text{for } (x, r) \in U \times (0, \infty).$$

Proof. We consider two cases.

Case $0 < t < \infty$. Take $\{e_i\}_{i=1}^n$ be the canonical basis of \mathbb{R}^n . Let us define the second differential quotients of u at point x by

$$\Delta_{he_i}^2 u(x) = \frac{u(x + he_i) + u(x - he_i) - 2u(x)}{|h|^2},$$

where $h \in \mathbb{R}$, $x + he_i$ and $x - he_i$ belong to U . Note that

$$-\underline{\Theta}(u, U \cap B_{|h|}(x))(x) \leq \Delta_{he_i}^2 u(x) \leq \overline{\Theta}(u, U \cap B_{|h|}(x))(x), \quad \forall x \in U,$$

which implies

$$|\Delta_{he_i}^2 u(x)| \leq \Theta(u, |h|)(x).$$

Together with $u_{e_i, e_i}(x) = \lim_{h \rightarrow 0} \Delta_{he_i}^2 u(x)$ and letting $r = |h|$, we have

$$|D_{e_i, e_i} u(x)| \leq \Theta(u, r)(x).$$

A simple calculation infers

$$\begin{aligned} D_{ij}u(x) = D_{e_i e_j}u(x) &= \frac{1}{2} \left(D_{e_i + e_j, e_i + e_j}u(x) - D_{e_i e_i}u(x) - D_{e_j e_j}u(x) \right) \\ &= \frac{1}{2} \left(2D_{\nu\nu}u(x) - D_{e_i e_i}u(x) - D_{e_j e_j}u(x) \right) \end{aligned}$$

where $\nu = (e_i + e_j)/\sqrt{2}$. Thereby

$$\|D^2u\| \lesssim \sum_{i,j=1}^n |D_{ij}u(x)| \lesssim \Theta(u, r)(x), \quad i, j = 1, \dots, n.$$

It follows that

$$\begin{aligned} \|D^2u\|_{L_\omega^{q,t}(U)} &= \left(q \int_0^\infty (\lambda^q \omega(\{x \in U : \|D^2u\| > \lambda\}))^{\frac{t}{q}} \frac{d\lambda}{\lambda} \right)^{\frac{1}{t}} \\ &\lesssim \left(q \int_0^\infty (\lambda^q \omega(\{x \in U : |\Theta(u, r)| > \lambda\}))^{\frac{t}{q}} \frac{d\lambda}{\lambda} \right)^{\frac{1}{t}} \\ &= \|\Theta(u, r)\|_{L_\omega^{q,t}(U)}. \end{aligned}$$

Case $t = \infty$. With the same argument as in the proof of [?, Proposition 1.1], we deduce that for any $\zeta \in C_c^\infty(U)$ (infinitely differentiable functions with compact support in U)

$$\left| \int_U u D_{ij} \zeta dx \right| \lesssim \int_U |\Theta(u, r)| |\zeta| dx = \|\Theta(u, r)\zeta\|_{L^{1,1}(U)} \leq \|\Theta(u, r)\|_{L^{q,\infty}(U)} \|\zeta\|_{L^{q',1}(U)}$$

where $q' = \frac{q}{q-1}$. Therefore, the strong doubling property in Lemma ?? gives

$$\|D^2u\|_{L_\omega^{q,\infty}(U)} \lesssim \|D^2u\|_{L^{q,\infty}(U)} \lesssim \|\Theta(u, r)\|_{L^{q,\infty}(U)} \lesssim \|\Theta(u, r)\|_{L_\omega^{q,\infty}(U)}.$$

□

In what follows, we collect the monotone increasing in s , open-end property and strong doubling property of A_s weight, for their detailed proofs see [?, Lemma 2.1] and [?, Lemma 3.4].

Lemma 2.3 *Let ω be an A_s weight for some $1 < s < \infty$ and $k > 0$. Then*

1. (increasing) $\omega \in A_p$ with $p \geq s$ and $[\omega]_p \leq [\omega]_s$;
2. (open-end) $\omega \in A_{s-\varepsilon_0}$ with small enough $\varepsilon_0 > 0$ depending on $n, s, [\omega]_s$ such that $s - \varepsilon_0 > 1$;
3. (strong doubling) Let E be a measurable subset of a ball $B \subset \mathbb{R}^n$. Then there exist two constants $\beta > 0, \nu \in (0, 1)$ depending only on $n, s, [\omega]_s$ such that

$$[\omega]_s^{-1} \left(\frac{|E|}{|B|} \right)^s \leq \frac{\omega(E)}{\omega(B)} \leq \beta \left(\frac{|E|}{|B|} \right)^\nu. \quad (2.4)$$

4. the translation of ω , $\tilde{\omega}(x) = \omega(x - y)$ is an A_s weight with $[\tilde{\omega}]_s = [\omega]_s$;
5. the truncation of ω , $\tilde{\omega}(x) = \min\{\omega, k\}$ is an A_s weight and satisfies

$$[\tilde{\omega}]_s \leq C_s [\omega]_s, \quad (2.5)$$

where $C_s = \max\{2, 2^{s-1}\}$.

One of the main tools used to prove Theorem ?? is the Hardy-Littlewood maximal function which controls the local behavior of a function. For a function $g(x) \in L^1_{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function of $g(x)$ is defined by

$$\mathcal{M}g(x) = \sup_{B_r(x) \subset \mathbb{R}^n} \int_{B_r(x)} |g(y)| dy.$$

For a function defined only on a bounded subset $U \subset \mathbb{R}^n$, we define $\mathcal{M}_U g := \mathcal{M}(g\chi_U)$. It is well known that \mathcal{M} is bounded on $L^{s,t}_\omega(\mathbb{R}^n)$ with $1 < s < \infty$, if and only if ω is an A_s weight (cf. [?, Lemma 3.11]).

We are now in a position to recalling an elementary characterization of functions in Lorentz spaces, which can easily be proved using methods in standard measure theory (cf. [?, Lemma 3.12]).

Lemma 2.4 *Let ω be an A_s weight for some $1 < s < \infty$ and g be a nonnegative measurable function in a bounded domain $U \subset \mathbb{R}^n$. Let $\gamma > 0$ and $N > 1$ be constants. Then for any $0 < q, t < \infty$, we have*

$$g \in L^{q,t}_\omega(U) \Leftrightarrow S := \sum_{k \geq 1} N^{tk} \omega(\{x \in U : g(x) > \gamma N^k\})^{\frac{t}{q}} < +\infty,$$

and moreover

$$C^{-1}S \leq \|g\|_{L^{q,t}_\omega(U)}^t \leq C(\omega(U)^{\frac{1}{q}} + S) \quad (2.6)$$

with constant $C = C(\gamma, N, t) > 0$. Analogously, for $0 < q < \infty$ and $t = \infty$ we have

$$C^{-1}T \leq \|g\|_{L^{q,\infty}_\omega(U)} \leq C(\omega(U)^{\frac{1}{q}} + T), \quad (2.7)$$

where T is the quantity

$$T := \sup_{k \geq 1} N^k \omega(\{x \in U : |g(x)| > \gamma N^k\})^{\frac{1}{q}}.$$

Finally, we state a summary of embedding relations in Lorentz spaces as follows (provided in [?, Proposition 3.9]).

Lemma 2.5 *Let U be a bounded measurable subset of \mathbb{R}^n , and ω be an A_s weight for some $1 < s < \infty$. Then the following holds.*

1. *If $|g|^r \in L^{q,t}_\omega(U)$ for some $0 < r < \infty$, then $g \in L^{rq,rt}_\omega(U)$ with $\|g\|_{L^{rq,rt}_\omega(U)}^r = \| |g|^r \|_{L^{q,t}_\omega(U)}$;*
2. *If $0 < t_1, t_2 \leq \infty$ and $0 < r < q < \infty$, then $L^{q,t_1}_\omega(U) \subset L^{q,t_2}_\omega(U)$;*
3. *If $0 < t_1 < t_2 \leq \infty$ and $0 < q < \infty$, then $L^{q,t_1}_\omega(U) \subset L^{q,t_2}_\omega(U) \subset L^{q,\infty}_\omega(U)$; moreover, if $1 < q < \infty$, then $L^{q,\infty}_\omega(U) \subset L^{q-\varepsilon}_\omega(U)$ for any $\varepsilon > 0$ such that $q - \varepsilon > 1$.*

3 The proof of Theorem ?? (i)

We begin this section with the weighted Lorentz estimate in ball $B_{\frac{1}{2}}$ and that in half ball $B^+_{\frac{1}{2}}$, respectively, for the viscosity solutions of (??).

Lemma 3.1 *Let $(q, t, p) \in (n, \infty) \times (0, \infty) \times [n, q)$ and ω be an $A_{\frac{q}{n}}$ weight.*

(i) *Assume that $f \in L^{q,t}_\omega(B_1)$. If there exist a small $\delta > 0$ and R depending on $n, \vartheta_1, \vartheta_2, q, t, [\omega]_{\frac{q}{n}}$ such that F satisfies the conditions H1-H3 in B_1 , then for any bounded L^p -viscosity solution u of*

$$F(D^2u, x) = f(x) \quad \text{in } B_1,$$

we have $D^2u \in L_\omega^{q,t}(B_{\frac{1}{2}})$ and

$$\|D^2u\|_{L_\omega^{q,t}(B_{\frac{1}{2}})} \leq C \left(\|f\|_{L_\omega^{q,t}(B_1)} + \|u\|_{L^\infty(B_1)} \right), \quad (3.1)$$

for some positive constant $C = C(n, \vartheta_1, \vartheta_2, q, t, [\omega]_{\frac{q}{n}})$.

(ii) Assume that $f \in L_\omega^{q,t}(B_1^+)$. If there exist a small $\delta > 0$ and R depending on $n, \vartheta_1, \vartheta_2, q, t, [\omega]_{\frac{q}{n}}$ such that F satisfies the conditions H1-H3 in B_1^+ , then for any bounded L^p -viscosity solution u of

$$\begin{cases} F(D^2u, x) = f(x), & \text{in } B_1^+, \\ u = 0, & \text{on } \Gamma_1, \end{cases}$$

we have $D^2u \in L_\omega^{q,t}(B_{\frac{1}{2}}^+)$ with the estimate

$$\|D^2u\|_{L_\omega^{q,t}(B_{\frac{1}{2}}^+)} \leq C \left(\|f\|_{L_\omega^{q,t}(B_1^+)} + \|u\|_{L^\infty(B_1^+)} \right). \quad (3.2)$$

for some positive constant $C = C(n, \vartheta_1, \vartheta_2, q, t, [\omega]_{\frac{q}{n}})$.

Proof. Since the proof of the interior weighted Lorentz estimates (??) is similar to that of (??), we only show the boundary estimate (??). For given point $x_0 \in \Gamma_{\frac{1}{2}}$, we take a small constant $r \in (0, \frac{1}{28\sqrt{n}})$, which will be determined later. Let us denote

$$\tilde{u}(x) := \frac{\delta u}{r^{-2}\delta\|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))} + \|f\|_{L_\omega^{q,t}(B_{14r\sqrt{n}}^+(x_0))}}, \quad \tilde{f}(x) := \frac{\delta f}{r^{-2}\delta\|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))} + \|f\|_{L_\omega^{q,t}(B_{14r\sqrt{n}}^+(x_0))}},$$

and define

$$\tilde{F}(D^2\tilde{u}, x) = \frac{F\left(\left(r^{-2}\delta\|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))} + \|f\|_{L_\omega^{q,t}(B_{14r\sqrt{n}}^+(x_0))}\right)D^2\tilde{u}, rx + x_0\right)}{r^{-2}\delta\|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))} + \|f\|_{L_\omega^{q,t}(B_{14r\sqrt{n}}^+(x_0))}},$$

$$\tilde{u}_r(x) = r^{-2}\tilde{u}(rx + x_0), \quad \tilde{f}_r(x) = \tilde{f}(rx + x_0), \quad \tilde{\omega}(x) = \omega(rx + x_0), \quad \tilde{\Omega} = \{r^{-1}x + x_0 : x \in \Omega\}.$$

Then we see that $\tilde{u}_r(x)$ is a bounded L^p -viscosity solution of

$$\begin{cases} \tilde{F}(D^2\tilde{u}_r, x) = \tilde{f}_r(x), & \text{in } B_{14\sqrt{n}}^+ \subset \tilde{\Omega}, \\ \tilde{u}_r = 0, & \text{on } \Gamma_{14\sqrt{n}}^+, \end{cases}$$

$\omega(x)$ is an $A_{\frac{q}{n}}$ weight, $\tilde{f}_r(x) \in L_\omega^{q,t}(\tilde{\Omega})$, $\partial\tilde{\Omega} \in C^{1,1}$, \tilde{F} satisfies hypotheses H1-H3, and

$$\|\tilde{u}_r\|_{L^\infty(B_{14\sqrt{n}}^+)} \leq 1, \quad \|\tilde{f}_r\|_{L^n(B_{14\sqrt{n}}^+)} \lesssim \|\tilde{f}_r\|_{L_\omega^{q,t}(B_{14\sqrt{n}}^+)} \lesssim \delta.$$

The proof of the second inequality comes from (??) below. Then from [?, Lemma 2.15], it follows that for a given small ϵ_0 satisfying $\epsilon_0^{\frac{1}{\nu}}\beta^{-\frac{1}{\nu}} \in (0, 1)$ there exists constants $\gamma > 0$ and $N > 1$ such that

$$|A_{k+1}| \leq \epsilon_0^{\frac{1}{\nu}}\beta^{-\frac{1}{\nu}}|A_k \cup B_k|,$$

where

$$\begin{cases} A_k = G_{N^k}(\tilde{u}_r, B_{14\sqrt{n}}^+) \cap Q_1^+; \\ B_k = \{x \in Q_1^+ : \mathcal{M}(\tilde{f}_r^n)(x) \geq (\gamma N^k)^n\}; \\ Q_1^+ := \left\{x \in \mathbb{R}^{n-1} : \max_{1 \leq i \leq n-1} |x_i| < \frac{1}{2}\right\} \times (0, 1), \end{cases}$$

$k = 0, 1, 2, \dots$. Using the strong doubling property in Lemma ?? we get

$$\begin{aligned}\omega(\mathbf{A}_{k+1}) &\leq \beta \left(\frac{|\mathbf{A}_{k+1}|}{|\mathbf{A}_k \cup \mathbf{B}_k|} \right)^\nu \omega(\mathbf{A}_k \cup \mathbf{B}_k) \\ &\leq \beta \left(\epsilon_0^{\frac{1}{\nu}} \beta^{-\frac{1}{\nu}} \right)^\nu \omega(\mathbf{A}_k \cup \mathbf{B}_k) \\ &= \epsilon_0 \omega(\mathbf{A}_k \cup \mathbf{B}_k) \\ &\leq \epsilon_0 \omega(\mathbf{A}_k) + \epsilon_0 \omega(\mathbf{B}_k)\end{aligned}$$

Thereby,

$$\begin{aligned}\sum_{k=1}^{\infty} N^{tk} \tilde{\omega}^{\frac{t}{q}}(\mathbf{A}_k) &\lesssim \sum_{k=1}^{\infty} (N^t \epsilon_1)^k \tilde{\omega}^{\frac{t}{q}}(\mathbf{A}_0) + \sum_{k=1}^{\infty} N^{tk} \left(\sum_{i=1}^k \epsilon_1^i \tilde{\omega}^{\frac{t}{q}}(\mathbf{B}_{k-i}) \right) \\ &\lesssim \sum_{k=1}^{\infty} (N^t \epsilon_1)^k \tilde{\omega}^{\frac{t}{q}}(Q_1^+) + \sum_{i=1}^{\infty} (N^t \epsilon_1)^i \left(\sum_{k=i}^{\infty} N^{t(k-i)} \tilde{\omega}^{\frac{t}{q}}(\mathbf{B}_{k-i}) \right) \\ &\lesssim \sum_{k=1}^{\infty} (N^t \epsilon_1)^k \left(\tilde{\omega}^{\frac{t}{q}}(Q_1^+) + \|\mathcal{M}(\tilde{f}_r^n)\|_{L_{\tilde{\omega}}^{\frac{t}{n}, \frac{t}{n}}(Q_1^+)}^{\frac{t}{n}} \right)\end{aligned}$$

where $\epsilon_1 = \max\{1, 2^{\frac{t}{q}-1}\} \epsilon_0^{\frac{t}{q}}$. Now we choosing ϵ_0 sufficiently small so that $N^t \epsilon_1 < 1$ to deduce that

$$\begin{aligned}\|\Theta(\tilde{u}_r, B_{\frac{1}{2}}^+)\|_{L_{\tilde{\omega}}^{q,t}(B_{\frac{1}{2}}^+)}^t &\lesssim \tilde{\omega}^{\frac{t}{q}}(B_{\frac{1}{2}}^+) + \sum_{k=1}^{\infty} N^{tk} \tilde{\omega}^{\frac{t}{q}} \left(\left\{ x \in B_{\frac{1}{2}}^+ : \Theta(u, B_{\frac{1}{2}}^+)(x) > N^k \right\} \right) \\ &\lesssim \tilde{\omega}^{\frac{t}{q}}(B_{\frac{1}{2}}^+) + \sum_{k=1}^{\infty} N^{tk} \tilde{\omega}^{\frac{t}{q}} \left(G_{N^k}(\tilde{u}_r, B_{\frac{1}{2}}^+) \right) \\ &\lesssim \tilde{\omega}^{\frac{t}{q}}(B_{\frac{1}{2}}^+) + \sum_{k=1}^{\infty} N^{tk} \tilde{\omega}^{\frac{t}{q}}(\mathbf{A}_k) \\ &\lesssim \tilde{\omega}^{\frac{t}{q}}(B_{\frac{1}{2}}^+) + \sum_{k=1}^{\infty} (N^t \epsilon_1)^k \left(\tilde{\omega}^{\frac{t}{q}}(Q_1^+) + \|\mathcal{M}(\tilde{f}_r^n)\|_{L_{\tilde{\omega}}^{\frac{t}{n}, \frac{t}{n}}(Q_1^+)}^{\frac{t}{n}} \right) \\ &\lesssim \tilde{\omega}^{\frac{t}{q}}(B_{\frac{1}{2}}^+) + \tilde{\omega}^{\frac{t}{q}}(Q_1^+) + \|\mathcal{M}(\tilde{f}_r^n)\|_{L_{\tilde{\omega}}^{\frac{t}{n}, \frac{t}{n}}(Q_1^+)}^{\frac{t}{n}} \\ &\lesssim 1 + \|\tilde{f}_r^n\|_{L_{\tilde{\omega}}^{\frac{t}{n}, \frac{t}{n}}(Q_1^+)}^{\frac{t}{n}}\end{aligned}$$

and this gives

$$\|\Theta(\tilde{u}_r, B_{\frac{1}{2}}^+)\|_{L_{\tilde{\omega}}^{q,t}(B_{\frac{1}{2}}^+)}^t \lesssim 1 + \|\tilde{f}_r^n\|_{L_{\tilde{\omega}}^{q,t}(Q_1^+)}^t \lesssim 1,$$

where we used $\|\tilde{f}_r^n\|_{L_{\tilde{\omega}}^{\frac{t}{n}, \frac{t}{n}}(Q_1^+)}^{\frac{t}{n}} = \|\tilde{f}_r^n\|_{L_{\tilde{\omega}}^{q,t}(Q_1^+)}^t$ for any $0 < r < \infty$. Using Lemma ?? yields

$$\|D^2 \tilde{u}_r\|_{L_{\tilde{\omega}}^{q,t}(B_{\frac{1}{2}}^+)} \leq C$$

for some positive constant $C = C(n, \vartheta_1, \vartheta_2, q, t, [\omega]_{\frac{q}{n}})$. Then recalling the definition of \tilde{u}_r it yields

$$\|D^2 u\|_{L_{\tilde{\omega}}^{q,t}(B_r^+(x_0))} \leq C \left(r^{-2} \|u\|_{L^\infty(B_{14r\sqrt{n}}^+(x_0))} + \|f\|_{L_{\tilde{\omega}}^{q,t}(B_{14r\sqrt{n}}^+(x_0))} \right). \quad (3.3)$$

Analogously, we can establish an interior estimate

$$\|D^2 u\|_{L_\omega^{q,t}(B_{\frac{r}{2}}(x_0))} \leq C \left(r^{-2} \|u\|_{L^\infty(B_{8r\sqrt{n}}(x_0))} + \|f\|_{L_\omega^{q,t}(B_{8r\sqrt{n}}(x_0))} \right) \quad \text{for any } x_0 \in B_{\frac{1}{2}}^+. \quad (3.4)$$

Take r sufficiently small so that $B_{\frac{1}{2}}^+$ can be covered by finite number of $B_{\frac{r}{2}}^+(x_0)$ for $x_0 \in \Gamma_{\frac{1}{2}}$ and $B_{\frac{r}{2}}(x_0)$ for $x_0 \in B_{\frac{1}{2}}^+$. Then (??) and (??) yields the desired estimate (??). \square

Proof of Theorem ?? (i). Based on the interior and boundary weighted Lorentz estimates in Lemma ??, we can use the standard flattening and covering arguments to prove Theorem ??.

Step 1: Existence and uniqueness of L^p -viscosity solution, $p \in [n, q)$. The open-end property of ω in Lemma ?? implies that for any $p \in [n, q)$ there exists a small $\varepsilon_0 = \varepsilon_0(n, p, q, [\omega]_{\frac{q}{n}})$ such that $\omega \in A_{\frac{q}{p}-\varepsilon_0}$ and $\frac{q}{p} - \varepsilon_0 > 1$. Setting $i_0 = \frac{q}{p} - \varepsilon_0$ we obtain $q > pi_0$. Then we derive

$$\begin{aligned} \int_{\Omega} |f|^p dx &= \int_{\Omega} |f|^p \omega^{\frac{1}{i_0}} \omega^{-\frac{1}{i_0}} dx \\ &\leq \left(\int_{\Omega} |f|^{pi_0} \omega dx \right)^{\frac{1}{i_0}} \left(\int_{\Omega} \omega^{-\frac{1}{i_0-1}} dx \right)^{\frac{i_0-1}{i_0}} \\ &\leq \frac{|\Omega| [\omega]_{i_0}^{\frac{1}{i_0}}}{\omega(\Omega)^{\frac{1}{i_0}}} \|f\|_{L_\omega^{pi_0}(\Omega)}^p \\ &\leq \frac{C |\Omega| [\omega]_{i_0}^{\frac{1}{i_0}}}{\omega(U)^{\frac{1}{i_0}}} \omega(\Omega)^{\frac{1}{i_0} - \frac{p}{q}} \|f\|_{L_\omega^{q,\infty}(\Omega)}^p \\ &\leq C |\Omega| \omega(\Omega)^{-\frac{p}{q}} [\omega]_{i_0}^{\frac{1}{i_0}} \|f\|_{L_\omega^{q,t}(\Omega)}^p \\ &\leq C(n, p, q, [\omega]_{\frac{q}{n}}, |\Omega|) \|f\|_{L_\omega^{q,t}(\Omega)}^p, \end{aligned} \quad (3.5)$$

which shows that $L_\omega^{q,t}$ can be continuously embedded into L^p for any $p \in [n, q)$. Hence, by virtue of [?, Theorem 4.6], there exists a unique L^p -viscosity solution $u \in W^{2,p}(\Omega)$ of (??) with

$$\|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}.$$

The restriction on p , i.e. $p > n > \frac{n}{2}$, also ensures that u is continuous because $W^{2,p}(\Omega) \hookrightarrow C(\Omega)$. So u is bounded due to the boundedness of domain Ω .

Step 2: Global weighted Lorentz estimate. Fixed any point $x_0 = (x_0, x'_0) = (x_0, x_{02}, \dots, x_{0n}) \in \partial\Omega$, we assume that

$$\Omega \cap B_r(x_0) = \{x \in \Omega : x_1 > \psi(x')\} \cap B_r(x_0),$$

for some $r > 0$ and some $C^{1,1}$ function $\psi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ satisfying $\frac{\partial \psi}{\partial x_i}(x'_0) = 0$ for any $i = 2, \dots, n$, and $\|D^2 \psi\|_{L^\infty(\mathbb{R}^{n-1})} < +\infty$. In order to flatten out the boundary near x_0 , we use the change of variables $y = \Phi(x)$ as follows:

$$\begin{cases} y_1 = x_1 - \psi(x') = \Phi^1(x), \\ y_i = x_i := \Phi^i(x), & \text{for } i = 2, 3, \dots, n; \end{cases}$$

satisfying $\Phi(x_0) = 0$. We define $\Phi = \Psi^{-1}$, then $x = \Psi(y)$. Choose a suitable $r > 0$ such that $B_1^+ \subset \Phi(\Omega \cap B_r(x_0))$. Define $\tilde{u}(y) = u(\Psi(y)) = u(x)$ for $y \in B_1^+$, and $\tilde{\omega} = \omega(\Psi(y))$ for $y \in \mathbb{R}^n$. Then it is readily checked that $\tilde{\omega} \in A_{\frac{q}{n}}$ and \tilde{u} is a bounded L^p -viscosity solution of

$$\begin{cases} \tilde{F}(D^2 \tilde{u}, y) = \tilde{f}, & \text{in } B_1^+ \\ \tilde{u} = 0, & \text{on } \Gamma_1; \end{cases}$$

where

$$\widetilde{F}(M, y) := F(D\Phi^T \circ \Psi M D\Phi \circ \Psi, \Psi(y)), \quad \widetilde{f}(y) := f(\Psi(y)).$$

Note that \widetilde{F} is convex in M and $\widetilde{F}(0, y) = 0$. Moreover, we see that $\Upsilon_{\widetilde{F}}(y, y_0) \leq C(\Phi)\Upsilon_F(\Psi(y), \Psi(y_0))$ for any $y, y_0 \in B_1^+$, and \widetilde{F} satisfies the conditions H2-H3. Therefore, using Lemma ?? (ii), it follows that

$$\|D^2\tilde{u}\|_{L_\omega^{q,t}(B_{\frac{1}{2}}^+)} \leq C\left(\|\tilde{f}\|_{L_\omega^{q,t}(B_1^+)} + \|\tilde{u}\|_{L^\infty(B_1^+)}\right).$$

Converting back to the original x -variables, we conclude

$$\|D^2u\|_{L_\omega^{q,t}(\Psi(B_{\frac{1}{2}}^+))} \leq C\left(\|f\|_{L_\omega^{q,t}(\Psi(B_1^+))} + \|u\|_{L^\infty(\Psi(B_1^+))}\right). \quad (3.6)$$

Set $V_0 := \Psi(B_{\frac{1}{2}}^+) \subset \Omega \cap B_r(x_0)$. Since $\partial\Omega$ is compact, there exist finitely many points $x_1, x_2, \dots, x_N \in \partial\Omega$, and open sets $V_i \subset \Omega \cap B_{r_i}(x_i)$ ($i = 1, \dots, N$) such that $\partial\Omega \subset \cup_{i=1}^N V_i$. Take $V_{N+1} \subset\subset \Omega$ so that $\Omega = \cup_{i=1}^{N+1} V_i$, and let $\{\xi_i\}_{i=1}^{N+1}$ be an associated partition of unity. Write $u = \sum_{i=1}^{N+1} \xi_i u_i$, then utilizing estimates (??) (with u_i in place of u) and the interior estimate (??), we have

$$\|D^2u\|_{L_\omega^{q,t}(\Omega)} \leq C\left(\|f\|_{L_\omega^{q,t}(\Omega)} + \|u\|_{L^\infty(\Omega)}\right),$$

which implies

$$\|u\|_{W^2L_\omega^{q,t}(\Omega)} \leq C\left(\|f\|_{L_\omega^{q,t}(\Omega)} + \|u\|_{L^\infty(\Omega)} + \|u\|_{L_\omega^{q,t}(\Omega)} + \|Du\|_{L_\omega^{q,t}(\Omega)}\right). \quad (3.7)$$

Next, we shall prove the desired estimate (??) by contradiction. Assume that there exist $\{u_k\}_{k=1}^\infty, \{f_k\}_{k=1}^\infty$ such that u_k is bounded L^p -viscosity solution of

$$\begin{cases} F(D^2u_k, x) = f_k, & \text{in } \Omega \\ u_k = 0, & \text{on } \partial\Omega, \end{cases}$$

satisfying

$$\|u_k\|_{W^2L_\omega^{q,t}(\Omega)} > k\|f_k\|_{L_\omega^{q,t}(\Omega)}, \quad (3.8)$$

for any $k \geq 1$. Without loss of generality, we may assume

$$\|u_k\|_{W^2L_\omega^{q,t}(\Omega)} = 1. \quad (3.9)$$

Then it follows from (??) that

$$\|f_k\|_{L_\omega^{q,t}(\Omega)} < \frac{1}{k} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (3.10)$$

Since $\{u_k\}_{k=1}^\infty$ is uniformly bounded in $W^2L_\omega^{q,t}(\Omega)$, there exist a subsequence, which be still denoted by $\{u_k\}_{k=1}^\infty$, and a function $u_0 \in W^2L_\omega^{q,t}(\Omega)$, such that

$$u_k \rightharpoonup u_0 \quad \text{in } W^2L_\omega^{q,t}(\Omega), \quad u_k \rightarrow u_0 \quad \text{in } L_\omega^{q,t}(\Omega), \quad \text{as } k \rightarrow \infty. \quad (3.11)$$

From [?, Proposition 1.5] it follows that u_0 is a bounded L^p -viscosity solution of

$$\begin{cases} F(D^2u_0, x) = 0, & \text{in } \Omega \\ u_0 = 0, & \text{on } \partial\Omega. \end{cases} \quad (3.12)$$

Accordingly, by the uniqueness of viscosity solutions to problem (??) (proved in Step 1), we see $u_0 \equiv 0$ in Ω . Moreover, (??) and (??) imply

$$f_k \rightarrow 0 \quad \text{in } L_\omega^{q,t}(\Omega), \quad u_k \rightarrow 0 \quad \text{in } L_\omega^{q,t}(\Omega), \quad \text{as } k \rightarrow \infty. \quad (3.13)$$

Since $W^2L_\omega^{q,t}(\Omega) \hookrightarrow W^2L^p(\Omega) \hookrightarrow C(\Omega)$ for $p \in [n, q]$ we have

$$\|u_k\|_{L^\infty(\Omega)} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad (3.14)$$

From [?, Theorem 4.5], we have

$$\|Du_k\|_{L^n(\Omega)} \lesssim \|f_k\|_{L^n(\Omega)} + \|u\|_{L^\infty(\Omega)} \lesssim \|f_k\|_{L_\omega^{q,t}(\Omega)} + \|u\|_{L^\infty(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

Letting $\mu := \omega(x)dx$, we see that

$$Du_k \rightarrow 0 \quad \mu - a.e. \text{ in } \Omega \quad \text{as } k \rightarrow \infty \quad (\text{up to subsequence})$$

which implies

$$Du_k \rightarrow 0 \quad \text{in } L_\omega^{q,t}(\Omega) \quad \text{as } k \rightarrow \infty. \quad (3.15)$$

Combining (??) (??) (??) (??) and (??), it yields

$$1 \lesssim \|f_k\|_{L_\omega^{q,t}(\Omega)} + \|u_k\|_{L^\infty(\Omega)} + \|u_k\|_{L_\omega^{q,t}(\Omega)} + \|Du_k\|_{L_\omega^{q,t}(\Omega)} \rightarrow 0 \quad \text{as } k \rightarrow \infty,$$

which is a contradiction. \square

4 The proof of Theorem ?? (ii)

Proof of Theorem ?? (ii). Here, we restrict our proof to the case $t \neq \infty$ since the proof for the case $t = \infty$ is similar. Let $(y, r) \in \Omega \times (0, d_0)$. For an $\varrho \in (0, \theta)$, we consider a weight function

$$\omega(x) = \min\{|x - y|^{-n+\theta-\varrho}, r^{-n+\theta-\varrho}\}.$$

The fifth property in Lemma ?? implies $\omega(x)$ is an $A_{\frac{q}{n}}$ weight and $[\omega]_{\frac{q}{n}}$ is bounded from above by a constant independent of y and r . Since $\omega = r^{-n+\theta-\varrho}$ on $B_r(y)$, it follows from Theorem ?? (i) that

$$\|D^2u\|_{L^{q,t}(B_r(y) \cap \Omega)} = r^{\frac{n-\theta+\varrho}{q}} \|D^2u\|_{L_\omega^{q,t}(B_r(y) \cap \Omega)} \leq Cr^{\frac{n-\theta+\varrho}{q}} \|f\|_{L_\omega^{q,t}(\Omega)}. \quad (4.1)$$

For $\lambda > 0$ we denote $F_\lambda = \{x \in \Omega : |f(x)| > \lambda\}$, then we can derive that

$$\begin{aligned} \|f\|_{L_\omega^{q,t}(\Omega)}^t &= q \int_0^\infty (\lambda^q \omega(\{x \in \Omega : |f(x)| > \lambda\}))^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\ &= q \int_0^\infty \left(\lambda^q \int_{F_\lambda} \omega(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\ &= q \int_0^\infty \left(\lambda^q \int_{\mathbb{R}^n} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\ &\leq N_2 q \int_0^\infty \left(\lambda^q \int_{B_r(y)} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\ &\quad + N_2 q \int_0^\infty \left(\lambda^q \int_{\mathbb{R}^n \setminus B_r(y)} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\ &=: I_1 + I_2. \end{aligned} \quad (4.2)$$

where $N_2 = \max\{1, 2^{\frac{t}{q}-1}\}$. In the sequel, we shall focus on the estimates of I_1 and I_2 , respectively. To estimate I_1 , on account of $\omega = r^{-n+\theta-\varrho}$ on $B_r(y)$, we get

$$\begin{aligned}
I_1 &= N_2 r^{\frac{(-n+\theta-\varrho)t}{q}} q \int_0^\infty \left(\lambda^q \int_{B_r(y)} \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&= N_2 r^{\frac{(-n+\theta-\varrho)t}{q}} q \int_0^\infty (\lambda^q |\{x \in B_r(y) \cap \Omega : |f(x)| > \lambda\}|)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&= N_2 r^{\frac{(-n+\theta-\varrho)t}{q}} \|f\|_{L^{q,t}(B_r(y) \cap \Omega)}^t \\
&= N_2 r^{\frac{-\varrho t}{q}} \left(r^{\frac{\theta-n}{q}} \|f\|_{L^{q,t}(B_r(y) \cap \Omega)} \right)^t \\
&\leq C r^{\frac{-\varrho t}{q}} \|f\|_{L^{q,t;\theta}(\Omega)}^t.
\end{aligned} \tag{4.3}$$

To estimate I_2 , we shall consider two cases.

Case 1: $0 < t \leq q$.

$$\begin{aligned}
I_2 &= N_2 q \int_0^\infty \left(\lambda^q \int_{\mathbb{R}^n \setminus B_r(y)} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&= N_2 q \int_0^\infty \left(\lambda^q \sum_{k=1}^\infty \int_{B_{2^k r}(y) \setminus B_{2^{k-1} r}(y)} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&\leq N_2 \sum_{k=1}^\infty q \int_0^\infty \left(\lambda^q \int_{B_{2^k r}(y) \setminus B_{2^{k-1} r}(y)} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda},
\end{aligned}$$

where we used the fact that

$$\left(\sum_{k=1}^\infty c_k \right)^{\frac{t}{q}} \leq \sum_{k=1}^\infty c_k^{\frac{t}{q}}$$

which holds for all nonnegative sequences $\{c_i\}$ due to $t \leq q$. Since for any $x \in B_{2^k r}(y) \setminus B_{2^{k-1} r}(y)$,

$$\omega(x) = |x - y|^{-n+\theta-\varrho} \leq (2^{k-1} r)^{-n+\theta-\varrho} = 2^{-(k-1)\varrho} r^{-n+\theta-\varrho},$$

it follows that

$$\begin{aligned}
I_2 &= N_2 \sum_{k=1}^\infty q \int_0^\infty \left(\lambda^q \int_{B_{2^k r}(y) \setminus B_{2^{k-1} r}(y)} 2^{-(k-1)\varrho} r^{-n+\theta-\varrho} \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&\leq N_2 \sum_{k=1}^\infty 2^{\frac{-(k-1)\varrho t}{q}} r^{\frac{(-n+\theta-\varrho)t}{q}} q \int_0^\infty \left(\lambda^q \int_{B_{2^k r}(y)} \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&= N_2 \sum_{k=1}^\infty 2^{\frac{-(k-1)\varrho t}{q}} r^{\frac{(-n+\theta-\varrho)t}{q}} q \int_0^\infty (\lambda^q |\{x \in B_{2^k r}(y) \cap \Omega : |f(x)| > \lambda\}|)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&= N_2 \sum_{k=1}^\infty 2^{\frac{-(k-1)\varrho t}{q}} r^{\frac{(-n+\theta-\varrho)t}{q}} \|f\|_{L^{q,t}(B_{2^k r}(y) \cap \Omega)}^t \\
&\leq C \left(\sum_{k=0}^\infty \left(2^{\frac{-\varrho t}{q}} \right)^k \right) r^{\frac{-\varrho t}{q}} \|f\|_{L^{q,t;\theta}(\Omega)}^t \\
&\leq C r^{\frac{-\varrho t}{q}} \|f\|_{L^{q,t;\theta}(\Omega)}^t.
\end{aligned} \tag{4.4}$$

Case 2: $t > q$. Using the fact that $\omega \leq r^{-n+\theta-\varrho}$ in \mathbb{R}^n , the inclusion $\{x \in (\mathbb{R}^n \setminus B_r(y)) \cap F_\lambda : \omega(x) > \beta\} \subset F_\lambda \cap B_{\frac{1}{\beta^{-n+\theta-\varrho}}}(y)$, and the Minkowski's integral inequality since $t > q$, we get

$$\begin{aligned}
I_2 &= N_2 q \int_0^\infty \left(\lambda^q \int_{\mathbb{R}^n \setminus B_r(y)} \omega(x) \chi_{F_\lambda}(x) dx \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&= N_2 q \int_0^\infty \left(\lambda^q \int_0^\infty |\{x \in (\mathbb{R}^n \setminus B_r(y)) \cap F_\lambda : \omega(x) > \beta\}| d\beta \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&\leq N_2 q \int_0^\infty \left(\lambda^q \int_0^{r^{-n+\theta-\varrho}} |F_\lambda \cap B_{\frac{1}{\beta^{-n+\theta-\varrho}}}(y)| d\beta \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \\
&\leq N_2 q \left(\int_0^{r^{-n+\theta-\varrho}} \left(\int_0^\infty \left(\lambda^q |F_\lambda \cap B_{\frac{1}{\beta^{-n+\theta-\varrho}}}(y)| \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \right)^{\frac{q}{t}} d\beta \right)^{\frac{t}{q}} \\
&\leq N_2 \left(\int_0^{r^{-n+\theta-\varrho}} \left(q \int_0^\infty \left(\lambda^q \left| \left\{ B_{\frac{1}{\beta^{-n+\theta-\varrho}}}(y) \cap \Omega : f(x) > \lambda \right\} \right| \right)^{\frac{t}{q}} \frac{d\lambda}{\lambda} \right)^{\frac{q}{t}} d\beta \right)^{\frac{t}{q}} \\
&\leq N_2 \|f\|_{L^{q,t,\theta}(\Omega)}^t \left(\int_0^{r^{-n+\theta-\varrho}} \beta^{-\frac{n-\theta}{-n+\theta-\varrho}} d\beta \right)^{\frac{t}{q}} \\
&\leq Cr^{\frac{-\varrho t}{q}} \|f\|_{L^{q,t,\theta}(\Omega)}^t. \tag{4.5}
\end{aligned}$$

Combining (??) and (??) deduces

$$I_2 \leq Cr^{\frac{-\varrho t}{q}} \|f\|_{L^{q,t,\theta}(\Omega)}^t.$$

Together with (??) (??) and (??), it yields

$$\|D^2 u\|_{L^{q,t}(B_r(y) \cap \Omega)} \leq Cr^{\frac{n-\theta}{q}} \|f\|_{L^{q,t,\theta}(\Omega)}.$$

Then dividing the both sides of the above inequality by $r^{\frac{n-\theta}{q}}$ and taking the supremum with respect to $y \in \Omega$ and $0 < r < d_0$, we obtain the desired estimate (??). \square

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