A C^0 INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD FOR FOURTH ORDER TOTAL VARIATION FLOW

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ABSTRACT. We consider the numerical solution of a fourth order total variation flow problem representing surface relaxation below the roughening temperature. Based on a regularization and scaling of the nonlinear fourth order parabolic equation, we perform an implicit discretization in time and a C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) discretization in space. We prove existence and uniqueness of a solution of the C^0IPDG approximation by a nonlinear analogue of the Lax-Milgram Lemma. This requires to show that the nonlinear operator associated with the C^0IPDG semilinear form is Lipschitz continuous and strongly monotone on bounded subsets of the underlying finite element space. The fully discrete problem can be interpreted as a parameter dependent nonlinear system with the discrete time as a parameter. It is solved by a predictor corrector continuation strategy featuring an adaptive choice of the time step sizes. A documentation of numerical results is provided illustrating the performance of the C^0IPDG method and the predictor corrector continuation strategy.

1. INTRODUCTION

Surface relaxation by surface diffusion is about the relaxation of a high symmetry crystalline surface on which a particular profile has been imprinted such that the typical length scale of the imposed profile is much larger than the lattice constant (dimension of unit cells in the crystal lattice). Therefore, surface relaxation is an important process in material sciences, in particular in the production of nanotechnology devices. The problem is to understand along which route the initial profile relaxes to a completely flat surface. One distinguishes between relaxation above and below the roughening temperature. Below the roughening temperature, the surface free energy has a cusp singularity. Several authors have suggested to model the dynamics by a total variation H^{-1} flow problem that can be formulated as a fourth order total variation flow (TVF) problem (cf., e.g., [10, 18, 25, 26, 31, 32, 34, 35]).

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Given a bounded domain $\hat{\Omega} \subset \mathbb{R}^2$ with boundary $\hat{\Gamma} = \partial\hat{\Omega}$, the total variation-$H^{-1}$ (TV-H^{-1}) minimization of the energy functional

$$E(w) = \beta \int_{\hat{\Omega}} |\nabla w| \; dx, \quad \beta > 0, \tag{1.1}$$

leads to the following fourth order total variation flow (TVF) problem

$$\frac{\partial w}{\partial \hat{t}} + \beta \Delta \nabla \cdot \frac{\nabla w}{|\nabla w|} = 0 \quad \text{in } \hat{Q} := \hat{\Omega} \times (0, \hat{T}), \tag{1.2a}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta \frac{\nabla w}{|\nabla w|} = \mathbf{n}_{\hat{\Gamma}} \cdot \nabla \nabla \Big(\nabla \cdot \frac{\nabla w}{|\nabla w|} \Big) = 0 \quad \text{on } \hat{\Sigma} := \hat{\Gamma} \times (0, \hat{T}), \tag{1.2b}$$

$$w(\cdot, 0) = w^0 \quad \text{in } \hat{\Omega}, \tag{1.2c}$$

where $\beta > 0$ is related to the mobility, $\hat{T} > 0$ is the final time, $\mathbf{n}_{\hat{\Gamma}}$ stands for the exterior unit normal at $\hat{\Gamma}$, and $w^0 \in L^2(\hat{\Omega})$ is some given initial data. The fourth order equation (1.2a) has to be interpreted as follows: On $H^{-1}(\hat{\Omega})$ we introduce an inner product according to

$$(w, z)_{-1,\hat{\Omega}} := (\nabla(-\Delta^{-1} w), \nabla(-\Delta^{-1} z)_{0,\hat{\Omega}},$$

where Δ^{-1} stands for the inverse of the Laplacian. For $E(w) = \beta \int_{\hat{\Omega}} |\nabla w| \; dx, w \in H^{-1}(\hat{\Omega})$, with $D(E) = \{ w \in H^{-1}(\hat{\Omega}) \mid E(w) < \infty \}$, the subdifferential

$$\partial_{H^{-1}} E(w) = \{ v \in H^{-1}(\hat{\Omega}) \mid (v, z - w)_{-1,\Omega} \le E(z) - E(w) \text{ for all } z \in H^{-1}(\hat{\Omega}) \}$$

is given by (cf., e.g., [22])

$$\partial_{H^{-1}} E(w) = \{ \Delta \nabla \cdot \boldsymbol{\xi} \mid \boldsymbol{\xi}(\hat{x}) \in \partial\Phi(\nabla w(\hat{x})) \},$$

where $\Phi(|\boldsymbol{\eta}|)$ and $\partial\Phi(|\boldsymbol{\eta}|)$ are given by

$$\Phi(\boldsymbol{\eta}) = \lambda^{-1} |\boldsymbol{\eta}|, \quad \partial\Phi(\boldsymbol{\eta}) = \left\{ \begin{array}{r} \beta\boldsymbol{\eta}/|\boldsymbol{\eta}| \text{ , if } \boldsymbol{\eta} \neq \mathbf{0} \\ \{ \boldsymbol{\tau} \in \mathbb{R}^2 \mid |\boldsymbol{\tau}| \le \beta \} \text{ , if } \boldsymbol{\eta} = \mathbf{0} \end{array} \right. . \tag{1.3}$$

We thus obtain

$$-\frac{\partial w}{\partial \hat{t}} \in \partial E_{H^{-1}}(w).$$

Initial-boundary value problems for fourth order TVF problems have been considered mainly from an analytical point of view (cf., e.g., [15, 16, 22, 23, 24].
Here, we consider the regularized TV-H^{-1} energy functional

$$E_{reg}(w) = \beta \int_{\hat{\Omega}} (\delta^2 + |\nabla w|^2)^{1/2} \quad w \in H^{-1}(\hat{\Omega}),$$

where $\delta > 0$ is a regularization parameter. This leads to the regularized fourth order TVF problem

$$\frac{\partial w}{\partial \hat{t}} + \beta \Delta \nabla \cdot (\delta^2 + |\nabla w|^2)^{-1/2} \nabla w) = 0 \quad \text{in } \hat{Q}, \tag{1.4a}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta (\delta^2 + |\nabla w|^2)^{-1/2} \nabla w) = \; 0 \quad \text{on } \hat{\Sigma}, \tag{1.4b}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta \nabla \Big(\nabla \cdot (\delta^2 + |\nabla w|^2)^{-1/2} \nabla w \Big) = 0 \quad \text{on } \hat{\Sigma},$$

$$w(\cdot, 0) = w^0 \quad \text{in } \hat{\Omega}. \tag{1.4c}$$

We further consider a scaling in both the time variable and the spatial variables according to

$$t = \delta \hat{t}, \quad x_i = \delta \hat{x}_i,\ 1 \le i \le 2. \tag{1.5}$$

Setting $T := \delta \hat{T}, \Omega := \delta \hat{\Omega}, \Gamma := \partial\Omega, Q := \Omega \times (0,T), \Sigma := \Gamma \times (0,T)$, and $u^0(x) = w^0(\delta^{-1}x)$, as well as

$$\omega(\boldsymbol{\nabla} u) := 1 + |\boldsymbol{\nabla} u|^2, \tag{1.6}$$

the scaled regularized fourth order TVF problem reads as follows

$$\frac{\partial u}{\partial t} + \beta\delta^2 \Delta \boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u)^{-1/2} \boldsymbol{\nabla} u) = 0 \quad \text{in } Q, \tag{1.7a}$$

$$\mathbf{n}_\Gamma \cdot \beta\delta^2 (\omega(\boldsymbol{\nabla} u)^{-1/2} \boldsymbol{\nabla} u) = \mathbf{n}_\Gamma \cdot \beta\delta^2 \boldsymbol{\nabla} \Big(\boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u)^{-1/2} \boldsymbol{\nabla} u) \Big) = 0 \quad \text{on } \Sigma, \tag{1.7b}$$

$$u(\cdot, 0) = u^0 \quad \text{in } \Omega. \tag{1.7c}$$

The numerical solution of the regularized fourth order TVF problem with periodic boundary conditions has been considered in [25] based on a mixed formulation of the implicitly in time discretized problem. At each time-step, this amounts to the solution of two second order elliptic PDEs by standard Lagrangian finite elements with respect to a triangulation of the computational domain Ω. On the other hand, Interior Penalty Discontinuous Galerkin (IPDG) methods for fourth order elliptic boundary value problems, fourth order and higher order polyharmonic parabolic initial-boundary value problems have been studied in [2, 3, 4, 8, 13, **?**, 14, 20, 21, 28, 29, 36, 41]. The advantage of the C^{0}IPDG approach is that it directly applies to the fourth order problem and thus only requires the numerical solution of one equation by using the same Lagrangian finite elements as in the mixed method.

Remark 1.1. *We note that another example for a TV-H^{-1} minimization problem is the minimization of the energy functional*

$$E(w, \hat{g}) = \int_{\hat{\Omega}} |\boldsymbol{\nabla} w| + \frac{\lambda}{2} \|w - \hat{g}\|^2_{H^{-1}(\hat{\Omega})}$$

which occurs in image recovery where w represents a true image, $\hat{g}$ describes a blurred and/or noisy image, and $\lambda > 0$ is a fidelity parameter (cf. [27, 30, 38, 39]*). The associated fourth order total variation flow (TVF) problem is given by the initial-boundary value problem*

$$\frac{\partial w}{\partial \hat{t}} + \lambda^{-1} \Delta \boldsymbol{\nabla} \cdot \frac{\boldsymbol{\nabla} w}{|\boldsymbol{\nabla} w|} + w - \hat{g} = 0 \quad \textit{in } \hat{Q}, \tag{1.8a}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \frac{\boldsymbol{\nabla} w}{|\nabla w|} = \mathbf{n}_{\hat{\Gamma}} \cdot \boldsymbol{\nabla} \Big(\boldsymbol{\nabla} \cdot \frac{\boldsymbol{\nabla} w}{|\boldsymbol{\nabla} w|} \Big) = 0 \quad \textit{on } \hat{\Sigma}, \tag{1.8b}$$

$$w(\cdot, 0) = w^0 \quad \textit{in } \hat{\Omega}. \tag{1.8c}$$

The paper is organized as follows: In section 2, we will perform a discretization in time of the regularized and scaled fourth order TVF problem and consider a reformulation in terms of the matrix of second order partial derivatives of the unknown. Section 3 is devoted to the derivation of the C^{0}IPDG approximation based on an appropriate choice of numerical flux functions. In section 4, we will prove the existence and uniqueness of a solution of the C^{0}IPDG approximation by an application of the nonlinear analogue of the Lax-Milgram Lemma. In particular, this requires to prove that the nonlinear operator associated with the C^{0}IPDG semilinear form is Lipschitz continuous and strongly monotone on bounded subsets of the underlying finite element space. Since the solution exhibits steep gradients at narrow interfaces between upper and lower facets developing around local extrema of the initial data, the appropriate choice of the time steps is a crucial issue for the convergence of Newton's method as a solver for the fully discrete system. In section 5, we will show that the fully discrete system can be written as a parameter dependent nonlinear system with the discrete time as a parameter. We suggest a predictor corrector continuation strategy with constant continuation as a prdictor and Newton's method as a corrector featuring an adaptive choice of the time step sizes. Finally, in section 6 we present numerical results illustrating the performance of the C^{0}IPDG method and the predictor corrector continuation strategy.

Throughout the paper we will use the following notations and basic results. For vectors $\underline{\mathbf{x}} = (x_1, \cdots, x_n)^T, \underline{\mathbf{y}} = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$ and for matrices $\underline{\underline{\mathbf{A}}} = (a_{ij})_{i,j=1}^n, \underline{\underline{\mathbf{B}}} = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ we denote by $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$ and $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}}$ the Euclidean inner product $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}} = \sum_{i=1}^n x_i y_i$ and the Frobenius inner product $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = \sum_{i,j=1}^n a_{ij} b_{ij}$. In particular, $|\underline{\mathbf{x}}| := (\underline{\mathbf{x}} \cdot \underline{\mathbf{x}})^{1/2}$ and $|\underline{\underline{\mathbf{A}}}| := (\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{A}}})^{1/2}$ refer to the Euclidean norm and the Frobenius norm, respectively. If $\underline{\underline{\mathbf{A}}} \in \mathbb{R}^{n\times n}$ is symmetric positive definite with eigenvalues $\lambda_i(\underline{\underline{\mathbf{A}}}), 1 \le i \le n$ and $\lambda_{max}(\underline{\underline{\mathbf{A}}}) := \max\limits_{1\le i\le n} \lambda_i(\underline{\underline{\mathbf{A}}})$, $\lambda_{min}(\underline{\underline{\mathbf{A}}}) := \min\limits_{1\le i\le n} \lambda_i(\underline{\underline{\mathbf{A}}})$ and if $\underline{\underline{\mathbf{B}}}, \underline{\underline{\mathbf{C}}} \in \mathbb{R}^{n\times n}$, then it holds (cf., e.g., [17])

$$(\underline{\underline{\mathbf{A}}} \cdot \underline{\underline{\mathbf{B}}}) : \underline{\underline{\mathbf{B}}} \ge \lambda_{min}(\underline{\underline{\mathbf{A}}})\ |\underline{\underline{\mathbf{B}}}|^2, \tag{1.9a}$$

$$|(\underline{\underline{\mathbf{A}}} \cdot \underline{\underline{\mathbf{B}}}) : \underline{\underline{\mathbf{C}}}| \le \lambda_{max}(\underline{\underline{\mathbf{A}}})\ |\underline{\underline{\mathbf{B}}}||\underline{\underline{\mathbf{C}}}|, \tag{1.9b}$$

$$\lambda_{max}(\underline{\underline{\mathbf{A}}}) \le |\underline{\underline{\mathbf{A}}}| \le \sqrt{n}\ \lambda_{max}(\underline{\underline{\mathbf{A}}}). \tag{1.9c}$$

We will further use standard notation from Lebesgue and Sobolev space theory (cf., e.g., [37]). In particular, for a bounded domain $D \subset \mathbb{R}^d, d \in \mathbb{N}$, we refer to $L^p(D), 1 \le p < \infty$, as the Banach space of p-th power Lebesgue integrable functions on D with norm $\|\cdot\|_{0,p,D}$ and to $L^\infty(D)$ as the Banach space of essentially bounded functions on D with norm $\|\cdot\|_{0,\infty,D}$. Moreover, we denote by $W^{s,p}(D), s \in \mathbb{R}_+, 1 \le p \le \infty$, the Sobolev spaces with norms $\|\cdot\|_{s,p,D}$. We note that for $p = 2$ the spaces $L^2(D)$ and $W^{s,2}(D) = H^s(D)$ are Hilbert spaces with inner products $(\cdot,\cdot)_{0,2,D}$ and $(\cdot,\cdot)_{s,2,D}$. In the sequel, we will suppress the subindex 2 and write $(\cdot,\cdot)_{0,D}, (\cdot,\cdot)_{s,D}$ and $\|\cdot\|_{0,D}, \|\cdot\|_{s,D}$ instead of $(\cdot,\cdot)_{0,2,D}, (\cdot,\cdot)_{s,2,D}$ and $\|\cdot\|_{0,2,D}, \|\cdot\|_{s,2,D}$. The space $W_0^{s,p}(D)$ is the closure of C_0^∞ with respect to the $\|\cdot\|_{s,p,D}$-norm. We refer to $W^{-s,p}(D), s \in \mathbb{R}_+, 1 \le p \le \infty$, as the dual of $W_0^{s,q}(D)$, where $1/p + 1/q = 1$. In particular, $H^{-s}(D) = (H_0^s(D))^*$.

2. Implicit time-discretization

For the numerical solution of the regularized fourth order TVF problem (1.7) we perform a discretization in time with respect to a partition of the time interval $[0, T]$ into subintervals $[t_{m-1}, t_m]$ of length $\tau_m := t_m - t_{m-1}$. Denoting by u^m some approximation of u at time t_m, for $1 \le m \le M$ we have to solve the problems

$$u^m - u^{m-1} + \tau_m \beta \delta^2 \Delta \boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = 0 \text{ in } \Omega, \tag{2.1a}$$

(2.1b)

$$\mathbf{n}_\Gamma \cdot \beta\delta^2 (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = \mathbf{n}_\Gamma \cdot \beta\delta^2 \boldsymbol{\nabla}\Big(\boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m)\Big) = 0 \text{ on } \Gamma.$$

Introducing the objective functional

$$J(v) := \frac{1}{2}\|v - u^{m-1}\|^2_{-1,\Omega} + \tau_m \beta\delta^2 \int_\Omega (1 + |\boldsymbol{\nabla} v|^2)^{1/2}\ dx, \tag{2.2}$$

it is easy to see that (2.1) is related to the necessary and sufficient optimality condition for the minimization problem

$$J(u^m) = \inf_{v \in H^{-1}(\Omega)} J(v), \tag{2.3}$$

which has a unique solution, since the objective functional J is strictly convex, coercive, and lower semicontinuous.
The fourth order equation (2.1a) can be reformulated in terms of the 2×2 matrix

$$D^2 u^m = \begin{pmatrix} \frac{\partial^2 u^m}{\partial x_1^2} & \frac{\partial^2 u^m}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u^m}{\partial x_1 \partial x_2} & \frac{\partial^2 u^m}{\partial x_2^2} \end{pmatrix}.$$

of second partial derivatives of u^m. We note that the divergence of a matrix-valued function $\underline{\underline{\mathbf{q}}} = (q_{ij})_{i,j=1}^2$ with row vectors $\underline{\mathbf{q}}^{(i)} = (q_{i1}, q_{i2})^T, 1 \le i \le 2$, is defined by means of

$$\boldsymbol{\nabla} \cdot \underline{\underline{\mathbf{q}}} := (\boldsymbol{\nabla} \cdot \underline{\mathbf{q}}^{(1)}, \boldsymbol{\nabla} \cdot \underline{\mathbf{q}}^{(2)})^T. \tag{2.4}$$

Theorem 2.1. *The fourth order equation* (2.1a) *is equivalent to*

$$u^m - u^{m-1} + \tau_m \beta\delta^2 \nabla \cdot \nabla \cdot (\omega(\nabla u^m)^{-3/2} \underline{\underline{\mathbf{M}}}(u^m) D^2 u^m) = 0, \tag{2.5}$$

where $\underline{\underline{\mathbf{M}}}(v)$ *stands for the matrix-valued function*

$$\underline{\underline{\mathbf{M}}}(v) := \begin{pmatrix} 1 + (\frac{\partial v}{\partial x_2})^2 & -\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} \\ -\frac{\partial v}{\partial x_1}\frac{\partial v}{\partial x_2} & 1 + (\frac{\partial v}{\partial x_2})^2 \end{pmatrix}. \tag{2.6}$$

Proof. We reformulate the second term on the left-hand side of (2.1a) according to

$$\Delta \boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \Big(\boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) \Big) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}(\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m). \tag{2.7}$$

Obviously, we have

$$\boldsymbol{\nabla}(\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \Big(\omega(\boldsymbol{\nabla} u^m)^{-1/2} \begin{pmatrix} \frac{\partial u^m}{\partial x_1} \\ \frac{\partial u^m}{\partial x_2} \end{pmatrix} \Big). \tag{2.8}$$

In particular,

$$\frac{\partial}{\partial x_1}\Big(\omega(\boldsymbol{\nabla} u^m)^{-1/2}\begin{pmatrix}\frac{\partial u^m}{\partial x_1}\\ \\ \frac{\partial u^m}{\partial x_2}\end{pmatrix}\Big) = \tag{2.9}$$

$$-\,\omega(\boldsymbol{\nabla} u^m)^{-3/2}(\frac{\partial u^m}{\partial x_1}\frac{\partial^2 u^m}{\partial x_1^2}+\frac{\partial u^m}{\partial x_2}\frac{\partial^2 u^m}{\partial x_1\partial x_2})\begin{pmatrix}\frac{\partial u^m}{\partial x_1}\\ \\ \frac{\partial u^m}{\partial x_2}\end{pmatrix} +$$

$$\omega(\boldsymbol{\nabla} u^m)^{-1/2}\begin{pmatrix}\frac{\partial^2 u^m}{\partial x_1^2}\\ \\ \frac{\partial^2 u^m}{\partial x_1\partial x_2}\end{pmatrix} =$$

$$\omega(\boldsymbol{\nabla} u^m)^{-3/2}\begin{pmatrix}(1+(\frac{\partial u^m}{\partial x_2})^2\frac{\partial^2 u^m}{\partial x_1^2}-\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2}\frac{\partial^2 u^m}{\partial x_1\partial x_2}\\ \\ (1+(\frac{\partial u^m}{\partial x_1})^2\frac{\partial^2 u^m}{\partial x_1\partial x_2}-\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2}\frac{\partial^2 u^m}{\partial x_1^2}\end{pmatrix},$$

and

$$\frac{\partial}{\partial x_2}\Big(\omega(\boldsymbol{\nabla} u^m)^{-1/2}\begin{pmatrix}\frac{\partial u^m}{\partial x_1}\\ \\ \frac{\partial u^m}{\partial x_2}\end{pmatrix}\Big) = \tag{2.10}$$

$$-\,\omega(\boldsymbol{\nabla} u^m)^{-3/2}(\frac{\partial u^m}{\partial x_1}\frac{\partial^2 u^m}{\partial x_1\partial x_2}+\frac{\partial u^m}{\partial x_2}\frac{\partial^2 u^m}{\partial x_2^2})\begin{pmatrix}\frac{\partial u^m}{\partial x_1}\\ \\ \frac{\partial u^m}{\partial x_2}\end{pmatrix} +$$

$$\omega(\boldsymbol{\nabla} u^m)^{-1/2}\begin{pmatrix}\frac{\partial^2 u^m}{\partial x_1\partial x_2}\\ \\ \frac{\partial^2 u^m}{\partial x_2^2}\end{pmatrix} =$$

$$\omega(\boldsymbol{\nabla} u^m)^{-3/2}\begin{pmatrix}(1+(\frac{\partial u^m}{\partial x_2})^2\frac{\partial^2 u^m}{\partial x_1\partial x_2}-\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2}\frac{\partial^2 u^m}{\partial x_1^2}\\ \\ (1+(\frac{\partial u^m}{\partial x_1})^2\frac{\partial^2 u^m}{\partial x_2^2}-\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2}\frac{\partial^2 u^m}{\partial x_1\partial x_2}\end{pmatrix}.$$

Using (2.9) and (2.10) in (2.8), it follows that

$$\boldsymbol{\nabla}(\omega(\boldsymbol{\nabla} u^m)^{-1/2}\nabla u^m) = \tag{2.11}$$

$$\omega(\boldsymbol{\nabla} u^m)^{-3/2}\begin{pmatrix}1+(\frac{\partial u^m}{\partial x_2})^2 & -\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2}\\ \\ -\frac{\partial u^m}{\partial x_1}\frac{\partial u^m}{\partial x_2} & 1+(\frac{\partial u^m}{\partial x_2})^2\end{pmatrix} D^2 u^m,$$

which can be written as

$$\boldsymbol{\nabla}(\omega(\boldsymbol{\nabla} u^m)^{-1/2}\nabla u^m) = \omega(\boldsymbol{\nabla} u^m)^{-3/2}\underline{\underline{\mathbf{M}}}(u^m)D^2 u^m. \tag{2.12}$$

□

Remark 2.1. *The matrix $\underline{\underline{\mathbf{M}}}(v)$ is symmetric positive definite with the eigenvalues*

$$\lambda_{min}(\underline{\underline{\mathbf{M}}}(v)) = 1, \quad \lambda_{max}(\underline{\underline{\mathbf{M}}}(v)) = 1 + |\boldsymbol{\nabla} v|^2. \tag{2.13}$$

For notational convenience we set

$$\underline{\underline{\mathbf{A}}}_1(v) := \omega(\boldsymbol{\nabla} v)^{-3/2}\underline{\underline{\mathbf{M}}}(v). \tag{2.14}$$

The weak formulation of (2.5) reads: Find

$$u^m \in V := \{v \in H^2(\Omega) \mid \mathbf{n}_\Gamma \cdot \beta\delta^2\omega(\boldsymbol{\nabla} v)^{-1/2}\boldsymbol{\nabla} v = 0 \text{ on } \Gamma\}$$

such that for all $v \in V$ it holds

$$(u^m - u^{m-1}, v)_{0,\Omega} + \tau_m\beta\delta^2 \int\limits_\Omega \left(\underline{\underline{\mathbf{A}}}_1(u^m)D^2u^m\right) : D^2v \, dx = 0. \tag{2.15}$$

Finally, we provide a mixed formulation of (2.5), because the derivation of the C^{0}IPDG method will be based on the discrete analogue of that mixed formulation. Introducing the matrix-valued function

$$\underline{\underline{\mathbf{p}}}^m := \omega(\boldsymbol{\nabla} u^m)^{-1/4}D^2u^m, \tag{2.16}$$

and the matrix

$$\underline{\underline{\mathbf{A}}}_2(v) := \omega(\boldsymbol{\nabla} v)^{-5/4}\underline{\underline{\mathbf{M}}}(v). \tag{2.17}$$

the mixed formulation of (2.1a),(2.1b) reads as follows

$$\underline{\underline{\mathbf{p}}}^m - \omega(\boldsymbol{\nabla} u^m)^{-1/4}D^2u^m = 0 \text{ in } \Omega, \tag{2.18a}$$

$$u^m - u^{m-1} + \tau_m\beta\delta^2\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{A}}}_2(u^m)\underline{\underline{\mathbf{p}}}(u^m) = 0 \text{ in } \Omega, \tag{2.18b}$$

$$\mathbf{n}_\Gamma \cdot \beta\delta^2\omega(\boldsymbol{\nabla} u^m)^{-1/2}\boldsymbol{\nabla} u^m = 0 \text{ on } \Gamma, \tag{2.18c}$$

$$\mathbf{n}_\Gamma \cdot \beta\delta^2\boldsymbol{\nabla}\cdot\underline{\underline{\mathbf{A}}}_2(u^m)\underline{\underline{\mathbf{p}}}^m = 0 \text{ on } \Gamma. \tag{2.18d}$$

3. C^0 Interior Penalty Discontinuous Galerkin approximation

Let $\mathcal{T}_h$ be a geometrically conforming, uniform simplicial triangulation of Ω. We denote by $\mathcal{E}_h^\Omega$ and $\mathcal{E}_h^\Gamma$ the set of edges of $\mathcal{T}_h$ in the interior of Ω and on the boundary Γ, respectively, and set $\mathcal{E}_h := \mathcal{E}_h^\Omega \cup \mathcal{E}_h^\Gamma$. For $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we denote by h_K and h_E the diameter of K and the length of E, and we set $h := \max(h_K \mid K \in \mathcal{T}_h)$. Due to the assumptions on $\mathcal{T}_h$ there exist constants $0 < c_R \le C_R$, $0 < c_Q \le C_Q$, and $0 < c_S \le C_S$ such that for all $K \in \mathcal{T}_h$ it holds

$$c_Rh_K \le h_E \le C_Rh_K, \quad E \in \mathcal{E}_h(\partial K), \tag{3.1a}$$

$$c_Qh \le h_K \le C_Qh, \tag{3.1b}$$

$$c_Sh_K^2 \le \text{meas}(K) \le C_Sh_K^2. \tag{3.1c}$$

For two quantities A and B we write $A \lesssim B$, if there exists a constant $C > 0$ independent of h such that $A \le CB$.

Denoting by $P_k(T), k \in \mathbb{N}$, the linear space of polynomials of degree $\leq k$ on T, for $k \in \mathbb{N}$ we define

$$V_h := \{v_h \in C^0(\bar{\Omega}) \mid v_h|_T \in P_k(T),\ T \in \mathcal{T}_h\}, \tag{3.2}$$

(3.3)

and note that $V_h \subset H^1(\Omega)$, but $V_h \not\subset H^2(\Omega)$. Further, we introduce

$$\underline{\underline{\mathbf{M}}}_h := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega)^{2\times 2} \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2\times 2},\ K \in \mathcal{T}_h\} \tag{3.4}$$

as the space of element-wise polynomial moment tensors.
For interior edges $E \in \mathcal{E}_h^\Omega$ such that $E = K_+ \cap K_-, K_\pm \in \mathcal{T}_h$ and boundary edges on Γ we introduce the average and jump of ∇v_h according to

$$\{\nabla v_h\}_E := \begin{cases} \frac{1}{2}\Big(\nabla v_h|_{E\cap K_+} + \nabla v_h|_{E\cap K_-}\Big)\ ,\ E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E\ ,\ E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{3.5a}$$

$$[\nabla v_h]_E := \begin{cases} \nabla v_h|_{E\cap K_+} - \nabla v_h|_{E\cap K_-}\ ,\ E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E\ ,\ E \in \mathcal{E}_h(\Gamma) \end{cases}. \tag{3.5b}$$

The average $\{\Delta v_h\}_E$ and jump $[\Delta v_h]_E$ are defined analogously. We further denote by $\mathbf{n}_E$ the unit normal vector on E pointing in the direction from K_+ to K_-. In the sequel, we will frequently use

$$\begin{aligned} |\{v_h w_h\}_E| &\leq \frac{1}{2}(|v_h|_{E_+}|w_h|_{E_+} + |v_h|_{E_-}|w_h|_{E_-}) \leq 2\{|v_h|\}_E\{|w_h|\}_E, \\ |[v_h w_h]_E| &\leq (|v_h|_{E_+}|w_h|_{E_+} + |v_h|_{E_-}|w_h|_{E_-}) \leq 4\{|v_h|\}_E\{|w_h|\}_E. \end{aligned} \tag{3.6}$$

and

$$[v_h w_h]_E = \{v_h\}_E[w_h]_E + [v_h]_E\{w_h\}_E b. \tag{3.7}$$

In order to motivate the C^{0}IPDG approximation we will follow the approach taken in [1] for second order elliptic boundary value problems. For $\underline{\underline{\mathbf{p}}}_h^m \in \underline{\underline{\mathbf{M}}}_h$ and $u_h^m \in V_h$ we consider (2.18a),(2.18b) elementwise, i.e.,

$$\underline{\underline{\mathbf{p}}}_h^m - \omega(\boldsymbol{\nabla} u_h^m)^{-1/4} D^2 u_h^m = 0, \tag{3.8a}$$

$$u_h^m - u_h^{m-1} + \tau_m \beta\delta^2 \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\cdot \underline{\underline{\mathbf{A}}}_2(u_h^m)\underline{\underline{\mathbf{p}}}_h^m = 0, \tag{3.8b}$$

in $K \in \mathcal{T}_h$ with $u_h^0 = Q_h u^0$, where $Q_h : L^2(\Omega) \to V_h$ denotes the L^2 projection onto V_h. We multiply (3.8a) by $\underline{\underline{\mathbf{q}}}_h \in \underline{\underline{\mathbf{M}}}_h$ and integrate over K:

$$\int_K \underline{\underline{\mathbf{p}}}_h^m : \underline{\underline{\mathbf{q}}}_h\ dx = \int_K (\omega(\boldsymbol{\nabla} u_h^m)^{-1/4} D^2 u_h^m) : \underline{\underline{\mathbf{q}}}_h\ dx. \tag{3.9}$$

We note that for two matrix-valued functions $\underline{\underline{\mathbf{p}}} = (p_{ij})_{i,j=1}^2$ and $\underline{\underline{\mathbf{q}}} = (q_{ij})_{i,j=1}^2$

$$\underline{\underline{\mathbf{p}}} : \underline{\underline{\mathbf{q}}} = \sum_{i,j=1}^2 p_{ij} q_{ij}.$$

In view of (2.11) Green's formula yields

(3.10)
$$\int_K (\omega(\nabla u_h^m)^{-1/4} D^2 u_h^m) : \underline{\underline{\mathbf{q}}}_h \ dx = \int_K D^2 u_h^m : (\omega(\nabla u_h^m)^{-1/4} \underline{\underline{\mathbf{q}}}_h) \ dx \ =$$
$$= -\int_K \nabla u_h^m \cdot \nabla \cdot (\omega(\nabla u_h^m)^{-1/4} \underline{\underline{\mathbf{q}}}_h) \ dx + \int_{\partial K} \omega(\nabla u_h^m)^{-1/4} \nabla u_h^m \cdot \underline{\underline{\mathbf{q}}}_h \mathbf{n}_{\partial K} \ ds.$$

On the other hand, we multiply (3.8b) by $v_h \in V_h$ and integrate over K:

$$\int_K (u_h^m - u_h^{m-1}) v_h \ dx + \tau_m \beta \delta^2 \int_K \nabla \cdot \nabla \cdot \underline{\underline{\mathbf{A}}}_2(u_h^m) \underline{\underline{\mathbf{p}}}_h^m v_h \ dx = 0. \tag{3.11}$$

Applying Green's formula twice, we obtain

$$\begin{aligned} \int_K \nabla \cdot \nabla \cdot \underline{\underline{\mathbf{A}}}_2(u_h^m) \underline{\underline{\mathbf{p}}}_h^m v_h \ dx = -\int_K \nabla \cdot \underline{\underline{\mathbf{A}}}_2(u_h^m) \underline{\underline{\mathbf{p}}}_h^m \cdot \nabla v_h \ dx \ + \\ \int_{\partial K} \mathbf{n}_{\partial K} \cdot \nabla \cdot \underline{\underline{\mathbf{A}}}_2(u_h^m) \underline{\underline{\mathbf{p}}}_h^m v_h \ ds = \int_K \underline{\underline{\mathbf{p}}}_h^m : \underline{\underline{\mathbf{A}}}_2(u_h^m) D^2 v_h) \ dx \ - \\ \int_{\partial K} \underline{\underline{\mathbf{A}}}_2(u_h^m) \underline{\underline{\mathbf{p}}}_h^m \mathbf{n}_{\partial k} \cdot \nabla v_h \ ds + \int_{\partial K} \mathbf{n}_{\partial K} \cdot \nabla \cdot \underline{\underline{\mathbf{A}}}_2(u_h^m) \underline{\underline{\mathbf{p}}}_h^m v_h \ ds. \end{aligned} \tag{3.12}$$

Summing over all $K \in \mathcal{T}_h$ in (3.10) and (3.12), we obtain the weak formulation of the mixed formulation (3.8a),(3.8b).
A general C^{0}DG approximation of (2.1b),(2.1b) is based on the weak formulation of the mixed formulation (3.8a),(3.8b) and characterized by numerical flux functions $\hat{\underline{\mathbf{u}}}_{\partial K}^m$ and $\hat{\underline{\mathbf{p}}}_{\partial K}^m$. We are looking for a pair $(u_h^m, \underline{\underline{\mathbf{p}}}_h^m) \in V_h \times \underline{\underline{\mathbf{M}}}_h$ such that for all $(v_h, \underline{\underline{\mathbf{q}}}_h) \in V_h \times \underline{\underline{\mathbf{M}}}_h$ it holds

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K \underline{\underline{\mathbf{p}}}_h^m : \underline{\underline{\mathbf{q}}}_h \ dx + \sum_{K \in \mathcal{T}_h} \int_K \nabla u_h^m \cdot \nabla \cdot (\omega(\nabla u_h^m)^{-1/4} \underline{\underline{\mathbf{q}}}_h) \ dx \ - \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{\underline{\mathbf{u}}}_{\partial K}^m \cdot \underline{\underline{\mathbf{q}}}_h \mathbf{n}_{\partial K} \ ds = 0, \end{aligned} \tag{3.13a}$$

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_K u_h^m v_h \ dx + \tau_m \beta \delta^2 \Big(\sum_{K \in \mathcal{T}_h} \int_K \underline{\underline{\mathbf{p}}}_h^m : \underline{\underline{\mathbf{A}}}_2(u_h^m) D^2 v_h \ dx \ - \\ \sum_{K \in \mathcal{T}_h} \int_{\partial K} \hat{\underline{\mathbf{p}}}_{\partial K}^m \cdot (\omega(\nabla u_h^m)^{-1/4} \nabla v_h \ ds \Big) = \sum_{K \in \mathcal{T}_h} \int_K u_h^{m-1} v_h \ dx. \end{aligned} \tag{3.13b}$$

In particular, for the C^{0}IPDG approximation the numerical flux functions $\hat{\underline{\mathbf{u}}}_{\partial K}^m$ and $\hat{\underline{\mathbf{p}}}_{\partial K}^m$ are given by

$$\hat{\underline{\mathbf{u}}}_{\partial K}^m|_E := \begin{cases} \{\omega(\nabla u_h^m)^{-1/4} \nabla u_h^m\}_E \ , \ E \in \mathcal{E}_h^\Omega \\ \mathbf{0} \ , \ E \in \mathcal{E}_h^\Gamma \end{cases} , \tag{3.14a}$$

$$\hat{\underline{\mathbf{p}}}_{\partial K}^m := \{\underline{\underline{\mathbf{A}}}_2(u_h^m) D^2 u_h^m\}_E - \alpha h_E^{-1} [\omega(\nabla u_h^m)^{-1/4} \nabla u_h^m]_E, \tag{3.14b}$$

(3.14c)

where $\alpha > 0$ is a penalty parameter. This particular choice of the numerical flux functions allows to eliminate $\underline{\underline{\mathbf{p}}}_h^m$ from the system (3.13a),(3.13b). In fact, if we choose $\underline{\underline{\mathbf{q}}}_h = \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2 v_h$ in (3.13a) and observe (2.12), we obtain

$$
\begin{aligned}
(3.15)\qquad & \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\nabla} u_h^m \cdot \boldsymbol{\nabla}\cdot(\omega(\boldsymbol{\nabla} u_h^m)^{-1/4}\underline{\underline{\mathbf{q}}}_h)\ dx \ = \\
& - \sum_{K\in\mathcal{T}_h}\int_K D^2 u_h^m : \underline{\underline{\mathbf{A}}}_1(u_h^m)D^2 v_h\ dx\ + \\
& \sum_{K\in\mathcal{T}_h}\int_{\partial K} \omega(\boldsymbol{\nabla} u_h^m)^{-1/4}\boldsymbol{\nabla} u_h^m\cdot \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2 v_h \mathbf{n}_{\partial K}\ ds \ = \\
& - \sum_{K\in\mathcal{T}_h}\int_K \Big(\underline{\underline{\mathbf{A}}}_1(u_h^m)D^2 u_h^m\Big) : D^2 v_h\ dx\ + \\
& \sum_{E\in\mathcal{E}_h^\Omega}\int_E \Big((\omega(\boldsymbol{\nabla} u_h^m)^{-1/4}\boldsymbol{\nabla} u_h^m)|_{E\cap K_+}\cdot \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2 v_h|_{E\cap K_+}\mathbf{n}_{E\cap K_+}\ + \\
& \qquad (\omega(\boldsymbol{\nabla} u_h^m)^{-1/4}\boldsymbol{\nabla} u_h^m)|_{E\cap K_-}\cdot \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2 v_h|_{E\cap K_+}\mathbf{n}_{E\cap K_-}\Big)\ ds\ + \\
& \sum_{E\in\mathcal{E}_h^\Gamma}\int_E (\omega(\boldsymbol{\nabla} u_h^m)^{-1/4}\boldsymbol{\nabla} u_h^m)|_E\cdot \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2 v_h|_E\mathbf{n}_E\ ds.
\end{aligned}
$$

Hence, in view of (3.14a) we find

$$
\begin{aligned}
(3.16)\qquad & \sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\nabla} u_h^m\cdot\boldsymbol{\nabla}\cdot \underline{\underline{\mathbf{A}}}_1(u_h^m)D^2 v_h\ dx\ - \\
& \sum_{K\in\mathcal{T}_h}\int_{\partial K} \hat{\underline{\mathbf{u}}}_{\partial K}^m\cdot D^2 v_h\mathbf{n}_{\partial K}\ ds\ = \\
& - \sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{A}}}_1(u_h^m)D^2 u_h^m : D^2 v_h\ dx\ + \\
& \sum_{E\in\mathcal{E}_h}\int_E [\omega(\boldsymbol{\nabla} u_h^m)^{-1/4}\boldsymbol{\nabla} u_h^m]_E\cdot\{\underline{\underline{\mathbf{A}}}_2(u_h^m)D^2 v_h\}_E\mathbf{n}_E\ ds.
\end{aligned}
$$

On the other hand, taking (3.14b) into account and using

$$
\int_E \Big((\underline{\mathbf{p}}\cdot\underline{\mathbf{q}})|_{E\cap K_+} + (\underline{\mathbf{p}}\cdot\underline{\mathbf{q}})|_{E\cap K_-}\Big)\ ds = \int_E \Big(\{\underline{\mathbf{p}}\}_E\cdot[\underline{\mathbf{q}}]_E + [\underline{\mathbf{p}}]_E\cdot\{\underline{\mathbf{q}}\}_E\Big)\ ds.
$$

it follows that

$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h^m : \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2v_h\ dx - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \hat{\underline{\mathbf{p}}}_{\partial K}^m\cdot\boldsymbol{\nabla}v_h\ ds\ = \tag{3.17}$$
$$\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{p}}}_h^m : \underline{\underline{\mathbf{A}}}_2(u_h^m)D^2v_h\ dx -$$
$$\sum_{E\in\mathcal{E}_h}\int_E \{\underline{\underline{\mathbf{A}}}_2(u_h^m)D^2u_h^m\}_E\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}u_h^m)^{-1/4}\boldsymbol{\nabla}v_h]_E\ ds$$
$$+\,\alpha\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E [\omega(\boldsymbol{\nabla}u_h^m)^{-1/4}\boldsymbol{\nabla}u_h^m]_E\cdot[\omega(\boldsymbol{\nabla}u_h^m)^{-1/4}\boldsymbol{\nabla}v_h]_E\ ds.$$

For $z\in V+V_h$ and $u,v\in\hat{V}+V_h$, where $\hat{V}:=V\cap H^{2+\gamma}(\Omega), 1/2<\gamma\le 1$, we introduce the mesh-dependent C^0IPDG form $a_h^{IP}(\cdot,\cdot;z):(\hat{V}+V_h)\times(\hat{V}+V_h)$ according to

$$a_h^{IP}(u,v;z) := \sum_{K\in\mathcal{T}_h}(\underline{\underline{\mathbf{A}}}_1(z)D^2u,D^2v)_{0,K} - \tag{3.18}$$
$$\sum_{E\in\mathcal{E}_h}(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(z)D^2u\}_E\mathbf{n}_E,\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}z)^{-1/4}\boldsymbol{\nabla}v]_E)_{0,E} -$$
$$\sum_{E\in\mathcal{E}_h}(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(z)D^2v\}_E\mathbf{n}_E,\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}z)^{-1/4}\boldsymbol{\nabla}u]_E)_{0,E} +$$
$$\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}(\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}z)^{-1/4}\boldsymbol{\nabla}u]_E,\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}z)^{-1/4}\boldsymbol{\nabla}v]_E)_{0,E}.$$

The C^0IPDG method for the approximation of (2.1a),(2.1b) requires the computation of $u_h^m\in V_h$ such that for all $v_h\in V_h$ it holds

$$(u_h^m,v_h)_{0,\Omega}+\tau_m\beta\delta^2a_h^{IP}(u_h^m,v_h;u_h^m)=(u_h^{m-1},v_h)_{0,\Omega},\quad v_h\in V_h. \tag{3.19}$$

4. Existence and uniqueness of a solution of the C^{0}IPDG approximation

The existence and uniqueness of a solution of the C^{0}IPDG approximation (3.19) can be shown using the following nonlinear analogue of the Lax-Milgram Lemma.

Theorem 4.1. *Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$ and associated norm $\|\cdot\|_V$ and let V^* be the dual space with norm $\|\cdot\|_{V^*}$. We denote by $\langle\cdot,\cdot\rangle_{V^*,V}$ the dual pairing between V^* and V. Let $A:V\to V^*$ be a nonlinear operator that is Lipschitz continuous on $B(0,R):=\{v\in V\mid \|v\|_V\le R\}, R>0$, i.e., there exists a constant $\Gamma(R)>0$ such that for all $v,w\in V$ it holds*

$$\|Av-Aw\|_{V_h^*}\le\Gamma(R)\ \|v_h-w_h\|_V. \tag{4.1}$$

Moreover, assume that $A:V\to V^*$ is strongly monotone, i.e., there exists a constant $\gamma>0$ such that for all $v,w\in V$ it holds

$$\langle Av-Aw,v-w\rangle_{V^*,V}\ge\gamma\ \|v-w\|_V^2. \tag{4.2}$$

Then, for any $\ell\in V^*$ the nonlinear equation

$$Au=\ell \tag{4.3}$$

has a unique solution $u\in V$.

Proof. Although the result is well-known (cf., e.g., Chapter 5.3 in [11] and the references therein), for completeness we will provide a proof.
We refer to $\tau : V^* \to V$ as the Riesz mapping, i.e.,

$$\langle \ell, v\rangle_{V^*,V} = (\tau\ell, v)_V, \quad \ell \in V^*,\ v \in V. \tag{4.4}$$

Then, $u \in V$ is a solution of (4.3) if and only if u is a fixed point of the nonlinear map $T : V \to V$ defined by means of

$$Tv := v - \rho(\tau A v - \tau\ell), \quad v \in V,\ \rho > 0.$$

Due to (4.4) we have

$$\|Tv - Tw\|_V^2 = \|v - w\|_V^2 - 2\rho\langle Av - Aw, v - w\rangle_{V^*,V} + \rho^2\|Av - Aw\|_{V^*}^2.$$

Now, using (4.1) and (4.2) it follows that

$$\|Tv - Tw\|_V^2 \le q\|v - w\|_V^2, \quad q := 1 - 2\rho\gamma + \rho^2\Gamma(R)^2.$$

For $\rho \in (0, 2\gamma/\Gamma(R)^2)$ we have $q < 1$ and hence, T is a contraction on $B(0,R)$. The Banach fixed point theorem asserts the existence and uniqueness of a fixed point. □

In order to apply the previous result to the C^0IPDG method (3.19), we introduce a mesh-dependent semi-norm $|\cdot|_{2,h,\Omega}$ and weighted norm $\|\cdot\|_{2,h,\Omega}$ on V_h according to

$$|v_h|_{2,h,\Omega} := \Big(\sum_{K\in\mathcal{T}_h}\int_K D^2 v_h : D^2 v_h\ dx + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_{E\in\mathcal{E}_h} |\mathbf{n}_E\cdot[\boldsymbol{\nabla} v_h]_E|^2\ ds\Big)^{1/2}, \tag{4.5a}$$

$$\|v_h\|_{2,h,\Omega} := \Big(\|v_h\|_{0,\Omega}^2 + |v_h|_{2,h,\omega,\Omega}^2\Big)^{1/2}. \tag{4.5b}$$

We further note that (3.19) can be written as the nonlinear equation

$$A_h^{DG} u_h^m = \ell_h, \tag{4.6}$$

where the nonlinear operator $A_h^{DG} : V_h \to V_h^*$ and the linear functional $\ell_h \in V_h^*$ are given by

$$\langle A_h^{DG} v_h, w_h\rangle_{V_h^*,V_h} := (v_h, w_h)_{0,\Omega} + \tau_m\beta\delta^2\ a_h^{DG}(v_h, w_h; v_h), \quad v_h, w_h \in V_h, \tag{4.7}$$

$$\ell_h(v_h) := (u_h^{m-1}, w_h)_{0,\Omega}, \quad w_h \in V_h. \tag{4.8}$$

For the proof of Lipschitz continuity on bounded sets and strong monotonicity of the nonlinear operator A_h^{DG} we need the inverse estimates (cf., e.g., [7, 11]):
For $p \in [1,\infty]$ and $\ell, m \in \mathbb{N}_0$ it holds

$$\|v_h\|_{m,p,K} \le \frac{C_{inv}}{\text{meas}(K)^{max(0,\frac{1}{2}-\frac{1}{p})}\ h_K^{m-\ell}}\|v_h\|_{\ell,K}, \quad v_h \in V_h, \tag{4.9}$$

where C_{inv} is a positive constant that only depends on k, ℓ, m, p and the shape regularity of the triangulation. We further need the trace inequality (cf., e.g., [33, 40]): For $p \in [1,\infty]$, $m \in \mathbb{N}_0$, and $K \in \mathcal{T}_h$ it holds

$$\|\boldsymbol{\nabla} v_h\|_{m,p,\partial K} \le C_T h_K^{-1/p}\|\boldsymbol{\nabla} v_h\|_{m,p,K}, \quad v_h \in V_h, \tag{4.10}$$

where C_T is a positive constant that only depends on k, m, p and the shape regularity of the triangulation. Moreover, we will frequently use the following Poincaré-Friedrichs inequality for piecewise H^2-functions (cf., e.g., [9])

$$\|\nabla v_h\|_{0,\Omega} \le C_{PF} |v_h|_{2,h,\Omega}, \quad v_h \in V_h, \tag{4.11}$$

where $C_{PF} > 0$ is a constant that only depends on the shape regularity of the triangulation.

In the sequel, we will frequently use some basic estimates for the weight function $\omega(\nabla v_h)$. In particular, for $\beta > 0$ and $v \in \hat{V} + V_h$ it holds

$$\omega(\nabla v)^{-\beta} = (1 + |\nabla v|^2)^{-\beta} \le 1, \tag{4.12a}$$

$$\omega(\nabla v)^{-(\beta+1)} |\nabla v| \le \omega(\nabla v)^{-(\beta+1)} (1 + |\nabla v|^2)^{1/2} \le \le \omega(\nabla v)^{-(\beta+1/2)} \le 1. \tag{4.12b}$$

Moreover, for $v, w \in \hat{V} + V_h$ and $\xi(s) := w + s(v - w)$, $s \in [0,1]$, it holds

$$\omega(\nabla v)^{-\beta} - \omega(\nabla w)^{-\beta} = -2\beta \int_0^1 \omega(\nabla \xi(s))^{-\beta-1} \nabla \xi(s) \cdot \nabla (v - w) \, ds, \tag{4.13a}$$

$$\omega(\nabla v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\nabla w)^{-\beta} \underline{\underline{\mathbf{M}}}(w) = \int_0^1 \Big(\omega(\nabla \xi(s))^{-\beta} \underline{\underline{\mathbf{F}}}(\xi(s); v - w) \, ds - 2\beta \int_0^1 \omega(\nabla \xi(s))^{-\beta-1} \nabla \xi(s) \cdot \nabla (v - w) \underline{\underline{\mathbf{M}}}(\xi(s)) \Big) \, ds, \tag{4.13b}$$

where the matrix $\underline{\underline{\mathbf{F}}}(v; w)$, $v, w \in \hat{V} + V_h$ is given by

$$\underline{\underline{\mathbf{F}}}(v; w) := \begin{pmatrix} 2 \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_2} & \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} \\ \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} & 2 \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1} \end{pmatrix}, \quad , v, w \in V + V_h. \tag{4.14}$$

An easy computation yields

$$|\underline{\underline{\mathbf{F}}}(v; w)|^2 \le 5 \, |\nabla v|^2 \, |\nabla w|^2. \tag{4.15}$$

It follows from (4.12b) and (4.13a) that

$$|\omega(\nabla v)^{-\beta} - \omega(\nabla w)^{-\beta}| \le 2\beta |\nabla (v - w)|, \tag{4.16a}$$

whereas in view of (2.13),(4.12b),(4.13b), and (4.15) we have

$$|\omega(\nabla v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\nabla w)^{-\beta} \underline{\underline{\mathbf{M}}}(w)| \le (2\beta + \sqrt{5}) |\nabla (v - w)|, \tag{4.16b}$$

We will first show that the nonlinear operator A_h^{DG} is Lipschitz continuous on the ball

$$B_h(0, R) := \{v_h \in V_h \mid \|v_h\|_{2,h,\Omega} \le R\}. \tag{4.17}$$

Theorem 4.2. *The nonlinear operator A_h^{DG} is Lipschitz continuous on the ball $B_h(0, R)$. In particular, there exists $\Gamma(h, R) > 0$ such that*

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} \le \Gamma(h, R) \, \|v_h - w_h\|_{2,h,\Omega}. \tag{4.18}$$

Proof. For $v_h, w_h \in B_h(0,R)$ we set $\xi_h := v_h - w_h$. In view of the definition (4.7) of the nonlinear operator A_h^{DG} we have

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} = \sup_{\|z_h\|_{2,h,\Omega}\le 1} |\langle A_h^{DG} v_h - A_h^{DG} w_h, z_h\rangle_{V_h^*,V_h}| = \sup_{\|z_h\|_{2,h,\Omega}\le 1} |(\xi_h, z_h)_{0,\Omega} + \Delta t \beta \delta^2 \Big(a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h)\Big). \tag{4.19}$$

According to the definition (3.18) of the semilinear form $a_h^{DG}(\cdot,\cdot;\cdot)$ we find

$$\begin{aligned} a_h^{DG}(v_h,z_h;v_h) - a_h^{DG}(w_h,z_h;w_h) \ = \\ \sum_{K\in\mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h) D^2 w_h\Big) : D^2 z_h \ dx \\ - \sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} z_h]_E \ - \\ \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} z_h]_E\Big) \ ds \\ - \sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 z_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} v_h]_E \ - \\ \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 z_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} w_h]_E\Big) \ ds \\ + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \Big(\mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} v_h]_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} z_h]_E \ - \\ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} w_h]_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} z_h]_E\Big) \ ds. \end{aligned} \tag{4.20}$$

We will estimate the four terms on the right-hand side of (4.20) separately.

(i) For the first term on the right-hand side of (4.20) we obtain

$$\begin{aligned} \sum_{K\in\mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h) D^2 w_h\Big) : D^2 z_h \ dx \ = \\ \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{A}}}_1(v_h) D^2 \xi_h : D^2 z_h \ dx}_{= I_1} + \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)\Big) D^2 w_h : D^2 z_h \ dx}_{= I_2}. \end{aligned}$$

In view of (2.13) and (4.12a) and using Hölder's inequality and the Cauchy-Schwarz inequality, we get the following upper bound for I_1:

$$\begin{aligned} |I_1| \le \sum_{K\in\mathcal{T}_h} \int_K |D^2\xi_h||D^2 z_h| \ dx \ \le \\ \sum_{K\in\mathcal{T}_h} \Big(\int_K |D^2\xi_h|^2 \ dx\Big)^{1/2} \Big(\int_K |D^2 z_h|^2 \ dx\Big)^{1/2} \ \le \\ \Big(\sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2 \ dx\Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \ dx\Big)^{1/2}. \end{aligned} \tag{4.21}$$

Likewise, using (2.13),(3.1c),(4.12a),(4.15),(4.16b), the inverse inequality (4.9), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.11), we can estimate

I_2 from above as follows:

$$
\begin{aligned}
|I_2| \le & \sum_{K\in\mathcal{T}_h} \int_K |\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)||D^2 w_h||D^2 z_h| \; dx \; \le \\
& (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \int_K |\boldsymbol{\nabla}\xi_h||D^2 w_h||D^2 z_h| \; dx \; \le \\
& (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K} \Big(\int_K |D^2 w_h|^2 \; dx \Big)^{1/2} \Big(\int_K |D^2 z_h|^2 \; dx \Big)^{1/2} \; \le \\
& (3+\sqrt{5}) c_S^{-1/2} C_{inv} R \sum_{K\in\mathcal{T}_h} h_k^{-1} \|\boldsymbol{\nabla}\xi_h\|_{0,K} \|D^2 z_h\|_{0,K} \; \le \\
& (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} R h^{-1} \Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \Big)^{1/2} \\
& \le (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R h^{-1} |\xi_h|_{2,h,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \Big)^{1/2}.
\end{aligned}
$$

Hence, setting $C_A^{(1)} := (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R$, we thus have

$$
|I_2| \le C_A^{(1)} h^{-1} |\xi_h|_{2,h,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \Big)^{1/2}. \tag{4.22}
$$

(ii) Setting $\tilde{\omega}(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} w_h) := \omega(\boldsymbol{\nabla} v_h)^{-1/4} - \omega(\boldsymbol{\nabla} w_h)^{-1/4}$, the second term on the right-hand side of (4.20) can be written as

$$
\begin{aligned}
& \sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} z_h]_E \; - \\
& \qquad \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} z_h]_E \Big) \; ds \; = \\
& \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 \xi_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} z_h]_E \; ds}_{=\, II_1} \; + \\
& \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{(\underline{\underline{\mathbf{A}}}_2(v_h) - \underline{\underline{\mathbf{A}}}_2(w_h)) D^2 w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} z_h]_E \; ds}_{=\, II_2} \; + \\
& \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}(w_h) D^2 w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\tilde{\omega}(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} w_h) \boldsymbol{\nabla} z_h]_E \; ds}_{=\, II_3}.
\end{aligned}
$$

Setting $E_1 := E_+, E_2 := E_-$, for $E \in \mathcal{E}_h(\Omega)$, and using (3.1a),(4.12a), and the trace inequality (4.10), for the first term II_1 we find

$$
\begin{aligned}
&|II_1| \le \sum_{E\in\mathcal{E}_h} \int_E |\{D^2\xi_h\}_E||\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E| \ ds \ \le \\
&\frac{1}{2}\sum_{E\in\mathcal{E}_h(\Omega)} \int_E \sum_{i=1}^2 |D^2\xi_h|_{E_i}||\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E| \ ds + \sum_{E\in\mathcal{E}_h(\Gamma)} \int_E |D^2\xi_h|||\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E| \ ds \ \le \\
&\sum_{E\in\mathcal{E}_h(\Omega)} \sum_{i=1}^2 h_E^{1/2}\Big(\int_E |D^2\xi_h|_{E_i}|^2 \ ds\Big)^{1/2} h_E^{-1/2}\Big(\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2} + \\
&\sum_{E\in\mathcal{E}_h(\Gamma)} \Big(\int_E |D^2\xi_h|^2 \ ds\Big)^{1/2} h_E^{-1/2}\Big(\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2} \ \le \\
&C_R^{1/2}\Big(\sum_{K\in\mathcal{T}_h} h_K\|D^2\xi_h\|_{0,\partial K}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2} \ \le \\
&C_R^{1/2}C_T\Big(\sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2}.
\end{aligned}
$$

We thus have

$$
|II_1| \le C_A^{(2)}\Big(\sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2}, \tag{4.23}
$$

where $C_A^{(2)} := C_R^{1/2}C_T$. In a similar way, using (2.13)(3.1a)-(3.1c),(4.12a),(4.15),(4.16b), the trace inequality (4.10), the inverse inequality (4.9), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.11), the second term II_2 can be estimated from above according to

$$
\begin{aligned}
|II_2| \le (\frac{1}{2}+\sqrt{5})C_R^{1/2}&\Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K}^2 h_K\int_{\partial K} |D^2 w_h|^2 \ ds\Big)^{1/2} \\
&\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2} \ \le \\
(\frac{1}{2}+\sqrt{5})C_R^{1/2}C_T&\Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K}^2 \int_K |D^2 w_h|^2 \ ds\Big)^{1/2} \\
&\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2} \ \le \\
(\frac{1}{2}+\sqrt{5})c_Q^{-1}c_S^{-1}C_{inv}C_R^{1/2}C_T R h^{-1}&\Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,K}^2\Big)^{1/2} \\
&\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2} \ \le \\
\le C_A^{(3)}h^{-1}|\xi_h|_{2,h,\Omega}&\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 \ ds\Big)^{1/2},
\end{aligned} \tag{4.24}
$$

where $C_A^{(3)} := (\frac{1}{2} + \sqrt{5}) c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$. In a similar way, for II_3 we obtain

$$|II_3| \le C_A^{(4)} h^{-1} |z_h|_{2,h,\Omega} \Big(\sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla} \xi_h]_E|^2 \, ds \Big)^{1/2}, \tag{4.25}$$

where $C_A^{(4)} := 2 c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$.

(iii) For the third term on the right-hand side of (4.20) we have

$$\begin{aligned}
&\sum_{E \in \mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 z_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} v_h]_E \, - \\
&\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 z_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} w_h]_E \Big) \, ds \ = \\
&\underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{(\underline{\underline{\mathbf{A}}}_2(v_h) - \underline{\underline{\mathbf{A}}}_2(w_h)) D^2 z_h\}_E \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} v_h]_E \, ds}_{= III_1} \ + \\
&\underbrace{\sum_{E \in \mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 z_h\}_E \mathbf{n}_E \ \mathbf{n}_E \cdot [\tilde{\omega}(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} w_h) \boldsymbol{\nabla} v_h]_E \, ds}_{= III_2} .
\end{aligned}$$

Both terms III_1 and III_2 can be estimated from above in much the same way as the corresponding terms for II. We obtain

$$|III_1| \le C_A^{(5)} h^{-1} |\xi_h|_{2,h,\Omega} \Big(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 \, ds \Big)^{1/2}, \tag{4.26}$$

where $C_A^{(5)} := (\frac{1}{2} + \sqrt{5}) c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$, and

$$|III_2| \le C_A^{(6)} h^{-1} |\xi_h|_{2,h,\Omega} \Big(\sum_{K \in \mathcal{T}_h} \int_K |D^2 z_h|^2 \, ds \Big)^{1/2}, \tag{4.27}$$

where $C_A^{(6)} := c_Q^{-1} c_S^{-1} C_{inv} C_{PF} C_R^{1/2} C_T R$.

(iv) Finally, for the fourth term on the right-hand side of (4.20) we get

$$
\begin{aligned}
(4.28)\quad & \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \Big(\mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla v_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla z_h]_E - \\
& \qquad \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla w_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla z_h]_E\Big)\ ds\ = \\
& \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla \xi_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla z_h]_E\ ds}_{=\ IV_1}\ + \\
& \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla w_h]_E\ \mathbf{n}_E\cdot[\tilde\omega(\nabla v_h,\nabla w_h)\nabla z_h]_E\ ds}_{=\ IV_2}\ + \\
& \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E\cdot[\tilde\omega(\nabla v_h,\nabla v_h)\nabla w_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla z_h]_E\ ds}_{=\ IV_3}\,.
\end{aligned}
$$

Using (3.1a),(4.12a), the trace inequality (4.10), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.11), for IV_1 we obtain

$$
\begin{aligned}
|IV_1| \le\ & 4\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1/2} \int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E||\mathbf{n}_E\cdot[\nabla z_h]_E|\ ds\ \le \\
& 4\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1/2}\Big(\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2 ds\Big)^{1/2} h_E^{-1/2}\Big(\int_E |\mathbf{n}_E\cdot[\nabla z_h]_E|^2 ds\Big)^{1/2} \le \\
& 4\alpha\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2 ds\Big)^{1/2} h_E^{-1/2}\Big(\sum_{E\in\mathcal{E}_h}\int_E h_E^{-1}|\mathbf{n}_E\cdot[\nabla z_h]_E|^2 ds\Big)^{1/2}.
\end{aligned}
$$

Hence, it follows that

$$
\begin{aligned}
(4.29)\quad & \\
& |IV_1|\ \le \\
& C_A^{(7)}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla\xi_h]_E|^2 ds\Big)^{1/2} h_E^{-1/2}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla z_h]_E|^2 ds\Big)^{1/2},
\end{aligned}
$$

where $C_A^{(7)} := 4\alpha$. Setting $K_1 := K_+$ and $K_2 := K_-$ for $E\in\mathcal{E}_h(\Omega), E = K_+\cap K_-$, and $K_1 = K_2 = K$ for $E\in\mathcal{E}_h(\Gamma), E\in\mathcal{E}_h(K\cap\Gamma)$, the term IV_2 can be estimated from above as follows:

$$
\begin{aligned}
|IV_2| \le 4\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\nabla w_h]_E|\{|\nabla\xi_h|\}_E|\mathbf{n}_E\cdot[\nabla z_h]_E| \le 2\alpha\sum_{i=1}^{2}\|\nabla\xi_h\|_{0,\infty,K_i} \\
\sum_{E\in\mathcal{E}_h(\Omega)} \Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\nabla w_h]_E|^2 ds\Big)^{1/2}\Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\nabla z_h]_E|^2 ds\Big)^{1/2}.
\end{aligned}
$$

Using (3.1b),(3.1c), the inverse inequality (4.9), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.11), for IV_1, we have

$$\sum_{i=1}^{2} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K_i} \le c_R^{-1}c_S^{-1}C_{inv}h^{-1}\sum_{i=1}^{2}\|\boldsymbol{\nabla}\xi_h\|_{0,K_i} \le$$
$$2c_R^{-1}c_S^{-1}C_{inv}h^{-1}\|\|\boldsymbol{\nabla}\xi_h\|_{0,\Omega} \le 2c_R^{-1}c_S^{-1}C_{inv}C_{PF}h^{-1}|\xi_h|_{2,h,\Omega}.$$

Hence, observing $\Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\boldsymbol{\nabla}w_h]_E|^2 ds\Big)^{1/2} \le R$, we obtain

$$|IV_2| \le C_A^{(8)}h^{-1}|\xi_h|_{2,h,\Omega}\Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2 ds\Big)^{1/2}. \tag{4.30}$$

where $C_A^{(8)} := 4\alpha c_R^{-1}c_S^{-1}C_{inv}C_{PF}R$. In the same way we get

$$|IV_3| \le C_A^{(9)}h^{-1}|\xi_h|_{2,h,\Omega}\Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2 ds\Big)^{1/2}. \tag{4.31}$$

where $C_A^{(9)} := C_A^{(8)}$.

Setting $C_A := \sum_{i=1}^{9} C_A^{(i)}$, it follows from (4.19)-(4.31) that

$$|\langle A_h^{DG}v_h - A_h^{DG}w_h, z_h\rangle_{V_h^*,V_h}| \le \max(1,\tau_m\beta\delta^2 C_A h^{-1})\ \|\xi_h\|_{2,h,\Omega}\|z_h\|_{2,h,\Omega},$$

which implies (4.18) with $\Gamma(h,R) := \max(1,\tau_m\beta\delta^2 C_A h^{-1})$. □

Next, we will show that under some assumption in terms of the time step size τ_m and the regularization parameter δ the nonlinear operator A_h^{DG} is strongly monotone on bounded sets.

Theorem 4.3. *Under the assumption that there exist constants* $0 < \kappa \ll 1$ *and* $C_\Delta > 0$ *such that*

$$\tau_m\delta^2 \le C_\Delta h^{4+\kappa}, \tag{4.32}$$

for sufficiently small $0 < h < 1$ *there exists* $\gamma(h,R) > 0$ *such that for* $v_h, w_h \in B_h(0,R)$ *it holds*

$$\langle A_h^{DG}v_h - A_h^{DG}w_h, v_h - w_h\rangle_{V_h^*,V_h} \ge \gamma(h,R)\|v_h - w_h\|_{2,h,\Omega}^2. \tag{4.33}$$

Proof. For $v_h, w_h \in B_h(0,R)$ we set $\xi_h := v_h - w_h$. Taking the definition (4.7) of the nonlinear operator A_h^{DG} into account, we have

$$\langle A_h^{DG}v_h - A_h^{DG}w_h, \xi_h\rangle_{V_h^*,V_h} = \tag{4.34}$$
$$\|z_h\|_{0,\Omega}^2 + \tau_m\beta\delta^2\Big(a_h^{DG}(v_h,\xi_h;v_h) - a_h^{DG}(w_h,\xi_h;w_h)\Big).$$

Recalling the definitions (2.14),(2.17) of $\underline{\underline{\mathbf{A}}}_1$ and $\underline{\underline{\mathbf{A}}}_2$, for the second term on the right-hand side of (4.34) it follows that

$$
\begin{aligned}
(4.35)\quad & a_h^{DG}(v_h,\xi_h;v_h) - a_h^{DG}(w_h,\xi_h;w_h) \; = \\
& \sum_{K\in\mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h)D^2v_h - \underline{\underline{\mathbf{A}}}_1(w_h))D^2w_h\Big) : D^2\xi_h \; dx \\
& - \sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2v_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E \; - \\
& \qquad \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\Big) \; ds \\
& - \sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2\xi_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E \; - \\
& \qquad \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2\xi_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E\Big) \; ds \\
& + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \Big(\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E \; \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E \; - \\
& \qquad \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E \; \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\Big) \; ds.
\end{aligned}
$$

As in the previous theorem, we will estimate the four terms on the right-hand side of (4.35) separately.

(i) For the first term we obtain

$$
\begin{aligned}
& \sum_{K\in\mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h)D^2v_h - \underline{\underline{\mathbf{A}}}_1(w_h)D^2w_h\Big) : D^2\xi_h \; dx \; = \\
& \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h : D^2\xi_h \; dx}_{=\,I_1} \; + \\
& \underbrace{\sum_{K\in\mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)\Big)D^2w_h : D^2\xi_h \; dx}_{=\,I_2}\,.
\end{aligned}
$$

As far as I_1 is concerned, due to (2.13) we have

$$
\int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h : D^2\xi_h \; dx \ge (1+\|\boldsymbol{\nabla} v_h\|_{0,\infty,K}^2)^{-3/2}\|D^2\xi_h\|_{0,K}^2.
$$

Using (3.1a),(3.1c), the inverse inequality (4.9), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.11), and observing $\|v_h\|_{2,h,\Omega} \le R$, we get

$$
\begin{aligned}
& \|\boldsymbol{\nabla} v_h\|_{0,\infty,K}^2 \le c_S^{-2}C_{inv}^2h_K^{-2}\|\boldsymbol{\nabla} v_h\|_{0,K}^2 \le c_Q^2c_S^{-2}C_{inv}^2h^{-2}\|\boldsymbol{\nabla} v_h\|_{0,\Omega}^2 \; \le \\
& c_Q^2c_S^{-2}C_{inv}^2C_{PF}^2h^{-2}\|v_h\|_{2,h,\Omega}^2 \le \gamma_M^{(0)}h^{-2},
\end{aligned}
$$

where $\gamma_M^{(0)} := c_Q^2c_S^{-2}C_{inv}^2C_{PF}^2R^2$. Observing $h \le 1$, it follows that

$$
(1+\|\boldsymbol{\nabla} v_h\|_{0,\infty,K}^2)^{-3/2} \ge h^3(h^2+\gamma_M^{(0)})^{-3/2} \ge h^3(1+\gamma_M^{(0)})^{-3/2} = \gamma_M^{(1)}h^3,
$$

where $\gamma_M^{(1)} := (1+\gamma_M^{(0)})^{-3/2}$. Hence, we obtain the following lower bound for I_1:

$$|I_1| \geq \gamma_M^{(1)} h^3 \sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|_{0,K}^2. \tag{4.36}$$

In order to estimate I_2 from above, we use (2.13),(3.1b),(3.1c),(4.15),(4.16b), Hölder's inequality, the inverse inequality (4.9), the Cauchy-Schwarz inequality, and observe $\|D^2 w_h\|_{0,K} \leq (\sum_K \|D^2 w_h\|_{0,K}^2)^{1/2} \leq R$:

$$\begin{aligned}
&|I_2| \leq (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \int_K |\boldsymbol{\nabla}\xi_h||D^2 w_h||D^2\xi_h| \; dx \;\leq \\
&(3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K} \Big(\int_K |D^2 w_h|^2 \; dx\Big)^{1/2} \Big(\int_K |D^2\xi_h|^2 \; dx\Big)^{1/2} \;\leq \\
&(3+\sqrt{5}) c_S^{-1} C_{inv} \sum_{K\in\mathcal{T}_h} h_K^{-2} \|\xi_h\|_{0,K} \|D^2 w_h\|_{0,K} \|D^2\xi_h\|_{0,K} \;\leq \\
&(3+\sqrt{5}) c_S^{-1} C_{inv}^2 R \sum_{K\in\mathcal{T}_h} h_K^{-2} \|\xi_h\|_{0,K} h_K^{-2} \|\xi_h\|_{0,K} \;\leq \\
&(3+\sqrt{5}) c_S^{-1} c_R^{-4} C_{inv}^2 R h^{-4} \sum_{K\in\mathcal{T}_h} \|\xi_h\|_{0,K}^2.
\end{aligned}$$

Hence, it follows that

$$|I_2| \leq C_B^{(1)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \tag{4.37}$$

where $C_B^{(1)} := (3+\sqrt{5}) c_S^{-1} c_R^{-4} C_{inv}^2 R$.

(ii) We now deal with the second term on the right-hand side of (4.35) which we rewrite as follows:

$$\begin{aligned}
&\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \;- \\
&\qquad\qquad \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E\Big) \; ds \;= \\
&\underbrace{\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 \xi_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \; ds}_{=\,II_1} \;+ \\
&\underbrace{\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{(\underline{\underline{\mathbf{A}}}_2(v_h) - \underline{\underline{\mathbf{A}}}_2(w_h)) D^2 w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \; ds}_{=\,II_2} \\
&+ \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\tilde{\omega}(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} w_h) \boldsymbol{\nabla}\xi_h]_E \; ds}_{=\,II_3},
\end{aligned}$$

where $\tilde{\omega}(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} w_h) := \omega(\boldsymbol{\nabla} v_h)^{-1/4} - \omega(\boldsymbol{\nabla} w_h)^{-1/4}$. In view of (2.13),(3.1b),(3.1c), (3.6),(4.12), Hölder's inequality, the Cauchy-Schwarz inequality, the trace inequality

(4.10), and the inverse inequality (4.9), we can estimate II_1 from above as follows:

$$
\begin{aligned}
&|II_1| \le 8(\frac{5}{2}+\sqrt{5}) \sum_{E\in\mathcal{E}_h} \int\limits_E \{|D^2\xi_h|\}_E\{|\boldsymbol{\nabla}\xi_h|\}_E \ ds \ \le \\
&8 \sum_{E\in\mathcal{E}_h} \Big(\int\limits_E \{|D^2\xi_h|^2\}_E \ ds\Big)^{1/2}\Big(\int\limits_E \{|\boldsymbol{\nabla}\xi_h|^2\}_E \ ds\Big)^{1/2} \ \le \\
&8 \sum_{K\in\mathcal{T}_h} \Big(\int\limits_{\partial K} |D^2\xi_h|^2 \ ds\Big)^{1/2}\Big(\int\limits_{\partial} K|\boldsymbol{\nabla}\xi_h|^2 \ ds\Big)^{1/2} \ = \\
&8c_R^{-1}h^{-1} \sum_{K\in\mathcal{T}_h} h_K^{1/2}\|D^2\xi_h\|_{0,\partial K} h_K^{1/2}\|\boldsymbol{\nabla}\xi_h\|_{0,\partial K} \le \\
&8c_R^{-1}C_T^2h^{-1}\Big(\sum_{K\in\mathcal{T}_h}\|D^2\xi_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,K}^2\Big)^{1/2} \ \le \\
&8c_R^{-4}c_S^{-2}C_{inv}^2C_T^2h^{-4}\sum_{K\in\mathcal{T}_h}\|\xi_h\|_{0,K}^2 \le 8c_R^{-4}c_S^{-2}C_{inv}^2C_T^2h^{-4}\|\xi_h\|_{0,\Omega}^2.
\end{aligned}
$$

Hence, we obtain

$$
|II_1| \le C_B^{(2)}h^{-4}\|\xi_h\|_{0,\Omega}^2, \tag{4.38}
$$

where $C_B^{(2)} := 8c_R^{-4}c_S^{-2}C_{inv}^2C_T^2$. Likewise, for II_2 it follows that

$$
\begin{aligned}
&|II_2| \le 8(\frac{5}{2}+\sqrt{5}) \sum_{E\in\mathcal{E}_h} \int\limits_E \{|\boldsymbol{\nabla}\xi_h|^2\}_E\{|D^2w_h|\}_E \ ds \ \le \\
&8(\frac{5}{2}+\sqrt{5}) \sum_{E\in\mathcal{E}_h} \Big(\int\limits_E \{|\boldsymbol{\nabla}\xi_h|^4\}_E \ ds\Big)^{1/2}\Big(\int\limits_E \{|D^2w_h|^2\}_E \ ds\Big)^{1/2} \ \le \\
&8(\frac{5}{2}+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \Big(\int\limits_{\partial K} |\boldsymbol{\nabla}\xi_h|^4 \ ds\Big)^{1/2}\Big(\int\limits_{\partial} K|D^2w_h|^2 \ ds\Big)^{1/2} \ = \\
&8(\frac{5}{2}+\sqrt{5})c_R^{-3/2}h^{-3/2}\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{\nabla}\xi_h\|_{0,4,\partial K}^2 h_K^{1/2}\|D^2w_h\|_{0,\partial K}^2 \le \\
&8(\frac{5}{2}+\sqrt{5})c_R^{-3/2}C_T^2h^{-3/2}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,4,K}^2\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\|D^2w_h\|_{0,K}^2\Big)^{1/2} \ \le \\
&8(\frac{5}{2}+\sqrt{5})c_R^{-2}c_S^{-2}C_{inv}C_T^2Rh^{-2}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,K}^4\Big)^{1/2} \ \le \\
&8(\frac{5}{2}+\sqrt{5})c_R^{-4}c_S^{-2}C_{inv}^2C_T^2Rh^{-4}\Big(\sum_{K\in\mathcal{T}_h}\|\xi_h\|_{0,K}^2.
\end{aligned}
$$

We thus obtain

$$
|II_2| \le C_B^{(3)}h^{-4}\|\xi_h\|_{0,\Omega}^2, \tag{4.39}
$$

where $C_B^{(3)} := 8(\frac{5}{2}+\sqrt{5})c_R^{-4}c_S^{-2}C_{inv}^2C_T^2$. Finally, II_3 can be bounded from above in much the same way as II_2. We get

$$
|II_3| \le C_B^{(4)}h^{-4}\|\xi_h\|^2, \tag{4.40}
$$

where $C_B^{(4)} := 8(\frac{1}{2}+\sqrt{5})c_R^{-4}c_S^{-2}C_{inv}^2 C_T^2$.

(iii) For the third term on the right-hand side of (4.34) we have

$$\sum_{E\in\mathcal{E}_h}\int_E \Big(\mathbf{n}_E\cdot\{\omega(\nabla v_h)^{-5/4}\underline{\underline{\mathbf{M}}}(v_h)D^2\xi_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla v_h]_E\ -$$
$$\mathbf{n}_E\cdot\{\omega(\nabla w_h)^{-5/4}\underline{\underline{\mathbf{M}}}(w_h)D^2\xi_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla w_h]_E\Big)\,ds\ =$$
$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{(\underline{\underline{\mathbf{A}}}_2(v_h)-\underline{\underline{\mathbf{A}}}_2(w_h))D^2\xi_h\}_E\mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla v_h]_E ds\,+}_{=\,III_1}$$
$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2\xi_h\}_E\mathbf{n}_E\ \mathbf{n}_E\cdot[\tilde{\omega}(\nabla v_h,\nabla w_h)\nabla v_h]_E ds\,.}_{=\,III_2}$$

Both terms can be estimated from above in a similar way as the corresponding terms in II. We obtain

$$|III_1| \le C_B^{(5)}h^{-4}\|\xi_h\|_{0,\Omega}^2, \quad |III_2| \le C_B^{(6)}h^{-4}\|\xi_h\|_{0,\Omega}^2, \tag{4.41}$$

where $C_B^{(5)} = C_B^{(6)} := 8(\frac{1}{2}+\sqrt{5})c_R^{-4}C_{inv}^2 C_T^2 R$.

(iv) For the fourth term on the right-hand side of (4.34) we obtain

$$\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \Big(\mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla v_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E\ -$$
$$\mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla w_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla\xi_h]_E\Big)\,ds\ = \tag{4.42}$$
$$\underbrace{\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla v_h)^{-1/4}\nabla\xi_h]_E\ ds\ +}_{=\,IV_1}$$
$$\underbrace{\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla w_h]_E\ \mathbf{n}_E\cdot[\tilde{\omega}(\nabla v_h,\nabla w_h)\nabla\xi_h]_E\ ds\ +}_{=\,IV_2}$$
$$\underbrace{\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \mathbf{n}_E\cdot[\tilde{\omega}(\nabla v_h,\nabla w_h)\nabla w_h]_E\ \mathbf{n}_E\cdot[\omega(\nabla w_h)^{-1/4}\nabla\xi_h]_E\ ds\,.}_{=\,IV_3}$$

In view of (3.7), the first term IV_1 can be further split according to

$$IV_1 = \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot \{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E [\boldsymbol{\nabla}\eta_h]_E \ \mathbf{n}_E \cdot \{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E [\boldsymbol{\nabla}\xi_h]_E \ ds}_{=\ IV_{11}} +$$

$$\underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E \{\boldsymbol{\nabla}\xi_h\}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E \{\boldsymbol{\nabla}\xi_h\}_E \ ds}_{=\ IV_{12}} \ +$$

$$\underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E \{\boldsymbol{\nabla}\xi_h\}_E \ \mathbf{n}_E \cdot \{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E [\boldsymbol{\nabla}\xi_h]_E \ ds}_{=\ IV_{13}} \ +$$

$$\underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E \cdot \{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E [\boldsymbol{\nabla}\xi_h]_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E \{\boldsymbol{\nabla}\xi_h\}_E \ ds}_{=\ IV_{14}}.$$

For IV_{11} we have

$$\begin{aligned} IV_{11} \ge \alpha & \sum_{E\in\mathcal{E}_h(\Omega)} (1 + \frac{1}{2}\sum_{i=1}^2 \|\boldsymbol{\nabla} v_h\|^2_{0,\infty,E_i})^{-1/2} h_E^{-1} |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds + \\ \alpha & \sum_{E\in\mathcal{E}_h(\Gamma)} (1 + \|\boldsymbol{\nabla} v_h\|^2)^{-1/2} h_E^{-1} |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds. \end{aligned}$$

Taking advantage of (3.1b),(3.1c), the inverse inequality (4.9), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.11), it follows that for $E \in \mathcal{E}_h(\partial K)$ it holds

$$\begin{aligned} & \|\boldsymbol{\nabla} v_h\|_{0,\infty,E} \le \|\boldsymbol{\nabla} v_h\|_{0,\infty,K} \le c_S^{-1} C_{inv} h_K^{-1} \|\boldsymbol{\nabla} v_h\|_{0,K} \ \le \\ & c_R^{-1} c_S^{-1} C_{inv} h^{-1} \|\boldsymbol{\nabla} v_h\|_{0,\Omega} \le c_R^{-1} c_S^{-1} C_{inv} C_{PF} h^{-1} \|v_h\|_{2,h,\Omega} \le c_R^{-1} c_S^{-1} C_{inv} C_{PF} h^{-1}, \end{aligned}$$

and hence

$$\begin{aligned} & (1 + \|\boldsymbol{\nabla} v_h\|^2_{0,\infty,E})^{-1/2} \ge (1 + c_R^{-2} c_S^{-2} C_{inv}^2 C_{PF}^2 h^{-2})^{-1/2} \ = \\ & (h^2 + c_R^{-2} c_S^{-2} C_{inv}^2 C_{PF}^2)^{-1/2} h \ge (1 + c_R^{-2} c_S^{-2} C_{inv}^2 C_{PF}^2)^{-1/2} h. \end{aligned}$$

Consequently, we obtain

$$IV_{11} \ge \alpha \gamma_M^{(2)} h \sum_{E\in\mathcal{E}_h} |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds, \tag{4.43}$$

where $\gamma_M^{(2)} := \alpha(1 + c_R^{-2} c_S^{-2} C_{inv}^2 C_{PF}^2)^{-1/2}$.
The remaining terms $IV_{1i}, 2 \le i \le 4$, can be estimated from above similarly as the corresponding terms in Theorem 4.2:

(4.44)

$$|IV_{12}| \le C_B^{(6)} h^{-4} \|\xi_h\|^2_{0,\Omega}, \quad |IV_{13}| \le C_B^{(7)} h^{-4} \|\xi_h\|^2_{0,\Omega}, \quad |IV_{14}| \le C_B^{(8)} h^{-4} \|\xi_h\|^2_{0,\Omega},$$

where $C_B^{(6)} = C_B^{(7)} := 2\alpha c_R^{-4} c_S^{-2} C_{inv}^2 C_T^2$ and $C_B^{(8)} := 2C_B^{(6)}$. The remaining two terms IV_2 and IV_3 can be estimated from above in the same way. Using (3.1a)-(3.1c),(4.12), the inverse inequality (4.9), the trace inequality (4.10), and the Cauchy-Schwarz inequality, we obtain

$$|IV_2| \le 4\alpha c_Q^{-1/2} c_R^{-1/2} C_N^{(2)} h^{-1/2} \sum_{E\in\mathcal{E}_h} h_E^{-1/2} \int\limits_E |\mathbf{n}_E \cdot [w_h]_E| |\boldsymbol{\nabla}\xi_h|^2 \, ds \ \le$$

$$4\alpha c_Q^{-1/2} c_R^{-1/2} C_N^{(2)} h^{-1/2} \sum_{E\in\mathcal{E}_h} h_E^{-1/2} \Big(\int\limits_E |\mathbf{n}_E \cdot \boldsymbol{\nabla} w_h|^2 \, ds \Big)^{1/2} \Big(\int\limits_E |\boldsymbol{\nabla}\xi_h|^4 \, ds \Big)^{1/2} \ =$$

$$4\alpha c_Q^{-1/2} c_R^{-3/2} C_N^{(2)} h^{-3/2} \Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int\limits_E |\mathbf{n}_E \cdot \boldsymbol{\nabla} w_h|^2 \, ds \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} h_K^2 \|\boldsymbol{\nabla}\xi_h\|_{0,4,\partial K}^4 \Big)^{1/2}$$

$$\le 4\alpha c_Q^{-1/2} c_R^{-3/2} C_N^{(2)} C_T R h^{-3/2} \Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,4,K}^4 \Big)^{1/2} \ \le$$

$$4\alpha c_Q^{-1/2} c_R^{-7/2} C_N^{(2)} C_{inv}^2 C_T R h^{-7/2} \Big(\sum_{K\in\mathcal{T}_h} \|\xi_h\|_{0,K}^4 \Big)^{1/2} \ \le$$

$$4\alpha c_Q^{-1/2} c_R^{-7/2} C_N^{(2)} C_{inv}^2 C_T R h^{-7/2} \sum_{K\in\mathcal{T}_h} \|\xi_h\|_{0,K}^2.$$

Hence, it follows that

$$|IV_2| \le C_B^{(9)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \tag{4.45}$$

where $C_B^{(9)} := 4\alpha c_Q^{-1/2} c_R^{-7/2} C_N^{(2)} C_{inv}^2 C_T R$. Moreover, we get

$$|IV_3| \le C_B^{(10)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \tag{4.46}$$

where $C_B^{(10)} := C_B^{(9)}$.
Setting $C_B := \sum_{i=1}^{10} C_B^{(i)}$ and observing (4.32), it follows from (4.34)-(4.46) that

$$\begin{aligned} &\langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h \rangle_{V_h^*, V_h} \ \ge \\ &(1 - \beta C_\Delta C_B h^\kappa) \ \|\xi_h\|_{0,\Omega}^2 + \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h^3 \ |\xi_h|_{2,h,\Omega}^2. \end{aligned} \tag{4.47}$$

We choose $h_{min} > 0$ such that

$$q := \beta C_\Delta C_B h_{min}^\kappa < 1 \quad \text{and} \quad \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h_{min}^3 < 1 - q. \tag{4.48}$$

Then, for $h \le h_{min}$ (4.33) follows from (4.48) with

$$\gamma(h, R) := \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h^3. \tag{4.49}$$

□

Corollary 4.1. *Under the assumption* (4.32) *and for sufficiently small grid size* h, *the* C^0*IPDG approximation* (3.19) *has a unique solution* $u_h^m \in V_h$.

Proof. Using the Lipschitz continuity (4.19) and the strong monotonicity (4.33) of the nonlinear operator A_h^{DG}, the result follows from the nonlinear analogue of the Lax-Milgram Lemma (Theorem 4.1). □

Remark 4.1. *Under the assumption* (4.32), *for sufficiently small* h *we have* $\Gamma(h,R) = 1$ *in Theorem 4.2 and the application of Theorem 4.1 for* $V = V_h$ *and* $A = A_h^{DG}$ *implies that the fixed point operator* T *is a contraction as long as*

$$\rho < 2\ \frac{\gamma(h,R)}{\Gamma(h,R)^2} = 2\ min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)})\ h^3. \tag{4.50}$$

In other words, the contraction property degenerates for $h \to 0$. *This reflects the very singular character of the fourth order total variation flow problem.*

5. A predictor corrector continuation strategy for the numerical solution of the C^{0}IPDG approximation

The solution $u(x,t), (x,t) \in Q$, of the fourth order total variation flow problem (1.2) is characterized by

- the formation of facets around local extrema of the initial data with steep gradients at the interfaces,
- a finite extinction time $t_{ext} > 0$, i.e., $u(x,t) = 0, x \in \Omega$, for $t \geq t_{ext}$.

The same behavior can be expected from the solution of the C^{0}IPDG approximation (3.19). In particular, the appropriate choice of the time step is a crucial issue when solving the nonlinear system resulting from (3.19). Therefore, it is more advantageous to work with a variable time step $\tau_m = t_m - t_{m-1}, 1 \leq m \leq M$, instead of a uniform time step $\Delta t = T/M, M \in \mathbb{N}$ and to choose τ_m such that convergence of a Newton-type method is guaranteed. This can be achieved by viewing (3.19) as a parameter dependent nonlinear system with the time as the parameter and to apply a predictor corrector continuation strategy featuring an adaptive choice of the time step sizes τ_m (cf., e.g., [12, 19]).

We assume $V_h = \text{span}\{\varphi_1, \cdots, \varphi_{N_h}\}, N_h \in \mathbb{N}$, such that

$$u_h^m = \sum_{j=1}^{N_h} u_j^m \varphi_j.$$

Setting $\mathbf{u}^m := (u_1^m, \cdots, u_{N_h}^m)^T$, the algebraic formulation of (3.19) leads to the nonlinear system

$$\mathbf{F}(\mathbf{u}^m, t_m) = \mathbf{0}, \tag{5.1}$$

where the components $\mathbf{F}_i, 1 \leq i \leq N_h$, are given by

$$\mathbf{F}_i(\mathbf{u}^m, t_m) = \sum_{j=1}^{N_h} u_j^m\ (\varphi_j, \varphi_i)_{0,\Omega} + \tau_m \beta \delta^2 a_h^{DG}(\sum_{j=1}^{N_h} u_j^m \varphi_j, \varphi_i; \sum_{j=1}^{N_h} u_j^m \varphi_j) - \sum_{j=1}^{N_h} u_j^{m-1}\ (\varphi_j, \varphi_i)_{0,\Omega}.$$

Given $\mathbf{u}^{m-1}$, the time step size $\tau_{m-1,0} = \tau_{m-1}$, and setting $k = 0$, where k is a counter for the predictor corrector steps, the predictor step for (5.1) consists of constant continuation leading to the initial guess

$$\mathbf{u}^{(m,k)} = \mathbf{u}^{m-1}, \quad t_m = t_{m-1} + \tau_{m-1,k}. \tag{5.2}$$

Setting $\nu = 0$ and $\mathbf{u}^{(m,k,\nu_1)} = \mathbf{u}^{(m,k)}$, for $\nu \leq \nu_{max}$, where $\nu_{max} > 0$ is a pre-specified maximal number, the Newton iteration

$$\mathbf{F}'(\mathbf{u}^{(m,k,\nu)}, t_m)\boldsymbol{\Delta}\mathbf{u}^{(m,k,\nu)} = -\mathbf{F}(\mathbf{u}^{(m,k,\nu)}, t_m), \tag{5.3}$$

serves as a corrector whose convergence is monitored by the contraction factor

$$\Lambda^{(m,k,\nu)} = \frac{\|\overline{\boldsymbol{\Delta}\mathbf{u}^{(m,k,\nu)}}\|}{\|\boldsymbol{\Delta}\mathbf{u}^{(m,k,\nu)}\|}, \tag{5.4}$$

where $\overline{\boldsymbol{\Delta}\mathbf{u}^{(m,k,\nu)}}$ is the solution of the auxiliary Newton step

$$\mathbf{F}'(\mathbf{u}^{(m,k,\nu)}, t_m)\overline{\boldsymbol{\Delta}\mathbf{u}^{(m,k,\nu)}} = -\mathbf{F}(\mathbf{u}^{(m,k,\nu+1)}, t_m). \tag{5.5}$$

If the contraction factor satisfies

$$\Lambda^{(m,k,\nu)} < \frac{1}{2}, \tag{5.6}$$

we set $\nu = \nu+1$. If $\nu > \nu_{max}$, both the Newton iteration and the predictor corrector continuation strategy are terminated indicating non-convergence. Otherwise, we continue the Newton iteration (5.3). If (5.6) does not hold true, we set $k = k+1$ and the time step is reduced according to

$$\tau_{m,k} = \max\Big(\frac{\sqrt{2}-1}{\sqrt{4\Lambda^{(m,\nu)}+1}-1}\ \tau_{m,k-1}, \tau_{min}\Big), \tag{5.7}$$

where $\tau_{min} > 0$ is some pre-specified minimal time step. If $\tau_{m,k} \geq \tau_{min}$, we go back to the prediction step (5.2). Otherwise, the predictor corrector strategy is stopped indicating non-convergence. The Newton iteration is terminated successfully, if for some $\nu^* > 0$ the relative error of two subsequent Newton iterates satisfies

$$\frac{\|\mathbf{u}^{(m,k,\nu^*)} - \mathbf{u}^{(m,k,\nu^*-1)}\|}{\|\mathbf{u}^{(m,k,\nu^*)}\|} < \varepsilon \tag{5.8}$$

for some pre-specified accuracy $\varepsilon > 0$. In this case, we set

$$\mathbf{u}^m = \mathbf{u}^{(m,k,\nu_1^*)} \tag{5.9}$$

and predict a new time step according to

$$\tau_m = \min\Big(\frac{(\sqrt{2}-1)\ \|\boldsymbol{\Delta}\mathbf{u}^{(m,k,0)}\|}{2\Lambda^{(m,k,0)}\ \|\mathbf{u}^{(m,k,0)} - \mathbf{u}^m\|}, \mathrm{amp}\Big)\ \tau_{m,k}, \tag{5.10}$$

where amp > 0 is a pre-specified amplification factor for the time step sizes. We set $m = m+1$ and begin new predictor corrector iterations for the time interval $[t_m, t_{m+1}]$.

The choice of the contraction factor (5.6), the choice of the reduced time step (5.7), and the choice of the enlarged time step (5.10) are motivated by the affine covariant convergence theory of Newton's method (cf., e.g., [12, 19]).

6. Numerical results

We have implemented the C^{0}IPDG method of section 3 along with the predictor corrector continuation strategy of section 5 for two examples. In both cases, we have chosen $\Omega = (0,1)^2$, $\beta = 1.0$, polynomial degree $k = 2$ and penalization parameter $\alpha = 200.0$ in the C^{0}IPDG method, and $\nu_{max} = 50, \varepsilon = 1.0 \cdot 10^{-3}$, and $\tau_{min} = 1.0 \cdot 10^{-8}$ for the predictor corrector continuation strategy.

Example 1: The first example is the same as in [25], where the initial data u^0 has been chosen according to

$$u^0(x_1,x_2) = x_1(x_1-1)x_2(x_2-1) - \frac{1}{36}.$$

The C^0IPDG approximation u_h^m has been computed for various regularization parameters δ and finite element mesh sizes h.
For $\delta = 2.5 \cdot 10^{-4}$ and $h = 1/10$, Figure 1 displays the initial data u_h^0 at time $t = 0.0$ (top left), and the computed solutions at times $t = 4.6 \cdot 10^{-6}$ (top right), $t = 2.6 \cdot 10^{-3}$ (bottom left), and $t = 1.2 \cdot 10^{-3}$ (bottom right). We see that the solution develops facets around local extrema of the initial data with a narrow interface featuring steep gradients in between. The extinction time, i.e., the time when the initial profile gets completely flat, is $t_{ext} = 1.1 \cdot 10^{-2}$.

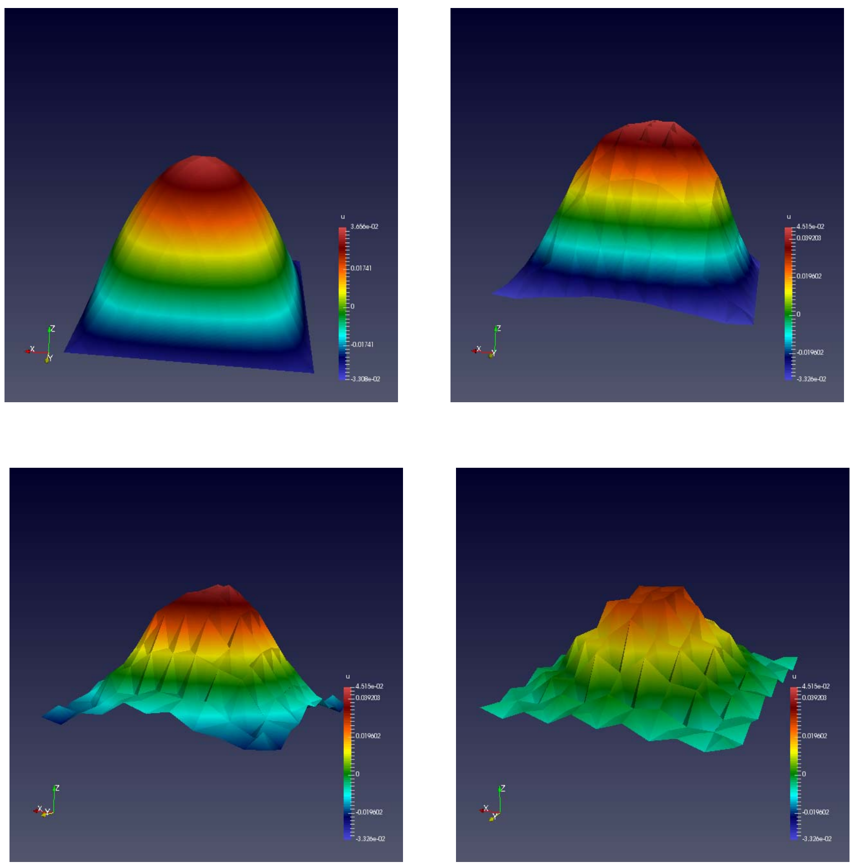

FIGURE 1. Example 1: Computed solution for $h = 1/10$ and $\delta = 2.5 \cdot 10^{-4}$ at initial time $t = 0$ sec (top left), at time $t = 4.6 \cdot 10^{-5}$ sec (top right), at time $t = 2.6 \cdot 10^{-3}$ sec, and at time $t = 1.2 \cdot 10^{-2}$ sec (bottom right).

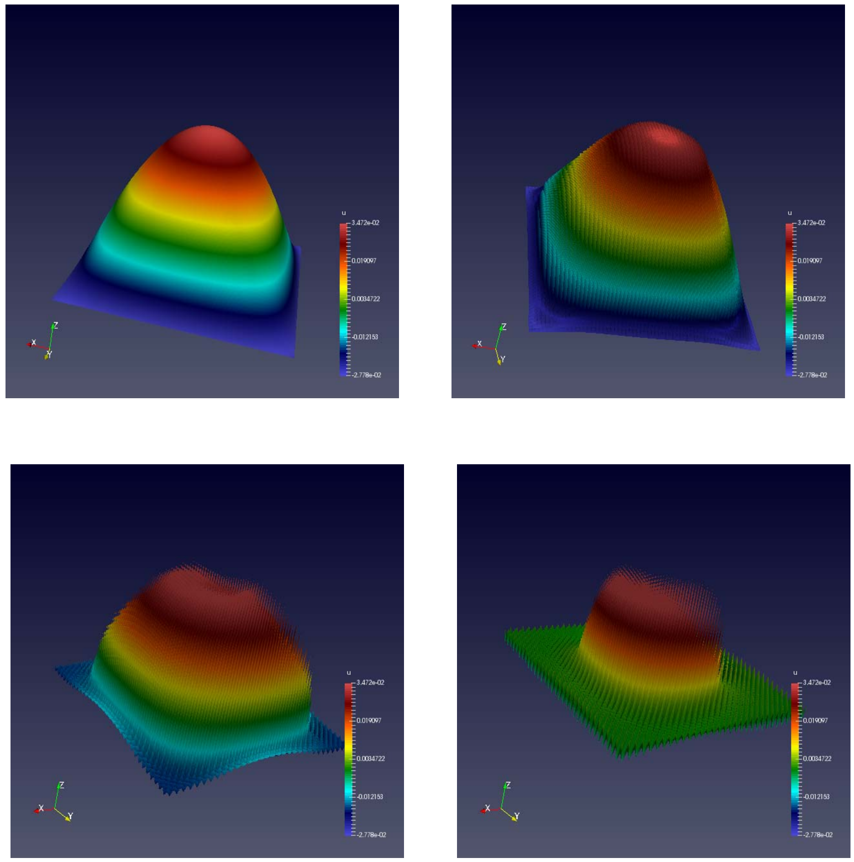

FIGURE 2. Example 1: Computed solution for $h = 1/64$ and $\delta = 7.5 \cdot 10^{-3}$ at initial time $t = 0$ sec (top left), at time $t = 6.9 \cdot 10^{-7}$ sec (top right), at time $t = 9.5 \cdot 10^{-5}$ sec, and at time $t = 1.3 \cdot 10^{-3}$ sec (bottom right).

For $\delta = 7.5 \cdot 10^{-3}$, and $h = 1/64$, Figure 2 shows the same behavior of the solution. However, due to the significantly smaller mesh size h the interface between the upper and lower facets is much better resolved. In this case, the extinction time turned out to be $t_{ext} = 2.2 \cdot 10^{-2}$.

The performance of the predictor corrector continuation strategy for $h = 1/10$ and $\delta = 2.5 \cdot 10^{-4}$ is shown in Figure 3 where the adaptive choice of the time steps τ_m is shown as a function of the iterations. We see that the time step sizes are gradually increasing in a step-like way where the individual steps correspond to the formation of the interface between the upper and the lower facets.
Figure 4 shows the corresponding results for the predictor corrector continuation strategy in case $h = 1/64$ and $\delta = 7.5 \cdot 10^{-3}$ where we started with the same initial time step as for $h = 1/10$ and $\delta = 2.5 \cdot 10^{-4}$. As we mentioned before, the convergence is expected to become a tougher issue for smaller mesh width h which is

reflected by Figure 4. At the beginning, the time step size drops significantly, then stays almost constant, and finally slightly increases shortly before the extinction time is reached.

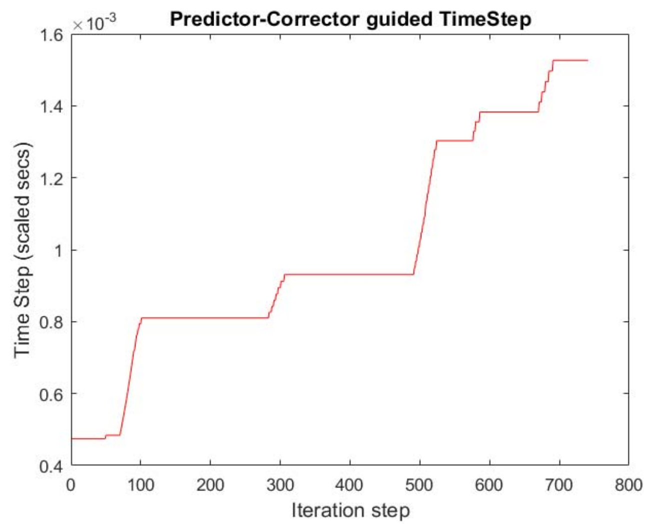

FIGURE 3. Example 1: Performance of the predictor corrector continuation strategy for $h = 1/10$ and $\delta = 2.5 \cdot 10^{-4}$. Adaptive choice of time steps τ_m.

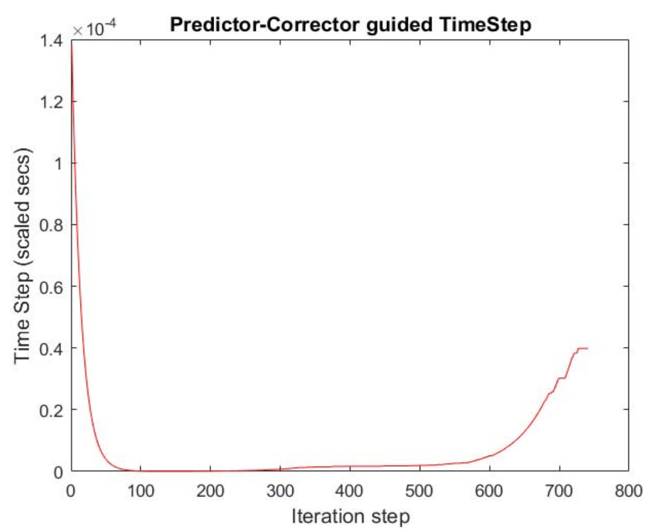

FIGURE 4. Example 1: Performance of the predictor corrector continuation strategy for $h = 1/64$ and $\delta = 7.5 \cdot 10^{-3}$. Adaptive choice of time steps τ_m.

Example 2: The initial profile u^0 has been chosen according to

$$u^0(x_1,x_2) \;=\; \begin{cases} x_1(\frac{1}{2}-x_1)(1-x_1)x_2(\frac{1}{2}-x_2)(1-x_2)-\frac{1}{32} & , 0 \le x_1 < \frac{1}{2}, 0 \le x_2 < \frac{1}{2} \\ x_1(x_1-\frac{1}{2})(1-x_1)x_2(\frac{1}{2}-x_2)(1-x_2)-\frac{1}{32} & , \frac{1}{2} \le x_1 \le 1, 0 \le x_2 < \frac{1}{2} \\ x_1(\frac{1}{2}-x_1)(1-x_1)x_2(x_2-\frac{1}{2})(1-x_2)-\frac{1}{32} & , 0 \le x_1 < \frac{1}{2}, \frac{1}{2} \le x_2 \le 1 \\ x_1(x_1-\frac{1}{2})(1-x_1)x_2(x_2-\frac{1}{2})(1-x_2)-\frac{1}{32} & , \frac{1}{2} \le x_1 \le 1, \frac{1}{2} \le x_2 \le 1 \end{cases} .$$

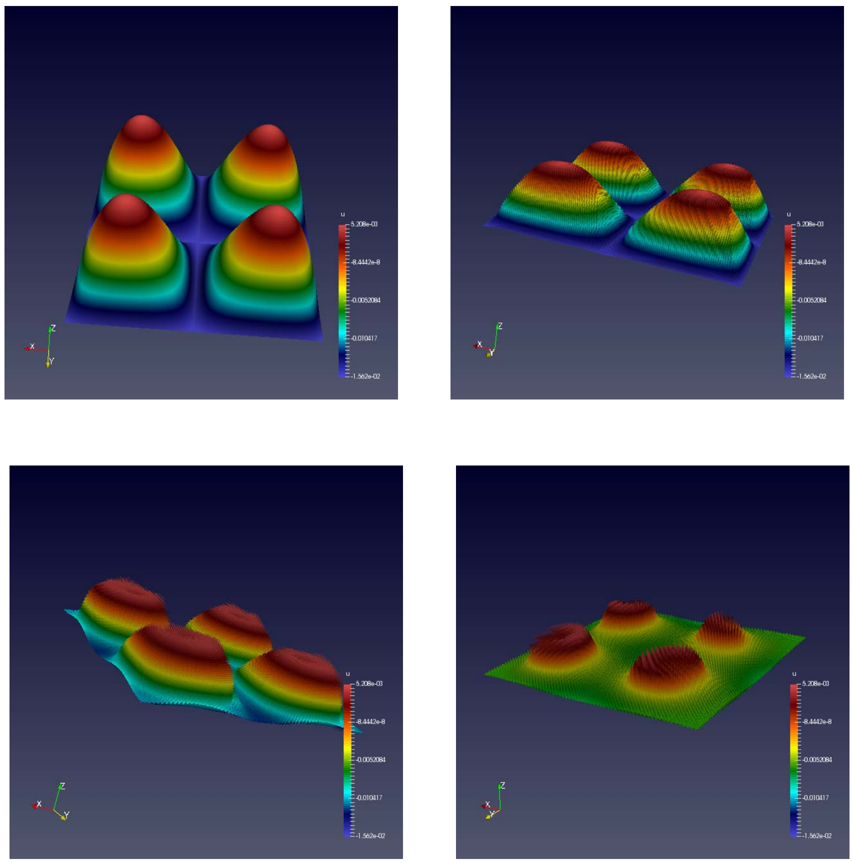

FIGURE 5. Example 2: Computed solution for $h = 1/128$ and $\delta = 4.5\cdot 10^{-3}$ at initial time $t = 0$ sec (top left), at time $t = 3.4\cdot 10^{-5}$ sec (top right), at time $t = 9.7 \cdot 10^{-3}$ sec (bottom left), and at time $t = 7.1 \cdot 10^{-2}$ sec (bottom right).

For $\delta = 4.5 \cdot 10^{-3}$ and $h = 1/128$, Figure 5 shows the route along which the initial profile becomes completely flat. We observe the development of four upper facets around the four maxima of the initial data and lower facets around the minima. Due to the small mesh size h, the formation of narrow interfaces with steep gradients between the upper and lower facets happens quickly. The extinction time is $t_{ext} = 1.2 \cdot 10^{-2}$.

Figure 6 displays the adaptive choice of the time steps. We see a similar behavior as in Example 1 for $h = 1764$ and $\delta = 7.5 \cdot 10^{-3}$.

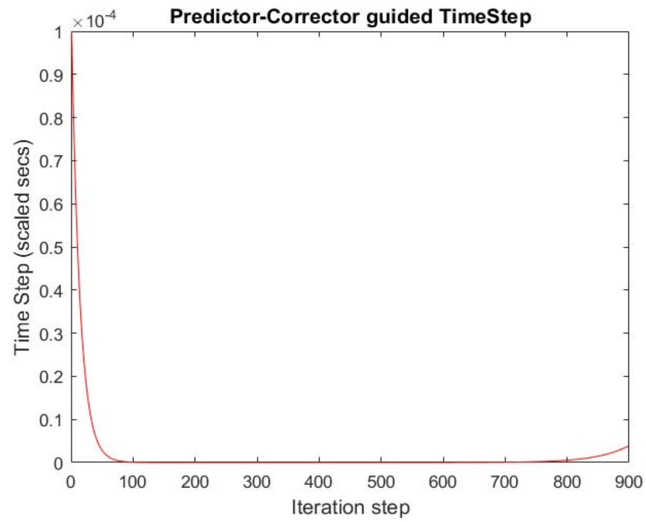

FIGURE 6. Example 2: Performance of the predictor corrector continuation strategy for $h = 1/128$ and $\delta = 4.5 \cdot 10^{-3}$. Adaptive choice of time steps τ_m.

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