Submitted exclusively to the London Mathematical Society
doi:10.1112/0000/000000

Némethi's division algorithm for zeta-functions of plumbed 3-manifolds

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Abstract

A polynomial counterpart of the Seiberg-Witten invariant associated with a negative definite plumbing 3-manifold has been proposed by earlier work of the authors. It is provided by a special decomposition of the zeta-function defined by the combinatorics of the manifold. In this article we give an algorithm, based on multivariable Euclidean division of the zeta-function, for the explicit calculation of the polynomial, in particular for the Seiberg–Witten invariant.

1. *Introduction*

The main motivation of the present article is to understand a multivariable division algorithm, proposed by A. Némethi (cf. [**24**], [**1**]), for the calculation of the normalized Seiberg–Witten invariant of a negative definite plumbed 3-manifold. The input is a multivariable zeta-function associated with the manifold and the output is a (Laurent) polynomial which, in particular, is a polynomial 'categorification' of the Seiberg–Witten invariant in the sense that the sum of its coefficients equals with the normalized Seiberg–Witten invariant. This polynomial was defined by the authors in [**13**] and called the polynomial part as a possible solution for the multivariable 'polynomial- and negative-degree part' decomposition problem for the zeta-function (cf. [**1**, **11**, **13**], see Section 2.4).

The one-variable algorithm goes back to the work of Braun and Némethi [**1**]. In that case the polynomial part is simply given by a division principle. However, in general, we show that in order to recover the multivariable polynomial part of [**13**] one constructs first a 'quotient' polynomial by division and then one has to modify the coefficients of its monomial terms with suitable multiplicity according to the corresponding exponents and the structure of the plumbing graph.

In the sequel, we give some details about the algorithm and state further results of the present note.

1.1.

Let M be a closed oriented plumbed 3-manifold associated with a connected negative definite plumbing graph Γ. Or, equivalently, M is the link of a complex normal surface singularity, and Γ is its dual resolution graph. Assume that M is a rational homology sphere, i.e. Γ is a tree and all the plumbed surfaces have genus zero. Let $\mathcal{V}$ be the set of vertices of Γ, δ_v be the valency of a vertex $v \in \mathcal{V}$, and we distinguish the following subsets: the set of *nodes* $\mathcal{N} := \{n \in \mathcal{V} : \delta_n \geq 3\}$ and the set of *ends* $\mathcal{E} = \{v \in \mathcal{V} : \delta_v = 1\}$.

2000 *Mathematics Subject Classification* Primary. 32S05, 32S25, 32S50, 20Mxx, 57M27 Secondary. 14Bxx, 32Sxx, 14J80, 57R57.

We are grateful to the referee for the very careful reading and suggestions. TL is supported by ERCEA Consolidator Grant 615655 – NMST and also by the Basque Government through the BERC 2014-2017 program and by Spanish excellence accreditation SEV-2013-0323. Partial financial support to ZsSz was provided by the 'Lendület' Grant of A. Stipsicz funded by the Hungarian Academy of Sciences.

We consider the plumbed 4-manifold $\widetilde{X}$ associated with Γ. Its second homology $L := H_2(\widetilde{X},\mathbb{Z})$ is a lattice, freely generated by the classes of 2-spheres $\{E_v\}_{v\in\mathcal{V}}$, endowed with the nondegenerate negative definite intersection form $(\,,\,)$. The second cohomology $L' := H^2(\widetilde{X},\mathbb{Z})$ is the dual lattice, freely generated by the (anti)dual classes $\{E^*_v\}_{v\in\mathcal{V}}$, where we set $(E^*_v, E_w) = -\delta_{vw}$, the negative of the Kronecker delta. The intersection form embeds L into L' and $H := L'/L \simeq H_1(M,\mathbb{Z})$. Denote the class of $l' \in L'$ in H by $[l']$. We denote by $\mathfrak{sw}^{norm}_h(M)$ the normalized Seiberg–Witten invariants of M indexed by the group elements $h \in H$, see Section 2.2.

The *multivariable zeta-function* associated with M (or Γ) was defined by Campillo, Delgado and Gusein-Zade [**6**, **7**] and Némethi [**21**] via the following formula

$$f(\mathbf{t}) = \prod_{v\in\mathcal{V}} (1-\mathbf{t}^{E^*_v})^{\delta_v-2}, \tag{1.1}$$

where $\mathbf{t}^{l'} := \prod_{v\in\mathcal{V}} t_v^{l_v}$ for any $l' = \sum_{v\in\mathcal{V}} l_v E_v \in L'$. (Notice that in a general situation of [**6**, **7**] the exponents of the brackets are defined by $-\chi(\dot{E}_v)$, the negative of the topological Euler characteristic of the 'smooth part' $\dot{E}_v$ of E_v, i.e., E_v minus intersection points with all other components of the exceptional divisor. Hence, in our case when the link M is a rational homology sphere, this equals $\delta_v - 2$.) One has a natural decomposition into its h-equivariant parts $f(\mathbf{t}) = \sum_{h\in H} f_h(\mathbf{t})$, see Subsection 2.3.1. For the purpose to decode the Seiberg–Witten invariants of M from f (c.f. Section 2.2), [**11**, Reduction theorem 5.4.2] has shown that the variables of f_h can be reduced to the variables of the nodes of the graph. Therefore, we restrict our discussions to the reduced zeta-functions defined by $f_h(\mathbf{t}_\mathcal{N}) = f_h(\mathbf{t})|_{t_v=1, v\notin\mathcal{N}}$. Here we introduce notation $\mathbf{t}^{l'}_\mathcal{N} := \prod_{n\in\mathcal{N}} t_n^{l_n}$.

1.2.

The multivariable polynomial part $P_h(\mathbf{t}_\mathcal{N})$ associated with $f_h(\mathbf{t}_\mathcal{N})$ (defined by [**13**, Formula (32)], see also Formula (2.6)) is mainly a combination of the one- and two-variable cases studied by [**1**] and [**11**] corresponding to the structure of the *orbifold graph* Γ^{orb}. The vertices of Γ^{orb} are the nodes of Γ and two of them are connected by an edge if the corresponding nodes in Γ are connected by a path which consists only vertices with valency $\delta_v = 2$. The main property of the polynomial part reads as $P_h(1) = \mathfrak{sw}^{norm}_h(M)$, see Section 2.4.

1.3. *Multivariable division algorithm*

On $L\otimes\mathbb{Q}$ we consider the following partial order: for any l_1, l_2 one writes $l_1 > l_2$ if $l_1 - l_2 = \sum_{v\in\mathcal{V}} \ell_v E_v$ with all $\ell_v > 0$. We introduce a multivariable division algorithm in Section 3.1, which provides a unique decomposition (Lemma 3.1)

$$f_h(\mathbf{t}_\mathcal{N}) = P^+_h(\mathbf{t}_\mathcal{N}) + f^{neg}_h(\mathbf{t}_\mathcal{N}),$$

where $P^+_h(\mathbf{t}_\mathcal{N}) = \sum_\beta p_\beta \mathbf{t}^\beta_\mathcal{N}$ is a Laurent polynomial such that $\beta \not< 0$ for every monomial and $f^{neg}_h(\mathbf{t}_\mathcal{N})$ is a rational function with negative degree in t_n for all $n \in \mathcal{N}$.

This newly introduced quotient $P^+_h(\mathbf{t}_\mathcal{N})$ in general is different from the earlier defined polynomial part $P_h(\mathbf{t}_\mathcal{N})$, however they are related. In Theorem 3.4 we show that the polynomial part $P_h(\mathbf{t}_\mathcal{N})$ can be recovered from the easier to compute quotient $P^+_h(\mathbf{t}_\mathcal{N})$ by taking its monomial terms with multiplicity $\mathfrak{s}$ involving the structure of Γ^{orb} (Definition 3.2). More precisely,

$$P_h(\mathbf{t}_\mathcal{N}) = \sum_\beta \mathfrak{s}(\beta) p_\beta \mathbf{t}^\beta_\mathcal{N}.$$

1.4. *Comparing the polynomial part and the quotient polynomial*

A consequence of the above algorithm (cf. Remark 3.5(i)) is that in general P_h is 'thicker' than P_h^+, in the sense that $\mathfrak{s}(\beta) \geq 1$ for all the exponents β of P_h^+. This motivates the study of their comparison on two different classes of graphs.

In the first case we assume that Γ^{orb} is a bamboo, that is, there are no vertices with valency greater or equal than 3. Notice that most of the examples considered in the aforementioned articles were taken from this class. We prove in Theorem 4.1 that for these graphs the two polynomials agree. Thus, the Seiberg–Witten invariant calculation is provided only by the division.

The second class is defined by a topological criterion: they are the graphs of the 3-manifolds $S^3_{-p/q}(K)$ obtained by $(-p/q)$-surgery along the connected sum K of some algebraic knots. We provide a concrete example of this class for which one has $P_h \neq P_h^+$ for some h, see Section 4.2. More precisely, Theorem 4.8 proves that if we look at part of the polynomials consisting of monomials for which the exponent of the variable associated with the 'central' vertex of the graph (cf. Subsection 4.3.2) is non-negative, then they agree. (See Subsection 4.3.4 for precise formulation.) In fact, by Proposition 4.10, for the canonical class $h = 0$ these are the only monomials, hence $P_0 = P_0^+$.

2. *Preliminaries*

2.1. *Links of normal surface singularities*

For more details regarding plumbed 3-manifolds, plumbing graphs and their relations with normal surface singularities see [**9**, **19**, **20**, **26**].

2.1.1. Let Γ be a connected negative definite plumbing graph with vertices $\mathcal{V} = \mathcal{V}(\Gamma)$. By plumbing disk bundles along Γ, we obtain a smooth 4–manifold $\widetilde{X}$ whose boundary is an oriented plumbed 3–manifold M. Γ can be realized as the dual graph of a good resolution $\pi : \widetilde{X} \to X$ of some complex normal surface singularity (X, o) and M is called the link of the singularity. In our study, we assume that M is a *rational homology sphere*, or, equivalently, Γ is a tree and all the genus decorations are zero.

Recall that $L := H_2(\widetilde{X}, \mathbb{Z}) \simeq \mathbb{Z}\langle E_v\rangle_{v\in\mathcal{V}}$ is a lattice, freely generated by the classes of the irreducible exceptional divisors $\{E_v\}_{v\in\mathcal{V}}$ (i.e. classes of 2-spheres), with a nondegenerate negative definite intersection form $I := [(E_v, E_w)]_{v,w\in\mathcal{V}}$. $L' := H^2(\widetilde{X}, \mathbb{Z}) \simeq Hom(L, \mathbb{Z})$ is the dual lattice, freely generated by the (anti)duals $\{E_v^*\}_{v\in\mathcal{V}}$. L is embedded in L' by the intersection form (which extends to $L \otimes \mathbb{Q} \supset L'$) and their finite quotient is $H := L'/L \simeq H^2(\partial\widetilde{X}, \mathbb{Z}) \simeq H_1(M, \mathbb{Z})$.

2.1.2. The *determinant* of a subgraph $\Gamma' \subseteq \Gamma$ is defined as the determinant of the negative of the submatrix of I with rows and columns indexed with vertices of Γ', and it will be denoted by $\det_{\Gamma'}$. In particular, $\det_\Gamma := \det(-I) = |H|$. We will also consider the following subgraphs: since Γ is a tree, for any two vertices $v, w \in \mathcal{V}$ there is a unique minimal connected subgraph (path connecting v and w) $[v, w]$ with vertices $\{v_i\}_{i=0}^k$ such that $v = v_0$, $w = v_k$ and v_iv_{i+1} are edges in the graph for $i = 0, \ldots, k-1$. Similarly, we also introduce notations $[v, w)$, $(v, w]$ and (v, w) for the complete subgraphs with vertices $\{v_i\}_{i=0}^{k-1}$, $\{v_i\}_{i=1}^k$ and $\{v_i\}_{i=1}^{k-1}$ respectively.

The inverse of I has entries $(I^{-1})_{vw} = (E_v^*, E_w^*)$, all entries are negative. Moreover, they can be computed using determinants of subgraphs as (cf. [**9**, page 83])

$$-(E_v^*, E_w^*) = \frac{\det_{\Gamma\setminus[v,w]}}{\det_\Gamma}. \tag{2.1}$$

2.1.3. We can consider the following partial order on $L\otimes\mathbb{Q}$: for any l_1, l_2 one writes $l_1 \geq l_2$ if $l_1 - l_2 = \sum_{v\in\mathcal{V}} \ell_v E_v$ with all $\ell_v \geq 0$. The Lipman (anti-nef) cone $\mathcal{S}'$ is defined by $\{l' \in L' : (l', E_v) \leq 0 \text{ for all } v\}$ and it is generated over $\mathbb{Z}_{\geq 0}$ by the elements E_v^*. We use notation $\mathcal{S}'_{\mathbb{R}} := \mathcal{S}' \otimes \mathbb{R}$ for the real Lipman cone.

2.1.4. Let $\widetilde{\sigma}_{can}$ be the *canonical spin^c-structure* on $\widetilde{X}$. Its first Chern class $c_1(\widetilde{\sigma}_{can}) = -K \in L'$, where K is the canonical class in L' defined by the adjunction formulas $(K + E_v, E_v) + 2 = 0$ for all $v \in \mathcal{V}$. The set of *spin^c*-structures $\mathrm{Spin}^c(\widetilde{X})$ of $\widetilde{X}$ is an L'-torsor, i.e. if we denote the L'-action by $l' * \widetilde{\sigma}$, then $c_1(l' * \widetilde{\sigma}) = c_1(\widetilde{\sigma}) + 2l'$. Furthermore, all the *spin^c*-structures of M are obtained by restrictions from $\widetilde{X}$. $\mathrm{Spin}^c(M)$ is an H-torsor, compatible with the restriction and the projection $L' \to H$. The *canonical spin^c-structure* σ_{can} of M is the restriction of the canonical *spin^c*-structure $\widetilde{\sigma}_{can}$ of $\widetilde{X}$. Hence, for any $\sigma \in \mathrm{Spin}^c(M)$ one has $\sigma = h * \sigma_{can}$ for some $h \in H$. For more details regarding *spin^c*-structures we refer to [**10**, page 415].

2.2. *Seiberg–Witten invariants of normal surface singularities*

For any closed, oriented and connected 3-manifold M we consider the *Seiberg–Witten invariant* $\mathfrak{sw} : \mathrm{Spin}^c(M) \to \mathbb{Q}$, $\sigma \mapsto \mathfrak{sw}_\sigma(M)$. In the case of rational homology spheres, it is the signed count of the solutions of the '3-dimensional' Seiberg–Witten equations, modified by the Kreck–Stolcz invariant (cf. [**15**, **28**]).

Since its calculation is difficult using the original definition, several topological/combinatorial interpretations have been invented in the last decades. E.g., [**28**] has shown that for rational homology spheres $\mathfrak{sw}(M)$ is equal with the Reidemeister–Turaev torsion normalized by the Casson–Walker invariant which, in some plumbed cases, can be expressed in terms of the graph and Dedekind–Fourier sums ([**16**, **26**]). Furthermore, there exist surgery formulas coming from homology exact sequences (e.g. Heegaard–Floer homology, monopole Floer homology, lattice cohomology, etc.), where the involved homology theories appear as categorifications of the (normalized) Seiberg–Witten invariant.

In the case when M is a rational homology sphere link of a normal surface singularity (X, o), different type of surgery ([**1**, **12**]) and combinatorial formulas ([**11**, **13**]) have been proved expressing the strong connection of the Seiberg–Witten invariant and the zeta-function/Poincaré series associated with M ([**22**]). This connection will be explained in the next section. Moreover, we emphasize that the Seiberg–Witten invariant plays a crucial role in the intimate relationship between the topology and geometry of normal surface singularities since it can be viewed as the topological 'analogue' of the geometric genus of (X, o), cf. [**26**].

For different purposes we may use different normalizations of the Seiberg–Witten invariant. If we are looking its relation with the geometric genus of the singularity (c.f. [**23**, Remark 3.2.8]), or, with the zeta-function f (c.f. [**11**, Corollary 5.2.1]) it is natural to consider the following: for any class $h \in H = L'/L$ we define the unique element $r_h \in L'$ characterized by $r_h \in \sum_v [0, 1)E_v$ with $[r_h] = h$, then

$$\mathfrak{sw}_h^{norm}(M) := -\frac{(K + 2r_h)^2 + |\mathcal{V}|}{8} - \mathfrak{sw}_{-h*\sigma_{can}}(M) \tag{2.2}$$

is called the *normalized Seiberg–Witten invariant* of M associated with $h \in H$.

2.3. *Zeta-functions and Poincaré series*

2.3.1. *Definitions and motivation* We have already defined in Section 1.1 the multivariable zeta-function $f(\mathbf{t})$ associated with the manifold M. Its multivariable Taylor expansion at the origin $Z(\mathbf{t}) = \sum_{l'} p_{l'} \mathbf{t}^{l'} \in \mathbb{Z}[[L']]$ is called the *topological Poincaré series*, where $\mathbb{Z}[[L']]$ is the $\mathbb{Z}[L']$-submodule of $\mathbb{Z}[[t_v^{\pm 1/|H|} : v \in \mathcal{V}]]$ consisting of series $\sum_{l' \in L'} a_{l'} \mathbf{t}^{l'}$ with $a_{l'} \in \mathbb{Z}$ for all

$l' \in L'$. It decomposes naturally into $Z(\mathbf{t}) = \sum_{h\in H} Z_h(\mathbf{t})$, where $Z_h(\mathbf{t}) = \sum_{[l']=h} p_{l'}\mathbf{t}^{l'}$. By Subsection 2.1.3, $Z(\mathbf{t})$ is supported in $\mathcal{S}'$, hence $Z_h(\mathbf{t})$ is supported in $(l' + L) \cap \mathcal{S}'$, where $l' \in L'$ with $[l'] = h$. This decomposition induces a decomposition $f(\mathbf{t}) = \sum_{h\in H} f_h(\mathbf{t})$ on the zeta-function level as well, where explicit formula for the rational function $f_h(\mathbf{t})$ is provided by [**14**, Theorem 5.0.1].

The zeta-function and its series were introduced by the work of Némethi [**21**], motivated by singularity theory. For a normal surface singularity (X, o) with fixed resolution graph Γ we may consider the divisorial Hilbert series $\mathcal{H}(\mathbf{t})$ (for more details see e.g. [**8**], [**6**] and [**23**, Section 2 and 3]) which can be connected with the topology of the link M by introducing the series $\mathcal{P}(\mathbf{t}) = -\mathcal{H}(\mathbf{t}) \cdot \prod_{v\in\mathcal{V}}(1 - t_v^{-1}) \in \mathbb{Z}[[L']]$. The point is that, for $h = 0$, $Z_0(\mathbf{t})$ serves as the 'topological candidate' for $\mathcal{P}(\mathbf{t})$: they agree for several class of singularities, e.g. for splice quotients (see [**23**]), which contain all the rational, minimally elliptic or weighted homogeneous singularities.

For more details regarding to this theory we refer to [**6**, **7**, **21**, **23**].

2.3.2. *Counting functions, Seiberg–Witten invariants and reduction* For any $h \in H$ we define the *counting function* Q_h of the coefficients of $Z_h(\mathbf{t}) = \sum_{[l']=h} p_{l'}\mathbf{t}^{l'}$ by $x \mapsto Q_h(x) := \sum_{l'\not\geq x,\,[l']=h} p_{l'}$. This sum is finite since $\{l' \in \mathcal{S}' : l' \not\geq x\}$ is finite by Subsection 2.1.3.

Its relation with the Seiberg–Witten invariant is given by a powerful result of Némethi [**22**] saying that if $x \in (-K + \mathrm{int}(\mathcal{S}')) \cap L$ then

$$Q_h(x) = \chi_{K+2r_h}(x) + \mathfrak{sw}_h^{norm}(M), \tag{2.3}$$

where $\chi_{K+2r_h}(x) := -(K + 2r_h + x, x)/2$. Thus, $Q_h(x)$ is a multivariable quadratic polynomial on L with constant term $\mathfrak{sw}_h^{norm}(M)$. Although Formula (2.3) only shows that Q_h is a polynomial on the shifted cone, the new approach of [**11**] is to construct Q_h as an Ehrhart type (quasi)polynomial on certain chambers associated with Z_h. More precisely, there exists a conical chamber decomposition of the real cone $\mathcal{S}'_{\mathbb{R}} = \cup_\tau \mathcal{C}_\tau$, a sublattice $\widetilde{L} \subset L$ and $l'_* \in \mathcal{S}'$ such that $Q_h(l')$ is a polynomial on $\widetilde{L} \cap (l'_* + \mathcal{C}_\tau)$, say $Q_h^{\mathcal{C}_\tau}(l')$. This allows to define the *multivariable periodic constant* ([**11**, Definition 4.4.1]) by $\mathrm{pc}^{\mathcal{C}_\tau}(Z_h) := Q_h^{\mathcal{C}_\tau}(0)$ associated with $h \in H$ and $\mathcal{C}_\tau$. Moreover, $Z_h(\mathbf{t})$ is rather special in the sense that all $Q_h^{\mathcal{C}_\tau}$ are equal for any $\mathcal{C}_\tau$. In particular, we say that there exists the periodic constant $\mathrm{pc}^{S'_{\mathbb{R}}}(Z_h) := \mathrm{pc}^{\mathcal{C}_\tau}(Z_h)$ associated with $S'_{\mathbb{R}}$, and in fact, it is equal to $\mathfrak{sw}_h^{norm}(M)$.

We also notice that Formula (2.3) has a geometric analogue which expresses the geometric genus of the complex normal surface singularity (X, o) from the series $\mathcal{P}(\mathbf{t})$ (cf. [**23**]).

[**11**, Reduction Theorem 5.4.2] has shown that from the point of view of Formula (2.3) the number of variables of the zeta-function (or Poincaré series) can be reduced to the number of nodes $|\mathcal{N}|$. Thus, if we define the *reduced zeta-function* and *reduced Poincaré series* by

$$f_h(\mathbf{t}_{\mathcal{N}}) = f_h(\mathbf{t})\,|_{t_v=1, v\notin\mathcal{N}} \qquad \text{and} \qquad Z_h(\mathbf{t}_{\mathcal{N}}) := Z_h(\mathbf{t})\,|_{t_v=1, v\notin\mathcal{N}},$$

then there exists the periodic constant of $Z_h(\mathbf{t}_{\mathcal{N}})$ associated with the projected real Lipman cone $\pi_{\mathcal{N}}(S'_{\mathbb{R}})$, where $\pi_{\mathcal{N}} : \mathbb{R}\langle E_v\rangle_{v\in\mathcal{V}} \to \mathbb{R}\langle E_v\rangle_{v\in\mathcal{N}}$ is the natural projection along the linear subspace $\mathbb{R}\langle E_v\rangle_{v\notin\mathcal{N}}$, and

$$\mathrm{pc}^{\pi_{\mathcal{N}}(S'_{\mathbb{R}})}(Z_h(\mathbf{t}_{\mathcal{N}})) = \mathrm{pc}^{S'_{\mathbb{R}}}(Z_h(\mathbf{t})) = \mathfrak{sw}_h^{norm}(M).$$

We set notation $\mathbf{t}_{\mathcal{N}}^x := \mathbf{t}^{\pi_{\mathcal{N}}(x)}$ for any $x \in L'$.

The above identity allows us to consider only the reduced versions in our study, which has several advantages: the number of reduced variables is drastically smaller, hence reduces the complexity of the calculations; reflects to the complexity of the manifold M (e.g. in case of Seifert 3-manifolds, it is enough to consider one variable); also, for special classes of singularities the reduced series can be compared with certain geometric series (or invariants), cf. [**21**].

2.4. *'Polynomial-negative degree part' decomposition*

2.4.1. *One-variable case* Let $s(t)$ be a one-variable rational function of the form $B(t)/A(t)$ with $A(t) = \prod_{i=1}^{d}(1-t^{a_i})$ and $a_i > 0$. Then by [**1**, Lemma 7.0.2] one has a unique decomposition $s(t) = P(t) + s^{neg}(t)$, where $P(t)$ is a polynomial and $s^{neg}(t) = R(t)/A(t)$ has negative degree, i.e. $\deg(R) < \deg(A)$, with vanishing periodic constant (the one-variable case was defined in [**27**, **29**]). Hence, the periodic constant $\mathrm{pc}(s)$ (associated with the Taylor expansion of s and the cone $\mathbb{R}_{\geq 0}$) equals $P(1)$. $P(t)$ is called the *polynomial part* while the rational function $s^{neg}(t)$ is called the *negative degree part* of the decomposition. The decomposition can be deduced easily by the following division on the individual rational fractions:

$$\frac{t^b}{\prod_i(1-t^{a_i})} = -\frac{t^{b-a_{i_0}}}{\prod_{i\neq i_0}(1-t^{a_i})} + \frac{t^{b-a_{i_0}}}{\prod_i(1-t^{a_i})} = \sum_{\substack{x_i\geq 1\\ \sum_i x_i a_i \leq b}} p_{(x_i)}\cdot t^{b-\sum_i x_i a_i} + \frac{\text{negative degree}}{\text{rational function}}, \tag{2.4}$$

for some coefficients $p_{(x_i)} \in \mathbb{Z}$.

2.4.2. *Multivariable case* The idea towards the multivariable generalization goes back to the theory developed in [**11**], saying that the counting functions associated with zeta-functions are Ehrhart-type quasipolynomials inside the chambers of an induced chamber-decomposition of $L\otimes\mathbb{R}$. Moreover, the previous one-variable division can be generalized to two-variable functions of the form $s(\mathbf{t}) = B(\mathbf{t})/(1-\mathbf{t}^{a_1})^{d_1}(1-\mathbf{t}^{a_2})^{d_2}$ with $a_i > 0$. In particular, for $f_h(\mathbf{t}_\mathcal{N})$ viewed as a function in variables t_n and $t_{n'}$, where $n, n' \in \mathcal{N}$ and there is an edge $\overline{nn'}$ connecting them in Γ^{orb}, where Γ^{orb} is the orbifold graph defined in Section 1.2. (For more details regarding two-variable division see [**11**, Section 4.5] and [**13**, Lemma 23]).

For more variables, the direct generalization using a division principle for the individual rational terms seems to be hopeless because the (Ehrhart) quasipolynomials associated with the counting functions can not be controlled inside the difficult chamber decomposition of $\mathcal{S}'_\mathbb{R}$.

Nevertheless, the authors in [**13**] have proposed a decomposition

$$f_h(\mathbf{t}_\mathcal{N}) = P_h(\mathbf{t}_\mathcal{N}) + f_h^-(\mathbf{t}_\mathcal{N}) \tag{2.5}$$

which defines the polynomial part as

$$P_h(\mathbf{t}_\mathcal{N}) = \sum_{\overline{nn'}\ edge\ of\ \Gamma^{orb}} P_h^{n,n'}(\mathbf{t}_\mathcal{N}) - \sum_{n\in\mathcal{N}} (\delta_{n,\mathcal{N}}-1)P_h^n(\mathbf{t}_\mathcal{N}), \tag{2.6}$$

where $\delta_{n,\mathcal{N}}$ is the number of neighbours of n in Γ^{orb}, $P_h^n(\mathbf{t}_\mathcal{N})$ for any $n\in\mathcal{N}$ are the polynomial parts given by the decompositions of $f_h(\mathbf{t}_\mathcal{N})$ as a one-variable function in t_n, while $P_h^{n,n'}(\mathbf{t}_\mathcal{N})$ are the polynomial parts viewed $f_h(\mathbf{t}_\mathcal{N})$ as a two-variable function in t_n and $t_{n'}$ for any $n, n' \in \mathcal{N}$ so that they are connected by an edge in Γ^{orb}. Then [**13**, Theorem 24] states the main property of the decomposition

$$P_h(1) = \mathfrak{sw}_h^{norm}(M). \tag{2.7}$$

3. *Decomposition by multivariable division and proof of the algorithm*

In this section we prove the main algorithm which expresses the multivariable polynomial part P of [**13**] in terms of the quotient polynomial P^+ which will be constructed in the sequel by the multivariable Euclidean division, and a certain multiplicity function.

3.1. *Multivariable Euclidean division*

We consider two Laurent polynomials $A(\mathbf{t}_\mathcal{N})$ and $B(\mathbf{t}_\mathcal{N})$ supported on the lattice $\pi_\mathcal{N}(L')$. The partial order $l_1 > l_2$ if $l_1 - l_2 = \sum_{v\in\mathcal{V}} \ell_v E_v$ with $\ell_v > 0$ for all $v\in\mathcal{V}$ on $L\otimes\mathbb{Q}$ induces a partial order on monomial terms and we assume that $A(\mathbf{t}_\mathcal{N})$ has a unique maximal monomial term with respect to this partial order denoted by $A_a\mathbf{t}_\mathcal{N}^a$ such that $a > 0$.

We introduce the following multivariable Euclidean division algorithm. We start with quotient $C = 0$ and remainder $R = 0$. For a monomial term $B_b\mathbf{t}_\mathcal{N}^b$ of $B(\mathbf{t}_\mathcal{N})$ if $b \not< a$ then we subtract $(B_b\mathbf{t}_\mathcal{N}^b/A_a\mathbf{t}_\mathcal{N}^a)\cdot A(\mathbf{t}_\mathcal{N})$ from $B(\mathbf{t}_\mathcal{N})$ and we add $B_b\mathbf{t}_\mathcal{N}^b/A_a\mathbf{t}_\mathcal{N}^a$ to the quotient $C(\mathbf{t}_\mathcal{N})$, otherwise we pass $B_b\mathbf{t}_\mathcal{N}^b$ from $B(\mathbf{t}_\mathcal{N})$ to the remainder $R(\mathbf{t}_\mathcal{N})$. By the assumption on $A(\mathbf{t}_\mathcal{N})$ the algorithm terminates in finite steps and gives a unique decomposition

$$B(\mathbf{t}_\mathcal{N}) = C(\mathbf{t}_\mathcal{N})\cdot A(\mathbf{t}_\mathcal{N}) + R(\mathbf{t}_\mathcal{N}) \tag{3.1}$$

such that $C(\mathbf{t}_\mathcal{N})$ is a supported on $\{l' \in \pi_\mathcal{N}(L') : l' \not< 0\}$ and $R(\mathbf{t}_\mathcal{N})$ is supported on $\{l' \in \pi_\mathcal{N}(L') : l' < a\}$.

The following decomposition generalizes the one- and two-variable cases.

Lemma 3.1. *For any $h \in H$ there exists a unique decomposition*

$$f_h(\mathbf{t}_\mathcal{N}) = P_h^+(\mathbf{t}_\mathcal{N}) + f_h^{neg}(\mathbf{t}_\mathcal{N}), \tag{3.2}$$

where $P_h^+(\mathbf{t}_\mathcal{N}) = \sum_{\beta\in\mathcal{B}_h} p_\beta \mathbf{t}_\mathcal{N}^\beta$ is a Laurent polynomial such that for each $\beta \in \mathcal{B}_h$ we have $\beta \not< 0$ and $f_h^{neg}(\mathbf{t}_\mathcal{N})$ is a rational function with negative degree in t_n for all $n \in \mathcal{N}$.

Proof. First of all, we use [**14**, Theorem 5.0.1] that for any $h \in H$ one can represent $f_h(\mathbf{t}_\mathcal{N})$ as a rational function in the following form

$$f_h(\mathbf{t}_\mathcal{N}) = \mathbf{t}_\mathcal{N}^{r_h}\cdot \sum_\ell b_\ell \mathbf{t}_\mathcal{N}^\ell \Big/ \prod_{n\in\mathcal{N}}(1-\mathbf{t}_\mathcal{N}^{a_n}),$$

where $\ell, a_n \in \mathbb{Z}\langle E_n\rangle_{n\in\mathcal{N}}$ so that $a_n = \lambda_n\pi_\mathcal{N}(E_n^*)$ for some $\lambda_n > 0$, $\ell \in \mathbb{R}_{\geq 0}\langle a_n\rangle_{n\in\mathcal{N}}$ and $b_\ell \in \mathbb{Z}$. (Since the shape of the formula is sufficient for our purpose, for its precise significance we refer to [**14**]). Note that $A(\mathbf{t}_\mathcal{N}) = \prod_{n\in\mathcal{N}}(1-\mathbf{t}_\mathcal{N}^{a_n})$ has a unique maximal term $(-1)^{|\mathcal{N}|}\mathbf{t}_\mathcal{N}^{\sum_{n\in\mathcal{N}} a_n}$ with $\sum_{n\in\mathcal{N}} a_n > 0$. Thus, by the above multivariable Euclidean division we can write

$$\mathbf{t}_\mathcal{N}^{r_h}\sum_\ell b_\ell \mathbf{t}_\mathcal{N}^\ell = P_h^+(\mathbf{t}_\mathcal{N})\cdot\prod_{n\in\mathcal{N}}(1-\mathbf{t}_\mathcal{N}^{a_n}) + R_h(\mathbf{t}_\mathcal{N}) \tag{3.3}$$

and we set $f_h^{neg}(\mathbf{t}_\mathcal{N}) := \frac{R_h(\mathbf{t}_\mathcal{N})}{\prod_{n\in\mathcal{N}}(1-\mathbf{t}_\mathcal{N}^{a_n})}$.

The uniqueness is followed by the assumptions on P_h^+ and f_h^{neg}. Indeed, if we assume $P_h^+ + f_h^{neg} = 0$ then f_h^{neg} is a polynomial and it has monomial terms with negative exponents in every variable, which contradicts to the assumption on P_h^+. Hence, they must be zero. □

3.2. *Multiplicity and relation to the polynomial part*

We will show that the multivariable polynomial part P_h can be computed from the multivariable quotient P_h^+ by taking its monomial terms with a suitable multiplicity. We start by defining on the set of nodes the following type of partial orders $\{\mathcal{N}, \succ\}$. Choose a node $n_0 \in \mathcal{N}$ and orient edges of Γ^{orb} (cf. Section 1.2) towards the direction of n_0. This induces a partial order on the set of nodes: $n \succ n'$ if there is an edge in Γ^{orb} connecting them, oriented from n to n'. Note that n_0 is the unique minimal node with respect to this partial order.

Definition 3.2. We fix a node $n_0 \in \mathcal{N}$, thus a partial order $\{\mathcal{N}, \succ\}$ corresponding to it. Associated with a monomial $\mathbf{t}_\mathcal{N}^\beta = \prod_{n\in\mathcal{N}} t_n^{\beta_n}$ we define first the following 'node' and 'edge' sign-functions. Set $\mathfrak{s}_n(\beta) := 1$ if $\beta_n \geq 0$ and 0 otherwise. For any $n, n' \in \mathcal{N}$ with $n \succ n'$ we define $\mathfrak{s}_{n\succ n'}(\beta) := 1$ if $\beta_n \geq 0$ and $\beta_{n'} < 0$, and 0 otherwise. Finally, these two sign-functions define the *multiplicity function* by the formula

$$\mathfrak{s}(\beta) = \mathfrak{s}_{n_0}(\beta) + \sum_{n \succ n'} \mathfrak{s}_{n \succ n'}(\beta).$$

REMARK 3.3.

(i) In fact, the multiplicity function $\mathfrak{s}$ does not depend on the chosen partial order $\{\mathcal{N}, \succ\}$. This can be checked easily for two partial orders with unique minimal nodes connected by an edge in Γ^{orb}.

(ii) There is another interpretation of the multiplicity $\mathfrak{s}(\beta)$: if we consider the maximal connected subgraphs $\Gamma_i^{orb}(\beta)$ of Γ^{orb} such that for any vertex n of $\Gamma_i^{orb}(\beta)$ one has $\beta_n \geq 0$, then $\mathfrak{s}(\beta)$ is the number of these subgraphs. Indeed, fix a partial order $\{\mathcal{N}, \succ\}$ with unique minimal node n_0. Then $\mathfrak{s}_{n_0}(\beta)$ contributes 1 to $\mathfrak{s}(\beta)$ exactly when there is a subgraph $\Gamma_i^{orb}(\beta)$ having n_0 as vertex. Moreover, $\mathfrak{s}_{n \succ n'}(\beta)$ contributes 1 to $\mathfrak{s}(\beta)$ precisely when n is the unique minimal node of a subgraph $\Gamma_i^{orb}(\beta)$.

THEOREM 3.4. *Consider the multivariable quotient $P_h^+(\mathbf{t}_{\mathcal{N}}) = \sum_{\beta \in \mathcal{B}_h} p_\beta \mathbf{t}_{\mathcal{N}}^\beta$ of f_h. Then the polynomial part defined in Formula (2.6) has the following form*

$$P_h(\mathbf{t}_{\mathcal{N}}) = \sum_{\beta \in \mathcal{B}_h} \mathfrak{s}(\beta) p_\beta \mathbf{t}_{\mathcal{N}}^\beta.$$

Proof. Recall that the polynomial part $P_h(\mathbf{t}_{\mathcal{N}})$ is defined in Formula (2.6) using the polynomials $P_h^{n'}(\mathbf{t}_{\mathcal{N}})$ and $P_h^{n,n'}(\mathbf{t}_{\mathcal{N}})$ for any $n, n' \in \mathcal{N}$ for which there exists an edge connecting them in Γ^{orb}. Moreover, $P_h^{n'}$ and $P_h^{n,n'}$ are results of one- and two-variable divisions in variables $t_{n'}$ and $t_n, t_{n'}$, while considering other variables as coefficients. These divisions can be deduced by the multivariable Euclidean division algorithm of Section 3.1 if we replace the partial order on $L \otimes \mathbb{Q}$ (defined in Subsection 2.1.3) by the corresponding projections '$<_{n'}$' and '$<_{n,n'}$'. That is, $a <_{n'} b$ and $a <_{n,n'} b$ if $a_{n'} < b_{n'}$ and $a_n < b_n$, $a_{n'} < b_{n'}$, respectively. Since $a \not<_{n'} b$ and $a \not<_{n,n'} b$ both imply $a \not< b$, the monomial terms of $P_h^{n'}$ and $P_h^{n,n'}$ can be found among monomial terms of P_h^+. More precisely, by choosing an arbitrary partial order on the nodes of type $\{\mathcal{N}, \succ\}$ with unique minimal node n_0 one can write

$$P_h^{n'}(\mathbf{t}_{\mathcal{N}}) = \sum_{\substack{\beta \in \mathcal{B}_h \\ \beta_{n'} \geq 0}} p_\beta \mathbf{t}_{\mathcal{N}}^\beta = \sum_{\beta \in \mathcal{B}_h} \mathfrak{s}_{n'}(\beta) p_\beta \mathbf{t}_{\mathcal{N}}^\beta,$$

$$P_h^{n,n'}(\mathbf{t}_{\mathcal{N}}) = \sum_{\substack{\beta \in \mathcal{B}_h \\ \beta_n \text{ or } \beta_{n'} \geq 0}} p_\beta \mathbf{t}_{\mathcal{N}}^\beta = \sum_{\beta \in \mathcal{B}_h} \big(\mathfrak{s}_{n'}(\beta) + \mathfrak{s}_{n \succ n'}(\beta)\big) p_\beta \mathbf{t}_{\mathcal{N}}^\beta \quad \text{if } n \succ n'.$$

Thus,

$$\begin{aligned} P_h(\mathbf{t}_{\mathcal{N}}) &= \sum_{n \succ n'} P_h^{n,n'}(\mathbf{t}_{\mathcal{N}}) - \sum_{n' \in \mathcal{N}} (\delta_{n',\mathcal{N}} - 1) P_h^{n'}(\mathbf{t}_{\mathcal{N}}) \\ &= \sum_{n \succ n'} \sum_{\beta \in \mathcal{B}_h} \big(\mathfrak{s}_{n'}(\beta) + \mathfrak{s}_{n \succ n'}(\beta)\big) p_\beta \mathbf{t}_{\mathcal{N}}^\beta - \sum_{n' \in \mathcal{N}} (\delta_{n',\mathcal{N}} - 1) \sum_{\beta \in \mathcal{B}_h} \mathfrak{s}_{n'}(\beta) p_\beta \mathbf{t}_{\mathcal{N}}^\beta \\ &= \sum_{\beta \in \mathcal{B}_h} \Big[\sum_{n \succ n'} \big(\mathfrak{s}_{n'}(\beta) + \mathfrak{s}_{n \succ n'}(\beta)\big) - \sum_{n' \in \mathcal{N}} (\delta_{n',\mathcal{N}} - 1) \mathfrak{s}_{n'}(\beta) \Big] p_\beta \mathbf{t}_{\mathcal{N}}^\beta \\ &= \sum_{\beta \in \mathcal{B}_h} \big[\mathfrak{s}_{n_0}(\beta) + \sum_{n \succ n'} \mathfrak{s}_{n \succ n'}(\beta)\big] p_\beta \mathbf{t}_{\mathcal{N}} = \sum_{\beta \in \mathcal{B}_h} \mathfrak{s}(\beta) p_\beta \mathbf{t}_{\mathcal{N}}, \end{aligned}$$

since $\#\{n \,|\, n \succ n'\} = \delta_{n',\mathcal{N}} - 1$ for $n' \neq n_0$ and $\#\{n \,|\, n \succ n_0\} = \delta_{n_0,\mathcal{N}}$. On the other hand, we emphasize that the final result does not depend on the chosen partial order $\{\mathcal{N}, \succ\}$ as the multiplicity function $\mathfrak{s}$ is already independent of it. □

Remark 3.5.

(i) For $\beta < 0$ we have $\mathfrak{s}(\beta) = 0$, while for $\beta \not< 0$ we have $\mathfrak{s}(\beta) \geq 1$. Hence, the multiplicity $\mathfrak{s}(\beta)$ is non-zero for every $\beta \in \mathcal{B}_h$, thus every monomial of P_h^+ appears in P_h.

(ii) Recall that the reduced Poincaré series $Z_h(\mathbf{t}_\mathcal{N})$ is the Taylor expansion at the origin of $f_h(\mathbf{t}_\mathcal{N})$, c.f. Subsection 2.3.1. On the other hand, the 'endless' multivariable Euclidean division (Section 3.1) can be thought as the expansion of $f_h(\mathbf{t}_\mathcal{N})$ at infinity. The 'endless' division (with stopping conditions) results P_h^+ from which we can recover P_h by the above theorem. Or, if we take each term of the expansion at infinity with multiplicity $\mathfrak{s}$ then we recover P_h, since terms with negative degree in each t_n have zero multiplicity.

4. *Comparison and examples for P and P^+*

The aim of this section is to compare the two polynomials $P_h(\mathbf{t}_\mathcal{N})$ and $P_h^+(\mathbf{t}_\mathcal{N})$, given by the two different decompositions, through crucial classes of negative definite plumbing graphs.

In case of the first class, when the orbifold graph is a bamboo, we will prove that the two polynomials agree. The second class is also motivated by singularity theory and contains the graphs of the manifolds $S^3_{-p/q}(K)$ where $K \subset S^3$ is the connected sum of algebraic knots. Although this class gives examples when the two polynomials do not agree, their structure can be understood using some specialty of these manifolds.

4.1. *The orbifold graph is a bamboo*

Let Γ be a negative definite plumbing graph with set of nodes $\mathcal{N} = \{n_1, \dots, n_k\}$. In this section we will assume that its orbifold graph Γ^{orb} is a *bamboo*, i.e. Γ^{orb} has no nodes.

Then we have the following result:

Theorem 4.1. *If the orbifold graph Γ^{orb} is a bamboo then $P_h(\mathbf{t}_\mathcal{N}) = P_h^+(\mathbf{t}_\mathcal{N})$ for any $h \in H$, i.e. every monomial term of $P_h^+(\mathbf{t}_\mathcal{N})$ appears in $P_h(\mathbf{t}_\mathcal{N})$ with multiplicity* 1.

Denote by $\mathfrak{v}_i := \pi_\mathcal{N}(E^*_{n_i})$ the projected vectors for all $i = 1, \dots, k$. Similarly as in the proof of Lemma 3.1, we use the result of [**14**, Theorem 5.0.1] concluding that in the case when Γ^{orb} is a bamboo we can write $f_h(\mathbf{t}_\mathcal{N})$ as linear combination of fractions of form $\dfrac{\mathbf{t}_\mathcal{N}^\alpha}{(1-\mathbf{t}_\mathcal{N}^{\lambda_1\mathfrak{v}_1})(1-\mathbf{t}_\mathcal{N}^{\lambda_k\mathfrak{v}_k})}$ for some $\alpha \in \mathbb{R}_{\geq 0}\langle \mathfrak{v}_i\rangle_{i=\overline{1,k}} \cap \mathbb{Z}\langle \pi_\mathcal{N}(E^*_v)\rangle_{v\in\mathcal{V}}$ and $\lambda_1, \lambda_k > 0$. (In fact, a consequence of [**14**] is that the h-equivariant parts f_h as rational functions have the same 'shape' as f, c.f. Formula (1.1), in their denominators consisting factors corresponding to nodes which are ends in Γ^{orb}.)

By the uniqueness of the decomposition (3.2) and Theorem 3.4 it is enough to prove the following proposition.

Proposition 4.2. *Let $\alpha \in \mathbb{R}_{\geq 0}\langle \mathfrak{v}_i\rangle_{i=\overline{1,k}} \cap \mathbb{Z}\langle \pi_\mathcal{N}(E^*_v)\rangle_{v\in\mathcal{V}}$ and consider the following fraction $\varphi(\mathbf{t}_\mathcal{N}) = \dfrac{\mathbf{t}_\mathcal{N}^\alpha}{(1-\mathbf{t}_\mathcal{N}^{\lambda_1\mathfrak{v}_1})(1-\mathbf{t}_\mathcal{N}^{\lambda_k\mathfrak{v}_k})}$, $\lambda_1, \lambda_k > 0$. Then for any monomial $\mathbf{t}_\mathcal{N}^\beta$ of the quotient φ^+ given by the decomposition $\varphi = \varphi^+ + \varphi^{neg}$ of Lemma 3.1 one has $\mathfrak{s}(\beta) = 1$.*

The main tool in the proof of the proposition will be the following lemma.

Lemma 4.3. *Under the assumption of the above proposition, for any* $\beta = \sum_{\ell=1}^{k} \beta_\ell E_{n_\ell} \in \alpha - \mathbb{R}_{\geq 0}\langle \mathfrak{v}_1, \mathfrak{v}_k \rangle$ *with not all* β_ℓ *negative we have*

$$\beta_1, \dots, \beta_{i-1} < 0 \leq \beta_i, \dots, \beta_j \geq 0 > \beta_{j+1}, \dots, \beta_k$$

for some $i, j \in \{1, \dots, k\}$.

We denote by $\mathcal{E}_i = \mathcal{E}_i(\alpha)$ the intersection $\{\gamma = \sum_{\ell=1}^{k} \gamma_\ell E_{n_\ell} \mid \gamma_i = 0\} \cap (\alpha - \mathbb{R}_{\geq 0}\langle \mathfrak{v}_1, \mathfrak{v}_k \rangle)$. Then for any fixed β satisfying the conditions of the lemma, we consider the parametric line $\beta(t) = t\beta + (1-t)\alpha$, $t \in \mathbb{R}$ connecting α to β, and denote by $\beta_i(t)$ the coordinates of $\beta(t)$. Whenever $\beta(t)$ crosses $\mathcal{E}_i$ as t goes from 0 to 1 the sign of $\beta_i(t)$ changes from positive to negative. Thus, the order in which $\beta(t)$ crosses $\mathcal{E}_i$ determines the order in which $\beta_i(t)$'s change sign, consequently determines the sign configuration of $\beta_i = \beta_i(1)$, $i = 1, \dots, k$.

Lemma 4.4. *Let* $\sigma_i = \sigma_i(\alpha)$ *and* $\tau_i = \tau_i(\alpha)$ *be such that* $\alpha - \sigma_i \mathfrak{v}_1 = (\alpha - \mathbb{R}_{\geq 0}\mathfrak{v}_1) \cap \mathcal{E}_i$ *and* $\alpha - \tau_i \mathfrak{v}_k = (\alpha - \mathbb{R}_{\geq 0}\mathfrak{v}_k) \cap \mathcal{E}_i$ *for any* $i = 1, \dots, k$. *If* $\alpha = a_\ell \mathfrak{v}_\ell$, $a_\ell > 0$ *for some* $\ell \in \{1, \dots, k\}$ *then we have*

$$0 < \sigma_1(\alpha) < \ldots < \sigma_\ell(\alpha) = \ldots = \sigma_k(\alpha) \quad \text{and} \quad \tau_1(\alpha) = \ldots = \tau_\ell(\alpha) > \ldots > \tau_k(\alpha) > 0.$$

Moreover, for general $\alpha \in \mathbb{R}_{\geq 0}\langle \mathfrak{v}_i \rangle_{i=\overline{1,k}}$ *one has* $\sigma_1(\alpha) \leq \ldots \leq \sigma_k(\alpha)$ *and* $\tau_1(\alpha) \geq \ldots \geq \tau_k(\alpha)$.

Proof. Note that we have additivity $\tau_i(\alpha' + \alpha'') = \tau_i(\alpha') + \tau_i(\alpha'')$ and $\sigma_i(\alpha' + \alpha'') = \sigma_i(\alpha') + \sigma_i(\alpha'')$, hence we may assume that $\alpha = a_\ell \mathfrak{v}_\ell$. Moreover, we will only prove the lemma for σ_i's. The intersection point $\alpha - \sigma_i \mathfrak{v}_1$ is characterized by $(\alpha - \sigma_i \mathfrak{v}_1, E^*_{n_i}) = 0$, whence

$$\sigma_i = \sigma_i(a_\ell \mathfrak{v}_\ell) = \frac{(a_\ell \mathfrak{v}_\ell, E^*_{n_i})}{(\mathfrak{v}_1, E^*_{n_i})} = a_\ell \frac{(E^*_{n_\ell}, E^*_{n_i})}{(E^*_{n_1}, E^*_{n_i})} > 0.$$

Therefore, it is enough to show that

$$\frac{(E^*_{n_\ell}, E^*_{n_i})}{(E^*_{n_1}, E^*_{n_i})} < \frac{(E^*_{n_\ell}, E^*_{n_{i+1}})}{(E^*_{n_1}, E^*_{n_{i+1}})}, \quad \forall\, i < \ell \quad \text{and} \quad \frac{(E^*_{n_\ell}, E^*_{n_i})}{(E^*_{n_1}, E^*_{n_i})} = \frac{(E^*_{n_\ell}, E^*_{n_{i+1}})}{(E^*_{n_1}, E^*_{n_{i+1}})}, \quad \forall\, i \geq \ell. \tag{4.1}$$

Recall that by (2.1) $(E^*_v, E^*_w) = -\dfrac{\det_{\Gamma \setminus [v,w]}}{\det_\Gamma}$ for any vertices v, w, hence (4.1) is equivalent to the following determinantal relations

$$\det_{\Gamma \setminus [n_1, n_i]} \det_{\Gamma \setminus [n_{i+1}, n_\ell]} - \det_{\Gamma \setminus [n_1, n_{i+1}]} \cdot \det_{\Gamma \setminus [n_i, n_\ell]} > 0, \quad \forall\, i < \ell, \tag{4.2}$$

and equality for $i \geq \ell$.

We use the technique of N. Duchon (cf. [**9**, Section 21]) to reduce (4.2) to the case when Γ is a bamboo. To do so, we can remove peripheral edges of a graph in order to simplify graph determinant computations. Removal of such an edge is compensated by adjusting the decorations of the graph. Let v be a vertex with decoration b_v and which is connected by an edge only to a vertex w with decoration b_w. If we remove this edge and replace the decoration of the vertex w by $b_w - b_v^{-1}$ then the resulting non-connected graph will be also negative definite and its determinant does not change. Using this technique we remove consecutively every edge on the legs of Γ, and denote the resulting decorated graph by Γ' which consists of a bamboo – connecting the nodes n_1 and n_k – and isolated vertices. Note that $\det_{\Gamma \setminus [n_i, n_j]} = \det_{\Gamma' \setminus [n_i, n_j]}$ for all $i, j = 1, \dots, k$. Moreover, (4.2) is equivalent with

$$\det_{\Gamma' \setminus [n_1, n_i]} \det_{\Gamma' \setminus [n_{i+1}, n_\ell]} - \det_{\Gamma' \setminus [n_1, n_{i+1}]} \cdot \det_{\Gamma' \setminus [n_i, n_\ell]} > 0, \quad \forall\, i < \ell, \tag{4.3}$$

and equality for $i \geq \ell$, respectively. From point of view of (4.3) we can forget about the isolated vertices of Γ', i.e. we may assume that Γ' is a bamboo. If we denote by $\det'_{[n_i, n_j]}$ the determinant of the graph $[n_i, n_j]$ as subgraph of (the bamboo) Γ' then for $i < \ell$ we have

$$\begin{aligned}\det_{\Gamma'\setminus[n_1,n_i]} \det_{\Gamma'\setminus[n_{i+1},n_\ell]} - \det_{\Gamma'\setminus[n_1,n_{i+1}]} \cdot \det_{\Gamma'\setminus[n_i,n_\ell]} = \\ \det'_{[n_1,n_{i+1})} \cdot \det'_{(n_i,n_k]} \cdot \det'_{(n_\ell,n_k]} - \det'_{[n_1,n_i)} \cdot \det'_{(n_{i+1},n_k]} \cdot \det'_{(n_\ell,n_k]} \\ = \det'_{[n_1,n_k]} \cdot \det'_{(n_i,n_{i+1})} \cdot \det'_{(n_\ell,n_k]},\end{aligned}$$

where the second equality uses the identity

$$\det'_{[n_1,n_{i+1})} \cdot \det'_{(n_i,n_k]} = \det'_{[n_1,n_k]} \cdot \det'_{(n_i,n_{i+1})} + \det'_{[n_1,n_i)} \cdot \det'_{(n_{i+1},n_k]}$$

from [**14**, Lemma 2.1.2]. Moreover, $\det'_{[n_1,n_k]} \cdot \det'_{(n_i,n_{i+1})} \cdot \det'_{(n_\ell,n_k]} > 0$ since any subgraph of the nondegenerate negative definite graph Γ' is nondegenerate negative definite, i.e. with strictly positive determinant. Therefore, all three factors of the product are positive themselves (note that $\det'_{(n_k,n_k]} = 1$). If $i \geq \ell$ then it is easy to see

$$\begin{aligned}\det_{\Gamma'\setminus[n_1,n_i]} \det_{\Gamma'\setminus[n_{i+1},n_\ell]} - \det_{\Gamma'\setminus[n_1,n_{i+1}]} \cdot \det_{\Gamma'\setminus[n_i,n_\ell]} = \\ \det'_{(n_i,n_k]} \cdot \det'_{[n_1,n_\ell)} \cdot \det'_{(n_{i+1},n_k]} - \det'_{(n_{i+1},n_k]} \cdot \det'_{[n_1,n_\ell)} \cdot \det'_{(n_i,n_k]} = 0.\end{aligned}$$

□

We also introduce additional notations $\mathcal{E}_0 = \mathcal{E}_0(\alpha) = \alpha - \mathbb{R}_{\geq 0}\mathfrak{v}_k$ and $\mathcal{E}_{k+1} = \mathcal{E}_{k+1}(\alpha) = \alpha - \mathbb{R}_{\geq 0}\mathfrak{v}_1$. Moreover, denote by $\varepsilon_{i,j} = \varepsilon_{i,j}(\alpha) = \mathcal{E}_i(\alpha) \cap \mathcal{E}_j(\alpha)$ the intersection points of segments $\mathcal{E}_i$ and $\mathcal{E}_j$.

Lemma 4.5. *For any $i = 0, \dots, k+1$ and $\alpha \in \mathbb{R}_{\geq 0}\langle \mathfrak{v}_\ell \rangle_{\ell=\overline{1,k}}$, on $\mathcal{E}_i(\alpha)$ the intersection points are in the following order: $\varepsilon_{i,0}(\alpha)$, …, $\varepsilon_{i,i-1}(\alpha)$, $\varepsilon_{i,i+1}(\alpha)$, …, $\varepsilon_{i,k+1}(\alpha)$.*

Proof. For $i = 0$ and $i = k+1$ the statement is immediate from Lemma 4.4. Notice that we have defined $\sigma_i = \sigma_i(\alpha)$ and $\tau_i = \tau_i(\alpha)$ such that $\varepsilon_{i,0} = \alpha - \tau_i \mathfrak{v}_k$ and $\varepsilon_{i,k+1} = \alpha - \sigma_i \mathfrak{v}_1$. If $t_{i,j} = t_{i,j}(\alpha) \in [0,1]$ such that $\varepsilon_{i,j} = (1 - t_{i,j})\varepsilon_{i,0} + t_{i,j}\varepsilon_{i,k+1}$, then we have to prove that (†) $t_{i,j}(\alpha) \leq t_{i,j+1}(\alpha)$ for all j.

The case $\alpha = a_\ell \mathfrak{v}_\ell$, $a_\ell > 0$ for some $\ell \in \{1, \dots, k\}$ follows directly from the first part of Lemma 4.4, resulting (‡) $t_{i,j}(a_\ell \mathfrak{v}_\ell) \leq t_{i,j+1}(a_\ell \mathfrak{v}_\ell)$ for all j. For general $\alpha = \sum_{\ell=1}^k a_\ell \mathfrak{v}_\ell$ one has the additivity $\varepsilon_{i,j}(\alpha' + \alpha'') = \varepsilon_{i,j}(\alpha') + \varepsilon_{i,j}(\alpha'')$ (as vectors), hence by definition we get

$$t_{i,j}(\alpha' + \alpha'') = \frac{t_{i,j}(\alpha')\sigma_i(\alpha') + t_{i,j}(\alpha'')\sigma_i(\alpha'')}{\sigma_i(\alpha') + \sigma_i(\alpha'')}, \tag{4.4}$$

which implies the inequalities (†) using the special cases (‡) for any ℓ. □

Lemma 4.6. *The bounded region $(\alpha - \mathbb{R}_{\geq 0}\langle \mathfrak{v}_1, \mathfrak{v}_k \rangle) \setminus \mathbb{R}_{<0}\langle E_n \rangle_{n \in \mathcal{N}}$ is the union of quadrangles between segments $\mathcal{E}_i, \mathcal{E}_{i+1}, \mathcal{E}_j, \mathcal{E}_{j+1}$ or triangles (degenerated cases). These polygons may intersect each other only at the boundary.*

Proof. The segments $\mathcal{E}_i$ divide $(\alpha - \mathbb{R}_{\geq 0}\langle \mathfrak{v}_1, \mathfrak{v}_k \rangle) \setminus \mathbb{R}_{<0}\langle E_n \rangle_{n \in \mathcal{N}}$ into convex polygons. By Lemma 4.5, we can assume that $[\varepsilon_{i,j}, \varepsilon_{i,j+1}]$ and $[\varepsilon_{i+1,j}, \varepsilon_{i,j}]$ are two faces at vertex $\varepsilon_{i,j}$ of such a polygon. Moreover, $\varepsilon_{i+1,j}$ and $\varepsilon_{i,j+1}$ must be also vertices of the polygon and another two faces must lie on segments $\mathcal{E}_{i+1}$ and $\mathcal{E}_{j+1}$. Hence, the segments $\mathcal{E}_i$, $\mathcal{E}_j$, $\mathcal{E}_{i+1}$, $\mathcal{E}_{j+1}$ form a convex polygon with vertices $\varepsilon_{i,j}$, $\varepsilon_{i+1,j}$, $\varepsilon_{i,j+1}$ and $\varepsilon_{i+1,j+1}$. The polygon can degenerate into triangles with vertices $\varepsilon_{i,j}$, $\varepsilon_{i,j+1}$ and $\varepsilon_{i+1,j+1}$, see e.g. Figure 1. □

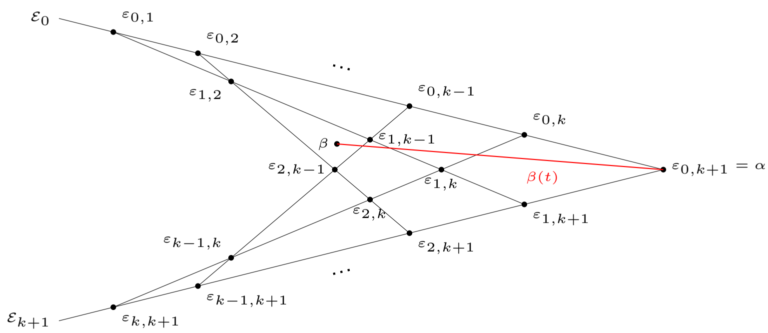


FIGURE 1. *The intersection points $\varepsilon_{i,j}$ and parametric line $\beta(t)$*

Proof of Lemma 4.3. Let $\beta \in (\alpha - \mathbb{R}_{\geq 0}\langle \mathfrak{v}_1, \mathfrak{v}_k\rangle) \setminus \mathbb{R}_{<0}\langle E_n\rangle_{n\in\mathcal{N}}$ be fixed. Consider the parametric line $\beta(t) = t\beta + (1-t)\alpha$ connecting β to the vertex α of the affine cone, c.f. Figure 1. The order in which $\beta(t)$ intersects the segments $\mathcal{E}_i$ as t goes from 0 to 1 tells us the order in which the coordinates $\beta_i(t)$ of $\beta(t)$ are changing signs.

In the beginning, every $\beta_i(t) > 0$ and $\beta(t)$ sits in the polygon with vertices $\alpha = \varepsilon_{0,k+1}$, $\varepsilon_{0,k}$, $\varepsilon_{1,k}$, $\varepsilon_{k+1,1}$, with sides lying on $\mathcal{E}_0$, $\mathcal{E}_{k+1}$, $\mathcal{E}_1$, $\mathcal{E}_k$. We also say that we have already intersected $\mathcal{E}_0$ and $\mathcal{E}_{k+1}$. Then $\beta(t)$ either intersects $\mathcal{E}_1$, hence $\beta_1(t)$ changes to $\beta_1(t) < 0$ and $\beta(t)$ arrives into the polygon $\varepsilon_{1,k+1}, \varepsilon_{1,k}, \varepsilon_{2,k}, \varepsilon_{2,k+1}$ with sides on $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_k$, $\mathcal{E}_{k+1}$, or, it intersects $\mathcal{E}_k$ implying that $\beta_k(t)$ becomes negative and $\beta(t)$ arrives into the polygon with sides on $\mathcal{E}_0$, $\mathcal{E}_1$, $\mathcal{E}_{k-1}$, $\mathcal{E}_k$. Therefore, we have crossed $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_{k+1}$ in the first, while $\mathcal{E}_0, \mathcal{E}_k, \mathcal{E}_{k+1}$ in the second case.

By induction, we assume that $\beta(t)$ lies in the polygon with sides $\mathcal{E}_i, \mathcal{E}_{i+1}, \mathcal{E}_j, \mathcal{E}_{j+1}$ for some t and it has already crossed $\mathcal{E}_0, \ldots, \mathcal{E}_i, \mathcal{E}_{j+1}, \ldots, \mathcal{E}_{k+1}$, that is $\beta_1(t), \ldots, \beta_i(t), \beta_{j+1}(t), \ldots, \beta_k(t) < 0$ and $\beta_{i+1}(t), \ldots, \beta_j(t) \geq 0$. Thus, $\beta(t)$ must intersect $\mathcal{E}_{i+1}$ or $\mathcal{E}_j$. Therefore, either $\beta_{i+1}(t)$ changes sign to $\beta_{i+1}(t) < 0$ and $\beta(t)$ arrives into the polygon with sides $\mathcal{E}_{i+1}, \mathcal{E}_{i+2}, \mathcal{E}_j, \mathcal{E}_{j+1}$, or $\beta_j(t)$ changes to $\beta_j(t) < 0$ and $\beta(t)$ arrives into the polygon with $\mathcal{E}_i, \mathcal{E}_{i+1}, \mathcal{E}_j, \mathcal{E}_{j-1}$. Hence, the induction stops after arriving to β and proves the desired configuration of signs. □

Proof of Proposition 4.2. If $p_\beta \mathbf{t}_{\mathcal{N}}^{\beta}$ is a monomial term of $\varphi^+(\mathbf{t}_{\mathcal{N}})$ then $\beta \in \alpha - \mathbb{R}_{\geq 0}\langle \mathfrak{v}_1, \mathfrak{v}_k\rangle$, moreover not all β_ℓ are negative and so we have sign configuration as in Lemma 4.3. To compute the multiplicity $\mathfrak{s}(\beta)$ we choose the ordering of nodes $n_\ell \succ n_{\ell+1}$ for all $\ell = 1, \ldots, k-1$. If $\beta_k \geq 0$ then $\mathfrak{s}_{n_k}(\beta) = 1$ and $\mathfrak{s}_{n_\ell \succ n_{\ell+1}}(\beta) = 0$ for all $\ell = 1, \ldots, k-1$, thus $\mathfrak{s}(\beta) = \mathfrak{s}_{n_k}(\beta) + \sum_{\ell=1}^{k-1} \mathfrak{s}_{n_\ell \succ n_{\ell+1}}(\beta) = 1$. If $\beta_k < 0$ then $\mathfrak{s}_{n_k}(\beta) = 0$ and $\mathfrak{s}_{n_\ell \succ n_{\ell+1}}(\beta) = 0$ for all ℓ except for $\ell = j$, for which $\beta_j \geq 0$ and $\beta_{j+1} < 0$, thus $\mathfrak{s}(\beta) = 1$ in this case too. □

4.2. *An example with higher multiplicities*

Consider the following negative definite plumbing graph Γ given by the left hand side of Figure 2. The associated plumbed 3-manifold is obtained by $(-7/2)$-surgery along the connected sum of three right handed trefoil knots in S^3. Its group $H \simeq \mathbb{Z}_7$ is cyclic of order 7, generated by the class $[E^*_{+1}]$, where E^*_{+1}, E^*_i and E^*_{ij} are the dual base elements in L' associated with the corresponding vertices shown by Figure 2. For simplicity, we set $\bar{l} := \pi_{\mathcal{N}}(l)$ for $l \in L \otimes \mathbb{Q}$ and use short notation (l_+, l_1, l_2, l_3) for $\bar{l} = l_+ E_+ + \sum_{i=1}^3 l_i E_i$.

By applying the Euclidean division algorithm from Section 3.1 to the full $f(\mathbf{t})$ as defined in (1.1), it turns out that every exponent $\beta = (\beta_+, \beta_1, \beta_2, \beta_3)$ appearing in $P^+(\mathbf{t}_{\mathcal{N}})$ can be written in the form $\beta = c_+ \bar{E}^*_+ + \sum_{i=1}^3 c_i \bar{E}^*_i - \sum_{i=1}^3 \sum_{j=1}^2 x_{ij} \bar{E}^*_{ij} - x_{+1} \bar{E}^*_{+1}$ for some $0 \leq c_+ \leq 2$, $0 \leq c_i \leq 1$ and $x_{ij}, x_{+1} \geq 1$. E.g, for the choice $c_+ = 2$, $c_i = 1$, $x_{+1} = x_{ij} = 1$ for $i \in \{1, 2\}$ and $x_{3j} = 2$ we get $\beta_0 = (-1/7, 1/7, 1/7, -34/7)$. Moreover, one can check that this is the only

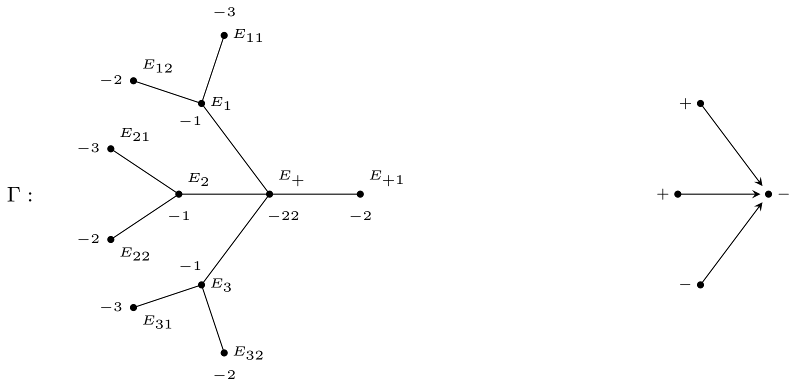


FIGURE 2. *The graph* Γ *and the chosen partial ordering on the nodes*

way to write β_0 in the above form. Therefore, the orientation (or partial order) given by the right hand side of Figure 2 implies that $\mathfrak{s}(\beta_0) = 2$. In fact, β_0 belongs to $P_6^+(\mathbf{t}_\mathcal{N})$, where we use $h \in \mathbb{Z}_7 = \{0, 1, \ldots, 6\}$ as a number to index P_h^+. Hence, by Theorem 3.4

$$P_6(\mathbf{t}_\mathcal{N}) \neq P_6^+(\mathbf{t}_\mathcal{N}).$$

We also emphasize that the exponents

$(-1/7, 1/7, 1/7, -34/7)$, $(-1/7, 1/7, -34/7, 1/7)$, $(-1/7, -34/7, 1/7, 1/7)$ with coefficient $p_\beta = 1$ and

$(-1/7, 1/7, 1/7, -27/7)$, $(-1/7, 1/7, -27/7, 1/7)$, $(-1/7, -27/7, 1/7, 1/7)$ with coefficient $p_\beta = -1$

(all of them present in $P_6^+(\mathbf{t}_\mathcal{N})$) are the only exponents with $\mathfrak{s}(\beta) = 2 > 1$. Hence, although the two polynomials may be different, it still holds that $P_h(1) = P_h^+(1) = \mathfrak{sw}_h^{norm}$ for any $h \in \mathbb{Z}_7$.

4.3. *On the 3-manifold* $S^3_{-p/q}(K)$

4.3.1. *Algebraic knots* Assume $K \subset S^3$ is an algebraic knot, i.e. it is the link of an irreducible plane curve singularity defined by the function germ $\mathfrak{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$.

The *Newton pairs* of K are the pairs of integers $\{(p_i, q_i)\}_{i=1}^r$, where $p_i \geq 2$, $q_i \geq 1$, $q_1 > p_1$ and $\gcd(p_i, q_i) = 1$. They are the exponents appearing naturally in the normal form of $\mathfrak{f}$. From topological point of view, it is more convenient to use the *linking pairs* $(p_i, a_i)_{i=1}^r$ (the decorations of the splice diagram, cf. [**9**]), which can be calculated recursively by

$$a_1 = q_1 \quad \text{and} \quad a_{i+1} = q_{i+1} + a_i p_i p_{i+1} \quad \text{for} \quad i \geq 1. \tag{4.5}$$

The set of intersection multiplicities of $\mathfrak{f}$ with all possible analytic germs is a *numerical semigroup* denoted by $\mathcal{M}_\mathfrak{f}$. Although its definition is analytic, $\mathcal{M}_\mathfrak{f}$ is described combinatorially by its Hilbert basis: $p_1 p_2 \cdots p_r$, $a_i p_{i+1} \cdots p_r$ for $1 \leq i \leq r-1$, and a_r. In fact, $|\mathbb{Z}_{\geq 0} \setminus \mathcal{M}_\mathfrak{f}| = \mu_\mathfrak{f}/2$ (cf. [**17**]), where $\mu_\mathfrak{f}$ is the Milnor number of $\mathfrak{f}$. The Frobenius number of $\mathcal{M}_\mathfrak{f}$ is $\mu_\mathfrak{f} - 1$, and for $\ell \leq \mu_\mathfrak{f} - 1$ one has the symmetry:

$$\ell \in \mathcal{M}_\mathfrak{f} \text{ if and only if } \mu_\mathfrak{f} - 1 - \ell \notin \mathcal{M}_\mathfrak{f}. \tag{4.6}$$

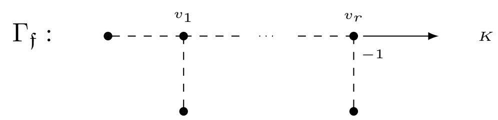
We emphasize that the integer $\delta_\mathfrak{f} := \mu_\mathfrak{f}/2$ is called the delta-invariant of $\mathfrak{f}$, which equals the minimal Seifert genus of the knot K.

The *Alexander polynomial* $\Delta(t)$ of K (normalized by $\Delta(1) = 1$) can be calculated in terms of the linking pairs via the formula

$$\Delta(t) = \frac{(1 - t^{a_1 p_1 p_2 \cdots p_r})(1 - t^{a_2 p_2 \cdots p_r}) \cdots (1 - t^{a_r p_r})(1 - t)}{(1 - t^{a_1 p_2 \cdots p_r})(1 - t^{a_2 p_3 \cdots p_r}) \cdots (1 - t^{a_r})(1 - t^{p_1 \cdots p_r})}. \tag{4.7}$$

It has degree $\mu_\mathfrak{f}$. On the other hand, $\Delta(t)/(1-t) = \sum_{\ell \in \mathcal{M}_\mathfrak{f}} t^\ell$ is the *monodromy zeta-function* of $\mathfrak{f}$ (cf. [**5**]), whose *polynomial part* can be given in terms of gaps of the semigroup: $P_\mathfrak{f}(t) = -\sum_{\ell \notin \mathcal{M}_\mathfrak{f}} t^\ell$ (cf. [**14**, Subsection 7.1.2]). Hence, the degree of $P_\mathfrak{f}(t)$ equals $\mu_\mathfrak{f} - 1$.

The *embedded minimal good resolution graph* of $\mathfrak{f}$ (or the minimal negative-definite plumbing graph of K) has the shape of



where the arrowhead, attached to the unique (-1)-vertex, represents the knot K. Its decorations can be calculated from the Newton pairs $\{(p_i, q_i)\}_i$ using e.g. [**9**], see also [**18**, Section 4.I]. The graph has an additional multiplicity decoration: the multiplicity of a vertex is the coefficient of the pullback-divisor of $\mathfrak{f}$ along the corresponding exceptional divisor, while the arrowhead has the multiplicity decoration 1. E.g., we set $m_\mathfrak{f} := a_r p_r$ to be the multiplicity of the (-1)–vertex.

Notice that the isotopy type of $K \subset S^3$ is completely characterized by any of the invariants highlighted above. For general references see [**2**], [**9**] and also the presentation of [**18**] and [**25**].

4.3.2. *The plumbing of* $S^3_{-p/q}(K)$ Let $p/q > 0$ ($p > 0$, $\gcd(p,q) = 1$) be a positive rational number and $\{K_j\}_{j=1}^\nu$ be a collection of algebraic knots. Then we consider the oriented 3-manifold $M = S^3_{-p/q}(K)$, obtained by $(-p/q)$-surgery along the connected sum $K = K_1 \# \cdots \# K_\nu \subset S^3$ of the knots K_j. All the invariants associated with K_j, listed in the previous section, will be indexed by j. E.g., the linking pairs of K_j will be denoted by $(p_i^{(j)}, a_i^{(j)})_{i=1}^{r_j}$, the Alexander polynomial by $\Delta^{(j)}(t)$ and $m^{(j)}$ stands for the multiplicity of the (-1)-vertex in the minimal plumbing graph of K_j as above. Set also $m := \sum_{j=1}^\nu m^{(j)}$.

The schematic picture of the plumbing graph Γ of the oriented 3–manifold $M = S^3_{-p/q}(K)$ has the form as shown in Figure 3 (see [**4**]), where the dash-lines represent strings of vertices. The

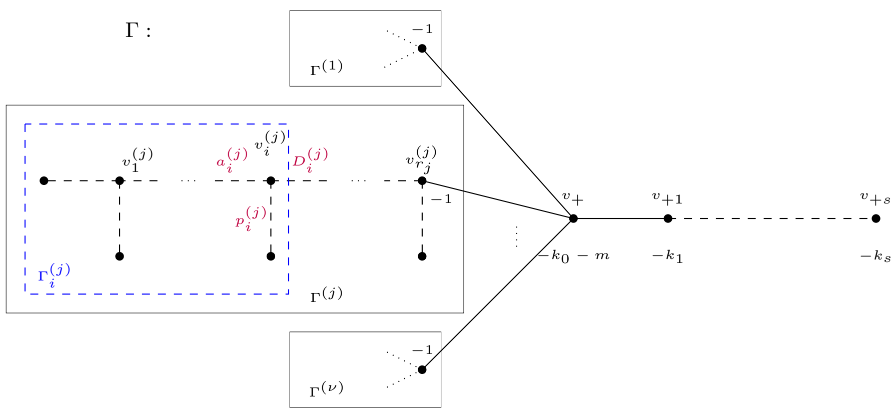


FIGURE 3. *Plumbing graph of* $S^3_{-p/q}(K)$

integers $k_0 \geq 1$ and $k_i \geq 2$ $(1 \leq i \leq s)$, in the decorations of the vertices v_{+i}, are determined by the Hirzebruch/negative continued fraction expansion

$$p/q = [k_0, \ldots, k_s] = k_0 - 1/(k_1 - 1/(\cdots - 1/k_s)\cdots).$$

We write E_+, E_{+i} and $E_i^{(j)}$ for the base elements corresponding to the vertices v_+, v_{+i} and $v_i^{(j)}$, respectively.

It is also known that the homology group $H = L'/L \simeq \mathbb{Z}_p$ is the cyclic group of order p, generated by $[E^*_{+s}]$ (for a complete proof see [**4**, Lemma 6]).

In Figure 3 we have put at the node $v_i^{(j)}$ its splice diagram decorations $a_i^{(j)}$, $p_i^{(j)}$ and $D_i^{(j)}$ defined in [**9**], i.e. the determinants of the respective connected subgraphs we get by deleting $v_i^{(j)}$ and its adjacent edges from Γ. For simplicity, we use notation $\Gamma_i^{(j)}$ for the subgraph spanned by the nodes $\{v_{i'}^{(j)}\}_{i'=1}^{i}$ and their corresponding end-vertices (inside the dashed rectangle in Figure 3). Hence, by definition one can write $D_i^{(j)} = \det(\Gamma \setminus \Gamma_i^{(j)})$. Also, set $\Gamma^{(j)} := \Gamma_{r_j}^{(j)}$ and its self-intersection decorations are the same as of the embedded minimal good resolution graph of K_j, we omit the decorations from the picture for simplicity.

In the next lemma we prove some useful formulas.

Lemma 4.7.
(i) $D_i^{(j)} = p + a_i^{(j)} p_i^{(j)} \left(p_{i+1}^{(j)} \cdots p_{r_j}^{(j)}\right)^2 q$, *for* $1 \leq i \leq r_j$;
(ii) $a_{i+1}^{(j)} D_i^{(j)} = q_{i+1}^{(j)} p + a_i^{(j)} p_i^{(j)} p_{i+1}^{(j)} D_{i+1}^{(j)}$, *for* $1 \leq i \leq r_j - 1$.

Proof. Let K'_j be the knot with Newton pairs $(p_{i'}^{(j)}, q_{i'}^{(j)})_{i'=i+1}^{r_j}$. The graph $\Gamma \setminus \Gamma_i^{(j)}$ is the plumbing graph of the manifold $S^3_{-p'/q}(K'_j \# \#_{j' \neq j} K_{j'})$ for some p' which can be computed as follows. The new linking pairs $(p_{i'}^{(j)}, \widetilde{a}_{i'}^{(j)})_{i'=i+1}^{r_j}$ can be calculated recursively using (4.5) and $\widetilde{a}_{i+1}^{(j)} = q_{i+1}^{(j)}$. Hence, we find the identity

$$\widetilde{a}_{r_j}^{(j)} = a_{r_j}^{(j)} - a_i^{(j)} p_i^{(j)} \left(p_{i+1}^{(j)} \cdots p_{r_j-1}^{(j)}\right)^2 p_{r_j}^{(j)},$$

which implies that the multiplicity $\widetilde{m}^{(j)}$ of the (-1)-vertex in the embedded graph of K'_j equals $\widetilde{a}_{r_j}^{(j)} p_{r_j}^{(j)} = m^{(j)} - a_i^{(j)} p_i^{(j)} \left(p_{i+1}^{(j)} \cdots p_{r_j}^{(j)}\right)^2$. Since the decoration on v_+ remains unchanged we must have for the Hirzebruch/negative continued fraction

$$p'/q = \left[k_0 + a_i^{(j)} p_i^{(j)} \left(p_{i+1}^{(j)} \cdots p_{r_j}^{(j)}\right)^2, k_1, \ldots, k_s\right] = p/q + a_i^{(j)} p_i^{(j)} \left(p_{i+1}^{(j)} \cdots p_{r_j}^{(j)}\right)^2.$$

Finally, note that $p' = D_i^{(j)}$ is the determinant of the graph $\Gamma \setminus \Gamma_i^{(j)}$ [**4**, Lemma 6]. This concludes the formula of (i) . The recursive identity of (ii) can be easily verified using (i). □

4.3.3. *Seiberg–Witten invariant via Alexander polynomials* We consider the product of the Alexander polynomials $\Delta(t) := \prod_j \Delta^{(j)}(t)$ with degree $\mu := \sum_j \mu^{(j)}$. By the known facts $\Delta(1) = 1$ and $\Delta'(1) = \mu/2$, we get a unique decomposition

$$\Delta(t) = 1 + (\mu/2)(t-1) + (t-1)^2 \cdot \mathcal{Q}(t)$$

for some polynomial with integral coefficients $\mathcal{Q}(t) = \sum_{i=0}^{\mu-2} \mathfrak{q}_i t^i$ of degree $\mu - 2$.

We remark that the coefficients of $\mathcal{Q}$ has many interesting arithmetical properties. E.g., notice that $\mathfrak{q}_0 = \mu/2$, $\mathfrak{q}_{\mu-2} = 1$ and $\mathfrak{q}_{\mu-2-i} = \mathfrak{q}_i + i + 1 - \mu/2$ for $0 \leq i \leq \mu - 2$, given by the symmetry of Δ. The explicit calculation of a general coefficient is rather hard, one can expect it to be connected with some counting function in a semigroup/affine monoid structure associated with the manifold M (cf. [**14**]). In particular, if $\nu = 1$ one can check that $\mathfrak{q}_i = \#\{n \notin \mathcal{M} \ : \ n > i\}$, where $\mathcal{M}$ is the semigroup of the unique algebraic knot K. More details and discussions about these coefficients can be found e.g. in [**3**].

We look at the decomposition $\mathcal{Q}(t) = \sum_{h \in \mathbb{Z}_p} \mathcal{Q}_h(t)$ where $\mathcal{Q}_h(t) := \sum_{i \geq 0} \mathfrak{q}_{[(ip+h)/q]} t^{[\frac{ip+h}{q}]}$ and consider the following normalization (different from Formula (2.2)) of the Seiberg–Witten invariants:

$$\widetilde{\mathfrak{sw}}_h^{norm}(M) := -\mathfrak{sw}_{-[hE^*_{+s}]*\sigma_{can}}(M) - ((K + 2hE^*_{+s})^2 + \mathcal{V})/8 \quad \text{for } 0 \le h < p. \tag{4.8}$$

Then the following identity is known by [**1**, **20**, **25**]:

$$\mathcal{Q}_h(1) = \widetilde{\mathfrak{sw}}_h^{norm}(M). \tag{4.9}$$

4.3.4. *On the structure of the polynomial part* For any $h \in \mathbb{Z}_p$ consider the decomposition

$$f_h(\mathbf{t}_{\mathcal{N}}) = P_h^+(\mathbf{t}_{\mathcal{N}}) + f_h^{neg}(\mathbf{t}_{\mathcal{N}}),$$

given by Lemma 3.1, i.e. $f_h^{neg}(\mathbf{t}_{\mathcal{N}})$ has negative degree in each variable and write $P_h^+(\mathbf{t}_{\mathcal{N}}) = \sum_{\beta\in\mathcal{B}_h} p_\beta \mathbf{t}_{\mathcal{N}}^\beta$ where $\beta = (\beta_v)_{v\in\mathcal{N}}$ and $\beta \not< 0$. Let β_+ be the E_+-coefficient of β and set $\mathcal{B} := \bigcup_h \mathcal{B}_h$ too. For any polynomial $\mathcal{P}(\mathbf{t}_{\mathcal{N}})$ we consider the decomposition $\mathcal{P}_{\beta_+\ge 0}(\mathbf{t}_{\mathcal{N}}) + \mathcal{P}_{\beta_+<0}(\mathbf{t}_{\mathcal{N}})$ so that the first part consists of those monomial terms for which $\beta_+ \ge 0$, and similarly, all the terms of the second part have $\beta_+ < 0$.

By definitions we have $P^+_{h,\beta_+\ge 0}(\mathbf{t}_{\mathcal{N}}) = P_h^{v_+}(\mathbf{t}_{\mathcal{N}})$, where $P_h^{v_+}$ is the result of a single variable division, see Subsection 2.4.2. Moreover, recall that Theorem 3.4 concludes that the monomial terms of $P_h(\mathbf{t}_{\mathcal{N}})$ are exactly of $P_h^+(\mathbf{t}_{\mathcal{N}})$ with multiplicities. Therefore, in general, the difference polynomial $\mathcal{D}_h(\mathbf{t}_{\mathcal{N}}) := P_h(\mathbf{t}_{\mathcal{N}}) - P_h^{v_+}(\mathbf{t}_{\mathcal{N}})$ consists of $P_{h,\beta_+<0}(\mathbf{t}_{\mathcal{N}})$ and the higher multiplicity terms ($\mathfrak{s}(\beta) \ge 2$) from $P_{h,\beta_+\ge 0}(\mathbf{t}_{\mathcal{N}})$.

However, in the next theorem we show that there are no monomial terms in $P_{h,\beta_+\ge 0}(\mathbf{t}_{\mathcal{N}})$ with $\mathfrak{s}(\beta) \ge 2$, i.e. $\mathcal{D}_h(\mathbf{t}_{\mathcal{N}}) = P_{h,\beta_+<0}(\mathbf{t}_{\mathcal{N}})$.

Theorem 4.8.

$$P_{h,\beta_+\ge 0}(\mathbf{t}_{\mathcal{N}}) = P^+_{h,\beta_+\ge 0}(\mathbf{t}_{\mathcal{N}}) = P_h^{v_+}(\mathbf{t}_{\mathcal{N}}).$$

Proof. First of all we may assume that $\nu \ge 2$ otherwise we have the situation of Section 4.1. We fix the orientation of Γ^{orb} towards to the node v_+ and consider its induced partial order $(\mathcal{N}, \succ)$ (see Section 3.2). For any $\beta \in \mathcal{B}$ for which $\beta_+ \ge 0$ one has $\mathfrak{s}_{v_+}(\beta) = 1$, thus by Theorem 3.4 we have to prove that for such a β we have $\mathfrak{s}_{v_i^{(j)} \succ v_{i+1}^{(j)}}(\beta) = 0$ for any $j \in \{1, \dots, \nu\}$ and $i \in \{1, \dots, r_j\}$. We set $v^{(j)}_{r_j+1} := v_+$. In order to see this, we prove that the sign configuration on the subgraphs $\Gamma^{(j)}$ behaves similarly as in Lemma 4.3. Thus, assuming $\beta_+ \ge 0$, for any j we show that

$$\beta_1^{(j)}, \dots, \beta_{i-1}^{(j)} < 0 \le \beta_i^{(j)}, \dots, \beta_{r_j}^{(j)}, \beta_+ \quad \text{for some } \ i \in \{1, \dots, r_j\}. \tag{4.10}$$

It is enough to show that Proposition 4.2 can be applied to the zeta-function $f(\mathbf{t}_{\mathcal{N}_j})$ reduced to the subset of nodes $\mathcal{N}_j$ consisting of $v_i^{(j)}$ and v_+ of Γ.

For a fixed j we construct a new plumbing graph Γ_{M_j} by deleting all the subgraphs $\Gamma^{(j')}$ and its adjacent edges in Γ for any $j' \neq j$ and modifying the decoration of v_+ into $-k_0 - m^{(j)}$. Then the new graph Γ_{M_j} is the plumbing graph of the manifold $M_j := S^3_{-p/q}(K_j)$. Or, if we look at Γ as the minimal good resolution graph of a normal surface singularity then one can obtain a new resolution graph by blowing down all the subgraphs $\Gamma^{(j')}$. In this resolution, the new exceptional divisor corresponding to the vertex v_+ is a rational curve with singular points and self-intersection $-k_0 - m^{(j)}$. If we disregard the singularities of this divisor then we obtain a normal surface singularity whose link is M_j and its minimal good resolution is Γ_{M_j}.

We distinguish the invariants of the new graphs in the following way: L_j denotes the lattice associated with Γ_{M_j} with base elements $E_{v,j}$, the dual lattice will be denoted by L'_j with base elements $E^*_{v,j}$. We identify $\mathcal{N}_j$ of Γ with the same set of vertices of Γ_{M_j} (notice that v_+ has degree two so it is no longer a node in Γ_{M_j}). Then one can also identify the base elements of $\pi_{\mathcal{N}_j}(L)$ and $\pi_{\mathcal{N}_j}(L_j)$. In particular, one can show that $\pi_{\mathcal{N}_j}(E^*_+) = \pi_{\mathcal{N}_j}(E^*_{+,j})$.

Using the above identifications, one can check the following identity

$$f(\mathbf{t}_{\mathcal{N}_j}) = f_j(\mathbf{t}_{\mathcal{N}_j}) \prod_{j'\neq j} \Delta^{(j')}(\mathbf{t}_{\mathcal{N}_j}^{E_+^*}),$$

by calculating explicitely the zeta-function $f(\mathbf{t}_{\mathcal{N}_j})$ and the zeta-function $f_j(\mathbf{t}_{\mathcal{N}_j})$ associated with Γ_{M_j} restricted to $\mathcal{N}_j$ using the Formula (1.1) and (2.1), and comparing the result with the Formula (4.7) for Alexander polynomials.

The only problem is that $\mathcal{N}_j$ contains v_+ which is no longer a node in Γ_{M_j}. Nevertheless, we can blow up the vertex v_+ and denote the new graph by Γ'_{M_j}. Then the newly created (-1)-vertex is connected to v_+ (if $q=1$ then we can create two such (-1)-vertices), hence v_+ becomes a node of Γ'_{M_j}. Using the natural identifications we have $\pi_{\mathcal{N}_j}(E^*_{+,j}) = \pi_{\mathcal{N}_j}(E^*_{b,j})$ where $E^*_{b,j}$ denotes the newly created dual base element. Moreover, one has $f_j(\mathbf{t}_{\mathcal{N}_j}) = f'_j(\mathbf{t}_{\mathcal{N}_j})$, where f'_j is associated with Γ'_{M_j}.

Finally, the rational function $f'_j(\mathbf{t}_{\mathcal{N}_j})\prod_{j'\neq j}\Delta^{(j')}(\mathbf{t}_{\mathcal{N}_j}^{E_+^*})$ is the sum of rational fractions as in Proposition 4.2 which implies the sign configuration (4.10) by Lemma 4.3. □

We notice that for the difference polynomial $\mathcal{D}_h(\mathbf{t}_{\mathcal{N}}) := P_h(\mathbf{t}_{\mathcal{N}}) - P_h^{v_+}(\mathbf{t}_{\mathcal{N}})$ one has

$$\mathcal{D}_h(1) = \mathfrak{sw}_h^{norm}(M) - \widetilde{\mathfrak{sw}}_h^{norm}(M) = \chi(r_{[hE^*_{+s}]}) - \chi(hE^*_{+s}),$$

where $\chi(l') := -(K+l',l')/2$ for any $l' \in L'$. This follows from (2.7), (4.9) and the fact that $P_h^{v_+}(t) = \mathcal{Q}_h(t)$, which is proven in [**1**, 8.1]. Thus, Theorem 4.8 implies that $P_{h,\beta_+<0}$ counts only the difference between the normalizations and the Seiberg–Witten information is contained in $P^+_{h,\beta_+\geq 0}$.

4.3.5. *Canonical case $h=0$* From geometric point of view we are mainly interested in the case when $h=0$, since $f_0(\mathbf{t}_{\mathcal{N}})$ is related with analytic Poincaré series associated with a normal surface singularity whose link is M (c.f. Section 2.3, e.g. in the case when $q=1$ the manifold $M = S^3_{-p}(K)$ may appear as the link of a superisolated singularity).

In this case one has $\mathcal{D}_0(1) = P_{0,\beta_+<0}(1) = 0$, although it is not clear whether there are some monomial terms appearing in $P_{0,\beta_+<0}$. This can indeed occur for $h\neq 0$ as shown by the example from Section 4.2. However, in the sequel we prove that for $h=0$ this is not the case, i.e.

$$P^+_{0,\beta_+<0}(\mathbf{t}_{\mathcal{N}}) = P_{0,\beta_+<0}(\mathbf{t}_{\mathcal{N}}) \equiv 0.$$

Thus, we have $P_0(\mathbf{t}_{\mathcal{N}}) = P_0^+(\mathbf{t}_{\mathcal{N}})$, in particular $P_0^+(1) = \mathfrak{sw}_0^{norm}(M)$.

LEMMA 4.9. *Let $\mathfrak{f}_i^{(j)}$ be the irreducible plane curve singularity with Newton pairs $(p_{i'}^{(j)}, q_{i'}^{(j)})_{i'=1}^{i}$ for any $1\leq j\leq \nu$ and $1\leq i\leq r_j$ and its associated semigroup will be denoted by $\mathcal{M}_{\mathfrak{f}_i^{(j)}}$. Then for any exponent $\beta = (\beta_+, \{\beta_i^{(j)}\}_{j=\overline{1,\nu};i=\overline{1,r_j}}) \in \mathcal{B}$ of the monomial terms in $P^+(\mathbf{t}_{\mathcal{N}})$ we have the following relations*

(i)

$$a_{i+1}^{(j)}\beta_i^{(j)} = a_i^{(j)}p_i^{(j)}\beta_{i+1}^{(j)} + q_{i+1}^{(j)}\ell^\beta_{\mathfrak{f}_i^{(j)}},$$

where $\ell^\beta_{\mathfrak{f}_i^{(j)}} \in \mathbb{Z}\setminus\mathcal{M}_{\mathfrak{f}_i^{(j)}}$ depending on β. In particular, for $i=r_j$ we set $a^{(j)}_{r_j+1} := 1$, $q^{(j)}_{r_j+1} := 1$ and $\beta^{(j)}_{r_j+1} := \beta_+$, hence the identity becomes $\beta^{(j)}_{r_j} = m^{(j)}\beta_+ + \ell^\beta_{\mathfrak{f}^{(j)}}$.

(ii)

$$\beta_i^{(j)} < a_i^{(j)}p_i^{(j)}\cdots p_{r_j}^{(j)}(\beta_+ + 1).$$

Proof. (i) By Formula (1.1) the zeta-function for our special plumbing graph Γ (c.f. Figure 3) has the form $f(\mathbf{t}_{\mathcal{N}}) = \big(1-\mathbf{t}_{\mathcal{N}}^{E_+^*}\big)^{\nu-1}\cdot\prod_{j,i}\big(1-\mathbf{t}_{\mathcal{N}}^{E_i^{(j)*}}\big)/\big(1-\mathbf{t}_{\mathcal{N}}^{E_{+s}^*}\big)\cdot\prod_j\prod_{v\in\mathcal{E}^{(j)}}\big(1-\mathbf{t}_{\mathcal{N}}^{E_v^{(j)*}}\big)$, where we use notation $\mathcal{E}^{(j)}$ for the set of end-vertices of $\Gamma^{(j)}$ and $E_v^{(j)*}$ are the dual base elements associated with them. Hence, applying the division algorithm of Section 3.1 for $f(\mathbf{t}_{\mathcal{N}})$ we can write the exponents of the quotient polynomial $P^+(\mathbf{t}_{\mathcal{N}})$ in the following form

$$\beta = k_+E_+^* + \sum_j\Big(\sum_i k_i^{(j)}E_i^{(j)*} - \sum_{v\in\mathcal{E}^{(j)}} x_v^{(j)}E_v^{(j)*}\Big) - x_+E_{+s}^* \tag{4.11}$$

for some integers $0\le k_+\le\nu-1$, $k_i^{(j)}\in\{0,1\}$, $x_v^{(j)}\ge 1$ and $x_+\ge 1$.

For any subgraph Γ' of Γ let us denote by $\beta^{\Gamma'}$ the partial sum considering only those terms from the right hand side of (4.11) which are associated with the nodes and end-vertices of Γ'. Recall that in Subsection 4.3.2 we have defined $\Gamma_i^{(j)}$ as the subgraph spanned by the nodes $\{v_{i'}^{(j)}\}_{i'=1}^{i}$ and their corresponding end-vertices. Then, we claim that

$$a_{i+1}^{(j)}\beta_i^{(j)} = a_{i+1}^{(j)}\cdot(\beta,-E_i^{(j)*}) = a_i^{(j)}p_i^{(j)}\beta_{i+1}^{(j)} + \Big(\frac{a_{i+1}^{(j)}D_i^{(j)}}{p_{i+1}^{(j)}D_{i+1}^{(j)}} - a_i^{(j)}p_i^{(j)}\Big)\cdot\beta_{i+1}^{\Gamma_i^{(j)}}, \tag{4.12}$$

where $\beta_{i+1}^{\Gamma_i^{(j)}} := (\beta^{\Gamma_i^{(j)}}, -E_{i+1}^{(j)*})$ is the $E_{i+1}^{(j)}$-component of $\beta^{\Gamma_i^{(j)}}$ (c.f. Section 1.1). Indeed, the first equality uses the fact that $\beta_i^{(j)}$ is the $E_i^{(j)}$-component of β, hence it can be calculated by the intersection $(\beta,-E_i^{(j)*})$. For the second equality we have applied to (4.11) the following identities coming from Formula 2.1: for any node or an end-vertex v of Γ one has that $a_{i+1}^{(j)}(E_v^*,E_i^{(j)*})$ equals to $\frac{a_{i+1}^{(j)}D_i^{(j)}}{p_{i+1}^{(j)}D_{i+1}^{(j)}}(E_v^*,E_{i+1}^{(j)*})$ in the case when v belongs to $\Gamma_i^{(j)}$, and equals to $a_i^{(j)}p_i^{(j)}(E_v^*,E_{i+1}^{(j)*})$ otherwise. (Notice that for the case $i=r_j$ we have to set $p_{i+1}^{(j)}=1$ too.) On the other hand, one can check from (4.11) that

$$\beta_{i+1}^{\Gamma_i^{(j)}} = \frac{p_{i+1}^{(j)}D_{i+1}^{(j)}}{p}\Big(\sum_{i'=1}^{i}\big(k_{i'}^{(j)}\cdot a_{i'}^{(j)}p_{i'}^{(j)}\cdots p_i^{(j)} - x_{u_{i'}}^{(j)}\cdot a_{i'}^{(j)}p_{i'+1}^{(j)}\cdots p_i^{(j)}\big) - x_{u_0}^{(j)}\cdot p_1^{(j)}\cdots p_i^{(j)}\Big), \tag{4.13}$$

where $u_{i'}=u_{i'}^{(j)}$ is the end-vertex connecting to $v_{i'}^{(j)}$ with the leg of determinant $p_{i'}^{(j)}$ and $u_0=u_0^{(j)}$ is the end-vertex connecting $v_1^{(j)}$ with the leg of determinant $a_1^{(j)}$, see Figure 3.

Now, the idea is that by (4.13) and (4.7) the quantity $\ell^{\beta}_{\mathfrak{f}_i^{(j)}} := p\beta_{i+1}^{\Gamma_i^{(j)}}/(p_{i+1}^{(j)}D_{i+1}^{(j)})$ can be viewed as an exponent coming from the division of the monodromy zeta-function of $\mathfrak{f}_i^{(j)}$. Hence, it is either negative or it is an exponent of the polynomial part of the monodromy zeta-function which implies $\ell^{\beta}_{\mathfrak{f}_i^{(j)}}\notin\mathcal{M}_{\mathfrak{f}_i^{(j)}}$ by [**14**, Section 7.1.2]. Therefore (4.12) transforms into

$$a_{i+1}^{(j)}\beta_i^{(j)} = a_i^{(j)}p_i^{(j)}\beta_{i+1}^{(j)} + \frac{a_{i+1}^{(j)}D_i^{(j)} - a_i^{(j)}p_i^{(j)}p_{i+1}^{(j)}D_{i+1}^{(j)}}{p}\ell^{\beta}_{\mathfrak{f}_i^{(j)}} = a_i^{(j)}p_i^{(j)}\beta_{i+1}^{(j)} + q_{i+1}^{(j)}\ell^{\beta}_{\mathfrak{f}_i^{(j)}},$$

where the second equality uses Lemma 4.7(ii).

(ii) According to the proof of part (i) and symmetry of $\mathcal{M}_{\mathfrak{f}_i^{(j)}}$ (4.6) we can write $\ell^{\beta}_{\mathfrak{f}_i^{(j)}} = \mu_{\mathfrak{f}_i^{(j)}} - 1 - s_{\mathfrak{f}_i^{(j)}}$ for some $s_{\mathfrak{f}_i^{(j)}}\in\mathcal{M}_{\mathfrak{f}_i^{(j)}}$. Therefore (i) implies $\beta_i^{(j)}\le(a_i^{(j)}p_i^{(j)}/a_{i+1}^{(j)})\beta_{i+1}^{(j)} + (q_{i+1}^{(j)}/a_{i+1}^{(j)})(\mu_{\mathfrak{f}_i^{(j)}}-1)$. Then applying this inequality recursively for all $\beta_{i+1}^{(j)},\dots,\beta_{r_j}^{(j)}$ one finds the following inequality

$$\beta_i^{(j)} \le a_i^{(j)}p_i^{(j)}\cdots p_{r_j}^{(j)}\beta_+ + \frac{q_{i+1}^{(j)}}{a_{i+1}^{(j)}}(\mu_{\mathfrak{f}_i^{(j)}}-1) + \sum_{i'=i+1}^{r_j}\frac{a_i^{(j)}p_i^{(j)}\cdots p_{i'-1}^{(j)}q_{i'+1}^{(j)}}{a_{i'}^{(j)}a_{i'+1}^{(j)}}(\mu_{\mathfrak{f}_{i'}^{(j)}}-1). \tag{4.14}$$

Then we use relations (4.5) for all $q^{(j)}_{i'+1}$, $i' \in \{i, \dots, r_j - 1\}$ (recall that $q^{(j)}_{r_j+1} := 1$) to get

$$\beta^{(j)}_i \le a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{r_j} \beta_+ + (\mu_{\mathfrak{f}^{(j)}_i} - 1) + \sum_{i'=i+1}^{r_j} \left(\frac{a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'-1}}{a^{(j)}_{i'}} (\mu_{\mathfrak{f}^{(j)}_{i'}} - 1) - \frac{a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'}}{a^{(j)}_{i'}} (\mu_{\mathfrak{f}^{(j)}_{i'-1}} - 1) \right). \tag{4.15}$$

We use a well-known recursive formula $\mu_{\mathfrak{f}^{(j)}_{i'}} = (a^{(j)}_{i'} - 1)(p^{(j)}_{i'} - 1) + p^{(j)}_{i'} \mu_{\mathfrak{f}^{(j)}_{i'-1}}$ (see e.g. [**18**, (4.13)]) for the Milnor numbers of $\mathfrak{f}^{(j)}_{i'}$, which can be rewritten for our purpose in the form

$$\frac{a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'-1}}{a^{(j)}_{i'}} (\mu_{\mathfrak{f}^{(j)}_{i'}} - 1) - \frac{a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'}}{a^{(j)}_{i'}} (\mu_{\mathfrak{f}^{(j)}_{i'-1}} - 1) = a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'} - a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'-1}.$$

Finally, this recursion can be applied repeatedly to (4.15) in order to deduce

$$\begin{aligned}\beta^{(j)}_i &\le a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{r_j} \beta_+ + \sum_{i'=i+1}^{r_j} (a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'} - a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{i'-1}) + \mu_{\mathfrak{f}^{(j)}_i} - 1 \\ &< a^{(j)}_i p^{(j)}_i \cdots p^{(j)}_{r_j} (\beta_+ + 1),\end{aligned}$$

where the second (strict) inequality uses [**18**, Theorem 4.12(a)], saying that $m_{\mathfrak{f}^{(j)}_i} - \mu_{\mathfrak{f}^{(j)}_i} + 1 \ge 2|\mathcal{V}(\Gamma^{(j)}_i)| - 1 > 0$. □

PROPOSITION 4.10. *For $h = 0$ one has $P_0(\mathbf{t}_{\mathcal{N}}) = P^+_0(\mathbf{t}_{\mathcal{N}}) = P^{v_+}_0(\mathbf{t}_{\mathcal{N}})$.*

Proof. By Theorems 3.4 and 4.8 we have to show that $P^+_{0,\beta_+<0}(\mathbf{t}_{\mathcal{N}}) \equiv 0$. Moreover, using the configuration of signs from the proof of Theorem 4.8, it needs to be proved that $\beta_+ < 0$ implies $\beta^{(j)}_i < 0$ for any $1 \le j \le \nu$ and $1 \le i \le r_j$. This is implied by part (ii) of Lemma 4.9, since $\beta_+, \beta^{(j)}_i \in \mathbb{Z}$ in the case $h = 0$. □

4.4. *Question about P^+*

We have shown an example in Section 4.2 in which $P_h(\mathbf{t}_{\mathcal{N}}) \ne P^+_h(\mathbf{t}_{\mathcal{N}})$ for some $h \in H$. Hence, by Theorem 3.4 the polynomial part P_h in general can be 'thicker' than P^+_h. Nevertheless, they have the same set of exponents for the monomials which object presumably plays an important role in geometrical applications.

On the other hand, the calculation of P^+_h is much more effective, therefore, it is natural to pose the question whether it can replace P_h as a polynomial part. More precisely, we ask the following:

Is it true in general that $P^+_h(1) = \mathfrak{sw}^{norm}_h$?

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