SHARP EXPONENTIAL LOCALIZATION FOR EIGENFUNCTIONS OF THE DIRAC OPERATOR

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ABSTRACT. We determine the fastest possible rate of exponential decay at infinity for eigenfunctions of the Dirac operator $\mathcal{D}_n + \mathbb{V}$, being $\mathcal{D}_n$ the massless Dirac operator in dimensions $n = 2, 3$ and $\mathbb{V}$ a matrix-valued perturbation such that $|\mathbb{V}(x)| \sim |x|^{-\epsilon}$ at infinity, for $\epsilon < 1$. Moreover, we provide explicit examples of solutions that have the prescripted decay, in presence of a potential with the related behaviour at infinity, proving that our results are sharp. This work is a result of *unique continuation from infinity*.

1. INTRODUCTION

In this paper we investigate the rate of exponential decay at infinity for eigenfunctions of the Dirac operator, i.e. solutions to

$$\mathcal{D}_n\psi + \mathbb{V}\psi = E\psi \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $E \in \mathbb{R}$, $\mathcal{D}_n$ is the massless Dirac operator in dimensions $n = 2, 3$, and $\mathbb{V}$ a matrix-valued perturbation such that $|\mathbb{V}(x)| \sim |x|^{-\epsilon}$ at infinity, with $\epsilon < 1$. We remark that the massive case and the case that $E \in \mathbb{C}$ are included if $\epsilon \leq 0$.

Since the Dirac operator is the matrix-valued square root of the Laplace operator, that is $\mathcal{D}_n^2 = \Delta\mathbb{I}_{2(n-1)}$, it is interesting to review the results on the analogous problem for the Laplace operator. Let u be a solution to

$$\Delta u + Vu = Eu \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

with $n \geq 2$, $E \in \mathbb{R}$, and assume that for some $\epsilon \in \mathbb{R}$ we have $|V(x)| \leq C|x|^{-\epsilon}$ for big $|x|$ and for some $C > 0$. In [22] it is shown that if $\epsilon \geq 1/2$ and $\exp[\tau|x|]u \in L^2(\mathbb{R}^n)$ for a big enough $\tau \gg 1$, then u has compact support. In [28], E. M. Landis conjectured that such phenomenon is more general, claiming that if V is bounded ($\epsilon = 0$) at infinity and $\exp[\tau|x|]u \in L^2(\mathbb{R}^n)$ for a big enough $\tau \gg 1$, then u must have compact support. This was disproved by Meshkov in [30]: by means of the appropriate Carleman estimate he proved that a general solution u to (1.2) in $\mathbb{R}^n$, $n \geq 2$, has compact support if $\exp[\tau|x|^{4/3}]u \in L^2(\mathbb{R}^n)$ for a big enough $\tau \gg 1$. Moreover he provided an intelligent example in $\mathbb{R}^2$ of a bounded potential V_M and a non-trivial function $u_M(x) \sim \exp[-C|x|^{4/3}]$ at infinity such that (1.2) holds true. It is important to underline that the functions u_M and V_M are *complex valued*: in fact the Carleman estimates

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can not distinguish between real or complex valued functions, and the question of the validity of Landis' conjecture was left open in the case of real-valued potentials V. The results in [30] were generalized to the case $\epsilon < 1/2$ in [9], where Cruz-Sampedro showed that u has compact support if $\exp[\tau|x|^p]u \in L^2(\mathbb{R}^n)$ for $\tau \gg 1$ big enough and $p = (4-2\epsilon)/3$, and provided an example in $\mathbb{R}^2$ of a non-trivial function with the optimal decay, in presence of a potential with the associated behaviour at infinity, in the style of [30]. In [15] Duyckaerts, Zuazua and Zhang observed that an easier construction for the examples with the critical decay can be done considering *vector-valued* solutions to (1.2): a $\mathbb{C}^4$–valued solution is constructed for (1.2) in $\mathbb{R}^3$ and when $\epsilon = 0$, but the matrix-valued potential V has a logaritmic growth at infinity, that is almost optimal in this context.

In [18] Escauriaza, Kenig, Ponce and Vega proved unique continuation properties for solutions of the evolution Schrödinger equation with a time dependent potential

$$i\partial_t u + \Delta u + V(x,t)u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,+\infty), \tag{1.3}$$

where $V \in L^\infty(\mathbb{R}^n \times (0,\infty))$, $V(x,t) = V_1(x,t) + V_2(x,t)$, V_2 is supported in $\{(x,t) : |x| \geq 1\}$, and for $C_1, C_2 > 0$ and $0 \leq \alpha < 1/2$

$$|V_1(x,t)| \leq \frac{C_1}{(1+|x|^2)^{\alpha/2}}, \quad -(\partial_r V_2(x,t))^- \leq \frac{C_2}{|x|^{2\alpha}}.$$

For $u \in C([0,\infty); L^2(\mathbb{R}^n))$ solution to (1.3) there exists a constant $\lambda_0 > 0$ such that if

$$\sup_{t\geq 0} \int_{\mathbb{R}^n} e^{\lambda_0 |x|^p} |u(x)|^2 \, dx < +\infty, \qquad \text{where } p = (4-2\alpha)/3,$$

then u vanishes. Some limit results are also provided when $\alpha = 1/2$. Moreover, their methods give immediately results in the stationary case for $\alpha < 1/2$, and the case $\alpha = 1/2$ is studied separately, by means of the appropriate Carleman estimate. For solutions to the evolution equation (1.3), properties of unique continuation from infinity have been investigated in many different papers, also in presence of electromagnetic perturbations, exploiting a connection with the Hardy uncertainty principle: we refer to [21, 19, 20, 2, 6, 7] and references therein.

A more quantitative approach was used in [4, 25, 26]: for u solution to (1.2), with u, V bounded and $u(0) = 1$, it was shown that for big $R \gg 1$ and $C_1, C_2 > 0$

$$M(R) := \inf_{|x_0|=R} \|u\|_{L^2(B_1(x_0))} \geq C_1 e^{-C_2 R^{4/3}\log R}.$$

This result was generalized in [10], where Davey considered the equation

$$-\Delta u + W\cdot\nabla u + Vu = \lambda u \quad \text{in } \mathbb{R}^n, \tag{1.4}$$

for $\lambda \in \mathbb{C}$, $|V(x| \lesssim (1+|x|^2)^{-N/2}, |W(x)| \lesssim (1+|x|^2)^{-P/2}$, $N, P \geq 0$. For u solution to (1.4) bounded and such that $u(0) \geq 1$, setting $\beta := \max(2-2P, (4-2N)/3)$, she showed that for big $R \gg 1$ and $C_1, C_2, C_3 > 0$

$$M(R) \geq \begin{cases} C_1 \exp[-C_2 R^\beta (\log R)^{C_3}] & \text{if } \beta > 1, \\ C_1 \exp[-C_2 R (\log R)^{C_3 \log\log R}] & \text{if } \beta < 1. \end{cases}$$

Moreover, explicit examples with the critical decay are provided: if V and W do not both decay too quickly, these examples are built in the style of [30], otherwise the constructions are simpler. The case $\beta = 1$ was treated in [29], where it is proved that for big $R \gg 1$ and $C_1, C_2, C_3 > 0$

$$M(R) \geq C_1 \exp[-C_2 R (\log R)^{C_3 (\log R)(\log\log\log R)(\log\log R)^{-2}}].$$

Finally, Davey showed in [11] that this estimate is sharp, providing an example in the style of [30].

The case of a real potential V has been finally addressed in [27], where a quantitative form of Landis' conjecture is proved in $\mathbb{R}^2$. Precisely, for u a real-valued solution of (1.2) for $E = 0$, $V \geq 0$ and $\|V\|_{L^\infty} \leq 1$, if $u(0) = 1$ and $|u(x)| \leq \exp[C_0|x|]$, then for a sufficiently large $R \gg 1$

$$\inf_{|x_0|=R} \sup_{|x-x_0|\leq 1} |u(x)| \geq \exp[CR\log R],$$

where C depends only on C_0. Similar estimates for equations with bounded magnetic potentials are also derived, i.e. for the equations $-\Delta u + W\cdot\nabla u + Vu = 0$ and $-\Delta u + \nabla(Wu) + Vu = 0$, and the corresponding estimates in exterior domains $\mathbb{R}^2 \setminus B_R(0)$ are provided. In [12] the results of [27] are generalized replacing the Laplace operator with a general operator $Lu := \operatorname{div}(A\nabla u)$, where A is real, symmetric and uniformly elliptic with Lipschitz continuous coefficients. Finally, in [13] Davey and Wang address the case of a general second order elliptic equation with singular lower order terms in $\mathbb{R}^2$.

Regarding the analogous question for the Dirac Operator, to the best of our knowledge, much less results are available. In [5], Boussaïd and Comech study the point spectrum of the nonlinear massive Dirac equation in any spatial dimension, linearized at one of the solitary wave solutions, and consider the presence of a bounded potential decaying at infinity in a weak sense. Thanks to the presence of the massive term, they show linear exponential decay for eigenfunctions and link explicitly the constant in the exponent with the mass, the associated eigenvalue in the gap of the spectrum and the ground state for the non-linear stationary equation. In the massless case a different behavior should be expected, as it is shown in [3] for the massless non-linear Dirac equation in 2D without considering the presence of potentials.

In order to state our results we remind the definition and some elementary properties of the Dirac operators $\mathcal{D}_n$, for $n = 2, 3$. In $\mathbb{R}^2$ we consider the two dimensional massless free Dirac operator

$$\mathcal{D}_2 := -i\sigma_1\partial_1 - i\sigma_2\partial_2 = \begin{pmatrix} 0 & -2i\partial_z \\ -2i\partial_{\bar z} & 0 \end{pmatrix},$$

where σ_j are the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.5}$$

We remark that $\mathcal{D}_2$ is self-adjoint in $L^2(\mathbb{R}^2;\mathbb{C}^2)$ with domain $H^1(\mathbb{R}^2;\mathbb{C}^2)$. We remind that $2\partial_z = \partial_1 - i\partial_2$, $2\partial_{\bar z} = \partial_1 + i\partial_2$, using the usual identification $(x,y) \in \mathbb{R}^2 \mapsto z \in \mathbb{C}$, with $z = x_1 + ix_2$, $\bar z = x_1 - ix_2$. In the following we also denote $r := |z| = \sqrt{x^2+y^2}$ and make use of polar coordinates in $\mathbb{R}^2$. The massless free Dirac operator in $\mathbb{R}^3$ is defined by

$$D_3 := -i\alpha\cdot\nabla = -i\sum_{j=1}^{3}\alpha_j\partial_j,$$

where

$$\alpha = (\alpha_1,\alpha_2,\alpha_3), \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad (j = 1,2,3),$$

and σ_j are the *Pauli matrices*, defined in (1.5). We remark that $\mathcal{D}_3$ is self-adjoint in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ with domain $H^1(\mathbb{R}^3;\mathbb{C}^4)$.

In Theorem 1.1 and Theorem 1.2 we state our results in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

Theorem 1.1. *Let $\psi \in H^1_{loc}(\mathbb{R}^2;\mathbb{C}^2)$ be a solution to*

$$\mathcal{D}_2\psi(z) + \mathbb{V}(z)\psi(z) = E\psi(z), \qquad \text{for a.a. } z \in \mathbb{R}^2, \tag{1.6}$$

where $E \in \mathbb{R}$ and $\mathbb{V} : \mathbb{R}^2 \to \mathbb{C}^{2\times 2}$ is such that for $\epsilon < 1$ and $\rho, C > 0$

$$|\mathbb{V}(z)| \leq C|z|^{-\epsilon}, \qquad \text{as } |z| > \rho. \tag{1.7}$$

Then there exists $\tau_0 > 0$ such that, if for all $\tau > \tau_0$

$$e^{\tau|x|^{2-2\epsilon}}\psi \in L^2(\mathbb{R}^2), \tag{1.8}$$

then ψ has compact support.

Theorem 1.2. *Let $\psi\ inH^1_{loc}(\mathbb{R}^3;\mathbb{C}^4)$ be a solution to*

$$\mathcal{D}_3\psi(x) + \mathbb{V}(x)\psi(x) = E\psi(x), \qquad \text{for a.a. } x \in \mathbb{R}^3, \tag{1.9}$$

where $E \in \mathbb{R}$ and $\mathbb{V} : \mathbb{R}^3 \to \mathbb{C}^{4\times 4}$ is such that for $\epsilon < 1$ and $C > 0$

$$|\mathbb{V}(z)| \leq C|z|^{-\epsilon}, \qquad \text{as } |z| > \rho \gg 1. \tag{1.10}$$

Then there exists $\tau_0 > 0$ such that, if for all $\tau > \tau_0$

$$e^{\tau|x|^{2-2\epsilon}}\psi \in L^2(\mathbb{R}^3), \tag{1.11}$$

then ψ has compact support.

Remark 1.3. We remark that Theorem 1.1 and Theorem 1.2 can be generalized to include the massive term in the case that $\epsilon \leq 0$, i.e. one can replace in the statements $\mathbb{V}$ with

$$\tilde{\mathbb{V}} := \mathbb{V} + m\beta = \mathbb{V} + m\begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad \mathbb{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

for $m \in \mathbb{R}$, since such term does not affect the validity of (1.7) and (1.10) respectively. Analogously, one can consider $E \in \mathbb{C}$.

Remark 1.4. For $\epsilon = 1$ the potential $\mathbb{V}$ is a critical perturbation of the Dirac operator: we expect to loose the exponential decay and that only polynomial behaviour at infinity can be recovered. For example (see [8, Remark 1.11]), in $\mathbb{R}^3$ and for $\mathbb{V}(x) = -1/|x|$ and $m > 0$, the ground state of the Dirac-Coulomb operator is the function

$$\psi_0(x) = \frac{e^{-m|x|}}{|x|}\begin{pmatrix} 1 \\ 1 \\ i\sigma\cdot\hat{x}\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix},$$

i.e. ψ_0 is solution to the equation $(-i\alpha\cdot\nabla + m\beta - \nu/|x|)\psi = 0$. In the massless case $m = 0$, ψ_0 is not in $L^2(\mathbb{R}^3)$. We refer to [8] for a description of the functions in the domain of $\mathcal{D}_3 + \mathbb{V}$: we hope to investigate the phenomena described in this paper in the case $\epsilon \geq 1$ in the future.

Remark 1.5. We observe that our results fit in the known theory available for the Laplace operator if we ask more regularity on the potential $\mathbb{V}$ and the function u. For example, let us consider $\mathbb{V} \in W^{1,\infty}(\mathbb{R}^3;\mathbb{C}^{4\times 4})$ and $u \in H^2_{loc}(\mathbb{R}^3;\mathbb{C}^4)$, solution to (1.9): then u is solution to

$$-\Delta u + \sum_{j=1}^{3} -i(\alpha_j\cdot\mathbb{V})\partial_j u - i(\alpha_j\partial_j\cdot\mathbb{V})u = 0.$$

Thanks to [10], u has compact support if $\exp[\tau|x|^2]u \in L^2(\mathbb{R}^3)$, for a big enough $\tau \gg 1$, in accordance with Theorem 1.2.

The main tools for the proof of Theorem 1.1 and Theorem 1.2 are the following Carleman Estimates. Although the Carleman estimates have been developed in the context of the Laplace operator, there have been many contributions for obtaining such inequalities for the Dirac operator, mainly for the applications to properties of strong unique continuation, we refer to [1, 24, 31, 16, 32] and references therein for further details.

Proposition 1.6. *Let $E, \tau \in \mathbb{R}$ and $a > 0$.*

i) For all $u \in C_c^\infty(\mathbb{R}^2; \mathbb{C}^2)$ the following holds:

$$\tau a^2 \int_{\mathbb{R}^2} |x|^{a-2} e^{2\tau|x|^a} |u(x)|^2 \, dx \leq \int_{\mathbb{R}^2} e^{2\tau|x|^a} |(\mathcal{D}_2 + E)u(x)|^2 \, dx. \tag{1.12}$$

ii) For all $u \in C_c^\infty(\mathbb{R}^3; \mathbb{C}^4)$ the following holds:

$$\tau a(a+1) \int_{\mathbb{R}^3} |x|^{a-2} e^{2\tau|x|^a} |u(x)|^2 \, dx \leq \int_{\mathbb{R}^3} e^{2\tau|x|^a} |(\mathcal{D}_3 + E)u(x)|^2 \, dx. \tag{1.13}$$

Remark 1.7. We remark that (1.12) is reminescent of the analogus Carleman estimate for the $\overline{\partial}$ operator, see [23, Theorem 15.1.1] and [14, Proposition 2.1]; (1.13) already appeared in [17, eq. (4.4)], but with a worse multiplicative constant at left hand side.

The following theorems show that Theorem 1.1 and Theorem 1.2 are sharp, respectively.

Theorem 1.8. *For all $\epsilon < 1$ there exist nontrivial functions*

$$u \in C^\infty(\mathbb{R}^2; \mathbb{C}^2), \quad \mathbb{V} \in C^\infty(\mathbb{R}^2; \mathbb{C}^{2\times 2}), \tag{1.14}$$

such that for all $z \in \mathbb{R}^2$

$$\mathcal{D}_2 u(z) = \mathbb{V}(z) u(z), \tag{1.15}$$

$$|u(z)| \leq C_1 e^{-C_2|z|^{2-2\epsilon}}, \tag{1.16}$$

$$|\mathbb{V}(z)| \leq C_3 |z|^{-\epsilon}, \tag{1.17}$$

for some $C_1, C_2, C_3 > 0$.

Theorem 1.9. *For all $\epsilon < 1$ there exist nontrivial functions*

$$u \in C^\infty(\mathbb{R}^3; \mathbb{C}^4), \quad \mathbb{V} \in C^\infty(\mathbb{R}^3; \mathbb{C}^{4\times 4}), \tag{1.18}$$

such that for all $x \in \mathbb{R}^3$

$$\mathcal{D}_3 u(x) = \mathbb{V}(x) u(x), \tag{1.19}$$

$$|u(x)| \leq C_1 e^{-C_2|x|^{2-2\epsilon}}, \tag{1.20}$$

$$|\mathbb{V}(x)| \leq C_3 (\log|x|)^3 |x|^{-\epsilon}, \tag{1.21}$$

for some $C_1, C_2, C_3 > 0$.

Remark 1.10. In [15] the Authors construct an explicit function in $\mathbb{R}^3$ with critical decay for the Laplace operator and for bounded potentials, moreover they suggest that an analogous construction can be done in $\mathbb{R}^2$: their examples are respectively $\mathbb{C}^4$ and $\mathbb{C}^2$–valued. The critical examples we construct in Theorem 1.8 and Theorem 1.9 are similar to the ones in [15] and somehow motivate this numerology. In detail, in the case $\epsilon = 0$ the function we build in Theorem 1.9 is similar to the example given in [15, Theorem 3.2]; the generalization to the case $\epsilon < 1$ is done in light of the approach of [9, Theorem 2].

Remark 1.11. In accordance with the established theory for the Laplace operator, in Theorem 1.8 and Theorem 1.9 the constructed potentials $\mathbb{V}$ are not symmetric and the functions u are complex-valued. It would be interesting to understand what is the sharp decay associated to symmetric potentials $\mathbb{V}$.

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2. Proof of Proposition 1.6

The proof of Proposition 1.6 is standard, but we will give it for the sake of completeness.

Proof. We prove in detail only *ii)*: the point *i)* can be proved ananlogously.

We set

$$b(x) := e^{\tau|x|^a}, \quad v(x) := b(x)u(x).$$

By a standard approximation procedure, to proving (1.13) it is enough to show that for all $v \in C_c^\infty(\mathbb{R}^3;\mathbb{C}^4)$ we have

$$\int_{\mathbb{R}^3} (\Delta \log b(x))|v(x)|^2\, dx \leq \int_{\mathbb{R}^3} \left|b(x)(\mathcal{D}_3 + E)b(x)^{-1}\, v(x)\right|^2 dx. \tag{2.1}$$

We observe that

$$(b\,(\mathcal{D}_3 + E)\, b^{-1})\, v =: \mathcal{S}v + \mathcal{A}v, \tag{2.2}$$

with

$$\begin{aligned} \mathcal{S}v &:= (\mathcal{D}_3 + E)v = -i\alpha\cdot\nabla v + Ev,\\ \mathcal{A}v &:= b(\mathcal{D}_3 b^{-1})v = -i(b\alpha\cdot\nabla b^{-1})v. \end{aligned}$$

We remark that $\mathcal{S}$ and $\mathcal{A}$ are respectively a symmetric and anti-symmetric operator. We observe that

$$\begin{aligned} \|(\mathcal{S}+\mathcal{A}])v\|_{L^2}^2 =& \|\mathcal{S}v\|_{L^2}^2 + \|\mathcal{A}v\|_{L^2}^2 + \langle \mathcal{S}v, \mathcal{A}v\rangle_{L^2} + \langle \mathcal{A}v, \mathcal{S}v\rangle_{L^2}\\ \geq& \langle(\mathcal{S}\mathcal{A} - \mathcal{A}\mathcal{S})v, v\rangle_{L^2} = \langle[\mathcal{S},\mathcal{A}]v, v\rangle_{L^2}. \end{aligned} \tag{2.3}$$

We compute the commutator $[\mathcal{S},\mathcal{A}]$: since

$$[E, -ib\alpha\cdot\nabla b^{-1}] = 0, \tag{2.4}$$

we get

$$[\mathcal{S},\mathcal{A}] = -(\alpha\cdot\nabla b)\cdot(\alpha\cdot\nabla b^{-1}) - b\alpha\cdot\nabla(\alpha\cdot\nabla b^{-1}).$$

Thanks to the following elementary property of the matrices σ_j:

$$(\sigma\cdot A)(\sigma\cdot B) := \Big(\sum_j \sigma_j\cdot A_j\Big)\Big(\sum_k \sigma_k\cdot B_k\Big) = A\cdot B\,\mathbb{I}_2 + i\sigma\cdot A\wedge B, \qquad \text{for all } A, B\in\mathbb{C}^3,$$

we have that

$$[\mathcal{S},\mathcal{A}] = \frac{\Delta b(x)}{b(x)} - \frac{|\nabla b(x)|^2}{b(x)^2} \tag{2.5}$$

and we conclude (2.1) thanks to (2.2), (2.3), (2.5). □

3. Proof of Theorem 1.1 and Theorem 1.2

We prove in detail only Theorem 1.1, the proof of Theorem 1.2 being analogous.

Let $a := 2 - 2\epsilon$. We divide the proof in two steps.

3.1. **Step 1.** We show that, for $\tau > 0$ big enough, for all $u \in C_c^\infty(\{|x| > \rho\})$ the following holds:

$$\frac{\tau a^2}{4} \int_{|x|>\rho} e^{2\tau|x|^a} |x|^{a-2} |u|^2 \, dx \le \int_{|x|>\rho} e^{2\tau|x|^a} |(\mathcal{D} + \mathbb{V} + E)u|^2 \, dx. \tag{3.1}$$

Indeed, thanks to Proposition 1.6, for $\tau, E \in \mathbb{R}$, $a = 2 - 2\epsilon > 0$, and $u \in C_c^\infty(\{|x| > \rho\})$, (1.7) and the elementary inequality $(a+b)^2 \le 2(a^2+b^2)$, for $a, b \in \mathbb{R}$, we have

$$\begin{aligned}\tau a^2 \int_{|x|>\rho} e^{2\tau|x|^a} |x|^{a-2} |u|^2 \, dx \le & 2\int_{|x|>\rho} e^{2\tau|x|^a} |(\mathcal{D} + \mathbb{V} + E)u|^2 \, dx + 2\int_{|x|>\rho} e^{2\tau|x|^a} |\mathbb{V}u|^2 \, dx \\ \le & 2\int_{|x|>\rho} e^{2\tau|x|^a} |(\mathcal{D} + \mathbb{V} + E)u|^2 \, dx + 2C^2 \int_{|x|>\rho} e^{2\tau|x|^a} |x|^{-2\epsilon} |u|^2 \, dx.\end{aligned}$$

Since u has compact support in $\{|x| > \rho\}$, the second term at right hand side is finite and can be subtracted from both sides. Thanks to the arbitrariness in the choice of $\tau \in \mathbb{R}$, we have that (3.1) holds for τ big enough.

3.2. **Step 2.** Thanks to an approximation process, (3.1) holds for all $u \in H^1_{loc}(\{|x| > \rho\})$.

Let h be a C^∞ function such that $h(r) = 0$ for $r \le 1$ and $h(r) = 1$ for $r \ge 2$, and $h_\rho(r) := h(r/\rho)$. Let moreover k be a C_c^∞ function such that $k(r) = 1$ for $r \le 1$ and $\operatorname{supp} k \subset \{|x| \le 2\}$, and $k_R(r) := k(r/R)$.

Set $u_R(x) := h_\rho(|x|) k_R(|x|) \psi(x)$ for all $x \in \mathbb{R}$: such function is in $H^1(\{|x| > \rho\})$. Since $(\mathcal{D}_n + \mathbb{V} - E)\psi = 0$, we get, for the appropriate $C > 0$,

$$\begin{aligned}&\frac{\tau a^2}{4} \int_{|x|>\rho} e^{2\tau|x|^a} |x|^{a-2} |h_\rho|^2 |k_R|^2 |\psi|^2 \, dx \le 2\int_{|x|>\rho} e^{2\tau|x|^a} |\mathcal{D}(h_\rho k_R)|^2 |\psi|^2 \, dx \\ &\le \frac{C}{\rho^2} \int_{|x|>\rho} e^{2\tau|x|^a} \chi_{\{\rho<|x|<2\rho\}} |\psi|^2 \, dx + \frac{C}{R^2} \int_{|x|>\rho} e^{2\tau|x|^a} \chi_{\{R<|x|<2R\}} |\psi|^2 \, dx.\end{aligned}$$

Letting $R \to +\infty$ and for the appropriate $C, C' > 0$, we have that

$$\begin{aligned}\tau a^2 \int_{|x|>\rho} e^{2\tau|x|^a} |x|^{a-2} |h_\rho|^2 |\psi|^2 \, dx \le & \frac{C}{\rho^2} \int_{|x|>\rho} e^{2\tau|x|^a} \chi_{\{\rho<|x|<2\rho\}} |\psi|^2 \, dx \\ \le & C' \int_{\rho<|x|<2\rho} e^{2\tau|x|^a} |x|^{-2} |\psi|^2 \, dx.\end{aligned}$$

From the monotonicity of $e^{\tau|x|^a}$, and since $|x|^{-2} \le |x|^{a-2}$

$$\tau a^2 e^{2\tau|2\rho|^a} \int_{|x|>2\rho} |x|^{a-2} |\psi|^2 \, dx \le C' e^{2\tau|2\rho|^a} \int_{\rho<|x|<2\rho} |x|^{a-2} |\psi|^2 \, dx.$$

Simplifying the term $e^{2\tau|2\rho|^a}$ in both sides, the right hand side is smaller than a constant independent of τ: from the arbitrariness of $\tau \in \mathbb{R}$ we get that $\psi \equiv 0$ in $\{|x| > 2\rho\}$, that is the thesis.

4. Proof of Theorem 1.8

The strategy of the proof of Theorem 1.8 is the following:

- we break down $\mathbb{R}^2$ in annuli $\{\rho_k \le |z| \le \rho_{k+1}\}$, for the appropriate $\rho_k > 0$, $\rho_k \to +\infty$; in such annuli we define the functions $\mathbf{E}_k$ (see Section 4.2);
- for $k \in \mathbb{N}$ big enough, we define the functions u_k and $\mathbb{V}_k$ in the annulus $\{\rho_k \le |z| \le \rho_{k+1}\}$ (see Section 4.3);
- we define the functions u and $\mathbb{V}$ in $\{|z| \le \rho_{k_0}\}$, for k_0 big enough (see Section 4.4);
- in $\{|z| \ge \rho_{k_0}\}$ we define u and $\mathbb{V}$ glueing together the functions u_k and $\mathbb{V}_k$, for $k \ge k_0$ (see Section 4.5);
- we check the behaviour at infinity of the function u (see Section 4.6).

4.1. **Notation.** In the following we will write, for sequences of real numbers $(A_k)_{k\in\mathbb{N}}$, $(B_k)_{k\in\mathbb{N}}$,

$$A_k = O(B_k)$$

when there exist constants C>0 and $\bar{k}$ such that

$$\forall k \ge \bar{k}, \quad |A_k| \le C|B_k|.$$

When A_k and B_k also depend on $r \in I$, being I an interval, the estimate is also assumed to be uniform with respect to $r \in I$. We will also use the notation

$$A_k \approx B_k \iff (A_k = O(B_k) \text{ and } B_k = O(A_k)).$$

4.2. **Preliminary definitions.** In this section we collect some definitions and elementary results we need in the proof.

4.2.1. *Definition of* $\delta, n_k, d_k, \rho_k, a_k$. Let

$$\delta := 1 - 2\epsilon. \tag{4.1}$$

Let n_0 be a large odd number, and for all $k \ge 0$ define the following quantities:

$$n_{k+1} := n_k + 2\lfloor n_k^{1/2} \rfloor, \quad d_k := \frac{n_{k+1} - n_k}{2}, \tag{4.2}$$

$$\rho_k := n_k^{\frac{1}{1+\delta}}, \quad \rho_{kj} := \rho_k + j\,\frac{\rho_{k+1} - \rho_k}{4}, \quad j = 1, \dots, 4, \tag{4.3}$$

$$a_0 := 1, \quad a_{k+1} := \rho_k^{2d_k} a_k. \tag{4.4}$$

It is easy to see that

$$d_k = n_k^{1/2} + O(1), \tag{4.5}$$

$$\rho_{k+1} - \rho_k = n_{k+1}^{\frac{1}{1+\delta}} - n_k^{\frac{1}{1+\delta}} = \frac{1}{1+\delta} n_k^{-\frac{\delta}{1+\delta}} (2n_k^{1/2} + O(1)) = \frac{2}{1+\delta}\rho_k^{\frac{1-\delta}{2}} + O(\rho_k^{-\delta}). \tag{4.6}$$

Finally, for $\rho_k \le r \le \rho_{k+1}$ we have that

$$\frac{a_k}{r^{n_k}} \approx \frac{a_{k+1}}{r^{n_{k+1}}}. \tag{4.7}$$

Indeed, denoted $g(r) := \log(r^{d_k}/\rho_k^{d_k})$ for $\rho_k \le r \le \rho_{k+1}$, thanks to (4.5) and (4.3) we have that

$$g'(r) = \frac{d_k}{r} = \frac{O(\rho_k^{\frac{1+\delta}{2}})}{r} = O(\rho_k^{-\frac{1-\delta}{2}}).$$

Observing that $g(\rho_k) = 0$ and $\rho_{k+1} - \rho_k = O(\rho_k^{\frac{1-\delta}{2}})$, we get that $g(r) = O(1)$, that is

$$r^{d_k} \approx \rho_k^{d_k}, \quad \text{for } \rho_k \le r \le \rho_{k+1}. \tag{4.8}$$

From (4.8) we get immediately (4.7).

4.2.2. *Definition of* $\boldsymbol{E}_k$. For $k \in \mathbb{N}$ big enough and $r := |z| \ge \rho_k$ we define

$$E_k(z) := \frac{a_k}{z^{n_k}},$$
$$\mathbf{E}_k(z) := \begin{pmatrix} E_k(z) \\ 0 \end{pmatrix} \text{ if } k \text{ is even}, \quad \mathbf{E}_k(z) := \begin{pmatrix} 0 \\ \overline{E_k(z)} \end{pmatrix} \text{ if } k \text{ is odd}.$$

We observe that for all $z \in \mathbb{R}^2 \setminus \{0\}$ and $k \in \mathbb{N}$:

$$\mathcal{D}\mathbf{E}_k(z) = 0. \tag{4.9}$$

We remark that (4.4) implies $|E_k(z)| = |E_{k+1}(z)|$ whenever $r = |z| = \rho_{k+1}$. Moreover, since $|E_k(z)| = a_k r^{-n_k}$ we have immediately from (4.7) that

$$E_k(z) \approx E_{k+1}(z), \quad \text{for } \rho_k \le |z| \le \rho_{k+1}. \tag{4.10}$$

4.3. **Definition of u_k and $\mathbb{V}_k$ in the annulus $\{\rho_k \le |z| \le \rho_{k+1}\}$.** For $k \in \mathbb{N}$ big enough, in this section we construct functions

$$\begin{aligned} &u_k \in C^\infty(\{z \in \mathbb{R}^2 \colon \rho_k \le |z| \le \rho_{k+1}\}; \mathbb{C}^2), \\ &V_k \in C^\infty(\{z \in \mathbb{R}^2 \colon \rho_k \le |z| \le \rho_{k+1}\}; \mathbb{C}^{2\times 2}), \end{aligned} \tag{4.11}$$

such that

$$\mathcal{D}u_k(z) = \mathbb{V}(z)u_k(z), \quad \text{for all } \rho_k \le |z| \le \rho_{k+1}, \tag{4.12}$$

and

$$u_k(z) = \begin{cases} \mathbf{E}_k(z) & \text{for } r \in [\rho_{k0}, \rho_{k1}], \\ \mathbf{E}_{k+1}(z) & \text{for } r \in [\rho_{k3}, \rho_{k4}], \end{cases} \tag{4.13}$$

for $\rho_k \le |z| \le \rho_{k+1}$

$$|u_k(z)| = O(a_k r^{-n_k}), \tag{4.14}$$

and (1.17) holds. In the proof in this section we assume k to be a odd integer: the case of even k can be treated analogously.

4.3.1. *Construction of the cut-off functions.* Let χ be a C^∞ non-decreasing function on $\mathbb{R}$ such that $\chi(s) = 0$ for $s \le 0$, $\chi(s) = 1$ for $s \ge 1$. For $\rho_k \le r \le \rho_{k+1}$ let

$$\chi_k(r) := \chi\left(\frac{4}{\rho_{k+1} - \rho_k}(r - \rho_{k1})\right), \quad \tilde{\chi}_k(r) := \chi\left(\frac{4}{\rho_{k+1} - \rho_k}(\rho_{k3} - r)\right). \tag{4.15}$$

Thanks to (4.6), we have that

$$\|\chi_k'\|_{L^\infty([\rho_k, \rho_{k+1}])} = \|\tilde{\chi}_k'\|_{L^\infty([\rho_k, \rho_{k+1}])} = O(\rho_k^{-\frac{1-\delta}{2}}). \tag{4.16}$$

4.3.2. *Definition of the functions u_k and $\mathbb{V}_k$.* For $\rho_k \leq |z| \leq \rho_{k+1}$, let

$$u_k(z) := \tilde{\chi}_k(r)\mathbf{E}_k(z) + \chi_k(r)\mathbf{E}_{k+1}(z) = \begin{pmatrix} \chi_k(r)\overline{E_{k+1}(z)} \\ \tilde{\chi}_k(r)\overline{E_k(z)} \end{pmatrix},$$

$$\mathbb{V}_k(z) := \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } r \in [\rho_{k0}, \rho_{k1}] \cup [\rho_{k3}, \rho_{k4}], \\ \begin{pmatrix} 0 & 0 \\ 0 & -2i\partial_{\bar{z}}(\chi_k(r))E_{k+1}(z)(\overline{E_k(z)})^{-1} \end{pmatrix} & \text{for } r \in [\rho_{k1}, \rho_{k2}], \\ \begin{pmatrix} -2i\partial_z(\tilde{\chi}(r))\overline{E_k(z)}(E_{k+1}(z))^{-1} & 0 \\ 0 & 0 \end{pmatrix} & \text{for } r \in [\rho_{k2}, \rho_{k3}]. \end{cases}$$

By construction (4.11), (4.12) and (4.13) are true; thanks to (4.10), (4.14) holds for a large enough k. Some more details are in order for checking condition (1.17).

If $r \in [\rho_{k0}, \rho_{k1}] \cup [\rho_{k3}, \rho_{k4}]$ (1.17) is trivially verified. In the case $r \in [\rho_{k1}, \rho_{k2}]$, thanks to (4.10), (4.16) and (4.1),

$$|\mathbb{V}_k(z)| = O(\rho_k^{-\frac{1-\delta}{2}}) = O(\rho_k^{-\epsilon}).$$

Observing that

$$\frac{\rho_{k+1}}{\rho_k} = O(1 + 4\rho_k^{-\frac{1+\delta}{2}}) = O(1), \tag{4.17}$$

we conclude (1.17) for $r \in [\rho_{k1}, \rho_{k2}]$ since

$$|\mathbb{V}_k(z)| = O(\rho_{k+1}^{-\epsilon}) = O(|z|^{-\epsilon}).$$

Finally (1.17) is proved analogously in the interval $[\rho_{k2}, \rho_{k3}]$.

4.4. **Definition of u in $\{|z| \leq \rho_{k_0}\}$.** Let k_0 be a large integer such that the arguments in the previous section hold for all $k \geq k_0$. Without loss of generality we can assume k_0 to be odd. For $|z| \leq \rho_{k_0}$ let

$$\psi(z) := \chi\left(\frac{4}{\rho_{k_0}}\left(|z| - \frac{\rho_{k_0}}{4}\right)\right), \quad \tilde{\psi}(z) := \chi\left(\frac{4}{\rho_{k_0}}\left(\frac{3\rho_{k_0}}{4} - |z|\right)\right),$$

with χ defined in Section 4.3.1 and

$$u(z) := \tilde{\psi}(z)\begin{pmatrix} a_{k_0} z^{n_{k_0}} \\ 0 \end{pmatrix} + \psi(z)\mathbf{E}_{k_0} = \begin{pmatrix} \tilde{\psi}(z) a_{k_0} z^{n_{k_0}} \\ \psi(z) a_{k_0} \bar{z}^{-n_{k_0}} \end{pmatrix}. \tag{4.18}$$

We remark that $\mathcal{D}u(z) = 0$ for $|z| \leq \rho_{k_0}/4$. Moreover, for

$$\mathbb{V}(z) := \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{for } |z| \in [0, \rho_{k_0}/4] \cup [3\rho_{k_0}/4, \rho_{k_0}], \\ \begin{pmatrix} -2i\partial_z(\psi(z))|z|^{-2n_{k_0}} & 0 \\ 0 & 0 \end{pmatrix} & \text{for } |z| \in [\rho_{k_0}/4, 2\rho_{k_0}/4], \\ \begin{pmatrix} 0 & 0 \\ 0 & -2i\partial_{\bar{z}}(\tilde{\psi}(z))|z|^{2n_{k_0}} \end{pmatrix} & \text{for } |z| \in [2\rho_{k_0}/4, 3\rho_{k_0}/4], \end{cases} \tag{4.19}$$

conditions (1.14) – (1.17) hold for the appropriate $C_1, C_2, C_3 > 0$.

4.5. **Definition of u in $\{|z| \geq \rho_{k_0}\}$.** For any $z \in \mathbb{R}^2$, $|z| \geq \rho_{k_0}$, let k be such that $|z| \in [\rho_k, \rho_{k+1}]$. We set

$$u(z) := u_k(z), \quad \mathbb{V}(z) := \mathbb{V}_k(z), \quad \text{for } |z| \in [\rho_k, \rho_{k+1}]. \tag{4.20}$$

It is easy to show that conditions (1.14), (1.15), (4.1), (1.17) hold, for the appropriate $C_3 > 0$.

4.6. **Decay of u at infinity.** We show in detail that (1.16) holds. Let $j \in \mathbb{N}$ big enough and $z \in \mathbb{R}^2$ such that $\rho_j \leq r = |z| \leq \rho_{j+1}$. Denoting $\frac{r}{\rho_k} = 1 + h$, thanks to (4.3) and (4.6) we have that $h = O(\rho_k^{-\frac{1+\delta}{2}})$ and $n_k h^2 = O(1)$. Using (4.14) we have that

$$\begin{aligned}\log|u(z)| - \log|u(\rho_k)| \leq& - n_k \log r + n_k \log \rho_k + O(1) = -n_k \log \frac{r}{\rho_k} + O(1)\\ \leq& - n_k h + \frac{n_k h^2}{2} + O(1) = -n_k h + O(1).\end{aligned}$$

Moreover, for $m(r) := e^{-\frac{r^{1+\delta}}{1+\delta}}$, we have

$$\log m(r) - \log m(\rho_k) = -\frac{r^{1+\delta}}{1+\delta} + \frac{\rho_k^{1+\delta}}{1+\delta} = -\frac{\rho_k^{1+\delta}}{1+\delta}\Big((1+h)^{1+\delta} - 1\Big) = -\rho_k^{1+\delta} h + O(1).$$

From the previous reasoning, we have immediately that

$$\log|u(z)| - \log|u(\rho_k)| \leq \log m(r) - \log m(\rho_k) + O(1). \tag{4.21}$$

From (4.21) we have moreover that for all k big enough

$$\log|u(\rho_k)| - \log|u(\rho_{k-1})| \leq \log m(\rho_k) - \log m(\rho_{k-1}) + O(1). \tag{4.22}$$

Adding (4.21) and all the (4.22), we have that

$$\log|u(z)| \leq \log m(r) + O(k).$$

From (4.6) we have

$$k = O(\rho_k^{\frac{1+\delta}{2}}),$$

and we can conclude immediately (1.16), for the appropriate $C_1, C_2 > 0$.

5. Proof of Theorem 1.9

The proof of Theorem 1.9 is very similar to the proof of Theorem 1.8, so it is just sketched underlining the main differences.

5.1. **Preliminary definitions.** In this section we collect some definitions and elementary results we need in the proof. We set

$$r = |x|, \quad \hat{x} = x/|x| \quad \text{and} \quad L = -ix \times \nabla \quad \text{for } x \in \mathbb{R}^3 \setminus \{0\}.$$

In the following we will often use polar coordinates.

For $l = 0, 1, \ldots$ and $m = -l, -l+1, \ldots, l$, let Y_l^m be the *spherical harmonics*

$$\begin{aligned}Y_l^m(\theta, \varphi) &= \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, e^{im\varphi}\, P_l^m(\cos\theta),\\ Y_l^{-m} &= (-1)^m \overline{Y_l^m},\end{aligned}$$

where P_l^m are the *associated Legendre polynomials*

$$P_l^m(x) = \frac{(-1)^m}{2^l l!}(1-x^2)^{m/2}\,\frac{d^{l+m}}{dx^{l+m}}(x^2-1)^l.$$

For $\kappa = \pm 1, \pm 2, \dots$, $j = |k| - 1/2 = 1/2, 3/2, \dots$, and $m_j = -j, -j+1, \dots, j$, let $\mathcal{Y}_{\kappa,m_j}$ be the *spinor harmonics*

$$\mathcal{Y}_{\kappa,m_j} = \begin{cases} \frac{1}{\sqrt{2\kappa-1}}\begin{pmatrix} \sqrt{\kappa - \frac12 + m_j}\,Y_{\kappa-1}^{m_j-1/2} \\ \sqrt{\kappa - \frac12 - m_j}\,Y_{\kappa-1}^{m_j+1/2}\end{pmatrix}, & \kappa \geq 1, \\ \frac{1}{\sqrt{1-2\kappa}}\begin{pmatrix} -\sqrt{\frac12 - \kappa - m_j}\,Y_{-\kappa}^{m_j-1/2} \\ \sqrt{\frac12 - \kappa + m_j}\,Y_{-\kappa}^{m_j+1/2}\end{pmatrix}, & \kappa \leq -1. \end{cases}$$

It is well known (see [34, Section 3.9.4] or [33, Section 4.6.4]) that

$$(\mathbb{I}_2 + \sigma\cdot L)\,\mathcal{Y}_{\kappa,m_j} = \kappa\,\mathcal{Y}_{\kappa,m_j}, \tag{5.1}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of the *Pauli matrices* (defined in (1.5)).

5.1.1. *Definition of $\delta, n_k, d_k, \rho_k, a_k$.* Let $\delta > -1$ as in (4.1). Let n_0 be a large odd number and for all $k \geq 0$ let n_k, d_k, ρ_k and a_k be defined as in (4.2) – (4.4).

5.1.2. *Definition of F_{n_k}.* For $m \in \mathbb{N}$ set

$$F_m(\theta,\phi) := c_m \mathcal{Y}_{-(m-1),\frac12}(\theta,\phi) = \frac{c_m}{\sqrt{4\pi}}\begin{pmatrix} -\sqrt{m-1}P_{m-1}^0(\cos\theta) \\ \frac{1}{\sqrt{m-1}}e^{i\phi}P_{m-1}^1(\cos\theta)\end{pmatrix}, \tag{5.2}$$

with $c_m > 0$ defined later (in (5.6)). We show that for all large odd m the functions F_m satisfy

$$Cm^{-\frac74} \leq |F_m(\theta,\phi)| \leq 1, \tag{5.3}$$

for some $C > 0$. Indeed, from (5.2) we have that

$$|F_m(\theta,\phi)|^2 = \frac{c_m^2}{4\pi}\left((m-1)|P_{m-1}^0(\cos\theta)|^2 + \frac{1}{m-1}|P_{m-1}^1(\cos\theta)|^2\right).$$

We remind here the following property of the associated Legendre polynomials, proved in [15, Lemma 3.8]: there exists $C > 0$ such that for all large $l \in 2\mathbb{N}$

$$\frac{1}{Cl^{3/2}} \leq l(l+1)|P_l^0(x)|^2 + |P_l^1(x)|^2 \leq Cl(l+1), \qquad \text{for all } x \in [-1,1]. \tag{5.4}$$

Choosing $l = m-1$, with an easy computation we get

$$\frac{1}{C(m-1)^{3/2}m} \leq \frac{4\pi}{c_m^2}|F_m(\theta,\phi)|^2 - \frac{1}{m(m-1)}|P_{m-1}^1(\cos\theta)|^2 \leq C(m-1).$$

From (5.4) and the last equation we have

$$\frac{1}{C(m-1)^{3/2}m} \leq \frac{4\pi}{c_m^2}|F_m(\theta,\phi)|^2 \leq C(m-1) + \frac{1}{m(m-1)}|P_{m-1}^1(\theta)|^2 \leq C'm, \tag{5.5}$$

for an appropriate $C' > 0$ and big enough m. We set

$$c_k := \sqrt{\frac{4\pi}{C'm}}, \tag{5.6}$$

and we conclude (5.3) for the appropriate $C > 0$, thanks to (5.5).

5.1.3. *Definition of $\boldsymbol{E}_k$.* For $k \in \mathbb{N}$ big enough and $r := |x| \geq \rho_k$ we define

$$E_k(x) := a_k r^{-n_k} F_{n_k}(\theta, \phi),$$
$$\mathbf{E}_k(x) := \begin{pmatrix} E_k(x) \\ 0 \end{pmatrix} \text{ if } k \text{ is even}, \quad \mathbf{E}_k(x) := \begin{pmatrix} 0 \\ E_k(x) \end{pmatrix} \text{ if } k \text{ is odd}.$$

An easy computation (see [33, eq. (4.104)]) and (5.1) show that for all $x \in \mathbb{R}^3 \setminus \{0\}$ and $k \in \mathbb{N}$

$$-i\sigma \cdot \nabla E_k(x) = -i\sigma \cdot \hat{x} \left(\partial_r + \frac{1}{r} - \frac{1 + \sigma \cdot L}{r} \right) E_k(x) = 0,$$

that gives immediately

$$-i\alpha \cdot \nabla \mathbf{E}_k(x) = 0.$$

Moreover, thanks to (4.7), (4.17) and (5.3) we have that for k big enough

$$\frac{|\mathbf{E}_{k+1}(x)|}{|\mathbf{E}_k(x)|} = O(n_k^{\frac{7}{4}}), \quad \frac{|\mathbf{E}_k(x)|}{|\mathbf{E}_{k+1}(x)|} = O(n_k^{\frac{7}{4}}). \tag{5.7}$$

for the appropriate $C > 0$.

5.2. **Definition of u_k and $\mathbb{V}_k$ in the annulus $\{\rho_k \leq |z| \leq \rho_{k+1}\}$.** For $k \in \mathbb{N}$ big enough, in this section we construct functions

$$\begin{aligned} &u_k \in C^\infty(\{x \in \mathbb{R}^3 \colon \rho_k \leq |x| \leq \rho_{k+1}\}; \mathbb{C}^4), \\ &\mathbb{V}_k \in C^\infty(\{x \in \mathbb{R}^3 \colon \rho_k \leq |x| \leq \rho_{k+1}\}; \mathbb{C}^{2\times 2}), \end{aligned} \tag{5.8}$$

such that

$$\mathcal{D}_3 u_k(z) = \mathbb{V}_k(z) u_k(z), \qquad \text{for all } \rho_k \leq |x| \leq \rho_{k+1}, \tag{5.9}$$

and

$$u_k(x) = \begin{cases} \mathbf{E}_k(x) & \text{for } r \in [\rho_{k0}, \rho_{k1}], \\ \mathbf{E}_{k+1}(x) & \text{for } r \in [\rho_{k3}, \rho_{k4}], \end{cases} \tag{5.10}$$

for $\rho_k \leq |x| \leq \rho_{k+1}$

$$|u_k(x)| = O(a_k r^{-n_k}), \tag{5.11}$$

and (1.21) holds. In the proof in this section we assume k to be a odd integer: the case of even k can be treated analogously.

5.2.1. *Construction of the cut-off functions.* Let χ be a real non-decreasing C^∞ function on $\mathbb{R}$ such that

$$\chi(s) = \begin{cases} 0 & s \leq 0, \\ e^{-1/s} & 0 \leq s \leq 1/2, \\ 1 & s \geq 1, \end{cases}$$

and for $\rho_k \leq r \leq \rho_{k+1}$ let χ_k and $\tilde{\chi}_k$ be defined as in (4.15). We remark that (4.16) holds.

5.2.2. *Definition of the functions u_k and $\mathbb{V}_k$.* For $\rho_k \leq r = |x| \leq \rho_{k+1}$, let

$$u_k(x) := \tilde{\chi}_k(r)\mathbf{E}_k(x) + \chi_k(r)\mathbf{E}_{k+1}(x) = \begin{pmatrix} \chi_k(r)E_{k+1}(x) \\ \tilde{\chi}_k(r)E_k(x) \end{pmatrix}.$$

Let

$$\mathbb{V}_k(x) := \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad r \in [\rho_{k0}, \rho_{k1}] \cup [\rho_{k3}, \rho_{k4}],$$

and for $r \in [\rho_{k1}, \rho_{k2}] \cup [\rho_{k2}, \rho_{k3}]$ let $\mathbb{V}_k(x) := \mathcal{D}_3 u_k(x)\, \overline{u_k(x)}^t / |u_k(x)|^2$, that is

$$\mathbb{V}_k(x) := \begin{cases} \frac{1}{|\chi_k E_{k+1}|^2 + |E_k|^2} \begin{pmatrix} 0 & 0 \\ 0 & -i\sigma\cdot\hat{x} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \chi_k' E_{k+1}\cdot \chi_k \overline{E_{k+1}}^t & \chi_k' E_{k+1} \cdot \overline{E_k}^t \end{pmatrix} & \text{for } r \in [\rho_{k1}, \rho_{k2}], \\ \frac{1}{|\chi_k E_{k+1}|^2 + |E_k|^2} \begin{pmatrix} -i\sigma\cdot\hat{x} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{\chi}_k' E_k \cdot \overline{E_{k+1}}^t & \tilde{\chi}_k' E_k \cdot \tilde{\chi}_k \overline{E_k}^t \\ 0 & 0 \end{pmatrix} & \text{for } r \in [\rho_{k2}, \rho_{k3}]. \end{cases}$$

By construction (5.8), (5.9) and (5.10) are true; thanks to (5.7), (5.11) holds for a large enough k.

Condition (1.21) holds obviously for $r \in [\rho_{k0}, \rho_{k1}] \cup [\rho_{k3}, \rho_{k4}]$. For the case $r \in [\rho_{k1}, \rho_{k2}]$ we need to distinguish various cases: let $s := 4(r - \rho_{k1})/(\rho_{k+1} - \rho_k)$ and let us consider first the case $s \in (0, (\log \rho_k)^{-3/2})$. We have that

$$\begin{aligned} |\mathbb{V}_{21}(x)| \leq & \frac{|\chi_k'(r)||E_{k+1}(x)||\chi_k(r)||E_{k+1}(x)|}{|\chi_k(r)E_{k+1}|^2 + |E_k(x)|^2} \\ \leq & \frac{|\chi_k'(r)||E_{k+1}(x)||\chi_k(r)||E_{k+1}(x)|}{2|\chi_k(r)||E_{k+1}(x)||E_k(x)|} = \frac{|\chi_k'(r)||E_{k+1}(x)|}{2|E_k(x)|} \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} |\mathbb{V}_{22}(x)| \leq & \frac{|\chi_k'(r)||E_{k+1}(x)||E_k(x)|}{|\chi_k(r)E_{k+1}|^2 + |E_k(x)|^2} \\ \leq & \frac{|\chi_k'(r)||E_{k+1}(x)||E_k(x)|}{|E_k(x)|^2} = \frac{|\chi_k'(r)||E_{k+1}(x)|}{|E_k(x)|}. \end{aligned} \tag{5.13}$$

We observe that

$$\chi_k'(r) = \frac{4}{\rho_{k+1} - \rho_k}\chi'(s) = O(\rho_k^{-\frac{1-\delta}{2}})\chi'(s). \tag{5.14}$$

In this interval χ has an explicit expression: since χ' is increasing we get for large k that

$$\chi_k'(r) \leq O(\rho_k^{-\frac{1-\delta}{2}})\chi'((\log \rho_k)^{-3/2}) = O(\rho_k^{-\frac{1-\delta}{2}})(\log \rho_k)^3 e^{-(\log \rho_k)^{3/2}}. \tag{5.15}$$

Thanks to (5.12), (5.13), (5.15) and (5.7) we have that for some $C > 0$ and for big k

$$\begin{aligned} |\mathbb{V}(x)| \leq & C|\chi_k'(r) n_k^{\frac{7}{4}}| \leq C\rho_k^{-\frac{1-\delta}{2}}(\log \rho_k)^3 e^{-(\log \rho_k)^{3/2}} n_k^{\frac{7}{4}} \\ \leq & C(\log \rho_k)^3 \rho_k^{-\frac{1-\delta}{2}} \leq C(\log|x|)^3 |x|^{-\epsilon}, \end{aligned} \tag{5.16}$$

using (4.1) in the last inequality.

If $s \geq (\log \rho_k)^{-3/2}$, we have

$$|\mathbb{V}_{21}(x)| \leq \frac{|\chi_k'(r)||E_{k+1}(x)||\chi_k(r)||E_{k+1}(x)|}{|\chi_k(r)E_{k+1}|^2 + |E_k(x)|^2} \leq \frac{|\chi_k'(r)|}{|\chi_k(r)|} \tag{5.17}$$

and

$$|\mathbb{V}_{22}(x)| \le \frac{|\chi_k'(r)||E_{k+1}(x)||E_k(x)|}{|\chi_k(r)E_{k+1}|^2 + |E_k(x)|^2} \le \frac{|\chi_k'(r)|}{2|\chi_k(r)|}. \tag{5.18}$$

If $s \in [(\log \rho_k)^{-3/2}, \frac{1}{2}]$, thanks to (5.17), (5.18), (5.14) and the explicit expression for χ_k, for large k we have that

$$|\mathbb{V}(x)| = O(\rho_k^{-\frac{1-\delta}{2}})s^{-2} \le C(\log \rho_k)^3 \rho_k^{-\frac{1-\delta}{2}} \le C(\log|x|)^3|x|^{-\epsilon}, \tag{5.19}$$

while in the case $s \in (1/2, 1)$, since χ is non-decreasing, we have that for some $C > 0$

$$|\mathbb{V}(x)| \le C\rho_k^{-\frac{1-\delta}{2}} \le C|x|^{-\epsilon}. \tag{5.20}$$

Gathering (5.16), (5.19) and (5.20) we have (1.21) for all $r \in [\rho_{k1}, \rho_{k2}]$. Finally, the case $r \in [\rho_{k2}, \rho_{k3}]$ is analogous and will be omitted in this proof.

5.3. **Conclusion of the proof.** The remaining part of the proof is analogous to the proof of Theorem 1.8: the construction of u in $\{|x| \le \rho_{k_0}\}$ is done as in Section 4.4, the construction of u in $\{|x| \ge \rho_{k_0}\}$ as in Section 4.5 and the study of the decay of u at infinity as in Section 4.6.

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