SEMIALGEBRAIC CMC SURFACES IN $\mathbb{R}^3$ WITH SINGULARITIES

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Abstract. In this paper we present a classification of a class of semialgebraic CMC surfaces in $\mathbb{R}^3$ that generalizes the recent classification made by Barbosa and do Carmo in 2016 (complete reference is in the paper), we show that a semialgebraic CMC surface in $\mathbb{R}^3$ with isolated singularities and suitable conditions on the singularities and of local connectedness is a plane or a finite union of round spheres and cylinders touching at the singularities. As a consequence, we obtain that a semialgebraic good CMC surface in $\mathbb{R}^3$ that is a topological manifold does not have isolated singularities and, moreover, it is a plane or a round sphere or a cylinder. A result in the case non-isolated singularities also is presented.

1. Introduction

The question of describing the minimal surfaces or, more generally, surfaces of constant mean curvature (CMC surfaces) is known in Analysis and Differential Geometry since the classical papers of Bernstein [4], Bombieri, De Giorgi and Giusti [8], Hopf [15] and Alexandrov [1]. Recently, in the paper [2], Barbosa and do Carmo showed that the connected algebraic smooth CMC surfaces in $\mathbb{R}^3$ are only the planes, round spheres and cylinders. A generalization of this result it was proven in Barbosa et al. [3], it was obtained the same conclusion in the case of connected globally subanalytic smooth CMC surfaces in $\mathbb{R}^3$.

When we know well about smooth sets in a certain category, it is natural to think about objects in this category that have singularities. For example, minimal submanifolds with singularities apply an important rule in the study of minimal submanifolds and there are already many works on minimal submanifolds with singularities see, for example, [5], [8], [9], [13], [20] and [24]. Let me remark that these papers are devoted to give conditions on the CMC surface with singularities such that their singularities are removable or to present examples with non-removable singularities.

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Thus, this paper is devoted to study and classify algebraic CMC surfaces in $\mathbb{R}^3$ with possibly non-removable singularities. For instance, it is easy find examples of algebraic surfaces with non-removable isolated singularities that have non-zero constant mean curvature, namely, finite unions of round spheres and cylinders touching at the singularities.

A first natural question is: Are there further examples?

The main aim of this paper is show that the answer to the question above is no, when we impose a suitable condition on the singularities and a suitable condition of local connectedness called here by LP connected (see definition 2.12).

As it was said before, some of the papers quoted above are devoted to give conditions on the CMC surface with singularities such that their singularities are removable. Look for minimal surfaces with removable singularities is also a subject studied in Complex Algebraic Geometry, since any complex analytic set is a minimal surface (possibly with singularities) (see p. 180 in [10]). A pioneer result in the topology of singular analytic surfaces is the Mumford's Theorem (see [19]) that in $\mathbb{C}^3$ this result can be formulated as follows: *if $X \subset \mathbb{C}^3$ is a complex analytic surface with an isolated singularity p and its link at p has trivial fundamental group, then X is smooth at p.* We can find some results related with Mumford's Theorem in [6], [21] and [22].

Let me describe how this paper is organized. Section 2 is presented some definitions and main properties used in the paper about semialgebraic sets. Section 3 is dedicated to show the main results of this paper. It is presented a classification in Theorem 3.1 of the semialgebraic CMC surfaces $X \subset \mathbb{R}^3$ with $\dim \mathrm{Sing}(X) < 1$ and the closure of each connected component of $X \setminus \mathrm{Sing}(X)$ is a good CMC surface (see definition 2.10) and LP connected. In particular, we can change the condition "regular" in ([2], Theorem 1.1) by "topological manifold with $\dim \mathrm{Sing}(X) < 1$" and we obtain still the same conclusion, i.e., it is proven that if a closed set $X \subset \mathbb{R}^3$ is topological manifold and a semialgebraic good CMC surface with $\dim \mathrm{Sing}(X) < 1$, then X is a plane or a round sphere or a cylinder (see Corollary 3.9). Moreover, it is given a classification of semialgebraic CMC surfaces $X \subset \mathbb{R}^3$ with $\dim \mathrm{Sing}(X) = 1$ when each connected component of $X \setminus \mathrm{Sing}(X)$ has smooth closure. Finally, it is proven a real version of the Mumford's Theorem.

2. Preliminaries

In this section, we make a brief exposition about semialgebraic sets. In order to know more about semialgebraic sets, for example, see [11], [7] and [25].

Definition 2.1 (Algebraic sets). *A subet $X \in \mathbb{R}^n$ is called* **algebraic** *if there are polynomials $p_1, \dots, p_k : \mathbb{R}^n \to \mathbb{R}$ such that $X = \{x \in \mathbb{R}^n;\ p_k(x) = \cdots = p_k(x) = 0\}$.*

Definition 2.2 (Semialgebraic sets). *A subet $X \in \mathbb{R}^n$ is called* **semialgebraic** *if X can be written as a finite union of sets of the form $\{x \in \mathbb{R}^n;\ p(x) = 0, q_1(x) >$*

$0, \ldots, q_k(x) > 0\}$, *where* $p, q_1, \ldots, q_k$ *are polynomials on* $\mathbb{R}^n$. *A function* $f : X \subset \mathbb{R}^n \to \mathbb{R}^p$ *is said to be a* **semialgebraic function** *if its graph is a semialgebraic set.*

It is clear of the definition that the complement or finite intersection of semialgebraic sets is still a semialgebraic set. Next results tell us more about semialgebraic sets.

Proposition 2.3 ([7], Proposition 2.2.2)**.** *The closure and the interior of a semi-algebraic set are semi-algebraic.*

Notation. Let p be a point in $\mathbb{R}^n$, $Y \subset \mathbb{R}^n$ and $\varepsilon > 0$. Then we denote the sphere with center p and radius ε by

$$\mathbb{S}^{n-1}_{\varepsilon_0}(p) := \{x \in \mathbb{R}^n;\ \|x - p\| = \varepsilon\},$$

the open ball with center p and radius ε by

$$B^n_{\varepsilon_0}(p) := \{x \in \mathbb{R}^n;\ \|x - p\| < \varepsilon\},$$

and the cone over Y with vertex p by

$$Cone_p(Y) := \{tx + p \in \mathbb{R}^n;\ x \in Y \text{ and } t \in [0, 1]\}.$$

Definition 2.4 (Analytic sets)**.** *A subset* $X \subset \mathbb{R}^n$ *is called* **analytic** *if for each* $x \in \mathbb{R}^n$ *there exists an open neighborhood* U *of* x *in* $\mathbb{R}^n$ *such that* $U \cap X$ *can be written as a finite union of sets of the form* $\{x \in \mathbb{R}^n \mid p_k(x) = \cdots = p_k(x) = 0\}$, *where* $p_1, \ldots, p_k$ *are analytic functions on* U.

Proposition 2.5 ([17], Corollary 2.9 and Theorem 2.10)**.** *Let* $Y \subset \mathbb{R}^n$ *be an analytic set and* $p \in Y$. *Suppose that there exists an open neighborhood* U *of* p *such that* $Y \cap U \setminus \{p\}$ *is smooth. For* $\varepsilon > 0$ *sufficiently small,* $Y \cap \mathbb{S}^{n-1}_{\varepsilon}(p)$ *is a smooth manifold and* $Y \cap \overline{B^n_{\varepsilon}(p)}$ *is homeomorphic to* $Cone_p(Y \cap \mathbb{S}^{n-1}_{\varepsilon}(p))$.

In this case, we denote the set $Y \cap \mathbb{S}^{n-1}_{\varepsilon}(p)$ by $link_p(Y)$ and it is called **the link of** X **at** p.

Definition 2.6. *Let* $X \subset \mathbb{R}^n$ *be a subset. The* **singular set** *of* X, *denoted by* $\mathrm{Sing}(X)$, *is the set of points* $x \in X$ *such that* $U \cap X$ *is not a smooth manifold for any open neighborhood* U *of* x.

Therefore, if X is a semialgebraic or an analytic subset, then $X \setminus \mathrm{Sing}(X)$ is a finite union of non-empty smooth semialgebraic manifolds (see [11] and [17]), let us denote them by $X_1, ..., X_k$. Thus, we define the **dimension** of X by $\dim X = \max \dim X_i$ and we say that X has **pure dimension**, if $\dim X_i = \dim X_j$ for all $i, j \in \{1, ..., k\}$.

Here we assume that the sets have pure dimension.

Definition 2.7. *Let $X \subset \mathbb{R}^n$ be a set and $x_0 \in \overline{X}$. We say that $v \in \mathbb{R}^n$ is a tangent vector of X at $x_0 \in \mathbb{R}^n$ if there are a sequence of points $\{x_i\} \subset X$ tending to x_0 and sequence of positive real numbers $\{t_i\}$ such that*

$$\lim_{i\to\infty} \frac{1}{t_i}(x_i - x_0) = v.$$

Let $C(X, x_0)$ denote the set of all tangent vectors of X at $x_0 \in \mathbb{R}^n$. We call $C(X, x_0)$ the **tangent cone** *of X at x_0.*

Definition 2.8. *Let $X \subset \mathbb{R}^n$ be a set and $x_0 \in \overline{X}$ be a non-isolated point. Suppose that $X \backslash \mathrm{Sing}(X)$ is a smooth manifold with dimension d. We denote by $\mathcal{N}(X, x_0)$ the subset of the Grasmannian $Gr(d, \mathbb{R}^n)$ of all d-dimensional linear subpaces $T \subset \mathbb{R}^n$ such that there is a sequence of points $\{x_i\} \subset X \setminus \mathrm{Sing}(X)$ tending to x_0 and $\lim T_{x_i}X = T$. We denote by $\widetilde{\mathcal{N}}(X, x_0)$ the subset of $\mathbb{R}^n$ given by the union of all $T \in \mathcal{N}(X, x_0)$.*

Definition 2.9. *Let $X \subset \mathbb{R}^n$ be a subset with $d = \dim X \setminus \mathrm{Sing}(X)$. We say that X* **has a good tangent cone at** *$p \in X$ if*

(i) *$C(X, p)$ has at least two different lines passing through the origin;*
(ii) *$\widetilde{\mathcal{N}}(X, p) \subsetneq \mathbb{R}^n$.*

When X has a good tangent cone at p for all $p \in X$, we say that X **has good tangent cones**.

Remind that if $X \subset \mathbb{R}^3$ is a smooth surface, there exists a smooth function $H : X \to \mathbb{R}$ called **mean curvature function** such that if X is locally expressed as the graph of a smooth function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$, then u satisfies the following PDE

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 2H(1 + u_x^2 + u_y^2)^{\frac{3}{2}}. \tag{1}$$

When H is a constant function, we say that X is a **smooth CMC surface** (with mean curvature H). Since is well known that any smooth CMC surface in $\mathbb{R}^3$ is an analytic set, in the following definition we generalize the notion of smooth CMC surface.

Definition 2.10. *We say that an analytic subset $X \subset \mathbb{R}^3$ is a* **CMC surface** *(with mean curvature H) if $X \setminus \mathrm{Sing}(X)$ is a smooth CMC surface (with mean curvature H). In addiction, if X has a good tangent cone at p for all $p \in X$, we say that X is a* **good CMC surface** *(with mean curvature H).*

Remark 2.11. *Since $\mathcal{N}(X, p) = \{T_pX\}$ and $C(X, p) = T_pX$ when p is a smooth point of X, then any smooth CMC surface is a good CMC surface.*

Definition 2.12. *We say that a subset $Y \subset \mathbb{R}^n$ is* **locally punctured connected** *or* **LP connected** *if for each $y \in Y$, there exists $\varepsilon_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$, $(Y \setminus \{y\}) \cap B^n_\varepsilon(y)$ has a unique connected component C with $y \in \overline{C}$.*

The definition of LP connected subset is essentialy more general than the definitioin of topological submanifold in $\mathbb{R}^n$, since any topological submanifold $Y^m \subset \mathbb{R}^n$ ($m > 1$) is LP connected and there exist LP connected sets that are not topological submanifolds, for example, the union of two transversal planes in $\mathbb{R}^3$.

3. CMC surfaces with singularities

Theorem 3.1. *Let $X \subset \mathbb{R}^3$ be a closed and connected semialgebraic set. Suppose that $\dim \mathrm{Sing}(X) < 1$ and the closure of each connected component of $X \setminus \mathrm{Sing}(X)$ is LP connected and has good tangent cones. If X is a CMC surface with mean curvature H, then we have the following:*

(1) if $H = 0$, then $\mathrm{Sing}(X) = \emptyset$ and X is a plane;

(2) if $H \neq 0$, then

(i) X is a round sphere or a cylinder, when $\mathrm{Sing}(X) = \emptyset$;

(ii) X is a finite union of round spheres and cylinders touching at the points of $\mathrm{Sing}(X)$, $\mathrm{Sing}(X) \neq \emptyset$.

Proof. Let $X_1, ..., X_m$ be the connected components of $X \setminus \mathrm{Sing}(X)$. Thus, $X = \bigcup_{i=1}^{m} \overline{X_i}$. Fixed $i \in \{1, ..., m\}$, it is enough to show that $Z := \overline{X}_i$ is a plane or a round sphere or a cylinder.

Claim 1. *Z is a topological manifold.*

Proof of the Claim 1. Let $p \in Z$. If $p \notin \mathrm{Sing}(X)$, it is clear that Z is topological manifold around p. Thus, we can assume that $p \in \mathrm{Sing}(X)$. Let $\varepsilon_0 > 0$ be a number that satisfies the definition 2.12 and Proposition 2.5. Then, $(Z \cap B^3_{\varepsilon_0}(p)) \setminus \{p\}$ is connected and $Z \cap \mathbb{S}^2_{\varepsilon_0}(p)$ is smooth (and compact). By Lemma 2.5, $Z \cap \mathbb{S}^2_{\varepsilon_0}(p)$ is connected and using the classification of compact smooth curves, we have that $Z \cap \mathbb{S}^2_{\varepsilon_0}(p)$ is homeomorphic to $\mathbb{S}^1$. Therefore, by Proposition 2.5, once again, $Y := Z \cap B^3_{\varepsilon_0}(p)$ is a topological manifold. □

Let $p \in Z$. By hypothesis Z has good tangent cones, then there exists a plane $H \subset \mathbb{R}^3$ such that $C(Z, p) \cap H^\perp = \{0\}$ and $T \cap H^\perp = \{0\}$ for all $T \in \mathcal{N}(Z, p)$. Let $\pi \colon \mathbb{R}^3 \to H \cong \mathbb{R}^2$ be the orthogonal projection.

Claim 2. *There exists an open neighborhood A of p in Z such that $\pi|_A$ is an open mapping.*

Proof of the Claim 2. This proof shares its structure with the proof of Lemma 3.3 in [18] and Lemma 4.1 in [16].

Since $\dim X < 1$, we can choose an open neighborhood U of p so small such that $T \cap H^\perp = \{0\}$ for all $T \in \mathcal{N}(Z, q)$ and for all $q \in U \cap Z$ and $Z \cap U \setminus \{p\}$ is smooth. In particular, $C(Z, q) = T_q Z$ for all $q \in Z \cap U \setminus \{p\}$ and $C(Z, q) \cap H^\perp = \{0\}$ for all $q \in Z \cap A$. We claim that π is then an open mapping on $A := Z \cap U$. To see this, fix $q \in A$. Since Z is locally closed, there is an $r > 0$ small enough such that

$V := \overline{B^3_r(q)} \cap Z \subset A$ is compact, and the topological boundary ∂V of V in Z lies on $\partial B^3_r(q)$, which implies that $q \notin \partial V$. Moreover, for a small enough $r > 0$, we can assume that

$$\pi(q) \notin \pi(\partial V)$$

otherwise, there exists a sequence of positive numbers $\{t_k\}$ tending to 0 such that for each k, there is a point $q_k \in Z \cap \partial \overline{B^3_{r_i}(q)}$ with $\pi(q_k) = \pi(q)$. So, extracting a subsequence if necessary, we can assume that $\lim \frac{1}{t_k}(q_k - q) = v \neq 0$. This implies that $v \in H^\perp \cap C(Z, q)$, which is a contradiction, since $C(Z, w) \cap H^\perp = \{0\}$ for all $w \in Z \cap A$. Since $\pi(\partial V)$ is a compact set, there is $s > 0$ such that

$$\overline{B^2_s(\pi(q))} \cap \pi(\partial V) = \emptyset. \tag{2}$$

It is enough to show that $\pi(q)$ is an interior point of $\pi(V)$ in H.

Thus, suppose by contradiction that $\pi(q)$ is not an interior point of $\pi(V)$. Then $B^2_\delta(\pi(q)) \not\subset \pi(V)$, for any $\delta > 0$. In particular, there is a point

$$x \in B^2_{s/2}(\pi(q)) \setminus \pi(V).$$

Since $x \notin \pi(V)$ and $\pi(V)$ is compact, for $t = dist(x, \pi(V))$, we have that $\overline{B^2_t(x)} \subset H$ intersects $\pi(V)$ while $B^2_t(x) \cap \pi(V) = \emptyset$. Moreover, since $q \in V$, we have $t \leq \|x - \pi(q)\| < s/2$, and if $y \in \overline{B^2_t(x)}$ then

$$\|y - \pi(q)\| \leq \|y - x\| + \|x - \pi(q)\| \leq t + s/2 < s,$$

which yields

$$\overline{B^2_t(x)} \subset \overline{B^2_s(\pi(q))}.$$

Thus $\overline{B^2_t(x)} \cap \pi(\partial V) = \emptyset$ by (2). Take $y' \in \overline{B^2_t(x)} \cap \pi(V)$ and $y \in \pi^{-1}(y') \cap V$. Note that $y \notin \partial V$, so y is an interior point of V, and hence $C(V, y) = C(Z, y)$.

Since $B^2_t(x) \cap \pi(V) = \emptyset$, no point of V is contained in the cylinder $C := \pi^{-1}(\overline{B^2_t(x)})$. This implies that $\ell \subset T_y \partial C$ for each line $\ell \subset C(Z, y)$ passing through the origin, which is a contradiction, since $C(Z, p)$ has at least two different lines passing through the origin, $C(Z, p) \cap H^\perp = \{0\}$ and $T_y \partial C$ is a hyperplane with $H^\perp \subset T_y \partial C$.

Therefore, $\pi(q)$ is an interior point of $\pi(V)$ and this finish the proof. □

Claim 3. *Z can be locally expressed as the graph of a continuous function.*

Proof of the Claim 3. Let $y \in Z$. After a translation, if necessary, we can assume that $y = 0$. By Claim 2, there exists a bounded neighborhood A of the origin in Z such that $\pi = \pi|_A : A \to \pi(A)$ is an open mapping and $\pi(A)$ is an open in $\mathbb{R}^2$. After a rotation, if necessary, we can suppose that π is the projection on the two first coordinates. Let $B \subset A$ be a bounded open subset of Z such that $K = \overline{B} \subset A$ and $0 \in B$. Let $f : \Omega := \pi(K) \to \mathbb{R}$ given by $f(x) = \max\{y_3 \in \mathbb{R}; (y_1, y_2, y_3) \in \pi^{-1}(x) \cap K\}$. Then, $f(0) = 0$ and $0 \in Graph(f)$.

We are going to show that f is a continuous function. Since Z is semialgebraic, we have that if $x \in \Omega$, then $\pi^{-1}(x) \cap K$ is finite and, in particular, $\pi^{-1}(x) \cap K$ is discrete. Let $\{x_k\}_{k\in\mathbb{N}} \subset \Omega$ with $x_k \to x \in \Omega$ and for each $k \in \mathbb{N}$ we define $\bar{x}_k = (x_k, f(x_k))$. Then, taking a subsequence, if necessary, we have that $\bar{x}_k \to \bar{x}' \in K$, since K is compact. But $\bar{x}' \in \pi^{-1}(x) \cap K$, then $\pi_3(\bar{x}') \leq \pi_3(\bar{x}) = f(x)$, where $\pi_3 : \mathbb{R}^3 \to \mathbb{R}$ is the projection on the third coordinate. Suppose that $\pi_3(\bar{x}') < \pi_3(\bar{x})$, then there exist B' and B disjoints open balls with center $\bar{x}'$ and $\bar{x}$, resp., such that $\pi(B' \cap Z)$ and $\pi(B \cap Z)$ are open subsets of $\mathbb{R}^2$. Hence, $V' := B' \cap Z$ is bellow of $V := B \cap Z$. Then, for k sufficiently large, $\bar{x}_k \in V'$ and $x_k \in \pi(V') \cap \pi(V)$. Therefore, $\pi^{-1}(x_k) \cap V$ is above of $\pi^{-1}(x_k) \cap V'$, but this is a contradiction. Thus f is continuous at x.

There exist an open neighborhood W of the origin in $\mathbb{R}^3$ and an open ball $B_r(0) \subset \mathbb{R}^2$ such that $Graph(f|_{B_r(0)}) = Z \cap W$. In fact, since Z is a topological manifold by Claim 1, there exists a homeomorphism $\varphi : W_1 \cap Z \to B_s(0) \subset \mathbb{R}^2$, where W_1 is an open neighborhood of the origin in $\mathbb{R}^3$ and using that f is a continuous function, we can take $r > 0$ such that $Graph(f|_{B_r(0)}) \subset W_1 \cap Z$. Moreover, let $\psi : B_r(0) \to W_1 \cap Z$ be the mapping given by $\psi(y) = (y, f(y))$, we have that $\varphi \circ \psi : B_r(0) \to B_s(0)$ is a continuous and injective map. Then, by the Invariance of Domain Theorem, $\varphi \circ \psi$ is an open mapping. In particular, $\psi(B_r(0))$ is an open set of Z and, therefore, there exists an open neighborhood W of the origin in $\mathbb{R}^3$ such that $\psi(B_r(0)) = Graph(f|_{B_r(0)}) = Z \cap W$. Then we finish the proof of the Claim 3. □

Claim 4. *Z is a smooth CMC surface.*

Proof of the Claim 4. Let $p \in Z$. It is enough to show that Z is smooth at p, i.e. $p \notin \mathrm{Sing}(Z)$. We can suppose that $p = 0$ and by Claim 3, we can assume that there are $r > 0$ small enough, a continuous function $u : B_r^2(0) \to \mathbb{R}$ and an open subset $W \subset \mathbb{R}^3$ such that $Graph(u) = Y \cap W$ and $\mathbb{R}^2 \times \{0\} \notin \mathcal{N}(X, 0)$. Therefore, u is smooth on $B_r^2(0) \setminus \{0\}$. Thus, by Theorem 3 in ([23], p. 168) (or Theorem in ([14], p. 170)), u is smooth on $B_r^2(0)$. Therefore, Z is a smooth at p. □

Now, we are ready to complete the proof of the theorem. By Theorem 3.2 in [3], Z is a plane, a round sphere or a cylinder, since Z is a smooth CMC surface and also a closed semialgebraic set. Thus, it is clear that X is a finite union of planes or a finite union of round spheres and a cylinders touching at the singularities. However, since X is connected and $\dim \mathrm{Sing}(X) < 1$, if X is a finite union of planes (in the case $H = 0$), then $m = 1$ and X is plane and, in particular, X is smooth. □

The hypothesis $\dim \mathrm{Sing}(X) < 1$ cannot be removed, since there exist CMC surfaces in $\mathbb{R}^3$ with non-isolated singularities that do not satisfy the thesis of Theorem 3.1, as we can see in the next example.

Example 3.2 (Enneper's minimal surface)**.** *Let X be the self-intersecting minimal surface generated using the Enneper-Weierstrass parameterization with $f = 1$ and*

$g = \mathrm{id}$ ([12], p. 93, Proposition 4). With some computations, we can see that X is algebraic and it is given by the following equation (see [26])

$$\left(\tfrac{y^2-x^2}{2}+\tfrac{2z^3}{9}+\tfrac{2z}{3}\right)^3-6z\left(\tfrac{y^2-x^2}{4}-\tfrac{z}{4}\left(x^2+y^2+\tfrac{8}{9}z^2\right)+\tfrac{2z}{9}\right)^2=0.$$

However, it is easy to use the proof of Theorem 3.1 to obtain the following result in the case of non-isolated singularities.

Theorem 3.3. *Let $X\subset\mathbb{R}^3$ be a closed connected semialgebraic set. Suppose that we can write $X\setminus \mathrm{Sing}(X)=\bigcup\limits_{k=1}^{r} X_k$ such that for each $k\in\{1,...,r\}$, the closure of X_k is smooth and X_k is a union of connected components of $X\setminus\mathrm{Sing}(X)$. If X is a CMC surface, then each $\overline{X_k}$ is a plane or a round sphere or a cylinder. In particular, X is a finite union of planes or a finite union of round spheres and cylinders touching at the points of $\mathrm{Sing}(X)$.*

Corollary 3.4. *Let $X\subset\mathbb{R}^3$ be a closed connected semialgebraic set. Suppose that the closure of each connected component of $X\setminus\mathrm{Sing}(X)$ is smooth. If X is a CMC surface, then X is a plane or a finite union of round spheres and cylinders touching at the points of $\mathrm{Sing}(X)$.*

Corollary 3.5. *Let $X\subset\mathbb{R}^3$ be a closed connected semialgebraic set. Suppose that the closure of each connected component of $X\setminus\mathrm{Sing}(X)$ is smooth, $\dim\mathrm{Sing}(X)=1$ and $\mathrm{Sing}(X)$ does not have isolated points. If X is a CMC surface, then $\mathrm{Sing}(X)$ is an union lines and X is a finite union of cylinders touching at the points of $\mathrm{Sing}(X)$.*

Corollary 3.6. *Let $X\subset\mathbb{R}^3$ be a closed connected semialgebraic set. Suppose that X is LP connected and $\dim\mathrm{Sing}(X)<1$. If X is a good CMC surface, then $\mathrm{Sing}(X)=\emptyset$ and X is a plane or a round sphere or a right circular cylinder.*

The hypothesis $\dim\mathrm{Sing}(X)<1$ cannot be removed, since there exist CMC surfaces in $\mathbb{R}^3$ with non-isolated singularities that are LP connected, as we can see in example 3.2. Another example is the following.

Example 3.7. *Let $X=\{x,y,z)\in\mathbb{R}^3;\, xy=0\}$. Then X is an algebraic set, a good CMC surface and LP connected.*

The hypothesis that $X\setminus\mathrm{Sing}(X)$ is LP connected cannot be removed as well, as we can see in the next example.

Example 3.8. *Let $X=\{x,y,z)\in\mathbb{R}^3;\, ((x-1)^2+y^2+z^2-1)((x+1)^2+y^2+z^2-1)=0\}$. Then X is an algebraic good CMC surface. However, for all $0<\varepsilon<1$, $X\setminus\{0\}$ has two connected components such that the closure of each one of them contains the origin.*

Corollary 3.9. *Let $X \subset \mathbb{R}^3$ be a closed and connected semialgebraic set. Suppose that X a is topological manifold and* $\dim \mathrm{Sing}(X) < 1$. *If X is a good CMC surface, then* $\mathrm{Sing}(X) = \emptyset$ *and X is a plane or a round sphere or a right circular cylinder.*

In particular, we obtain the Theorem 1.1 in [2].

Corollary 3.10 ([2], Theorem 1.1 and Proposition 4.1)**.** *Let $X \subset \mathbb{R}^3$ be a connected algebraic set. If X is a smooth CMC surface, then X is a plane or a round sphere or a cylinder.*

3.1. **A real version of Mumford's Theorem.** Since $X = \{(x,y,z) \in \mathbb{R}^3;\ z^3 = x^3y + xy^3\}$ is an analytic surface with an isolated singularity at 0 and such that $\pi_1(link_0(X)) \cong \mathbb{Z}$ and since any complex analytic set is a minimal surface, then the correct assuptions that we need to impose in a real version of the Mumford's Theorem should be an analytic set $X \subset \mathbb{R}^3$ that is a minimal surface with isolated singularities and $\pi_1(link_p(X)) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$. Thus, we obtain the following real version of the Mumford's Theorem.

Theorem 3.11. *Let $X \subset \mathbb{R}^3$ be a good CMC surface. Suppose that* $\dim \mathrm{Sing}(X) < 1$. *If $H_1(link_p(X)) \cong \mathbb{Z}$ then X is smooth at p.*

Proof. Let $\varepsilon > 0$ be a number that satisfies Proposition 2.5. Thus, $X \cap \mathbb{S}^2_\varepsilon(p)$ is a closed and smooth set and the connected components $Y_1, ..., Y_r$ of $X \cap \overline{B^2_\varepsilon(p)} \setminus \{p\}$ are closed manifolds. By classification of compact smooth curves, for each $i \in \{1, ..., r\}$, we have that $Y_i \cap \mathbb{S}^2_\varepsilon(p)$ is homeomorphic to $\mathbb{S}^1$. Then,

$$H_1(link_p(X)) \cong \mathbb{Z}^r.$$

Then, $r = 1$ and by using the same argument used in the proof of Theorem 3.1, we obtain that $Y_i \cup \{p\} = X \cap B_\varepsilon(p)$ is a smooth CMC surface. □

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