Lorentz estimates for the gradient of weak solutions to elliptic obstacle problems with partially BMO coefficients *

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February 5, 2018

Abstract

We prove global Lorentz estimates for variable power of the gradient of weak solution to linear elliptic obstacle problems with small partially BMO coefficients over a bounded nonsmooth domain. Here, we assume that the leading coefficients are measurable in one variable and have small BMO semi-norms in the other variables, variable exponents $p(x)$ satisfy log-Hölder continuity, and the boundary of domains are so-called Reifenberg flat. This is a natural outgrowth of the classical Calderón-Zygmund estimates to a variable power of the gradient of weak solutions in the scale of Lorentz spaces for such variational inequalities beyond the Lipschitz domain.

2010 Mathematics Subject Classification: 35J86, 35B65, 35R05.

Keywords: elliptic obstacle problems; variable power for the gradient of weak solution; Lorentz spaces; partial BMO coefficients; Reifenberg flat domains.

1 Introduction

The main purpose of this present article is to attain a possibility of global estimates of variable exponent power of the gradient in the framework of the Lorentz spaces to weak solutions for the following variational inequalities. Let Ω be a bounded domain in $\mathbb{R}^d$ for $d \geq 2$, with its rough boundary $\partial\Omega$ beyond the Lipschitz category specified later. For given ψ obstacle function with

$$\psi \in W^{1,2}(\Omega) \quad \text{and} \quad \psi \leq 0 \quad \text{a.e. on } \partial\Omega,$$

we define the admissible set $\mathcal{A}$ by

$$\mathcal{A} = \left\{\phi \in W_0^{1,2}(\Omega) : \phi \geq \psi \text{ a.e. in } \Omega\right\}.$$

Note that $\mathcal{A}$ is nonempty due to $\psi^+ \in \mathcal{A}$. Here we are interested in the elliptic obstacle problems by minimizing the energy functional $\mathcal{J}[u] = \int_\Omega (A(x)\nabla u \cdot \nabla u + \mathbf{f} \cdot \nabla u)dx$ in the Sobolev spaces $u \in W_0^{1,2}(\Omega)$ satisfying the admissible condition $u \in \mathcal{A}$. This leads to the following variational inequalities in the weak sense that for the functions $u \in W_0^{1,2}(\Omega)$ lying in $\mathcal{A}$ such that

$$\int_\Omega A(x)\, Du \cdot D(\phi - u)dx \geq \int_\Omega \mathbf{f} \cdot D(\phi - u)dx \quad \text{for all} \quad \phi \in \mathcal{A}, \tag{1.1}$$

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where the coefficient $A(x)$ is an $d \times d$ matrix satisfying uniform ellipticity, and the nonhomogeneous term $\mathbf{f} \in L^2(\Omega, \mathbb{R}^d)$. Such function $u \in \mathcal{A}$ is called a weak solution if it satisfies the variational inequalities (1.1). In the context, we mainly focus on the Calderón-Zygmund type estimates of $|Du|^{p(x)}$ in the scale of Lorentz spaces $L^{(\gamma,q)}$ to weak solutions of variational inequalities (1.1) by imposing optimal regular conditions on the leading coefficients $A(x)$ and the boundary of domains $\partial\Omega$, which implies that

$$|D\psi|^{p(x)},\ |\mathbf{f}|^{p(x)}\ \in L^{(\gamma,q)}(\Omega) \Longrightarrow |Du|^{p(x)} \in L^{(\gamma,q)}(\Omega) \tag{1.2}$$

for every real valued function $p(x)$ with locally log-Hölder continuity in Ω, $\gamma \in [1, \infty)$ and $q \in (0, \infty]$.

An optimal regularity is always important for mathematics and physics in the classical functional frame with minimal regular given datum, for example, in the Lebesgue spaces L^p and Sobolev spaces $W^{1,p}$ with p as a fixed constant in $(1, \infty)$. In recent decades, many extensive researches have been made in the field of the variable exponent Lebesgue and Sobolev spaces, $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$ with $k \geq 1$ (cf. [4, 10, 12, 16]), since the pioneering work of Zhikov in [35]. Indeed, for some materials with inhomogeneities such as electrorheological fluids, this is not enough for energy with constant exponent, but rather the exponent p should be able to vary. These variable exponent Lebesgue, Sobolev and Lorentz spaces rather than the classical Sobolev spaces and Lorentz spaces are involved in the area of non-Newtonian fluids, as the underlying integral energy is naturally arising in the modelling of electrorheological fluids [28]. The other areas of the application of variable exponent spaces include elastic mechanics [35], porous medium [5], and image restoration [14]. Generally speaking, various physical phenomena with strong anisotropy are well described by the variable exponent spaces. This leads us to the study of partial differential equations in the setting of variable exponent Lebesgue, Sobolev and Lorentz spaces.

Nowadays the classical Calderón-Zygmund theory with constant exponent for elliptic obstacle problems has been widely studied, for instance, we can refer to Byun et al's papers [8, 9] and the references therein for the global L^p regularity to various irregular obstacle problems over a nonsmooth domain. Furthermore, Lorentz spaces are a two-parameter scale of spaces which refine Lebesgue spaces (cf. [27]) and there is a large of literature on the topic of Lorentz regularity, see [3, 6, 7, 25, 34, 32, 33]. Baroni [6, 7] obtained Lorentz estimates for evolutionary p-Laplacian systems and obstacle parabolic p-Laplacian respectively, by using the large-M-inequality principle introduced by Acerbi-Mingione [2]. Meanwhile, Mengesha-Phuc [25] established the gradient estimates in weighted Lorentz spaces for quasilinear p-Laplacian based on a rather different geometrical approach. Later, Zhang-Zhou [34] extended the result of [25] to the quasilinear elliptic $p(x)$-Laplacian equations also using a geometrical argument, Adimurthil-Phuc [3] proved that global Lorentz and Lorentz-Morrey estimates below the natural exponent for quasilinear equations, and Zhang-Zheng [32, 33] studied with Lorentz estimates for fully nonlinear parabolic and elliptic equations with small BMO nonlinearities, and weighted Lorentz estimates of the Hessian of strong solution for nondivergence linear elliptic equations with partially BMO coefficients. Our aim of this paper is inspired by two aspects. One is that recently more attention has been paid to a systematic study on the regularity estimates in the variable exponent Sobolev spaces for divergence and non-divergence elliptic problems, see [10, 11]. Another is that the new definition is available to our aim for Lorentz spaces with variable exponent powers proposed by Kempka-Vybíral in [21].

Motivated by these recent papers above-mentioned, we are interested in minimizing regular requirements to the variational inequalities (1.1) imposed on the coefficients and the boundary of domain, under which the gradient of the weak solution is integrable as the nonhomogeneous term and the gradient of the obstacle functions in the setting of the generalized Lorentz spaces with variable exponent powers $p(x)$. Our investigation is to attain an optimal natural extension of such elliptic variational inequalities (1.1) from L^p-regularity or $L^{(p,q)}$-regularity with constant exponents to the setting of variable exponents. It is an obvious observation that a uniformly ellipticity on the coefficients is not enough to ensure the kind of regularity we mentioned above. To this end, it is necessary to impose some suitable minimal regular assumptions on the coefficients

$A(x)$ and geometric restriction on the boundary of the domain under the assumption that the given variable exponent $p(x)$ has log-Hölder continuity. A recent notable achievement is that Kim-Krylov [22] recently got a unified approach to consider the L^p solvability to linear elliptic and parabolic problems with partially VMO coefficients. Later, these results were generalized to divergence form linear elliptic and parabolic equations/systems with (variably) partially BMO/VMO coefficients by Dong and Kim [17, 18, 19]. Also, Byun et al in [9] attained a global Calderón-Zygmund estimate to linear elliptic obstacle problems with small partially BMO coefficients over the nonsmooth domain by way of rather different geometrical approaches. More precisely, in this article we consider the variational inequalities (1.1) over the Reifenberg flat domain with the leading coefficients being only measurable in one variable, which allow this way quite arbitrary discontinuities in that direction, while being small BMO with respect to the remaining $(d-1)$-variables. In fact, this is a typical situation closely related to the equation of linear elastic laminates [15] and composite materials [24] which have been widely applied to various fields. In addition, we suppose that the boundary of non-smooth domain is flat in the sense of Reifenberg introduced in [29], which is well approximated by the two hyperplanes at each point at each scale. As we know, the class of Reifenberg flat domain contains the domains with rough fractal boundaries. To the best knowledge of the authors of this paper, this is the first time to consider the regularity in the category of the Lorentz spaces with variable exponents for the weak solution of variational inequalities (1.1) under the minimal regular assumptions on the leading coefficients and the boundary of domain. We would like to mention that if the leading coefficients $A(x)$ are only measurable, then there could not exist a unique solution to linear elliptic problems even in a very generalized sense. In 1963, Meyers' counterexample in [26] demonstrates, the gradient of weak solutions to elliptic equations corresponding to highly oscillatory coefficients cannot be expected to have higher integrability irrespective of the regularity of the data $f(x)$. Therefore, requiring the coefficients to satisfy small partially BMO condition not only is necessary to achieve higher integrability, but also is the weakest conditions so far even in the Lebesgue spaces L^p with constant exponents.

Note that the variational inequalities (1.1) are concerned with the Lorentz space with the variable exponent powers $p(x)$ of the gradient of weak solution, so that the techniques from harmonic analysis like the Calderón-Zygmund operator, the maximal function operator and the sharp maximal function operator might not be suitable for our estimates. Instead, we would like to point out that a key ingredient in our argument due to the order $p(\cdot)$ being a variable function, which is highly influenced for the variable exponent Sobolev spaces by Byun et al's works [10, 11, 12]. This argument is motivated from so-called maximal function free technique in [2]. To this end, an important point of our approach is to make use of the modified Vitali type covering argument on the upper-level set

$$\left\{x \in B_R(x_0) \cap \Omega : |Du|^{\frac{2p(x)}{\inf\{p(x): x \in B_{2R}(x_0) \cap \Omega\}}} > \lambda\right\}$$

with an increasing level for λ sufficiently large, for each point $x_0 \in \overline{\Omega}$ and for some size R sufficiently small, to derive its proper power decay estimate, see Lemma 3.2 in [10]. This present paper focus on considering the estimate of the variable exponent powers $p(x)$ for the gradient of weak solution in the scale of Lorentz spaces to the obstacle problems (1.1). Therefore, another key ingredient in the generalized Lorentz spaces is to make use of the modified version of the classic Hardy's inequality and the reverse Hölder inequality, see Lemma 3.5 and 3.6 in [6].

Finally, we would like to remark that the obstacle problems provide a basic analysis tool in the study of variational inequalities and free boundary problems [23] for various PDEs, which are deeply involved in various geometric and potential theory problems such as capacities of sets or minimal surfaces. In addition, these also arise naturally in the classical elasticity theory, see [13, 30]. Therefore, our problem also provides a natural extension of Byun et al's works in [10, 12] which only studied elliptic equations without obstacles in the framework of the classical Sobolev and variable Sobolev spaces, respectively.

The rest of this paper is organized as follows: In Section 2 we introduce some related notations and basic facts. By imposing optimal assumptions on $p(\cdot), A(x)$ and the boundary of domain Ω, we finally state our main results. Section 3 is devoted to establishing some technical tools and auxiliary results. Finally, the main result is proved in Section 4.

2 Notations and main result

The section is devoted to introducing some basic notations, facts and stating our main result concerning the variational inequalities (1.1). First of all, let us recall some well-known notations concerning the Lorentz spaces and log-Hölder continuity of $p(x)$. Lorentz spaces were introduced as the refined generalization of classical Lebesgue spaces.

Definition 2.1 *Let $\mathcal{D}$ be an open subset in $\mathbb{R}^d$. The Lorentz space $L^{(\gamma,q)}(\mathcal{D})$ with $\gamma \in [1, +\infty)$ and $q \in (0, +\infty)$, is the set of measurable functions $g : \mathcal{D} \to \mathbb{R}$ such that*

$$\|g\|_{L^{(\gamma,q)}(\mathcal{D})} := \left(\gamma \int_0^\infty \left(\lambda^\gamma |\{x \in \mathcal{D} : |g(x)| > \lambda\}|\right)^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda}\right)^{\frac{1}{q}} < +\infty.$$

For $q = \infty$ the space $L^{(\gamma,\infty)}(\mathcal{D})$ is set to be the usual Marcinkiewicz space with quasinorm

$$\|g\|_{L^{(\gamma,\infty)}(\mathcal{D})} := \sup_{\lambda>0} \left(\lambda^\gamma |\{x \in \mathcal{D} : |g(x)| > \lambda\}|\right)^{\frac{1}{\gamma}} < +\infty.$$

We remark that if $\gamma = q$ then the Lorentz space $L^{(\gamma,\gamma)}(\mathcal{D})$ is nothing but classical Lebesgue space $L^\gamma(\mathcal{D})$, which is equivalently defined by

$$\|g\|_{L^\gamma(\mathcal{D})} = \left(\int_{\mathcal{D}} |g(x)|^\gamma \, dx\right)^{\frac{1}{\gamma}} < +\infty.$$

We would like to mention that Baroni in [6, 7] has studied a local Lorentz regularity of the gradient for weak solutions of nonlinear elliptic and parabolic problems with small BMO coefficients based on the approach of the large-M-inequality principle [2].

Note that the main point in this paper is that the exponent $p(x)$ is a variable function. The basic regularity assumption on variable exponent $p(\cdot)$ is so-called log-Hölder continuity, which ensure most basic operation available. Indeed, Sharapudinov [31] was the first to consider the regularity of the exponent function $p(x)$ with a local log-Hölder continuity, and from then it is usual hypothesis for harmonic analysis and theory of PDEs. Let $\mathcal{D}$ be a measurable set of $\mathbb{R}^d$ and $p(x) : \mathcal{D} \to [1, \infty)$ be a bounded measurable function.

Definition 2.2 *We say that $p(x)$ is locally log-Hölder continuous, denote it by $p(x) \in LH_0(\mathcal{D})$, if there exist constants C_0 and $\delta > 0$ such that for all $x, y \in \mathcal{D}$ with $|x - y| < \delta$, one has*

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x - y|)}.$$

The $p(x)$ is said to be so-called log-Hölder continuous at infinity, denote by $p(x) \in LH_\infty(\mathcal{D})$, if there exist constants C_∞ and p_∞ such that for all $x \in \mathcal{D}$,

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

If $p(x)$ is log-Hölder continuous locally and at infinity, we denote it by writing $p(x) \in LH(\mathcal{D})$.

It is also worth to mention that log-Hölder continuity in the variable exponent is unavoidable, if we want to treat the regularity results in the generalized lorentz spaces with variable exponent for elliptic and parabolic problems, see [1, 10, 11, 12, 16] and the references therein. In what follows, we assume that $p(x) : \mathcal{D} \to \mathbb{R}$ is any log-Hölder continuous function, which implies that there exist positive constants γ_1 and γ_2 such that

$$2 < \gamma_1 \leq p(x) \leq \gamma_2 < \infty \qquad \forall x \in \mathcal{D} \tag{2.1}$$

and

$$|p(x) - p(y)| \leq \omega(|x - y|) \qquad \forall x, y \in \mathcal{D}, \tag{2.2}$$

where $\omega : [0, \infty) \to [0, \infty)$ is a modulus of continuity of $p(x)$. Without loss of generality, we suppose that ω is a nondecreasing continuous function with $\omega(0) = 0$, and $\limsup_{r\to 0} \omega(r) \log\left(\frac{1}{r}\right) < \infty$. With the above assumptions in hand, it is clear that $p(x) \in LH_0(\mathcal{D})$ and there exists a positive number A such that

$$\omega(r) \log\left(\frac{1}{r}\right) \leq A \Longleftrightarrow r^{-\omega(r)} \leq e^A \qquad \forall r \in (0, 1). \tag{2.3}$$

It is rather important and ubiquitous in the context for the log-Hölder continuity condition (2.3) involved regularity of the exponent function to study various variable exponent problems. Generally speaking, the log-Hölder condition plays a central role in harmonic analysis on variable Lebesgue and Sobolev spaces, which ensures that the Hardy-Littlewood maximal operator is still bounded within the framework of the generalized Lebesgue spaces, a mollification argument is working, variable Sobolev embedding theorem and Poincaré inequalities are available. In addition, a key ingredient in the main proof concerning variable exponent problems is usually so-called perturbation approach by various local comparisons with these problems of constant local maximal and minimal exponents p^+ and p^-, which also leads to an indispensable constant controlled by the log-Hölder condition (2.3), for more details see main proof in §4.

Now, we are supposed the coefficients matrix $A(x) = \left(a^{ij}(x)\right) : \Omega \to \mathbb{R}^{d\times d}$ to be uniform boundedness and ellipticity, which means that there exist $0 < \nu \leq \Lambda < \infty$ such that

$$\nu|\xi|^2 \leq A(x)\xi \cdot \xi \leq \Lambda|\xi|^2 \qquad \forall x \in \Omega, \xi \in \mathbb{R}^d. \tag{2.4}$$

We are in a position to introduce our principal assumptions on the coefficients $A(x)$ and the geometric structure of the boundary $\partial\Omega$ of domain. To this end, let us recall some notations useful later. For any fixed point $x = (x_1, \cdots, x_d) = (x_1, x') \in \mathbb{R}^d$ with $x' = (x_2, \cdots, x_d)$, we set

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \qquad B'_r(x') = \{y' \in \mathbb{R}^{d-1} : |x' - y'| < r\}$$

and

$$Q_r(x) = (x_1 - r, x_1 + r) \times B'_r(x').$$

For convenience, in the context we write $B_r = B_r(0)$, $B'_r = B'_r(0)$. We denote the average of f on Q_r with $r > 0$ by

$$\bar{f}_{Q_r} = \fint_{Q_r} f(x)\, dx = \frac{1}{|Q_r|} \int_{Q_r} f(x)\, dx,$$

where $|Q_r|$ is d-dimensional Lebesgue measure of Q_r; and also denote the $(d-1)$-dimensional average only with respect to x' by

$$\bar{f}_{B'_r}(x_1) = \fint_{B'_r} f(x_1, x')\, dx' = \frac{1}{|B'_r|} \int_{B'_r} f(x_1, x')\, dx',$$

where $|B'_r|$ is $(d-1)$-dimensional Lebesgue measure of B'_r.

Assumption 2.3 *We say that* (A, Ω) *with* $A(x) = (a_{ij}(x))$ *for all* $i, j = 1, 2, \cdots, d$*; is* (δ, R_0)*-vanishing of codimension* 1 *if for any* $x_0 \in \Omega$ *and for every number* $r \in (0, R_0]$ *with*

$$dist(x_0, \partial\Omega) = \min_{z \in \partial\Omega} dist(x_0, z) > \sqrt{2}r,$$

there exists a coordinate system depending on x_0 *and* r*, whose variables still denoted by* $x = (x_1, x')$*, such that in the new coordinate system* x_0 *is the origin and*

$$\fint_{Q_r(x_0)} |A(x) - \bar{A}_{B'_r(x'_0)}(x_1)|^2 \, dx \le \delta^2. \tag{2.5}$$

while, for any $x_0 \in \Omega$ *and for every number* $r \in (0, R_0]$ *with*

$$dist(x_0, \partial\Omega) = \min_{z \in \partial\Omega} dist(x_0, z) = dist(x_0, z_0) \le \sqrt{2}r$$

for some $z_0 \in \partial\Omega$*, there exists a coordinate system depending on* x_0 *and* r*, whose variables still denoted by* $x = (x_1, x')$*, such that in the new coordinate system* z_0 *is the origin,*

$$B_{3r}(z_0) \cap \{x_1 \ge 3\delta r\} \subset B_{3r}(z_0) \cap \Omega \subset B_{3r}(z_0) \cap \{x_1 \ge -3\delta r\} \tag{2.6}$$

and

$$\fint_{Q_{3r}(z_0)} |A(x) - \bar{A}_{B'_{3r}(z'_0)}(x_1)|^2 \, dx \le \delta^2, \tag{2.7}$$

where $A(x)$ *is a zero-extension from* $Q_{3r} \cap \Omega$ *to* Q_{3r}*, the parameters* $\delta > 0$ *and* R_0 *will be specified later.*

Remark 2.4 *We say that* Ω *is* (δ, R_0)*-Reifenberg flat if (2.6) holds in the new coordinate system. It is worth noticing that if* Ω *is* (δ, R_0)*-Reifenberg flat, we obtain the following measure density condition:*

$$\sup_{0 < r \le R_0} \sup_{x_0 \in \partial\Omega} \frac{|B_r(x_0)|}{|\Omega \cap B_r(x_0)|} \le \Big(\frac{2}{1-\delta}\Big)^d,$$

$$A|B_r(x_0)| \le |B_r(x_0) \cap \Omega| \le (1 - A)|B_r(x_0)| \qquad \forall x_0 \in \partial\Omega. \tag{2.8}$$

This implies that the boundary $\partial\Omega$ *satisfies the so-called A-type domain, namely, for the ball* $B_r(x_0)$ *of radius* r *with centered at* x_0 *there exists a positive constant* $A \in (0, 1)$ *such that the Lebesgue measurable of* $B_r(x_0) \cap \Omega$ *is comparable to that of* $B_r(x_0)$*. As a consequence, A-type domain guarantees a quantified higher integrability of the gradient of weak solutions of the variable problems (1.1) near the boundary based on Gehring-Giaquinta-Modica Lemma, see [6, 20, 23].*

We would like to point out that at each point y and scale r, the coefficient $A(x)$ is allowed to be merely measurable in one variable, depending on the point and the scale, but it has a small oscillation in all the other $(d-1)$ variables. Moreover, $A(x)$ has a small mean oscillation in the flat direction of the boundary near the boundary. Indeed, the δ-Reifenberg flat domain is so irregular that its boundary might be fractal and it goes beyond the Lipschitz one, which is meaningful in the area of geometric measure theory only if δ is small enough. We here point out that $R_0 > 0$ can be selected in an arbitrary way due to the scaling invariant of Lemma 3.6 below. Moreover, δ can be selected later in our main proof in a universal way so that it depends only on the basic structural constants like $d, \nu, \Lambda, \gamma_1, \gamma_2$ and $\omega(\cdot)$.

Finally, we are ready to present the main result in this paper.

Theorem 2.5 *Let $p(x)$ be a variable exponent with range $2 < \gamma_1 = \inf_\Omega p(x) \le \gamma_2 = \sup_\Omega p(x) < \infty$, q be a constant exponent defined in $(0, \infty]$ and $R_0 > 0$. Then there exists a positive constant δ such that if for all $(A(x), \Omega)$ is (δ, R_0)-vanishing of codimension 1 shown as* ***Assumption 2.3****, for all $p(x)$ satisfying log-Hölder continuity, $|D\psi|^{p(x)}$ and $|f|^{p(x)}$ belonging to $L^{(\gamma,q)}(\Omega)$, then each weak solution $u \in \mathcal{A}$ of variational inequalities (1.1) satisfies $|Du|^{p(x)} \in L^{(\gamma,q)}(\Omega)$ with the estimate*

$$\||Du|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} \le C\Big(\||f|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} + \||D\psi|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} + 1\Big)^{\frac{\gamma_2}{\gamma_1}}, \tag{2.9}$$

where $C = C(d, \nu, \Lambda, \omega(\cdot), \gamma, q, \gamma_1, \gamma_2, \delta, R_0, |\Omega|)$, $\gamma \in [1, \infty)$ and $q \in (0, \infty]$.

3 Preliminaries

Throughout the paper, we denote by $C_i(d, \nu, \Lambda, \cdots)$ for $i = 1, 2, \cdots$, a universal constant depending only on prescribed quantities and possibly varying from line to line. First, let us collect some preliminary results, which is useful in our main proof, see [6, Section 3.2] and [25, Proposition 3.9].

Proposition 3.1 *Let $\mathcal{D}$ be a bounded measurable subset of $\mathbb{R}^d$. Then the following holds:*

1) *If $0 < q_1, q_2 \le \infty$ and $1 \le \gamma_1 < \gamma_2 < \infty$, then $L^{(\gamma_2,q_2)}(\mathcal{D}) \subset L^{(\gamma_1,q_1)}(\mathcal{D})$ with the estimate*

$$\|g\|_{L^{(\gamma_1,q_1)}(\mathcal{D})} \le C(\gamma_1, \gamma_2, q_1, q_2, |\mathcal{D}|)\, \|g\|_{L^{(\gamma_2,q_2)}(\mathcal{D})}. \tag{3.1}$$

2) *If $1 \le \gamma < \infty$ and $0 < q_1 < q_2 \le \infty$, then $L^{(\gamma,q_1)}(\mathcal{D}) \subset L^{(\gamma,q_2)}(\mathcal{D}) \subset L^{(\gamma,\infty)}(\mathcal{D})$ with the estimate*

$$\|g\|_{L^{(\gamma,q_2)}(\mathcal{D})} \le C(\gamma, q_1, q_2)\|g\|_{L^{(\gamma,q_1)}(\mathcal{D})}. \tag{3.2}$$

3) *If for some $0 < \sigma < \infty$, $|g|^\sigma \in L^{(\gamma,q)}(\mathcal{D})$, then $g \in L^{(\sigma\gamma,\sigma q)}(\mathcal{D})$ with the estimate*

$$\||g|^\sigma\|_{L^{(\gamma,q)}(\mathcal{D})} = \|g\|^\sigma_{L^{(\sigma\gamma,\sigma q)}(\mathcal{D})}. \tag{3.3}$$

4) *If $f, g \in L^{(\gamma,q)}(\mathcal{D})$, then $f + g \in L^{(\gamma,q)}(\mathcal{D})$ with the estimate*

$$\|f + g\|_{L^{(\gamma,q)}(\mathcal{D})} \le C(\gamma, q)\left(\|f\|_{L^{(\gamma,q)}(\mathcal{D})} + \|g\|_{L^{(\gamma,q)}(\mathcal{D})}\right). \tag{3.4}$$

In what follows, we shall show some technical tools. The first inequality we need is a variant of the classical Hardy's inequality, whose proof can be found in [6].

Lemma 3.2 *Let $f : [0, +\infty) \to [0, +\infty)$ be a measurable function such that*

$$\int_0^\infty f(\alpha)\, d\alpha < \infty,$$

then for any $\sigma \ge 1$ and for any $\tau > 0$ there holds

$$\int_0^\infty \alpha^\tau \Big(\int_\alpha^\infty f(\beta) d\beta\Big)^\sigma \frac{d\alpha}{\alpha} \le C \int_0^\infty \alpha^\tau (\alpha f(\alpha))^\sigma \frac{d\alpha}{\alpha}, \tag{3.5}$$

where $C = C(\sigma, \tau)$.

The following reverse-Hölder inequality is also classical consequence originated from the famous Gehring-Giaquinta-Modica Lemma, also see [6]. More precisely, we have

Lemma 3.3 *Let $h : [0,+\infty) \to [0,+\infty)$ be a non-increasing measurable function, $\sigma_1 \le \sigma_2 \le \infty$ and $\tau > 0$. If $\sigma_2 < \infty$, then*

$$\Big(\int_\alpha^\infty (\beta^\tau h(\beta))^{\sigma_2} \frac{d\beta}{\beta}\Big)^{\frac{1}{\sigma_2}} \le \varepsilon \alpha^\tau h(\alpha) + C\Big(\int_\alpha^\infty (\beta^\tau h(\beta))^{\sigma_1} \frac{d\beta}{\beta}\Big)^{\frac{1}{\sigma_1}} \tag{3.6}$$

for every $\varepsilon \in (0,1)$ and for any $\alpha \ge 0$, where $C = C(\tau, \varepsilon, \sigma_1, \sigma_2)$. If $\sigma_2 = \infty$ then

$$\sup_{\beta > \alpha} \beta^\tau h(\beta) \le C\alpha^\tau h(\alpha) + C\Big(\int_\alpha^\infty (\beta^\tau h(\beta))^{\sigma_1} \frac{d\beta}{\beta}\Big)^{\frac{1}{\sigma_1}}, \tag{3.7}$$

where $C = C(\tau, \sigma_1)$.

In the process of main proof, we also make use of the following iterating lemma, which can be found in [20].

Lemma 3.4 *Let ϕ be a bounded nonnegative function on $[r_1, r_2]$. Suppose that for any s_1, s_2 with $0 < r_1 \le s_1 \le s_2 \le r_2$,*

$$\phi(s_1) \le \theta_1 \phi(s_2) + \frac{P_1}{(s_2 - s_1)^{\theta_2}} + P_2, \tag{3.8}$$

where the constants $P_1, P_2 \ge 0$, $0 < \theta_1 < 1$ and $\theta_2 > 0$. Then there holds

$$\phi(s_1) \le C\Big(\frac{P_1}{(s_2 - s_1)^{\theta_2}} + P_2\Big)$$

for some positive constant $C = C(\theta_1, \theta_2)$.

According to the classical L^2 solvability to the variational inequalities (1.1) in line with the Lax-Milgram theory, there exists a unique weak solution $u \in \mathcal{A}$ of (1.1) such that the following lemma holds, for details also see [9].

Lemma 3.5 *There is a unique weak solution $u \in \mathcal{A}$ to the variational inequalities (1.1) such that we have the estimate*

$$\|Du\|_{L^2(\Omega)} \le C\Big(\|\boldsymbol{f}\|_{L^2(\Omega)} + \|D\psi\|_{L^2(\Omega)}\Big), \tag{3.9}$$

where $C = C(d, \nu, \Lambda)$.

Now, let us employ the fact that the obstacle problem here considered is invariant under scaling and normalization. Then, the following property is an immediate consequence by straightforward computations, see Lemma 2.4 in [9].

Lemma 3.6 *Fixed $M > 1$ and $0 < \rho < 1$, we define*

$$\tilde{A}(x) = A(\rho x), \quad \tilde{u}(x) = \frac{u(\rho x)}{M\rho}, \quad \tilde{\psi}(x) = \frac{\psi(\rho x)}{M\rho}, \quad \tilde{\boldsymbol{f}}(x) = \frac{\boldsymbol{f}(\rho x)}{M}$$

and the set $\tilde{\Omega} = \{x/\rho : x \in \Omega\}$. Then we have
(1) *If $u \in \mathcal{A}$ is the weak solution to the variational inequalities (1.1) in Ω, then*

$$\tilde{u} \in \tilde{\mathcal{A}} = \{\phi \in W_0^{1,2}(\tilde{\Omega}) : \phi \ge \tilde{\psi}, \ a.e. \ in \ \tilde{\Omega}\}$$

is the weak solution to the variational inequalities

$$\int_{\tilde{\Omega}} \tilde{A}(x) D\tilde{u} \cdot D(\tilde{\phi} - \tilde{u})\, dx \ge \int_{\tilde{\Omega}} \tilde{\boldsymbol{f}} \cdot D(\tilde{\phi} - \tilde{u})\, dx, \ \forall \ \tilde{\phi} \in \tilde{\mathcal{A}}. \tag{3.10}$$

(2) *$\tilde{A}(x)$ satisfies the basic condition (2.4) with the same constants ν and Λ. Moreover, the regularity assumption 1 is invariant with the dilated scale R_0/ρ.*

Finally, we end this section by presenting a necessary auxiliary result concerning a higher integrability result for (1.1) in the interior and the boundary version, see [20]. This relies on the generalized reverse Hölder inequality first originating from Gehring-Giaquinta-Modica Lemma, and the boundary setting by using the (A)-condition of (δ, R_0)-Reifenberg flat domain, see Remark 2.4. For the setting of any boundary point, we set

$$\Omega_r := B_r \cap \Omega \qquad \text{and} \qquad \partial_w\Omega_r := B_r \cap \partial\Omega \qquad \text{for any} \quad r > 0.$$

Lemma 3.7 *(1) Let $u \in \mathcal{A}$ be a weak solution of (1.1) in $Q_{4r} \subset \Omega$ for any $r > 0$. Suppose $|f|, |D\psi| \in L^\gamma(Q_{4r})$ for some $\gamma > 2$, then there exists a small positive constant σ_1 such that for all $\sigma \leq \sigma_1$,*

$$\fint_{Q_r} |Du|^{2(1+\sigma)} dx \leq C\left(\left(\fint_{Q_{2r}} |Du|^2 dx\right)^{(1+\sigma)} + \fint_{Q_{2r}} \left(|F|^2 + |D\psi|^2\right)^{(1+\sigma)} dx\right) \tag{3.11}$$

for some positive constant $C = C(d, \nu, \Lambda, \gamma)$.
(2) Suppose Ω is a (δ, R_0)-Reifenberg flat domain. Let $u \in \mathcal{A}$ be a weak solution of (1.1) in Ω_{4r} and $u = 0$ on $\partial_w\Omega_{4r}$ with

$$Q^+_{4r} \subset \Omega_{4r} \subset Q_{4r} \cap \{x_1 > -8\delta r\}$$

for any $0 < r < R_0$. If $|f|, |D\psi| \in L^\gamma(\Omega_{4r})$ for some $\gamma > 2$, then there exists a small positive constant σ_2 such that for all $\sigma \leq \sigma_2$,

$$\fint_{\Omega_r} |Du|^{2(1+\sigma)} dx \leq C\left(\left(\fint_{\Omega_{2r}} |Du|^2 dx\right)^{(1+\sigma)} + \fint_{\Omega_{2r}} \left(|F|^2 + |D\psi|^2\right)^{(1+\sigma)} dx\right) \tag{3.12}$$

for some positive constant $C = C(d, \nu, \Lambda, \gamma, \delta, R_0)$.

4 Proof of main result

In this section, we focus on the proof of main Theorem 2.5. First, let us begin this section with the a priori assumption that the unique weak solution $u \in \mathcal{A}$ of the variational inequalities (1.1) satisfies

$$\||Du|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} < \infty. \tag{4.1}$$

We also assume that $(A(x), \Omega)$ is (δ, R_0)-vanishing of codimension 1, where $R_0 \leq 1$ is a given number while δ is to be determined later. Let $R \in \left(0, R_0/(|\Omega| + 1)\right)$ and $x_0 \in \overline{\Omega}$ to be fixed, we localize our interest in the region $\Omega_{2R}(x_0)$, and write

$$2 < \gamma_1 \leq p^- = \inf_{\Omega_{2R}(x_0)} p(x) \leq p^+ = \sup_{\Omega_{2R}(x_0)} p(x) \leq \gamma_2 < \infty \tag{4.2}$$

and

$$\lambda_0 = \fint_{\Omega_{2R}(x_0)} |Du|^{\frac{2p(x)}{p^-}} dx + \frac{1}{\delta}\Big(\fint_{\Omega_{2R}(x_0)} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}} + 1)^\eta dx\Big)^{\frac{1}{\eta}} > 1, \tag{4.3}$$

where $\eta > 1$ and small $\delta > 0$ will be specified later. We would like to remark that the δ-flatness Reifenberg condition (2.6) for the boundary of domain is meaningful in the area of geometric measure theory, only if δ is small enough, see [29]. For any τ_1, τ_2 with $1 \leq \tau_1 < \tau_2 \leq 2$, we denote an upper-level set by

$$E(\lambda) = \Big\{x \in \Omega_{\tau_1 R}(x_0) : |Du|^{\frac{2p(x)}{p^-}} > \lambda\Big\},$$

where λ large enough such that

$$\lambda > \Big(\frac{16}{7}\Big)^d \Big(\frac{248}{\tau_2 - \tau_1}\Big)^d \lambda_0. \tag{4.4}$$

We observe from the upper-level set that

$$\Omega_r(y) \subset \Omega_{2R}(x_0), \quad \forall y \in E(\lambda) \text{ and } 0 < r \le (\tau_2 - \tau_1)R.$$

Fix any point $y \in E(\lambda)$, we consider a continuous function $\Phi_y(r)$ defined by

$$\Phi_y(r) = \fint_{\Omega_r(y)} |Du|^{\frac{2p(x)}{p^-}} dx + \frac{1}{\delta}\Big(\fint_{\Omega_r(y)} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}})^\eta dx\Big)^{\frac{1}{\eta}}, \quad 0 < r \le (\tau_2 - \tau_1)R. \tag{4.5}$$

The Lebesgue differentiation theorem implies that

$$\lim_{r\to 0} \Phi_y(r) > \lambda \qquad \text{for almost every } y \in E(\lambda).$$

On the other hand, if $(\tau_2 - \tau_1)R/124 \le r \le (\tau_2 - \tau_1)R$, then by using the fact that $\eta > 1$ and the Reifenberg flat boundary satisfies measure density condition see Remark 2.4, we have

$$\begin{aligned}
\Phi_y(r) &\le \frac{|\Omega_{2R}(x_0)|}{|\Omega_r(y)|} \fint_{\Omega_{2R}(x_0)} |Du|^{\frac{2p(x)}{p^-}} dx + \Big(\frac{|\Omega_{2R}(x_0)|}{|\Omega_r(y)|}\Big)^{\frac{1}{\eta}} \frac{1}{\delta}\Big(\fint_{\Omega_{2R}(x_0)} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}})^\eta dx\Big)^{\frac{1}{\eta}} \\
&\le \frac{|\Omega_{2R}(x_0)|}{|\Omega_r(y)|} \Big(\fint_{\Omega_{2R}(x_0)} |Du|^{\frac{2p(x)}{p^-}} dx + \frac{1}{\delta}\Big(\fint_{\Omega_{2R}(x_0)} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}})^\eta dx\Big)^{\frac{1}{\eta}}\Big) \\
&\le \frac{|B_r(y)|}{|\Omega \cap B_r(y)|} \frac{|B_{2R}(x_0)|}{|B_r(y)|} \lambda_0 \le \Big(\frac{2}{1-\delta}\Big)^d \Big(\frac{248}{\tau_2 - \tau_1}\Big)^d \lambda_0 \le \Big(\frac{16}{7}\Big)^d \Big(\frac{248}{\tau_2 - \tau_1}\Big)^d \lambda_0.
\end{aligned}$$

Putting the above formula into assumption (4.4), it yields

$$\Phi_y(r) < \lambda, \quad \forall y \in E(\lambda) \text{ and } \forall r \in [(\tau_2 - \tau_1)R/124, (\tau_2 - \tau_1)R].$$

Consequently, we conclude that for almost every $y \in E(\lambda)$, there exists an $r_y = r(y) \in (0, (\tau_2 - \tau_1)R/124)$ such that

$$\Phi_y(r_y) = \lambda \quad \text{and} \quad \Phi_y(r) < \lambda \qquad \forall r \in (r_y, (\tau_2 - \tau_1)R]. \tag{4.6}$$

Then we infer the following lemma from the well-known Vitali covering lemma due to the property of r_y.

Lemma 4.1 *Let λ satisfy (4.4). Then there exists a disjoint family $\{\Omega_{r_{y_i}}(y_i)\}_{i=1}^\infty$ with $y_i \in E(\lambda)$ and $r_{y_i} \in (0, (\tau_2 - \tau_1)R/124)$ such that*

$$\Phi_{y_i}(r_{y_i}) = \lambda \quad \textit{and} \quad \Phi_{y_i}(r) < \lambda, \qquad \textit{for all } r \in (r_{y_i}, (\tau_2 - \tau_1)R]$$

and

$$E(\lambda) \subset \bigcup_{i=1}^\infty \Omega_{5r_{y_i}}(y_i).$$

Lemma 4.2 *Under the same hypothesis as in Lemma 4.1, we have*

$$\begin{aligned}
|\Omega_{r_{y_i}}(y_i)| \quad &\le \quad C\Big(|\Omega_{r_{y_i}}(y_i) \cap E(\lambda/4)| + \frac{1}{(\varsigma\lambda)^\eta} \int_{\varsigma\lambda}^\infty \mu^\eta |\{x \in \Omega_{r_{y_i}}(y_i) : |f|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu} \\
&+ \frac{1}{(\varsigma\lambda)^\eta} \int_{\varsigma\lambda}^\infty \mu^\eta |\{x \in \Omega_{r_{y_i}}(y_i) : |D\psi|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu}\Big),
\end{aligned}$$

where $\varsigma = \delta/6$ and $C = C(d, \nu, \Lambda)$.

Proof. With Lemma 4.1 in hand, we then have that one of the following results must hold,

$$\fint_{\Omega_{r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}}\,dx \geq \frac{\lambda}{3}, \quad \fint_{\Omega_{r_{y_i}}(y_i)} |\mathbf{f}|^{\frac{2p(x)}{p^-}\eta}\,dx \geq \Big(\frac{\delta\lambda}{3}\Big)^{\eta} \quad \text{and} \quad \fint_{\Omega_{r_{y_i}}(y_i)} |D\psi|^{\frac{2p(x)}{p^-}\eta}\,dx \geq \Big(\frac{\delta\lambda}{3}\Big)^{\eta}.$$

For the first setting, let take any

$$0 < \epsilon < \min\Big\{\frac{\gamma_1(1+\sigma_0)}{\gamma_1+\omega(2R)} - 1, \frac{\gamma p^-}{2} - 1\Big\}, \tag{4.7}$$

where $\sigma_0 = \min\{\sigma_1, \sigma_2\}$ is the same as Lemma 3.7 which is concerned with the higher integrability of Du, and $\gamma \in [1, \infty)$. It yields the following inequality:

$$\frac{p(x)}{p^-}(1+\epsilon) = \Big(1 + \frac{p(x)-p^-}{p^-}\Big)(1+\epsilon) \leq \Big(1 + \frac{\omega(2R)}{\gamma_1}\Big)(1+\epsilon) < 1+\sigma_0 \leq \gamma_1.$$

Then, by an additivity of the integral with respect to the domain and Hölder inequality we get

$$\begin{aligned}\frac{\lambda}{3}|\Omega_{r_{y_i}}(y_i)| &\leq \int_{\Omega_{r_{y_i}}(y_i)\cap E(\lambda/4)} |Du|^{\frac{2p(x)}{p^-}}\,dx + \int_{\Omega_{r_{y_i}}(y_i)\setminus E(\lambda/4)} |Du|^{\frac{2p(x)}{p^-}}\,dx \\ &\leq |\Omega_{r_{y_i}}(y_i)\cap E(\lambda/4)|^{1-\frac{1}{1+\epsilon}}\Big(\int_{\Omega_{r_{y_i}}(y_i)\cap E(\lambda/4)} |Du|^{\frac{2p(x)}{p^-}(1+\epsilon)}\,dx\Big)^{\frac{1}{1+\epsilon}} + \frac{\lambda}{4}|\Omega_{r_{y_i}}(y_i)|,\end{aligned}$$

which yields

$$\lambda|\Omega_{r_{y_i}}(y_i)|^{1-\frac{1}{1+\epsilon}} \leq C_1|\Omega_{r_{y_i}}(y_i)\cap E(\lambda/4)|^{1-\frac{1}{1+\epsilon}}\Big(\fint_{\Omega_{r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}(1+\epsilon)}\,dx\Big)^{\frac{1}{1+\epsilon}}. \tag{4.8}$$

Furthermore, on the basis of a higher integrability of the gradient to weak solutions for the variational inequalities (1.1) in lines with Lemma 3.7, we obtain

$$\Big(\fint_{\Omega_{r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}(1+\epsilon)}\,dx\Big)^{\frac{1}{1+\epsilon}} \leq C_2\Bigg(\fint_{\Omega_{2r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}}dx + \Big(\fint_{\Omega_{2r_{y_i}}(y_i)} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}})^{(1+\epsilon)}dx\Big)^{\frac{1}{1+\epsilon}}\Bigg).$$

Now we take $\eta = 1+\epsilon$, using Lemma 4.1 yields

$$\Big(\fint_{\Omega_{r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}(1+\epsilon)}\,dx\Big)^{\frac{1}{1+\epsilon}} \leq C_3\lambda.$$

Putting the above formula into (4.8), we have

$$|\Omega_{r_{y_i}}(y_i)| \leq C_4|\Omega_{r_{y_i}}(y_i)\cap E(\lambda/4)|. \tag{4.9}$$

For the second setting, we have

$$\begin{aligned}\Big(\frac{\delta\lambda}{3}\Big)^{\eta}|\Omega_{r_{y_i}}(y_i)| &\leq \int_{\Omega_{r_{y_i}}(y_i)} |\mathbf{f}|^{\frac{2p(x)}{p^-}\eta}\,dx \\ &= \eta\int_0^{\infty} \mu^{\eta}|\{x\in\Omega_{r_{y_i}}(y_i) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}|\frac{d\mu}{\mu} \\ &\leq (\varsigma\lambda)^{\eta}|\Omega_{r_{y_i}}(y_i)| + \eta\int_{\varsigma\lambda}^{\infty} \mu^{\eta}|\{x\in\Omega_{r_{y_i}}(y_i) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}|\frac{d\mu}{\mu}.\end{aligned}$$

Taking $\varsigma = \delta/6$, we get

$$|\Omega_{r_{y_i}}(y_i)| \le \frac{C_5}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta |\{x\in\Omega_{r_{y_i}}(y_i) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu}. \tag{4.10}$$

We now estimate the third setting in a similar way just as doing it in the second setting, and conclude that

$$|\Omega_{r_{y_i}}(y_i)| \le \frac{C_6}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta |\{x\in\Omega_{r_{y_i}}(y_i) : |D\psi|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu}. \tag{4.11}$$

Finally, putting the three cases (4.9),(4.10) and (4.11) together, this completes the proof of Lemma 4.2. □

For any fixed point y_i and the scale r_{y_i}, there are now two possible cases. One is the interior case that $B_{20r_{y_i}}(y_i) \subseteq \Omega$. The other is the boundary case that $B_{20r_{y_i}}(y_i) \not\subseteq \Omega$. We first look at the interior case. Since $A(x)$ is (δ, R_0)-vanishing of codimension one, we assume that in a new coordinate system $(x_1, \cdots, x_d)$, the origin is y_i and

$$\fint_{Q_{20r_{y_i}}(y_i)} |A(x) - \bar{A}_{B'_{20r_{y_i}}(y'_i)}(x_1)|^2 dx \le \delta^2. \tag{4.12}$$

For convenience, we write

$$p_i^- = \inf_{x\in Q_{20r_{y_i}}(y_i)} p(x) \quad \text{and} \quad p_i^+ = \sup_{x\in Q_{20r_{y_i}}(y_i)} p(x).$$

From Lemma 4.1 and the definition of $\Phi_{y_i}(r_{y_i})$, we have

$$\fint_{Q_{20r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}} dx \le \lambda \qquad \text{and} \qquad \Big(\fint_{Q_{20r_{y_i}}(y_i)} (\,|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}}\,)^\eta dx\Big)^{\frac{1}{\eta}} \le \delta\lambda. \tag{4.13}$$

By the re-scaling transformation and perturbation approach based on a local comparison we obtain

$$\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx \le C_0 \lambda^{\frac{p^-}{p_i^+}} \quad \text{and} \quad \fint_{Q_{20r_{y_i}}(y_i)} (\,|\mathbf{f}|^2 + |D\psi|^2\,)\, dx \le C_0 \lambda^{\frac{p^-}{p_i^+}} \delta^{\frac{\gamma_1}{\gamma_2}} \tag{4.14}$$

for some constant $C_0 \ge 1$ independent of i, and γ_1, γ_2 shown as (4.2). In fact, let $A_0 = \||\mathbf{f}|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} + \||D\psi|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} + 1 \ge 1$. A direct computation yields that

$$\begin{aligned}
\Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{p_i^+ - p_i^-} &= \Big(\frac{1}{|Q_{20r_{y_i}}(y_i)|}\Big)^{p_i^+ - p_i^-}\Big(\int_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{p_i^+ - p_i^-} \\
&\le C_1\Big(\frac{1}{40r_{y_i}}\Big)^{d\omega(40r_{y_i})}\Big(\int_\Omega |Du|^2 dx\Big)^{p_i^+ - p_i^-} \\
&\le C_2\Big(\int_\Omega |Du|^2 dx\Big)^{p_i^+ - p_i^-},
\end{aligned} \tag{4.15}$$

where we used (2.3) based on the log-Hölder condition in the last inequality. On the other hand, by making use of the standard L^2 estimates from Lemma 3.5 and Proposition 3.1 due to $\frac{\gamma\gamma_1}{2} > 1$ for $\gamma \in [1, \infty)$ and $q \in (0, \infty]$, we obtain

$$\begin{aligned}
\int_\Omega |Du|^2 dx &\le C_3\left(\int_\Omega |\mathbf{f}|^2 dx + \int_\Omega |D\psi|^2 dx\right) \\
&\le C_4\left(\int_\Omega |\mathbf{f}|^{\frac{2p(x)}{\gamma_1}} dx + \int_\Omega |D\psi|^{\frac{2p(x)}{\gamma_1}} dx + |\Omega|\right) \\
&\le C_5\left(\||\mathbf{f}|^{\frac{2p(x)}{\gamma_1}}\|_{L^{(\frac{\gamma\gamma_1}{2},\frac{q\gamma_1}{2})}(\Omega)} + \||D\psi|^{\frac{2p(x)}{\gamma_1}}\|_{L^{(\frac{\gamma\gamma_1}{2},\frac{q\gamma_1}{2})}(\Omega)} + |\Omega|\right) \\
&= C_5\Big(\||\mathbf{f}|^{p(x)}\|^{\frac{2}{\gamma_1}}_{L^{(\gamma,q)}(\Omega)} + \||D\psi|^{p(x)}\|^{\frac{2}{\gamma_1}}_{L^{(\gamma,q)}(\Omega)} + |\Omega|\Big),
\end{aligned}$$

which leads to

$$\int_{\Omega} |Du|^2 dx \leq C_6 A_0 \Big(1 + |\Omega|\Big).$$

Then we conclude

$$\Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{p_i^+ - p_i^-} \leq C_2 \left(C_6\, A_0 \Big(1 + |\Omega|\Big)\right)^{p_i^+ - p_i^-}.$$

Recalling $\Big(|\Omega| + 1\Big) < \frac{R_0}{R} \leq \frac{1}{R} \leq \frac{1}{40 r_{y_i}}$ and (2.3), it yields

$$\Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{p_i^+ - p_i^-} \leq C_7\, A_0^{p_i^+ - p_i^-} \Big(\frac{1}{40 r_{y_i}}\Big)^{\omega(40 r_{y_i})} \leq C_8\, A_0^{p_i^+ - p_i^-},$$

which follows from $A_0 \geq 1$ that

$$\Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{\frac{p_i^+ - p_i^-}{p_i^+}} \leq C_9\, A_0^{\frac{p_i^+ - p_i^-}{p_i^+}} \leq C_9\, A_0. \tag{4.16}$$

Using (4.16) and Jensen inequality yields

$$\begin{aligned}
\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx &= \Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{\frac{p_i^+ - p_i^-}{p_i^+}} \Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{\frac{p_i^-}{p_i^+}} \\
&\leq C_9 A_0 \Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^2 dx\Big)^{\frac{p_i^-}{p_i^+}} \\
&\leq C_9 A_0 \Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^{\frac{2p_i^-}{p^-}} dx\Big)^{\frac{p^-}{p_i^+}} \\
&\leq C_9 A_0 \Big(\fint_{Q_{20r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}} dx + 1\Big)^{\frac{p^-}{p_i^+}}.
\end{aligned}$$

Since $\lambda > 1$, by (4.13) we get the first desired inequality in (4.14) by taking $C_0 = 2C_9 A_0$.

Likewise, we derive that

$$\begin{aligned}
\fint_{Q_{20r_{y_i}}(y_i)} \Big(|\mathbf{f}|^2 + |D\psi|^2\Big) dx &\leq C_0 \left(\fint_{Q_{20r_{y_i}}(y_i)} \Big(|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}}\Big) dx + 1\right)^{\frac{p^-}{p_i^+}} \\
&\leq C_0 \left(\Big(\fint_{Q_{20r_{y_i}}(y_i)} \Big(|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}}\Big)^{\eta} dx\Big)^{\frac{1}{\eta}} + 1\right)^{\frac{p^-}{p_i^+}} \\
&\leq C_0 \Big(\delta\lambda + \delta\lambda_0\Big)^{\frac{p^-}{p_i^+}} \leq C_0 \lambda^{\frac{p^-}{p_i^+}} \delta^{\frac{\gamma_1}{\gamma_2}},
\end{aligned}$$

where we also employ (4.13) in the third inequality.

Define

$$\tilde{u}_i(x) = \frac{u(5r_{y_i}x)}{5r_{y_i}\sqrt{C_0 \lambda^{\frac{p^-}{p_i^+}}}}, \qquad \tilde{\psi}_i(x) = \frac{\psi(5r_{y_i}x)}{5r_{y_i}\sqrt{C_0 \lambda^{\frac{p^-}{p_i^+}}}}, \qquad \tilde{\mathbf{f}}_i(x) = \frac{\mathbf{f}(5r_{y_i}x)}{\sqrt{C_0 \lambda^{\frac{p^-}{p_i^+}}}}, \qquad \tilde{A}_i(x) = A(5r_{y_i}x).$$

By using Lemma 3.6, we get that $\tilde{u}_i(x) \in \tilde{\mathcal{A}}_i := \left\{\phi_i \in W^{1,2}(Q_4) : \phi_i \geq \tilde{\psi}_i, \text{ a.e. in } Q_4\right\}$ is a weak solution of

$$\int_{Q_4} \tilde{A}_i(x) D\tilde{u}_i \, D(\tilde{\phi}_i - \tilde{u}_i)\, dx \geq \int_{Q_4} \tilde{\mathbf{f}}_i \, D(\tilde{\phi}_i - \tilde{u}_i)\, dx \qquad \text{for all } \tilde{\phi}_i \in \tilde{\mathcal{A}}_i.$$

Moreover, using (4.12) and (4.14), by a straightforward computation we have

$$\fint_{Q_4} |A(x) - \bar{A}_{B'_4}(x_1)|^2 \leq \delta^2, \quad \fint_{Q_4} |D\tilde{u}|^2 dx \leq 1 \qquad \text{and} \qquad \fint_{Q_4} (|\tilde{\mathbf{f}}|^2 + |D\tilde{\psi}|^2)\, dx \leq \delta^{\frac{\gamma_1}{\gamma_2}}.$$

Thus, by Lemma 4.3 and Lemma 4.4 in [8], we find that for any $\varepsilon > 0$, there exist a constant $\delta > 0$ and a function $\tilde{v}_i \in W^{1,2}(Q_2)$ to be a weak solution of

$$\operatorname{div}(\tilde{A}_{iB'_2}(x_1) D\tilde{v}_i) = 0 \qquad \text{in} \quad Q_2,$$

such that

$$\fint_{Q_2} |D(\tilde{u}_i - \tilde{v}_i)|^2 dx \leq \varepsilon \qquad \text{and} \qquad \|D\tilde{v}_i\|^2_{L^\infty(Q_1)} \leq N_0$$

with a constant N_0 being independent of i. Scaling back and denoting v_i by the translated function of

$$\tilde{v}_i(x) = \frac{v_i(5r_{y_i}x)}{5r_{y_i}\sqrt{C_0 \lambda^{\frac{p^-}{p_i^+}}}},$$

we conclude that

$$\fint_{Q_{10r_{y_i}}(y_i)} |D(u - v_i)|^2 dx \leq C_0 \lambda^{\frac{p^-}{p_i^+}} \varepsilon \qquad \text{and} \qquad \|Dv_i\|^2_{L^\infty(Q_{5r_{y_i}}(y_i))} \leq N_1 \lambda^{\frac{p^-}{p_i^+}} \tag{4.17}$$

for some constant $N_1 = N_0 C_0 \geq 1$, being independent of i.

We next consider the boundary case that $dist\{y_i, \partial\Omega\} = |y_i - y_0| \leq 20 r_{y_i}$ for $y_0 \in \partial\Omega$. Let us recall that $r_{y_i} < (\tau_2 - \tau_1)R/124 < R_0/124$ and the geometry (2.6) of the boundary of Reifenberg flat domain, we have the following property: for any point y_0 on the boundary of Ω, there exists a coordinate system $\{x_1, \cdots, x_d\}$ with the origin lining somewhere in $\Omega_{60r_{y_i}\delta}(y_0)$, such that in this new coordinate system one has

$$Q^+_{60r_{y_i}} \subset \Omega_{60r_{y_i}} \subset Q_{60r_{y_i}} \cap \{x_1 > -120 r_{y_i}\delta\}$$

and

$$\fint_{Q_{60r_{y_i}}} |A(x) - \bar{A}_{B'_{60r_{y_i}}}(x_1)|^2 dx \leq \delta^2.$$

Let us now select δ so small such that $0 < \delta < \frac{1}{60}$, which yields $|y_i| < (20+1) r_{y_i} = 21 r_{y_i}$ and

$$\Omega_{5r_{y_i}}(y_i) \subset \Omega_{26r_{y_i}} \subset \Omega_{60r_{y_i}} \subset \Omega_{96r_{y_i}}(y_i). \tag{4.18}$$

We write

$$p_i^- = \inf_{x \in \Omega_{96r_{y_i}}(y_i)} p(x) \quad \text{and} \quad p_i^+ = \sup_{x \in \Omega_{96r_{y_i}}(y_i)} p(x).$$

By Lemma 4.1, we derive that

$$\fint_{\Omega_{60r_{y_i}}} |Du|^{\frac{2p(x)}{p^-}} dx \le \frac{|\Omega_{96r_{y_i}}(y_i)|}{|\Omega_{60r_{y_i}}|} \fint_{\Omega_{96r_{y_i}}(y_i)} |Du|^{\frac{2p(x)}{p^-}} dx \le C_{10}\,\lambda$$

and

$$\Big(\fint_{\Omega_{60r_{y_i}}} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}})^{\eta} dx\Big)^{\frac{1}{\eta}} \le \Big(\frac{|\Omega_{96r_{y_i}}(y_i)|}{|\Omega_{60r_{y_i}}|}\Big)^{\frac{1}{\eta}} \Big(\fint_{\Omega_{96r_{y_i}}(y_i)} (|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}})^{\eta} dx\Big)^{\frac{1}{\eta}} \le C_{10}\delta\lambda,$$

where the constant C_{10} depends only on d. Once we have the above uniform bounds, one can find in the same sprit as in the interior case that

$$\fint_{\Omega_{60r_{y_i}}} |Du|^2 dx \le C_{11}\lambda^{\frac{p^-}{p_i^+}} \quad \text{and} \quad \fint_{\Omega_{60r_{y_i}}} (|\mathbf{f}|^2 + |D\psi|^2)\, dx \le C_{11}\lambda^{\frac{p^-}{p_i^+}} \delta^{\frac{\gamma_1}{\gamma_2}}$$

for some constant $C_{11} = 2C_{10}A_0 \ge 1$, being independent of i. In a similar way that we have used for the interior case, also see Lemma 4.7 in [8], for any $\varepsilon \in (0, 1)$ there exist a small positive number δ and a function $v_i \in W^{1,2}(Q^+_{52r_{y_i}})$ such that

$$\fint_{\Omega_{52r_{y_i}}} |D(u - \bar{v}_i)|^2\, dx \le C_{11}\lambda^{\frac{p^-}{p_i^+}}\varepsilon \qquad \text{and} \qquad \|D\bar{v}_i\|^2_{L^\infty(\Omega_{26r_{y_i}})} \le N_2\lambda^{\frac{p^-}{p_i^+}} \tag{4.19}$$

for some constant $N_2 = N_2C_{11} \ge 1$, being independent of i. Here we have extended v_i by zero from $Q^+_{52r_{y_i}}$ to $Q_{52r_{y_i}}$ and also denote it by $\bar{v}_i$. Let us write $N = \max\{N_1, N_2\}$ being large enough, which is independent of the index i. For convenience, we also write

$$A = (4N)^{\frac{\gamma_2}{\gamma_1}} > 1, \quad B = \Big(\frac{16}{7}\Big)^d \Big(\frac{248}{\tau_2 - \tau_1}\Big)^d. \tag{4.20}$$

Lemma 4.3 *Let $R_0 > 0$. For any fixed $0 < \varepsilon < 1$, we can find a small constant $\delta > 0$ such that if $(A(x), \Omega)$ is (δ, R_0)-vanishing codimension 1 and $u \in \mathcal{A}$ is a weak solution of (1.1), then for any $1 \le \tau_1 < \tau_2 \le 2$ we have*

$$\begin{aligned} |E(A\lambda)| \;\le\; & C\varepsilon\Bigg(|\Omega_{\tau_2 R}(x_0) \cap E(\lambda/4)| \\ & + \frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^{\infty} \mu^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |f|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu} \\ & + \frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^{\infty} \mu^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |D\psi|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu}\Bigg) \end{aligned}$$

for all $\lambda > B\lambda_0$, $\varsigma = \delta/6$ and $C = C(d, \nu, \Lambda, \omega(\cdot), \gamma_1, \gamma_2, R_0)$.

Proof. By Lemma 4.1 and the fact that $E(A\lambda) \subset E(\lambda)$ for $A > 1$ as the above-mentioned in (4.20), we obtain that $\{\Omega_{5r_{y_i}}(y_i)\}$ cover almost all $E(A\lambda)$. Thus,

$$\begin{aligned} |E(A\lambda)| \;=\; & |\{x \in \Omega_{\tau_1 R}(x_0) : |Du|^{\frac{2p(x)}{p^-}} > A\lambda\}| \\ \le\; & \sum_{i=1}^{\infty} |\{x \in \Omega_{5r_{y_i}}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}| \\ \le\; & \sum_{\text{i: interior case}} |\{x \in \Omega_{5r_{y_i}}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}| \\ & + \sum_{\text{i: boundary case}} |\{x \in \Omega_{5r_{y_i}}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}|. \end{aligned} \tag{4.21}$$

For the interior estimate, using the fact that $|Du|^2 \leq 2|D(u-v_i)|^2 + 2|Dv_i|^2$, $\Omega_{5r_{y_i}}(y_i) = Q_{5r_{y_i}}(y_i)$ and Eq. (4.17), we find that

$$
\begin{aligned}
& |\{x \in Q_{5r_{y_i}}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}| \\
\leq\; & |\{x \in Q_{5r_{y_i}}(y_i) : |D(u-v_i)|^2 > N_1 \lambda^{\frac{p^-}{p_i^+}}\}| + |\{x \in Q_{5r_{y_i}}(y_i) : |Dv_i|^2 > N_1 \lambda^{\frac{p^-}{p_i^+}}\}| \\
\leq\; & \frac{1}{N_1 \lambda^{\frac{p^-}{p_i^+}}} \int_{Q_{10r_{y_i}}(y_i)} |D(u-v_i)|^2 \, dx \\
\leq\; & \frac{C_0 \lambda^{\frac{p^-}{p_i^+}} \varepsilon |Q_{10r_{y_i}}(y_i)|}{N_0 C_0 \lambda^{\frac{p^-}{p_i^+}}} \\
\leq\; & C_1 \varepsilon |Q_{10r_{y_i}}(y_i)|,
\end{aligned}
$$

which implies

$$
|\{x \in \Omega_{5r_{y_i}}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}| \leq C_2 \varepsilon |Q_{r_{y_i}}(y_i)|. \tag{4.22}
$$

For the boundary case, we carry out the same procedure in (4.21) with Eq. (4.19) to discover that

$$
|\{x \in \Omega_{26r_{y_i}} : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}| \leq C_3 \varepsilon |\Omega_{r_{y_i}}|.
$$

Then, using the measure density condition (2.8) and the geometry of Reifenberg flatness (4.18), we conclude that

$$
|\{x \in \Omega_{5r_{y_i}}(y_i) : |Du|^2 > (A\lambda)^{\frac{p^-}{p(x)}}\}| \leq C_4 \varepsilon |\Omega_{r_{y_i}}(y_i)|. \tag{4.23}
$$

Inserting (4.22) and (4.23) into (4.21) to get

$$
|E(A\lambda)| \quad \leq \quad C_5 \varepsilon \sum_{i=1}^{\infty} |\Omega_{r_{y_i}}(y_i)|,
$$

where $C_5 = \max\{C_2, C_4\}$. Furthermore, by using Lemma 4.2 we obtain

$$
\begin{aligned}
|E(A\lambda)| \quad \leq \quad & C_5 \varepsilon \sum_{i=1}^{\infty} \Bigg(|\Omega_{r_{y_i}}(y_i) \cap E(\lambda/4)| \\
& + \frac{1}{(\varsigma\lambda)^{\eta}} \int_{\varsigma\lambda}^{\infty} \mu^{\eta} |\{x \in \Omega_{r_{y_i}}(y_i) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu} \\
& + \frac{1}{(\varsigma\lambda)^{\eta}} \int_{\varsigma\lambda}^{\infty} \mu^{\eta} |\{x \in \Omega_{r_{y_i}}(y_i) : |D\psi|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu} \Bigg)
\end{aligned}
$$

for all $\lambda > B\lambda_0$ and $\varsigma = \delta/6$. Note that $\{\Omega_{r_{y_i}}(y_i)\}$ are non-overlapping in $\Omega_{\tau_2 R}(x_0)$, then it follows the required result. □

Proof of Theorem 2.5. We part it two steps: we first attain the estimate under the assumption of (4.1); then we prove the assumption (4.1).

Step 1. Let us first establish a global estimate to the variational inequalities (1.1) under the a priori assumption (4.1). In the case $0 < q < \infty$, thanks to (3.3) in Proposition 3.1 in hand we have

$$\begin{aligned} \||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_1 R}(x_0))} &= \||Du|^{\frac{2p(x)}{p^-}}\|^{\frac{p^-}{2}q}_{L^{(\frac{\gamma p^-}{2},\frac{qp^-}{2})}(\Omega_{\tau_1 R}(x_0))} \\ &= \frac{\gamma p^-}{2}\int_0^\infty \left(\alpha^{\frac{\gamma p^-}{2}}|\{x\in\Omega_{\tau_1 R}(x_0): |Du|^{\frac{2p(x)}{p^-}} > \alpha\}|\right)^{\frac{q}{\gamma}}\frac{d\alpha}{\alpha}. \end{aligned}$$

By using change of variables, a direct calculation shows that

$$\begin{aligned} \||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_1 R}(x_0))} &= \frac{\gamma p^-}{2}A^{\frac{qp^-}{2}}\int_0^\infty \lambda^{\frac{qp^-}{2}}|\{x\in\Omega_{\tau_1 R}(x_0): |Du|^{\frac{2p(x)}{p^-}} > A\lambda\}|^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda} \\ &= \frac{\gamma p^-}{2}A^{\frac{qp^-}{2}}\int_0^{B\lambda_0} \lambda^{\frac{qp^-}{2}}|E(A\lambda)|^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda} + \frac{\gamma p^-}{2}A^{\frac{qp^-}{2}}\int_{B\lambda_0}^\infty \lambda^{\frac{qp^-}{2}}|E(A\lambda)|^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda} \\ &:= I_1 + I_2. \end{aligned} \tag{4.24}$$

To estimate I_1, inserting A and B into (4.20) it yields

$$I_1 \le C_0|\Omega_{2R}(x_0)|^{\frac{q}{\gamma}}\left(AB\lambda_0\right)^{\frac{qp^-}{2}} \le C_1\frac{|\Omega_{2R}(x_0)|^{\frac{q}{\gamma}}}{(\tau_2-\tau_1)^{\frac{dq\gamma_2}{2}}}\left(\lambda_0\right)^{\frac{qp^-}{2}}.$$

For the estimate of I_2, using Lemma 4.3, then for any $0 < \varepsilon < 1$ we have

$$\begin{aligned} I_2 \le\ & C_2\,\varepsilon^{\frac{q}{\gamma}}\int_{B\lambda_0}^\infty \lambda^{\frac{qp^-}{2}}\Bigg(|\Omega_{\tau_2 R}(x_0)\cap E(\lambda/4)| \\ & + \frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta|\{x\in\Omega_{\tau_2 R}(x_0): |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}|\frac{d\mu}{\mu} \\ & + \frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta|\{x\in\Omega_{\tau_2 R}(x_0): |D\psi|^{\frac{2p(x)}{p^-}} > \mu\}|\frac{d\mu}{\mu}\Bigg)^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda}. \end{aligned}$$

Note that

$$\left(A_1 + A_2 + A_3\right)^m \le \max\{3^{m-1}, 1\}\left(A_1^m + A_2^m + A_3^m\right),$$

for any $A_i > 0$, $i = 1, 2, 3$ and $m > 0$, therefore we obtain

$$\begin{aligned} I_2 \le\ & C_3\varepsilon^{\frac{q}{\gamma}}\Bigg(\int_{B\lambda_0}^\infty \lambda^{\frac{qp^-}{2}}|\Omega_{\tau_2 R}(x_0)\cap E(\lambda/4)|^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda} \\ & + \int_{B\lambda_0}^\infty \lambda^{\frac{qp^-}{2}}\Big(\frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta|\{x\in\Omega_{\tau_2 R}(x_0): |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}|\frac{d\mu}{\mu}\Big)^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda} \\ & + \int_{B\lambda_0}^\infty \lambda^{\frac{qp^-}{2}}\Big(\frac{1}{(\varsigma\lambda)^\eta}\int_{\varsigma\lambda}^\infty \mu^\eta|\{x\in\Omega_{\tau_2 R}(x_0): |D\psi|^{\frac{2p(x)}{p^-}} > \mu\}|\frac{d\mu}{\mu}\Big)^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda}\Bigg) \\ :=\ & C_3\varepsilon^{\frac{q}{\gamma}}\left(I_{21} + I_{22} + I_{23}\right). \end{aligned} \tag{4.25}$$

To estimate I_{21}, a simple computation yields

$$I_{21} \le \int_0^\infty \lambda^{\frac{qp^-}{2}}|\Omega_{\tau_2 R}(x_0)\cap E(\lambda/4)|^{\frac{q}{\gamma}}\frac{d\lambda}{\lambda} \le C_4\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))}.$$

To estimate I_{22}, we examine it in two cases.
Case 1. If $q \geq \gamma$, by using the Hardy inequality showed in Lemma 3.2 and the change of variable we get

$$\begin{aligned} I_{22} &\leq C_5 \int_0^\infty \lambda^{\frac{qp^-}{2}} \Big(\frac{1}{(\varsigma\lambda)^\eta} (\varsigma\lambda)^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \varsigma\lambda\}| \Big)^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} \\ &= C_5 \int_0^\infty \lambda^{\frac{qp^-}{2}} |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \varsigma\lambda\}|^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} \\ &\leq C_6 \, \||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))}. \end{aligned}$$

Case 2. $0 < q < \gamma$, we use the reverse Hölder inequality in Lemma 3.3, and get

$$\begin{aligned} &\Big(\int_{\varsigma\lambda}^\infty \mu^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}| \frac{d\mu}{\mu} \Big)^{\frac{q}{\gamma}} \\ \leq\ & \Big((\varsigma\lambda)^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \varsigma\lambda\}| \Big)^{\frac{q}{\gamma}} + C_7 \int_{\varsigma\lambda}^\infty \Big(\mu^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}| \Big)^{\frac{q}{\gamma}} \frac{d\mu}{\mu}, \end{aligned}$$

which yields

$$\begin{aligned} I_{22} &\leq \int_0^\infty \lambda^{\frac{qp^-}{2}} \Big(\frac{1}{(\varsigma\lambda)^\eta} (\varsigma\lambda)^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \varsigma\lambda\}| \Big)^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} \\ &\quad + C_7 \int_0^\infty \lambda^{\frac{qp^-}{2}} \Big(\frac{1}{(\varsigma\lambda)^\eta} \Big)^{\frac{q}{\gamma}} \int_{\varsigma\lambda}^\infty \Big(\mu^\eta |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \mu\}| \Big)^{\frac{q}{\gamma}} \frac{d\mu}{\mu} \frac{d\lambda}{\lambda} \\ &\leq C_8 \int_0^\infty \lambda^{\frac{qp^-}{2}} |\{x \in \Omega_{\tau_2 R}(x_0) : |\mathbf{f}|^{\frac{2p(x)}{p^-}} > \varsigma\lambda\}|^{\frac{q}{\gamma}} \frac{d\lambda}{\lambda} \\ &\leq C_9 \, \||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))}, \end{aligned}$$

where we also employ the Hardy inequality of Lemma 3.2 in the second inequality. For the estimate of I_{23}, using the same way as the estimate I_{22} above, then it follows that

$$I_{23} \leq C_{10} \, \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))}.$$

Putting I_{21}, I_{22}, I_{23} into (4.25) it implies

$$I_2 \leq C_{11} \varepsilon^{\frac{q}{\gamma}} \Big(\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} + \||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} + \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} \Big).$$

Putting the estimates I_1 and I_2 into (4.24), we deduce that

$$\begin{aligned} \||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_1 R}(x_0))} &\leq C_{12} \varepsilon^{\frac{q}{\gamma}} \||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} + C_{12} \frac{|\Omega_{2R}(x_0)|^{\frac{q}{\gamma}}}{(\tau_2 - \tau_1)^{\frac{dq\gamma_2}{2}}} \big(\lambda_0\big)^{\frac{qp^-}{2}} \\ &\quad + C_{12} \Big(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} + \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} \Big). \end{aligned}$$

Setting $\Phi(\tau_i) = \||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_i R}(x_0))}$ for $i = 1, 2$. Let δ be sufficiently small in Lemma 4.3, and we now select a sufficiently small $\varepsilon > 0$ such that

$$0 < C_{12} \varepsilon^{\frac{q}{\gamma}} < \frac{1}{2},$$

which yields

$$\begin{aligned}\Phi(\tau_1) &\le \frac{1}{2}\Phi(\tau_2) + C_{12}\frac{|\Omega_{2R}(x_0)|^{\frac{q}{\gamma}}}{(\tau_2-\tau_1)^{\frac{dq\gamma_2}{2}}}\Big(\lambda_0\Big)^{\frac{qp^-}{2}} \\ &\quad + C_{12}\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))} + \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{\tau_2 R}(x_0))}\right)\end{aligned}$$

for any τ_1, τ_2 with $1 \le \tau_1 < \tau_2 \le 2$. We then apply the iterating Lemma 3.4, and derive

$$\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_R(x_0))} \le C_{13}|\Omega_{2R}(x_0)|^{\frac{q}{\gamma}}\Big(\lambda_0\Big)^{\frac{qp^-}{2}} + C_{13}\Big(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{2R}(x_0))} + \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{2R}(x_0))}\Big).$$

Now, we recall the definition of λ_0 in (4.3) and obtain

$$\begin{aligned}\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_R(x_0))} &\le C_{14}|\Omega_{2R}(x_0)|^{\frac{q}{\gamma}}\Big(\fint_{\Omega_{2R}(x_0)}|Du|^{\frac{2p(x)}{p^-}}dx + \frac{1}{\delta}\Big(\fint_{\Omega_{2R}(x_0)}(|\mathbf{f}|^{\frac{2p(x)}{p^-}} + |D\psi|^{\frac{2p(x)}{p^-}}+1)^\eta dx\Big)^{\frac{1}{\eta}}\Big)^{\frac{qp^-}{2}} \\ &\quad + C_{14}\Big(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{2R}(x_0))} + \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{2R}(x_0))}\Big). \qquad (4.26)\end{aligned}$$

To estimate the first item on the right-hand side of (4.26), noticing that

$$\frac{2p^+}{p^-} = 2(1+\frac{p^+-p^-}{p^-}) \le 2(1+\frac{\omega(2R)}{\gamma_1}) < 2(1+\sigma_0) \le \gamma\gamma_1 \le \gamma p^-, \qquad (4.27)$$

where $\sigma_0 = \min\{\sigma_1, \sigma_2\}$ is the same as Lemma 3.7 for $\gamma \in [1, \infty)$. Then, it yields

$$\begin{aligned}\Big(\fint_{\Omega_{2R}(x_0)}|Du|^{\frac{2p(x)}{p^-}}dx\Big)^{\frac{qp^-}{2}} &\le \Big(\fint_{\Omega_{2R}(x_0)}|Du|^{\frac{2p^+}{p^-}}dx + 1\Big)^{\frac{qp^-}{2}} \\ &\le C_{15}\bigg(\Big(\fint_{\Omega_{4R}(x_0)}|Du|^2dx\Big)^{\frac{p^+}{p^-}} + \fint_{\Omega_{4R}(x_0)}\Big(|\mathbf{f}|^2+|D\psi|^2\Big)^{\frac{p^+}{p^-}}dx + 1\bigg)^{\frac{qp^-}{2}}, \qquad (4.28)\end{aligned}$$

where we have employed reverse Hölder inequality in the last inequality. By using the standard L^2 estimate as Lemma 3.5, and the embedding inequality due to $\frac{\gamma\gamma_1}{2} > 1$ for $\gamma \in [1, \infty)$ and $q \in (0, \infty]$,we have

$$\begin{aligned}\bigg(\Big(\fint_{\Omega_{4R}(x_0)}|Du|^2dx\Big)^{\frac{p^+}{p^-}}\bigg)^{\frac{qp^-}{2}} &\le \Big(\frac{1}{|\Omega_{4R}(x_0)|}\Big)^{\frac{qp^+}{2}}\bigg(\int_\Omega|Du|^2dx\bigg)^{\frac{qp^+}{2}} \\ &\le C_{16}\Big(\frac{1}{|\Omega_{4R}(x_0)|}\Big)^{\frac{qp^+}{2}}\bigg(\int_\Omega|\mathbf{f}|^2\,dx + \int_\Omega|D\psi|^2\,dx\bigg)^{\frac{qp^+}{2}} \\ &\le C_{17}\Big(\frac{1}{|\Omega_{4R}(x_0)|}\Big)^{\frac{qp^+}{2}}\bigg(\int_\Omega|\mathbf{f}|^{\frac{2p(x)}{\gamma_1}}\,dx + \int_\Omega|D\psi|^{\frac{2p(x)}{\gamma_1}}\,dx + |\Omega|\bigg)^{\frac{qp^+}{2}} \\ &\le C_{18}\Big(\frac{1}{|\Omega_{4R}(x_0)|}\Big)^{\frac{qp^+}{2}}\bigg(\||\mathbf{f}|^{\frac{2p(x)}{\gamma_1}}\|_{L^{(\frac{\gamma\gamma_1}{2},\frac{q\gamma_1}{2})}(\Omega)} + \||D\psi|^{\frac{2p(x)}{\gamma_1}}\|_{L^{(\frac{\gamma\gamma_1}{2},\frac{q\gamma_1}{2})}(\Omega)} + 1\bigg)^{\frac{qp^+}{2}} \\ &= C_{18}\Big(\frac{1}{|\Omega_{4R}(x_0)|}\Big)^{\frac{qp^+}{2}}\bigg(\||\mathbf{f}|^{p(x)}\|^{\frac{2}{\gamma_1}}_{L^{(\gamma,q)}(\Omega)} + \||D\psi|^{p(x)}\|^{\frac{2}{\gamma_1}}_{L^{(\gamma,q)}(\Omega)} + 1\bigg)^{\frac{q\gamma_1}{2}\frac{p^+}{\gamma_1}} \\ &\le C_{19}\Big(\frac{1}{|\Omega_{4R}(x_0)|}\Big)^{\frac{qp^+}{2}}\bigg(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)} + \||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)} + 1\bigg)^{\frac{\gamma_2}{\gamma_1}}. \qquad (4.29)\end{aligned}$$

In a similar way as above, we also get the following estimate. In fact, again use (4.27) for the embedding theory and employ the fact that $0 < R < 1$ to obtain

$$
\begin{aligned}
&\left(\fint_{\Omega_{4R}(x_0)}\left(|\mathbf{f}|^2+|D\psi|^2\right)^{\frac{p^+}{p^-}}dx\right)^{\frac{qp^-}{2}}\\
\le\ & C_{20}\left(\fint_{\Omega_{4R}(x_0)}\left(|\mathbf{f}|^{\frac{2p(x)}{p^-}}+|D\psi|^{\frac{2p(x)}{p^-}}\right)^{\frac{p^+}{p^-}}dx+1\right)^{\frac{qp^-}{2}}\\
\le\ & C_{21}\left(\frac{1}{|\Omega_{4R}(x_0)|}\right)^{\frac{qp^-}{2}}\left(\||\mathbf{f}|^{\frac{2p(x)}{p^-}}\|_{L^{\frac{p^+}{p^-}}(\Omega)}+\||D\psi|^{\frac{2p(x)}{p^-}}\|_{L^{\frac{p^+}{p^-}}(\Omega)}+1\right)^{\frac{p^+}{p^-}\frac{qp^-}{2}}\\
\le\ & C_{22}\left(\frac{1}{|\Omega_{4R}(x_0)|}\right)^{\frac{qp^+}{2}}\left(\||\mathbf{f}|^{\frac{2p(x)}{p^-}}\|_{L^{(\frac{\gamma p^-}{2},\frac{qp^-}{2})}(\Omega)}+\||D\psi|^{\frac{2p(x)}{p^-}}\|_{L^{(\frac{\gamma p^-}{2},\frac{qp^-}{2})}(\Omega)}+1\right)^{\frac{qp^+}{2}}\\
\le\ & C_{23}\left(\frac{1}{|\Omega_{4R}(x_0)|}\right)^{\frac{qp^+}{2}}\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}}. \qquad (4.30)
\end{aligned}
$$

Putting (4.29) and (4.30) into (4.28), it deduces

$$
\left(\fint_{\Omega_{2R}(x_0)}|Du|^{\frac{2p(x)}{p^-}}dx\right)^{\frac{qp^-}{2}} \le C_{24}\left(\frac{1}{|\Omega_{4R}(x_0)|}\right)^{\frac{qp^+}{2}}\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}}. \qquad (4.31)
$$

Now we are in a position to estimate the second item on the right-hand side of (4.26). Since $\eta = 1+\epsilon$ for sufficiently small $\epsilon > 0$ as the limit of (4.7), we have $2\eta < \gamma p^-$. Then, for any $\gamma \in [1,\infty)$ by the embedding inequality it follows that

$$
\begin{aligned}
&\left(\fint_{\Omega_{2R}(x_0)}(|\mathbf{f}|^{\frac{2p(x)}{p^-}}+|D\psi|^{\frac{2p(x)}{p^-}}+1)^{\eta}dx\right)^{\frac{qp^-}{2\eta}}\\
\le\ & C_{25}\Big(\frac{1}{|\Omega_{2R}(x_0)|}\Big)^{\frac{qp^-}{2\eta}}\left(\||\mathbf{f}|^{p(x)}\|_{L^{\frac{2\eta}{p^-}}(\Omega_{2R}(x_0))}+\||D\psi|^{p(x)}\|_{L^{\frac{2\eta}{p^-}}(\Omega_{2R}(x_0))}+1\right)^{q}\\
\le\ & C_{25}\Big(\frac{1}{|\Omega_{2R}(x_0)|}\Big)^{\frac{qp^+}{2}}\left(\||\mathbf{f}|^{p(x)}\|_{L^{\frac{2\eta}{p^-}}(\Omega)}+\||D\psi|^{p(x)}\|_{L^{\frac{2\eta}{p^-}}(\Omega)}+1\right)^{q}\\
\le\ & C_{26}\Big(\frac{1}{|\Omega_{2R}(x_0)|}\Big)^{\frac{qp^+}{2}}\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+1\right). \qquad (4.32)
\end{aligned}
$$

Putting (4.31) and (4.32) into (4.26), we have

$$
\begin{aligned}
&\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_R(x_0))}\\
\le\ & C_{27}\left(|\Omega_R(x_0)|^{\frac{1}{\gamma}-\frac{\gamma_2}{2}}\right)^q\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}}. \qquad (4.33)
\end{aligned}
$$

The rest of Step 1 is to use the standard finite covering argument to obtain the global estimate. In fact, since $\overline{\Omega}$ is compactness in $\mathbb{R}^d$, there exist finitely many points $x_0^k \in \overline{\Omega}$, $k = 1,2,\cdots,N$ and the corresponding R_k such that $\Omega = \bigcup_{k=1}^N \Omega_{R_k}(x_0^k)$. Therefore,

$$
\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)} \le \sum_{k=1}^N \||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega_{R_k}(x_0^k))}.
$$

Now, thanks to the estimate (4.33) it yields

$$\begin{aligned}
\||Du|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)} &\le C_{27}\sum_{k=1}^N\left(|\Omega_{R_k}(x_0^k)|^{\frac{1}{\gamma}-\frac{\gamma_2}{2}}\right)^q\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}}\\
&\le C_{28}\left(\||\mathbf{f}|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|^q_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}},
\end{aligned}$$

where C_{28} is a constant depending only on $d,\nu,\Lambda,\omega(\cdot),\gamma,q,\delta,R_0$ and $|\Omega|$. The proof of $q=\infty$ is even simple. Here for briefness we omit it, which may also refer to Section 5.4 in [6] for the case of $q=\infty$.

Step 2. The remainder of our proof for Theorem 2.5 is to remove this assumption $|Du|^{p(x)}\in L^{(\gamma,q)}(\Omega)$ via an approximation procedure. To do this, let $\{|\mathbf{f}_k|^{p(x)}\}_{k=1}^\infty$ and $\{|D\psi_k|^{p(x)}\}_{k=1}^\infty$ be two sequences in $C_0^\infty(\Omega)$ converging to $|\mathbf{f}|^{p(x)}$ and $|D\psi|^{p(x)}$ in $L^{(\gamma,q)}(\Omega)$. It is clear that $|\mathbf{f}_k|$ and $|D\psi_k|\in L^{(\gamma\gamma_2,q\gamma_2)}(\Omega)$. According to the earlier work [9] and the facts that the constant Lorentz space is a interpolation space of Lebesgue spaces, and the obstacle problems under the considering is linear, then the unique weak solution

$$u_k\in\mathcal{A}_k=\{\phi_k\in W_0^{1,2}(\Omega):\phi_k\ge\psi_k \text{ a.e. in } \Omega\}$$

of the following variational inequalities

$$\int_\Omega A(x)\,Du_k\,D(\phi_k-u_k)dx\ge\int_\Omega \mathbf{f}_k\,D(\phi_k-u_k)dx\quad\text{for all}\quad \phi_k\in\mathcal{A}$$

satisfies a global gradient estimate in $L^{(\gamma\gamma_2,q\gamma_2)}(\Omega)$ under the assumption that $(A(x),\Omega)$ is (δ,R_0)-vanishing of codimension one. Thus, we have

$$|Du_k|\in L^{(\gamma\gamma_2,q\gamma_2)}(\Omega)\Longrightarrow |Du_k|^{p(x)}\in L^{(\gamma,q)}(\Omega),$$

due to (3.3). As a consequence of the interpolation space, we have

$$\begin{aligned}
\||Du_k|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)} &\le C\left(\||\mathbf{f}_k|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}+\||D\psi_k|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}}\\
&\le C\left(\||\mathbf{f}|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}},
\end{aligned}$$

where C is independent of k. From this estimate we observe that there exists $\bar{u}$ with $|D\bar{u}|^{p(x)}\in L^{(\gamma,q)}(\Omega)$ which is the weak limit of $\{u_k\}_{k=1}^\infty$ in $\mathcal{A}_k$ such that

$$\||D\bar{u}|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}\le C\left(\||\mathbf{f}|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}+\||D\psi|^{p(x)}\|_{L^{(\gamma,q)}(\Omega)}+1\right)^{\frac{\gamma_2}{\gamma_1}}.$$

Then it is easy to check that this $\bar{u}$ is the weak solution of the original problem (1.1). So by the uniqueness, we conclude that $u=\bar{u}$ almost everywhere in Ω. This completes the approximation procedure. □

5 Conclusions

This paper extends the classical Calderón-Zygmund theory to the refined estimates in the Lorentz spaces for variable exponent power of the gradient of weak solutions for elliptic obstacle problems. We would like to remark that the usual harmonic analysis like the Calderón-Zygmund operator, the maximal function operator and the sharp maximal function operator are not suitable for our estimates due to $p(x)$ being a variable function. Instead, our argument is motivated from the so-called maximal function free technique. Two things are deserved to be mentioned. One is our minimal regular assumptions which is concerned with elliptic obstacle problems (1.1) with partially BMO coefficients over the bounded non-smooth domain. Another is our refined conclusion that we show a regularity in Lorentz spaces for the variable exponent powers of the gradients of its weak solution. To the best knowledge of the authors of this paper, this is the first time in the category of the Lorentz spaces to consider the regularity of variable exponent powers of the gradient of weak solution for variational inequalities under the weakest conditions on coefficients and boundaries. We would also like to point out that there are a few of difficulties to deal with regularity of the gradient in Lorentz spaces with variable exponents to weak solutions of variational inequalities (1.1). We believe that our work here have independent interests.

Acknowledgement
The authors express their thanks to the referee for their useful comments. This work is supported by the National Natural Science Foundation of China (No. 11371050) and NSFC-ERC (No. 11611530539). The second author was also supported by ERCEA Advanced Grant 2014 669689-HADE, by the MINECO project MTM2014-53850-P, by Basque Government project IT-641-13 and also by the Basque Government through the BERC 2014-2017 program and by Spanish Ministry of Economy and Competitiveness MINECO: B-CAM Severo Ochoa excellence accreditation SEV-2013-0323.

Funding
Not applicable

Competing interests
The author declare that they have no competing interests.

Author's contributions
The authors contributed equally to the writing of this paper. The authors read and approved the final manuscript.

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