

# On a system of equations for the normal fluid-condensate interaction in a Bose gas

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## Abstract

The existence of global solutions for a system of differential equations is proved, and some of their properties are described. The system involves a kinetic equation for quantum particles. It is a simplified version of a mathematical description of a weakly interacting dilute gas of bosons in the presence of a condensate near the critical temperature.

## 1 Introduction

We consider the existence and properties of radially symmetric weak solutions to the following system of differential equations:

$$\begin{cases} \frac{\partial F}{\partial t}(t, p) = n(t)I_3(F(t))(p) & t > 0, p \in \mathbb{R}^3, \\ n'(t) = -n(t) \int_{\mathbb{R}^3} I_3(F(t))(p) dp & t > 0, \end{cases} \quad (1.1)$$

where

$$I_3(F(t))(p) = \iint_{(\mathbb{R}^3)^2} [R(p, p_1, p_2) - R(p_1, p, p_2) - R(p_2, p_1, p)] dp_1 dp_2, \quad (1.3)$$

$$\begin{aligned} R(p, p_1, p_2) &= [\delta(|p|^2 - |p_1|^2 - |p_2|^2) \delta(p - p_1 - p_2)] \times \\ &\times [F_1 F_2 (1 + F) - (1 + F_1)(1 + F_2)F], \end{aligned} \quad (1.4)$$

and we denote  $F = F(t, p)$  and  $F_\ell = F(t, p_\ell)$  for  $\ell = 1, 2$ .

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The system (1.1), (1.2) is motivated by the mathematical description of a weakly interacting dilute gas of bosons. Given such a gas at equilibrium, if its temperature is below the so-called critical temperature  $T_c$ , a macroscopic density of bosons, called a condensate, appears at the lowest quantum state (cf.[17]). A description of the system of particles out of equilibrium at zero temperature has also been rigorously obtained ([7]). The system (1.1), (1.2) is more directly related to a gas out of equilibrium and at non zero temperature. The equations that, in the physic's literature, describe a gas in such a situation have not been the object of a mathematical proof; they have rather been deduced on the basis of physical arguments (cf. [10], [11, 30], [28] for example). We are particularly interested in the kinetic description of the interaction between the condensate and the particles in the dilute gas, when most of the particles are still in the gas, and so when the system is at a temperature close to  $T_c$ .

### 1.1 The Nordheim equation

The kinetic equation consistently used to describe the evolution of the distribution function for a spatially homogeneous, weakly interacting dilute gas of bosons of momentum  $p_1$  is

$$\frac{\partial F}{\partial t}(t, p_1) = I_4(F(t))(p_1) \quad t > 0, p_1 \in \mathbb{R}^3, \quad (1.5)$$

where

$$I_4(F(t))(p_1) = \iiint_{(\mathbb{R}^3)^3} q(F) d\nu(p_2, p_3, p_4), \quad (1.6)$$

$$q(F) = F_3 F_4 (1 + F_1)(1 + F_2) - F_1 F_2 (1 + F_3)(1 + F_4), \quad (1.7)$$

$$d\nu(p_1, p_2, p_3) = 2a^2 \pi^{-3} \delta(p_1 + p_2 - p_3 - p_4) \times \\ \delta(E(p_1) + E(p_2) - E(p_3) - E(p_4)) dp_2 dp_3 dp_4. \quad (1.8)$$

sometimes called Nordheim equation ([22]), (cf. for example [10], [11], [28]). We are assuming that the particles have mass  $m = 1/2$  and  $E(p)$  denotes the energy of a particle of momentum  $p$ . The constant  $a$  is the scattering length that parametrizes the Fermi pseudopotential of scattering. In the absence of condensate, the energy of the particles is taken to be  $E(p) = |p|^2$ .

For a condensed Bose gas, it is necessary to include the collisions involving the condensate. A kinetic equation is derived in [6] and [15] describing such processes. More recently, [30] extended the treatment to a trapped Bose gas by including Hartree-Fock corrections to the energy of the excitations, and have derived coupled kinetic equations for the distribution functions of the normal and superfluid components. Later on the results were generalized to low temperatures in [12] using the Bogoliubov-Popov approximation

to describe the energy particle. The system is as follows

$$\begin{cases} \frac{\partial F}{\partial t}(t, p) = I_4(F(t))(p) + 32a^2 n(t) \tilde{I}_3(F(t))(p) & t > 0, p \in \mathbb{R}^3, \\ n'(t) = -n(t) \int_{\mathbb{R}^3} \tilde{I}_3(F(t))(p) dp & t > 0. \end{cases} \quad (1.9)$$

(cf. [6], [11], [15] for a deduction based on physic's arguments). The term  $I_4(F)$  is exactly as in (1.6) and the constant  $32a^2$  comes from the approximation of the transition probability:  $|\mathcal{M}(p, p_1, p_2)|^2 \approx 32a^2 n(t)$ . The integral collision  $\tilde{I}_3$  is given by an expression similar to (1.3), (1.4) but where the corresponding terms  $\tilde{R}(p, p_1, p_2)$  are as follows,

$$\begin{aligned} \tilde{R}(p, p_1, p_2) &= [\delta(E(p) - E(p_1) - E(p_2))\delta(p - p_1 - p_2)] \times \\ &\times [F(p_1)F(p_2)(1 + F(p)) - (1 + F(p_1))(1 + F(p_2))F(p)]. \end{aligned} \quad (1.11)$$

In presence of a condensate, the energy  $E(t, p)$  of the particles at time  $t$  is now taken as  $E(t, p) = \sqrt{|p|^4 + 16a n(t)|p|^2}$ , where  $n(t)$  is the condensate density ([3], [11]). Once equation (1.9) has been obtained, the equation (1.10) is just what is needed in order to ensure that the total number of particles  $n(t) + \int_{\mathbb{R}^3} F(t, p) dp$  in the system is constant in time.

We are particularly interested in a situation where most of the particles are in the gas, and the condensate density  $n$  is very small. The energy of the particles is then usually approximated as  $E(t, p) \approx |p|^2 + 4a\pi n(t)$  (cf.[11]). In all what follows we need the strongest simplification  $E(t, p) \approx |p|^2$  to have the collision integral  $I_3$  in (1.3).

Moreover, in the problem (1.1), (1.2) only the term that in the equation (1.9) describes the interactions involving one particle of the condensate has been kept. The term  $I_4$ , the same as in equation (1.5), that only considers interactions between particles in the gas, has been dropped. The term  $I_4$  has been studied with detail to prove the existence of solutions to the Nordheim equation (1.5) and describe some of their properties. The problem (1.1), (1.2) only takes into account the collision processes involving a particle of the condensate.

Since we are only concerned with radial solutions  $(F, n)$  of (1.1), (1.2), a very natural independent variable is  $x = |p|^2$ . But this introduces a jacobian and then, the most suitable quantity is not always  $f(x) = F(p)$  but may be sometimes  $\sqrt{x}f(x)$ .

## 1.2 The term $I_4$ and the Nordheim equation

The local existence of bounded solutions for Nordheim equation (1.5) was proved in [5]. Global existence of bounded solutions has been proved in [16] for bounded and suitably small initial data. The existence of radially symmetric weak solutions was first proved in [18] for all initial data  $f_0$  in the space of nonnegative radially symmetric measures on  $[0, \infty)$ .

For radially symmetric solutions  $F(p) = f(x)$ ,  $x = |p|^2$ , the expression of the Nordheim equation simplifies because it is possible to perform the angular variables in the collision integral. After rescaling the time variable  $t$  (in order to absorb some constants), the Nordheim equation reads:

$$\frac{\partial f}{\partial t}(t, x_1) = J_4(f(t))(x_1), \quad t > 0, \quad x_1 \geq 0, \quad (1.12)$$

where

$$J_4(f)(x_1) = \iint_{[0, \infty)^2} \frac{w(x_1, x_2, x_3)}{\sqrt{x_1}} q(f)(x_1, x_2, x_3) dx_2 dx_3, \quad (1.13)$$

$$q(f) = (1 + f_1)(1 + f_2)f_3f_4 - (1 + f_3)(1 + f_4)f_1f_2, \quad (1.14)$$

$$w(x_1, x_2, x_3) = \min \{ \sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, \sqrt{x_4} \}, \quad x_4 = (x_1 + x_2 - x_3)_+. \quad (1.15)$$

The factor  $\frac{w}{\sqrt{x_1}}$  in the collision integral comes from the angular integration of the Dirac's delta of the energies  $|p_\ell|^2$ .

If we denote  $\mathcal{M}_+([0, \infty))$  the space of positive and finite Radon measures on  $[0, \infty)$ , and define for all  $\alpha \in \mathbb{R}$

$$\mathcal{M}_+^\alpha([0, \infty)) = \{G \in \mathcal{M}_+([0, \infty)) : M_\alpha(G) < \infty\}, \quad (1.16)$$

$$M_\alpha(G) = \int_{[0, \infty)} x^\alpha G(x) dx \quad (\text{moment of order } \alpha), \quad (1.17)$$

the definition of weak solution introduced in [18] is the following.

**Definition 1.1** (Weak radial solutions of (1.5)). Let  $G$  be a map from  $[0, \infty)$  into  $\mathcal{M}_+^1([0, \infty))$  and consider  $f$  defined as  $\sqrt{x}f(t) = G(t)$ . We say that  $f$  is a weak radial solution of (1.5) if  $G$  satisfies:

$$\forall t > 0 : G(t) \in \mathcal{M}_+^1([0, \infty)), \quad (1.18)$$

$$\forall T > 0 : \sup_{0 \leq t < T} \int_{[0, \infty)} (1 + x)G(t, x) dx < \infty, \quad (1.19)$$

$$\forall \varphi \in C_b^{1,1}([0, \infty)) : \int_{[0, \infty)} \varphi(x)G(t, x) dx \in C^1([0, \infty)), \quad (1.20)$$

$$\frac{d}{dt} \int_{[0, \infty)} \varphi(t, x)G(t, x) dx = \mathcal{Q}_4(\varphi, G(t)), \quad (1.21)$$

$$\begin{aligned} \mathcal{Q}_4(\varphi, G) = & \iiint_{[0, \infty)^3} \frac{G_1 G_2 G_3}{\sqrt{x_1 x_2 x_3}} w \Delta \varphi \, dx_1 dx_2 dx_3 + \\ & + \frac{1}{2} \iiint_{[0, \infty)^3} \frac{G_1 G_2}{\sqrt{x_1 x_2}} w \Delta \varphi \, dx_1 dx_2 dx_3 \end{aligned} \quad (1.22)$$

$$\Delta \varphi(x_1, x_2, x_3) = \varphi(x_4) + \varphi(x_3) - \varphi(x_2) - \varphi(x_1), \quad (1.23)$$

$$w(x_1, x_2, x_3) = \min \{ \sqrt{x_1}, \sqrt{x_2}, \sqrt{x_3}, \sqrt{x_4} \}, \quad x_4 = (x_1 + x_2 - x_3)_+. \quad (1.24)$$

For all initial data  $f_0$  such that  $G_0 = \sqrt{x}f_0 \in \mathcal{M}_+^1([0, \infty))$ , the existence of a weak solution was proved in [18]. The moments of order zero and one of  $G$  were shown to be constant in time. It was shown in [20] that a definition equivalent to Definition 1.1 would be to impose  $\varphi(0) = 0$  to the test functions in Definition 1.1 and impose the conservation of mass on  $G(t)$  for all  $t > 0$ . Further properties of the solutions, such as the gain of moments, asymptotic behavior, were obtained in a series of articles [18, 19, 20, 21].

It is proved in Proposition 2.1 below that if the measure  $G$  is written as  $G(t) = n(t)\delta_0 + g(t)$ , where  $n(t) = G(t, \{0\})$ , then for all  $\varphi \in C_b^{1,1}([0, \infty))$  the term  $\mathcal{Q}_4(\varphi, G)$  may be decomposed as follows:

$$\mathcal{Q}_4(\varphi, G(t)) = \mathcal{Q}_4(\varphi, g(t)) + n(t)\mathcal{Q}_3(\varphi, g(t)), \quad (1.25)$$

where

$$\begin{aligned} \mathcal{Q}_4(\varphi, g) &= \iiint_{(0, \infty)^3} \frac{g_1 g_2 g_3}{\sqrt{x_1 x_2 x_3}} w \Delta \varphi \, dx_1 dx_2 dx_3 \\ &\quad + \frac{1}{2} \iiint_{(0, \infty)^3} \frac{g_1 g_2}{\sqrt{x_1 x_2}} w \Delta \varphi \, dx_1 dx_2 dx_3, \end{aligned} \quad (1.26)$$

$$\mathcal{Q}_3(\varphi, g) = \mathcal{Q}_3^{(2)}(\varphi, g) - \mathcal{Q}_3^{(1)}(\varphi, g), \quad (1.27)$$

$$\mathcal{Q}_3^{(2)}(\varphi, g) = \iint_{(0, \infty)^2} \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} g(x)g(y) \, dx dy, \quad (1.28)$$

$$\mathcal{Q}_3^{(1)}(\varphi, g) = \int_{(0, \infty)} \frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}} g(x) \, dx, \quad (1.29)$$

$$\Lambda(\varphi)(x, y) = \varphi(x + y) + \varphi(|x - y|) - 2\varphi(\max\{x, y\}), \quad (1.30)$$

$$\mathcal{L}_0(\varphi)(x) = x(\varphi(0) + \varphi(x)) - 2 \int_0^x \varphi(y) \, dy. \quad (1.31)$$

It was also proved in [18] that as  $t \rightarrow \infty$ , the measure  $G$  converges in the weak sense of measures to one of the measures:

$$G_{\beta, \mu, C} = \frac{\sqrt{x}}{e^{\beta x - \mu} - 1} + C\delta_0, \quad \beta > 0, \quad \mu \leq 0, \quad C \geq 0 \quad (1.32)$$

where the constants  $C$  and  $\mu$  are such that  $C\mu = 0$ .

When  $C = 0$  and  $\mu \leq 0$ , the function  $F_{\beta, \mu, 0}(p) = |p|^{-1}G_{\beta, \mu, 0}(|p|^2)$  is an equilibrium of the Nordheim equation (1.5) because  $q(F_{\beta, \mu, 0})d\nu \equiv 0$ . When  $C > 0$  and  $\mu = 0$ , then  $F_{\beta, 0, C}(p) = |p|^{-1}G_{\beta, 0, C}(|p|^2)$  is an equilibria of (1.9) because  $q(f_{\beta, 0, 0}) \equiv 0$  and  $R(p, p', p'') \equiv 0$  for all  $(p, p', p'') \in (\mathbb{R}^3)^3$  for  $f_{\beta, 0, 0}$ , where  $R(p, p', p'')$  is defined in (1.4). It was proved in [18] that  $F_{\beta, \mu, C}$  is a weak solution of the Nordheim equation (1.12) if and only if  $\mu C = 0$ .

On the other hand, it was proved in [8] that, given any  $N > 0$ ,  $E > 0$  there exists initial data  $f_0 \in L^\infty(\mathbb{R}_+; (1+x)^\gamma)$  with  $\gamma > 3$ , satisfying

$$\int_{\mathbb{R}^+} f_0(x) \sqrt{x} \, dx = N, \quad \int_{\mathbb{R}^+} f_0(x) \sqrt{x^3} \, dx = E,$$

and such that there exists a global weak solution  $f$  and positive times  $0 < T_* < T^*$  such that:

$$\sup_{0 < t \leq T_*} \|f(t, \cdot)\|_{L^\infty(\mathbb{R}^+)} < \infty, \quad \sup_{T_* < t \leq T^*} \int_{\{0\}} \sqrt{x} f(t, x) dx > 0. \quad (1.33)$$

Property (1.33) shows that the solution  $G = \sqrt{x}f$  of (1.18)–(1.24) is a bounded function on the time interval  $[0, T^*)$  and a Dirac mass is formed at the origin at some time  $T_0$  between  $T_*$  and  $T^*$ . After that time  $T_0$ , the solution  $G$  is such that  $G(t, \{0\}) > 0$ .

In the simplified description of the physical system of particles that we are using, where only the radial density  $G$  of particles of momentum  $p$  is considered, the description of the physical Bose-Einstein condensate can just be given by a Dirac measure at the origin.

Notwithstanding the similarity of these two phenomena, the extent to which the first one is a truthful mathematical description of the second is not clear. Nevertheless, we refer to the term  $n(t)\delta_0$  that appears in finite time in some of the weak solutions of the Nordheim equation as “condensate”, with some abuse of language.

### 1.3 The term $I_3$ in radial variables.

The results briefly presented in the previous sub Section describe some of the properties of the weak solutions to the Nordheim equation in terms of the measure  $G$ . In particular, the weak convergence of  $G$  to the measures defined in (1.32) shows what is the limit of  $G(t, \{0\})$  as  $t \rightarrow \infty$ . To understand better the dynamics of  $G(t, \{0\})\delta_0$  and its interaction with  $G(t) - G(t, \{0\})\delta_0$  it seems suitable to write  $G(t) = G(t, \{0\})\delta_0 + g(t)$  and consider the system (1.9), (1.10).

For radially symmetric functions  $F(p) = f(x)$ ,  $x = |p^2|$ , the system (1.1), (1.2) reads, after a suitable time rescaling to absorb some constants:

$$\begin{cases} \frac{\partial f}{\partial t}(t, x) = \frac{n(t)}{\sqrt{x}} J_3(f(t))(x) & t > 0, x > 0, \end{cases} \quad (1.34)$$

$$\begin{cases} n'(t) = -n(t) \int_0^\infty J_3(f(t))(x) dx & t > 0, \end{cases} \quad (1.35)$$

where

$$\begin{aligned} J_3(f)(x) = & \int_0^x \left( f(x-y)f(y) - f(x)[1 + f(x-y) + f(y)] \right) dy + \\ & + 2 \int_x^\infty \left( f(y)[1 + f(y-x) + f(x)] - f(y-x)f(x) \right) dy. \end{aligned} \quad (1.36)$$

(cf. [26] and [29] for the isotropic system that also contains the term  $J_4(f)$ , that comes from  $I_4$  in (1.9)). Notice that

$$\begin{aligned} & \int_0^\infty J_3(f(t))(x)dx \\ &= \int_0^\infty \int_0^\infty \left( f(t, x)f(t, y) - f(t, x+y)[1 + f(t, x) + f(t, y)] \right) dx dy \end{aligned} \quad (1.37)$$

whenever the integral in the right hand side is finite, for example, if  $f \in L^1(\mathbb{R}_+, (1+x)dx)$ . In that case we also have,

$$\int_0^\infty J_3(f(t))(x)dx = M_1(f(t)). \quad (1.38)$$

The factor  $x^{-1/2}$  in the right hand side of (1.34) comes from the angular integration of the Dirac's measure of energies of  $I_3$ , just as the  $\frac{w}{\sqrt{x_1}}$  term of (1.13) in  $I_4$ . But since  $\frac{w}{\sqrt{x_1}}$  is a bounded function, it appears that the operator  $I_3$  is more singular than  $I_4$  for small values of  $x$ .

If we denote  $F(t, p) = f(t, |p|^2) = |p|^{-1}g(t, |p|^2)$  and  $x = |p|^2$ , from the original motivation of the Nordheim equation it is very natural to expect

$$\int_{\mathbb{R}^3} F(t, p)dp = 2\pi \int_0^\infty f(t, x)\sqrt{x}dx = 2\pi \int_0^\infty g(t, x)dx < \infty,$$

(that corresponds to the number of particles in the normal fluid), and

$$\int_{\mathbb{R}^3} F(t, p)|p|^2dp = 2\pi \int_0^\infty f(t, x)x^{3/2}dx = 2\pi \int_0^\infty g(t, x)x dx < \infty,$$

(corresponding to the total energy in the system). But there is no particular reason to expect

$$\int_{\mathbb{R}^3} F(t, p)\frac{dp}{|p|} = 2\pi \int_0^\infty f(t, x)dx = 2\pi \int_0^\infty g(t, x)\frac{dx}{\sqrt{x}} < \infty.$$

Without that last condition, the convergence of the integrals in the term  $I_3(F(t))$  (cf. (1.3), (1.4)), or in (1.34), (1.36), is delicate. That difficulty is usually avoided using a suitable weak formulation.

If we suppose that  $f = x^{-1/2}g \in L^1(\mathbb{R}_+, (1+x)dx)$ , and multiply the equation (1.34) by  $\sqrt{x}\varphi$ , we obtain by Fubini's Theorem,

$$\frac{d}{dt} \int_{[0, \infty)} \varphi(x)g(t, x)dx = n(t)\tilde{\mathcal{Q}}_3(\varphi, g(t)) \quad \forall \varphi \in C_b^1([0, \infty)), \quad (1.39)$$

where

$$\tilde{\mathcal{Q}}_3(\varphi, g) = \mathcal{Q}_3^{(2)}(\varphi, g) - \tilde{\mathcal{Q}}_3^{(1)}(\varphi, g), \quad (1.40)$$

$$\tilde{\mathcal{Q}}_3^{(1)}(\varphi, g) = \int_{(0, \infty)} \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} g(x)dx, \quad (1.41)$$

$$\mathcal{L}(\varphi)(x) = x\varphi(x) - 2 \int_0^x \varphi(y)dy. \quad (1.42)$$

Notice that, by (1.27),

$$\mathcal{Q}_3(\varphi, g) = \tilde{\mathcal{Q}}_3(\varphi, g) - \varphi(0)M_{1/2}(g). \quad (1.43)$$

A natural weak formulation for  $G = n(t)\delta_0 + g$  is then obtained by adding (1.35) to (1.39). We then define a weak radially symmetric solution of the Problem (1.1), (1.2) as follows.

**Definition 1.2** (Weak radial solution of (1.1), (1.2)). Consider a map  $G : [0, T) \rightarrow \mathcal{M}_+^1([0, \infty))$  for some  $T \in (0, \infty]$ , that we decompose as follows:

$$\forall t \in [0, T) : \quad G(t) = n(t)\delta_0 + g(t), \quad \text{where } n(t) = G(t, \{0\});$$

and define  $F(t, p) = |p|^{-1}g(t, |p|^2)$  for all  $t > 0$  and  $p \in \mathbb{R}^3$ . We say that  $(F, n)$  is a weak radial solution of (1.1), (1.2) on  $(0, T)$  if:

$$\forall T' \in (0, T) : \quad \sup_{0 \leq t < T'} \int_{[0, \infty)} (1+x)G(t, x)dx < \infty, \quad (1.44)$$

$$\forall \varphi \in C_b^1([0, \infty)) : \quad t \mapsto \int_{[0, \infty)} \varphi(x)G(t, x)dx \in W_{loc}^{1, \infty}([0, T)), \quad (1.45)$$

and for a.e.  $t \in (0, T)$

$$\frac{d}{dt} \int_{[0, \infty)} \varphi(x)G(t, x)dx = n(t)\mathcal{Q}_3(\varphi, g(t)) \quad \forall \varphi \in C_b^1([0, \infty)), \quad (1.46)$$

where  $\mathcal{Q}_3(\varphi, g)$  is defined by (1.27)-(1.29).

We show in Proposition 2.1 that the Definition 1.2 substantially coincides with the Definition 1.1 of radial weak solution of (1.5) when the term  $\mathcal{Q}_4(\varphi, g)$  in (2.1) is dropped (cf. Remark 2.2). As a consequence, the measures  $f_{\beta, 0, C}(p)$  defined above are weak radial solutions of (1.1), (1.2) (cf. Proposition 2.3).

## 1.4 Main results

The existence of weak radial solutions for the Cauchy problem associated with the system (1.1), (1.2) is given in the following Theorem.

**Theorem 1.3** (Existence result). *Suppose that  $G_0 \in \mathcal{M}_+^1([0, \infty))$  satisfies  $G_0(\{0\}) > 0$ , and define  $F_0(p) = |p|^{-1}g_0(|p|^2)$ , where  $g_0 = G_0 - G_0(\{0\})\delta_0$ . Then, there exists a weak radial solution  $(F, n)$  of (1.1), (1.2) on  $(0, \infty)$  such that  $F(t, p) = |p|^{-1}g(t, |p|^2)$ , where  $G = n\delta_0 + g$  satisfies:*

$$G \in C([0, \infty), \mathcal{M}_+^1([0, \infty))), \quad G(0) = G_0 \quad (1.47)$$

and:



(i)  $G$  conserves the total number of particles  $N$  and energy  $E$ :

$$M_0(G(t)) = M_0(G_0) = N \quad \forall t \geq 0, \quad (1.48)$$

$$M_1(G(t)) = M_1(G_0) = E \quad \forall t \geq 0. \quad (1.49)$$

(ii) For all  $\alpha \geq 3$ , if  $M_\alpha(G_0) < \infty$ , then  $G \in C((0, \infty), \mathcal{M}_+^\alpha([0, \infty)))$  and

$$M_\alpha(G(t)) \leq \left( M_\alpha(G_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau(t) \right)^{\frac{\alpha-1}{2}} \quad \forall t > 0, \quad (1.50)$$

$$\text{where } \tau(t) = \int_0^t G(s, \{0\}) ds. \quad (1.51)$$

(iii) For all  $\alpha \geq 3$ ,

$$M_\alpha(G(t)) \leq C(\alpha, E) \left( \frac{1}{1 - e^{-\gamma(\alpha, E)\tau(t)}} \right)^{2(\alpha-1)} \quad \forall t > 0, \quad (1.52)$$

where  $\tau(t)$  is given by (1.51), and the constants  $C(\alpha, E)$  and  $\gamma(\alpha, E)$  are defined in Theorem 3.1.

(iv) If  $\alpha \in (1, 3]$  and

$$E > C(\alpha) N^{5/3}, \quad (1.53)$$

$$\text{where } C(\alpha) = \begin{cases} \left( \frac{(2^\alpha - 2)(\alpha + 1)}{(\alpha - 1)} \right)^{\frac{2}{3}} & \text{if } \alpha \in (1, 2], \\ (\alpha(\alpha + 1))^{\frac{2}{3}} & \text{if } \alpha \in (2, 3], \end{cases} \quad (1.54)$$

then  $M_\alpha(G(t))$  is a decreasing function on  $(0, \infty)$ .

The next result is a property satisfied by all the weak radial solutions of (1.1), (1.2).

**Theorem 1.4.** *Let  $G_0$  be as in Theorem 1.3, and  $G$  a weak radial solution of (1.1), (1.2). Then for all  $T > 0$ ,  $R > 0$  and  $\alpha \in (-\frac{1}{2}, \infty)$ ,*

$$\begin{aligned} & \int_0^T G(t, \{0\}) \int_{(0, R]} x^\alpha G(t, x) dx dt \leq \\ & \leq \frac{2R^{\frac{1}{2} + \alpha}}{1 - (\frac{2}{3})^{\frac{1}{2} + \alpha}} \left( \int_0^T G(t, \{0\}) dt \right)^{\frac{1}{2}} \left( \frac{\sqrt{E}}{2} \int_0^T G(t, \{0\}) dt + \sqrt{N} \right). \end{aligned} \quad (1.55)$$

The only possible algebraic behavior for such a measure  $G$  near the origin is then  $x^{-1/2}$ .

**Remark 1.5.** The functions  $F_{\beta,0,C}$  defined above are weak radial solutions of (1.1), (1.2) for all  $\beta > 0$  and  $C \geq 0$  (cf. Proposition 2.3). Since

$$\int_{(0,\infty)} x^\alpha G_{\beta,0,C} dx < \infty \iff \alpha > -1/2,$$

the estimate (1.55) can not hold for all radial weak solutions if  $\alpha \leq -1/2$ .

In the next two results we describe the evolution of the measure at the origin  $n(t) = G(t, \{0\})$  by taking the limit  $\varepsilon \rightarrow 0$  in the weak formulation (1.46) for test functions  $\varphi_\varepsilon$  as follows:

**Remark 1.6.** Given  $\varphi \in C_b^1([0, \infty))$  nonnegative, convex, with  $\varphi(0) = 1$  and  $\lim_{x \rightarrow \infty} \sqrt{x}\varphi(x) = 0$ , denote  $\varphi_\varepsilon(x) = \varphi(x/\varepsilon)$  for  $\varepsilon > 0$ . Notice that for any  $G \in \mathcal{M}_+([0, \infty))$ ,

$$G(\{0\}) = \lim_{\varepsilon \rightarrow 0} \int_{[0, \infty)} \varphi_\varepsilon(x) dG(x). \quad (1.56)$$

The standard example is  $\varphi_\varepsilon(x) = (1 - x/\varepsilon)_+^2$ .

**Theorem 1.7.** *Let  $G$  be the solution of (1.46) obtained in Theorem 1.3, with initial data  $G_0 \in \mathcal{M}_+^1([0, \infty))$  such that  $N = M_0(G_0) > 0$ ,  $E = M_1(G_0) > 0$  and  $G_0(\{0\}) > 0$ . Denote  $G(t) = n(t)\delta_0 + g(t)$ , with  $n(t) = G(t, \{0\})$ . Then  $n$  is right continuous and a.e. differentiable on  $[0, \infty)$ . Moreover, there exists a positive measure  $\mu$  on  $[0, \infty)$  whose cumulative distribution function is given by*

$$\mu((0, t]) = \lim_{\varepsilon \rightarrow 0} \int_0^t n(s) \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, g(s)) ds \quad (1.57)$$

for any  $\varphi_\varepsilon$  as in Remark 1.6, and such that:

$$n(t) - n(0) + \int_0^t n(s) M_{1/2}(g(s)) ds = \mu((0, t]) \quad \forall t > 0. \quad (1.58)$$

**Theorem 1.8.** *Let  $G$  and  $\mu$  be as in Theorem 1.7. Then*

$$0 < \mu((0, t]) < \infty \quad \forall t > 0. \quad (1.59)$$

The measure  $\mu$  in (1.57) depends on the atomic part of  $g$ , and on the behaviour of  $g$  at the origin (it seems to be actually related with its moment of order  $-1/2$  c.f. Proposition 6.4 and Remark 6.5). This measure  $\mu$  appears as a source term in the equation (1.58) for  $n$ . Given the function  $n$ , the equation (1.34) satisfied by  $g$  on  $(0, \infty)$  has also a natural weak formulation by itself. In terms of  $g(t)$ , where  $g(t) = G(t) - G(t, \{0\})\delta_0$  and  $\sqrt{x}f(t, x) = g(t, x)$  it reads

$$\frac{d}{dt} \int_{[0,\infty)} \varphi(x)g(t,x)dx = n(t)\mathcal{Q}_3(\varphi, g(t)), \quad \forall \varphi \in C_b^1([0,\infty)), \varphi(0) = 0. \quad (1.60)$$

In the next result we describe the relation between a weak solution  $(F, n)$  of (1.1), (1.2), where  $F(t, p) = |p|^{-1}g(t, |p|^2)$ ,  $G(t) = n(t)\delta_0 + g(t)$ ,  $n(t) = G(t, \{0\})$ , and a pair  $(g, n)$  where  $g$  is a weak radial solution of the equation (1.1) and  $n$  satisfies (1.2).

**Theorem 1.9.** *Suppose that  $G \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  is such that  $G(0) = G_0 \in \mathcal{M}_+^1([0, \infty))$  with  $G_0(\{0\}) > 0$ , and denote  $G(t) = n(t)\delta_0 + g(t)$  with  $n(t) = G(t, \{0\})$ .*

- (i) *If  $(F, n)$  is a weak radial solution of (1.1), (1.2) and  $F(t, p) = |p|^{-1}g(t, |p|^2)$ , then  $n$  is given by (1.58), (1.57), and  $g$  satisfies (1.60) for a.e.  $t > 0$ .*
- (ii) *On the other hand, if  $g$  satisfies (1.44), (1.45) and (1.60) for some nonnegative bounded function  $n$ , then the limit in (1.57) exists. If  $n$  also satisfies*

$$n(t) = n(0) + \lim_{\varepsilon \rightarrow 0} \int_0^t n(s)\mathcal{Q}_3^{(2)}(\varphi_\varepsilon, g(s))ds - \int_0^t n(s)M_{1/2}(g(s))ds \quad (1.61)$$

*and  $F(t, p) = |p|^{-1}g(t, |p|^2)$ , then  $(F, n)$  is a weak radial solution of (1.1), (1.2).*

If in the Definition 1.2 only test functions satisfying  $\varphi(0) = 0$  are taken, it becomes necessary to introduce some other condition to the system. Otherwise the system would be reduced to find  $g$  satisfying (1.45)–(1.46) for a given function  $n(t)$  and for test functions such that  $\varphi(0) = 0$ . If we impose just the conservation of mass, we prove below (Corollary 1.10) that we recover a solution that satisfies the Definition 1.2.

**Corollary 1.10.** *If  $g$  satisfies (1.44), (1.45) and (1.60) for some nonnegative bounded function  $n = n(t)$  such that*

$$n(t) + \int_{(0,\infty)} g(t, x)dx = \text{constant} \quad (1.62)$$

*and  $F(t, p) = |p|^{-1}g(t, |p|^2)$ , then  $(F, n)$  is a weak radial solution of (1.1), (1.2).*

In our last result we show that, under some sufficient conditions, the condensate density  $n(t)$  tends to zero as  $t \rightarrow \infty$ , fast enough to be integrable.

**Theorem 1.11.** *Suppose that  $G_0 \in \mathcal{M}_+^1([0, \infty))$  satisfies  $G_0(\{0\}) > 0$  and let  $(F, n)$  be the weak radial solution of (1.1), (1.2) obtained in Theorem 1.3.*

Let us call  $N = M_0(G_0)$  and  $E = M_1(G_0)$ . If condition (1.53), (1.54) hold for some  $\alpha \in (1, 3]$ , then, for all  $t_0 > 0$ ,

$$\int_{t_0}^{\infty} n(t)dt \leq M_{\alpha}(G(t_0))C(N, E, \alpha) \quad (1.63)$$

for some explicit constant  $C(N, E, \alpha)$  given in (7.1), and

$$\lim_{t \rightarrow \infty} n(t) = 0. \quad (1.64)$$

**Remark 1.12.** The quantity  $E/N^{5/3}$  has a very precise interpretation in physical terms. Suppose that  $T$  is the temperature of a system of particles at equilibrium with total number of particles  $N$  and total energy  $E$ . And denote  $T_c$  the critical temperature, that is the temperature at which the ground state of the system becomes macroscopically occupied. Then:

$$\frac{E}{N^{5/3}} = b \frac{T}{T_c}, \quad \text{where} \quad b = \frac{3}{(2\pi)^{1/3}} \frac{\zeta(5/2)}{\zeta(3/2)^{5/3}}.$$

and condition (1.53) implies

$$\frac{T}{T_c} = \frac{1}{b} \frac{E}{N^{5/3}} > \frac{C(\alpha)}{b}.$$

The function  $C(\alpha)/b$  is continuous and strictly increasing on  $[1, 3]$  and its limit as  $\alpha \rightarrow 1^+$  is  $\log(16)^{2/3}/b \approx 4.48403$ . Condition (1.53) means that, when at equilibrium, the system of particles would be at a temperature clearly above the critical temperature. Anyway, the solution  $F$  of the problem (1.1), (1.2) may be far from any real distribution of particles of the original system of particles.

## 1.5 Some arguments of the proofs.

It is very natural to make the following change of variables in problem (1.46). Given  $G(t) = n(t)\delta_0 + g(t)$ , where  $n(t) = G(t, \{0\})$ , we define

$$H(\tau) = G(t), \quad \text{where} \quad \tau = \int_0^t n(s)ds. \quad (1.65)$$

In terms of  $H$ , (1.46) reads

$$\frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) H(\tau, x) dx = \mathcal{Q}_3(\varphi, H(\tau)) \quad \forall \varphi \in C_b^1([0, \infty)). \quad (1.66)$$

To obtain a measure  $H$  that satisfies (1.66), we first find  $h$  satisfying

$$\frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) h(\tau, x) dx = \tilde{\mathcal{Q}}_3(\varphi, h(\tau)) \quad \forall \varphi \in C_b^1([0, \infty)), \quad (1.67)$$

where  $\tilde{\mathcal{Q}}_3$  is given in (1.40)–(1.42). Then we define  $H$  as

$$H(\tau) = h(\tau) - \left( \int_0^\tau M_{1/2}(h(\sigma)) d\sigma \right) \delta_0. \quad (1.68)$$

By (1.43), the measure  $H$  will satisfy (1.66).

As it will be seen in Section 3, all the arguments are much simpler and clear in the equation for  $H$  than in the equation for  $G$ . In particular, the measure  $\lambda$ , that corresponds to the measure  $\mu$  of Theorem 1.7, appears as the Lebesgue-Stieltjes measure associated to  $m(\tau) = h(\tau, \{0\})$ .

The proofs of Theorem 1.3 and Theorem 1.4 make great use of the change of variables (1.65). Several of our arguments will need the measure  $h(\tau)$  to satisfy only one inequality in (1.67). This requires the following:

**Definition 1.13.** A function  $h : [0, \infty) \rightarrow \mathcal{M}_+([0, \infty))$  is said to be a super solution of (1.67) if

$$\begin{cases} \forall \varphi \in C_b^1([0, \infty)) \text{ nonnegative, convex and decreasing :} \\ \frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) h(\tau, x) dx \geq \mathcal{Q}_3^{(2)}(\varphi, h(\tau)) \quad a.e. \tau > 0. \end{cases} \quad (1.69)$$

The operator  $\mathcal{Q}_3^{(2)}$  is considered in [13] and [14], where a problem similar to (1.67) is studied, with  $\tilde{\mathcal{Q}}_3$  replaced by  $\mathcal{Q}_3^{(2)}$  and for which, the property of instantaneous condensation is proved. We extend this result to the solutions  $h$  of the problem (1.67) with the whole  $\tilde{\mathcal{Q}}_3$ , using similar arguments (monotonicity, convexity of test functions) and taking care of the linear term.

Theorem 1.3 is deduced from the corresponding existence result of  $h$ , that is proved using very classical arguments: regularization of the problem, fixed point, a priori estimates and passage to the limit. Then, the delicate point is to invert the change of variables (1.65) in order to obtain a global in time nonnegative solution  $G$ .

The Plan of the article is the following. In Section 2 we prove Proposition 2.1. Section 3 is devoted to the proof of the existence of the measure  $H$ . In Section 4 we obtain several properties of  $h(\tau, \{0\})$ . In Section 5 we prove Theorem 1.3 (existence for the measure  $G$ ) and Theorem 1.4. The contents of Section 6 are the proofs of Theorem 1.7, Theorem 1.8, Theorem 1.9 and Corollary 1.10. Finally in Section 7 we prove Theorem 1.11. Several technical results are presented in an Appendix.

## 2 On weak formulations.

We deduce first a detailed expression of the weak formulation of (1.12) for a radial measure  $G$ .

**Proposition 2.1.** *Let  $G$  satisfy (1.18)–(1.24) for some  $T > 0$ , and write  $G(t) = n(t)\delta_0 + g(t)$ , where  $n(t) = G(t, \{0\})$ . Then, for all  $\varphi \in C_b^{1,1}([0, \infty))$  and for all  $t \in (0, T)$ :*

$$\frac{d}{dt} \int_{[0, \infty)} \varphi(x) G(t, x) dx = \mathcal{Q}_4(\varphi, g(t)) + n(t) \mathcal{Q}_3(\varphi, g(t)), \quad (2.1)$$

where  $\mathcal{Q}_4(\varphi, g)$  and  $\mathcal{Q}_3(\varphi, g)$  are defined in (1.26)–(1.31).

**Remark 2.2.** If the term  $\mathcal{Q}_4(\varphi, g)$  in (2.1) is dropped, we recover the equation (1.46) that defines a radial weak solution of (1.1), (1.2).

**Proof of Proposition 2.1.** We may rewrite  $\mathcal{Q}_4(\varphi, G)$  in (1.22) as

$$\mathcal{Q}_4(\varphi, G) = \iiint_{[0, \infty)^3} \Phi_\varphi dG_1 dG_2 dG_3 + \frac{1}{2} \iiint_{[0, \infty)^3} \sqrt{x_3} \Phi_\varphi dG_1 dG_2 dx_3,$$

where  $\Phi_\varphi$  is as in Lemma 8.11, and we have used notation  $dG$  instead of  $Gdx$ . Then we decompose  $[0, \infty)^3 = (0, \infty)^3 \cup A \cup P$ , where, for  $\{i, j, k\} = \{1, 2, 3\}$ ,

$$\begin{aligned} A &= \{(x_1, x_2, x_3) \in \partial[0, \infty)^3 : x_i = x_j = 0, x_k > 0\} \cup \{(0, 0, 0)\}, \\ P &= \{(x_1, x_2, x_3) \in \partial[0, \infty)^3 : x_i = 0, (x_j, x_k) \in (0, \infty)^2\}. \end{aligned}$$

Let  $\varphi \in C_b^{1,1}([0, \infty))$ . By (8.34) in Lemma 8.10 and the definition (8.35) of  $W$ , it follows that  $\Phi_\varphi \equiv 0$  on  $A$ . Hence, recalling the definition (1.26) of  $\mathcal{Q}_4(\varphi, g)$  and the definition of  $\Phi_\varphi$  in Lemma 8.10, we have

$$\mathcal{Q}_4(\varphi, G) = \mathcal{Q}_4(\varphi, g) + \iiint_P \Phi_\varphi dG_1 dG_2 dG_3 + \frac{1}{2} \iiint_P \sqrt{x_3} \Phi_\varphi dG_1 dG_2 dx_3. \quad (2.2)$$

We now study the integral over  $P$  for the cubic and the quadratic terms in (2.2).

(a) The cubic term. Since  $\Phi_\varphi$  is symmetric in the  $x_1, x_2$  variables, and  $\Phi_\varphi$  is uniformly continuous on  $[0, \infty)^3$  by Lemma 8.11, then

$$\begin{aligned} \iiint_P \Phi_\varphi dG_1 dG_2 dG_3 &= 2 \iiint_{\{x_2=0, x_1>0, x_3>0\}} \Phi_\varphi dG_1 dG_2 dG_3 \\ &\quad + \iiint_{\{x_3=0, x_1>0, x_2>0\}} \Phi_\varphi dG_1 dG_2 dG_3 \\ &= 2G(t, \{0\}) \iint_{(0, \infty)^2} \Phi_\varphi(x_1, 0, x_3) dG_1 dG_3 \\ &\quad + G(t, \{0\}) \iint_{(0, \infty)^2} \Phi_\varphi(x_1, x_2, 0) dG_1 dG_2. \quad (2.3) \end{aligned}$$

Using now the definition of  $\Phi_\varphi$ , we have

$$\begin{aligned}
& 2 \iint_{(0,\infty)^2} \Phi_\varphi(x_1, 0, x_3) dG_1 dG_3 \\
&= 2 \iint_{\{x_1 > x_3 > 0\}} [\varphi(x_1 - x_3) + \varphi(x_3) - \varphi(0) - \varphi(x_1)] \frac{dG_1 dG_3}{\sqrt{x_1 x_3}} \\
&= \iint_{(0,\infty)^2} [\varphi(|x_1 - x_3|) + \varphi(\min\{x_1, x_3\}) - \varphi(0) - \varphi(\max\{x_1, x_3\})] \frac{dG_1 dG_3}{\sqrt{x_1 x_3}}.
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
& \iint_{(0,\infty)^2} \Phi_\varphi(x_1, x_2, 0) dG_1 dG_2 \\
&= \iint_{(0,\infty)^2} [\varphi(x_1 + x_2) + \varphi(0) - \varphi(\min\{x_1, x_2\}) - \varphi(\max\{x_1, x_2\})] \frac{dG_1 dG_2}{\sqrt{x_1 x_2}}.
\end{aligned} \tag{2.5}$$

Notice in (2.4) that  $\varphi(|x_1 - x_3|) + \varphi(\min\{x_1, x_3\}) - \varphi(0) - \varphi(\max\{x_1, x_3\}) = 0$  on the diagonal  $\{x_1 = x_3 > 0\}$ . Then, using (2.4) (changing the labels  $x_3$  by  $x_2$ ) and (2.5) in (2.3), and recalling the definition (1.30) of  $\Lambda(\varphi)$ , we obtain

$$\iiint_P \Phi_\varphi dG_1 dG_2 dG_3 = G(t, \{0\}) \iint_{(0,\infty)^2} \frac{\Lambda(\varphi)(x_1, x_2)}{\sqrt{x_1 x_2}} dG_1 dG_2. \tag{2.6}$$

(b) The quadratic term. Again, by the symmetry of  $\Phi_\varphi$  in  $x_1, x_2$ , and the continuity of  $\Phi_\varphi$  on  $[0, \infty)^3$ , we obtain

$$\begin{aligned}
\frac{1}{2} \iiint_P \sqrt{x_3} \Phi_\varphi dG_1 dG_2 dx_3 &= \iiint_{\{x_2=0, x_1>0, x_3>0\}} \sqrt{x_3} \Phi_\varphi dG_1 dG_2 dx_3 \\
&= G(t, \{0\}) \iint_{(0,\infty)^2} \sqrt{x_3} \Phi_\varphi(x_1, 0, x_3) dG_1 dx_3 \\
&= G(t, \{0\}) \iint_{\{x_1 > x_3 > 0\}} \frac{\Delta\varphi(x_1, 0, x_3)}{\sqrt{x_1}} dG_1 dx_3 \\
&= -G(t, \{0\}) \int_{(0,\infty)} \frac{\mathcal{L}_0(\varphi)(x_1)}{\sqrt{x_1}} dG_1,
\end{aligned} \tag{2.7}$$

where  $\mathcal{L}_0(\varphi)$  is given in (1.31). Using (2.6) and (2.7) in (2.2), the result follows.  $\square$

**Proposition 2.3.** *For all  $C > 0$  and all  $\beta > 0$ , the measure  $f_{\beta,0,C}$  is a radial weak solutions of (1.1), (1.2).*

**Proof.** By Proposition 2.1,

$$\mathcal{Q}_4(\varphi, G_{\beta,0,C}) = \mathcal{Q}_4(\varphi, G_{\beta,0,0}) + C \mathcal{Q}_3(\varphi, G_{\beta,0,0}).$$

We already know by Theorem 5 of [18] that  $\mathcal{Q}_4(\varphi, G_{\beta,0,C}) = 0$  for all  $\varphi \in C^{1,1}([0, \infty))$ . Since  $\mathcal{Q}_4(\varphi, G_{\beta,0,0}) \equiv \mathcal{Q}_4(\varphi, G_{\beta,0,C})$ , we deduce  $\mathcal{Q}_4(\varphi, G_{\beta,0,0}) = 0$  for all  $\varphi \in C^{1,1}([0, \infty))$ . Then, since  $C > 0$ ,

$$\mathcal{Q}_3(\varphi, G_{\beta,0,0}) = 0 \quad \forall \varphi \in C^{1,1}([0, \infty)).$$

□

### 3 Existence of solutions $H$ to (1.66)

The main result of this Section is the following,

**Theorem 3.1.** *Let  $h_0 \in \mathcal{M}_+^1([0, \infty))$  with  $N = M_0(h_0) > 0$  and  $E = M_1(h_0) > 0$ . Then, there exists  $h \in C((0, \infty), \mathcal{M}_+^\alpha([0, \infty)))$  for any  $\alpha \geq 1$ , that satisfies the following properties: for all  $\varphi \in C_b^1([0, \infty))$*

$$(i) \quad \tau \mapsto \int_{[0, \infty)} \varphi(x) h(\tau, x) dx \in W_{loc}^{1, \infty}([0, \infty)), \quad (3.1)$$

$$(ii) \quad \frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) h(\tau, x) dx = \tilde{\mathcal{Q}}_3(\varphi, h(\tau)) \quad a.e. \tau > 0, \quad (3.2)$$

$$(iii) \quad h(0) = h_0, \quad (3.3)$$

$$(iv) \quad M_0(h(\tau)) \leq \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^2 \quad \forall \tau \geq 0, \quad (3.4)$$

$$(v) \quad M_1(h(\tau)) = E \quad \forall \tau \geq 0, \quad (3.5)$$

$$(vi) \quad \text{For all } \alpha \geq 3, \text{ if } M_\alpha(h_0) < \infty, \text{ then}$$

$$M_\alpha(h(\tau)) \leq \left( M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau \right)^{\frac{\alpha-1}{2}} \quad \forall \tau \geq 0, \quad (3.6)$$

$$(vii) \quad M_\alpha(h(\tau)) \leq C(\alpha, E) \left( \frac{1}{1 - e^{-\gamma(\alpha, E)\tau}} \right)^{2(\alpha-1)} \quad \forall \alpha \geq 3, \forall \tau > 0, \quad (3.7)$$

where  $C = C(\alpha, E)$  is the unique positive root of the algebraic equation

$$2^{\alpha-2}(\alpha+1)E^{\frac{2\alpha+3}{2(\alpha-1)}}(1+C) = C^{\frac{2\alpha-1}{2(\alpha-1)}}, \quad (3.8)$$

and  $\gamma = \gamma(\alpha, E)$ :

$$\gamma = \frac{1}{2(\alpha+1)} \left( \frac{C}{E} \right)^{\frac{1}{2(\alpha-1)}}. \quad (3.9)$$

The proof of Theorem 3.1 is in three steps. In the first, a regularized problem is solved (Theorem 3.6). Then, using an approximation argument, a solution is obtained that satisfies (3.1)–(3.6) but not yet (3.7) (Theorem 3.4). The Theorem 3.1 is proved with a second approximation argument on the initial data.

As a Corollary, we obtain the measure  $H$  (not necessarily positive).



**Corollary 3.2.** Suppose that  $h_0 \in \mathcal{M}_+^1([0, \infty))$  with  $N = M_0(h_0) > 0$  and  $E = M_1(h_0) > 0$ , consider  $h$  given by Theorem 3.1, and define, for  $\tau \geq 0$

$$H(\tau) = h(\tau) - \left( \int_0^\tau M_{1/2}(h(\sigma)) d\sigma \right) \delta_0. \quad (3.10)$$

Then  $H \in C([0, \infty), \mathcal{M}^1([0, \infty)))$  and for all  $\tau \in [0, \infty)$  and  $\varphi \in C_b^1([0, \infty))$ :

$$(i) \quad \tau \mapsto \int_{[0, \infty)} \varphi(x) H(\tau, x) dx \in W_{loc}^{1, \infty}([0, \infty)), \quad (3.11)$$

$$(ii) \quad \frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) H(\tau, x) dx = \mathcal{Q}_3(\varphi, H(\tau)) \quad a.e. \tau > 0, \quad (3.12)$$

$$(iii) \quad H(0) = h_0, \quad (3.13)$$

$$(iv) \quad M_0(H(\tau)) = N \quad \forall \tau \geq 0, \quad (3.14)$$

$$(v) \quad M_1(H(\tau)) = E \quad \forall \tau \geq 0, \quad (3.15)$$

$$(vi) \quad \forall \alpha \geq 3, \text{ if } M_\alpha(h_0) < \infty \text{ then, for all } \tau > 0,$$

$$M_\alpha(H(\tau)) \leq \left( M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau \right)^{\frac{\alpha-1}{2}}, \quad (3.16)$$

$$(vii) \quad M_\alpha(H(\tau)) \leq C(\alpha, E) \left( \frac{1}{1 - e^{-\gamma(\alpha, E)\tau}} \right)^{2(\alpha-1)}, \quad \forall \alpha \geq 3, \quad (3.17)$$

where the constants  $C(\alpha, E)$  and  $\gamma(\alpha, E)$  are defined in Theorem 3.1.

**Remark 3.3.** Under the hypothesis that all the moments of the initial data  $h_0$  are bounded it is easy to obtain the estimate (3.7) using the weak formulation (3.2). However, it is not so easy using the regularized weak formulation (3.21) below. For that reason, we first want to obtain a solution  $h$  satisfying (3.2) with an initial data with bounded moments of all order.

### 3.1 A first result.

**Theorem 3.4.** For any  $h_0 \in \mathcal{M}_+^1([0, \infty))$  with  $N = M_0(h_0)$  and  $E = M_1(h_0)$ , there exists  $h \in C([0, \infty), \mathcal{M}_+^1([0, \infty)))$  that satisfies (3.1)–(3.6).

The proof of Theorem 3.4 is made in two steps. We first solve a regularised version of (3.2). Then, in a second step, we use an approximation argument. More precisely, we consider the following cutoff:

**Cutoff 3.5.** For every  $n \in \mathbb{N}$  let  $\phi_n \in C_c([0, \infty))$  be such that  $\text{supp } \phi_n = [0, n+1]$ ,  $\phi_n(x) \leq x^{-1/2}$  for all  $x > 0$  and  $\phi_n(x) = x^{-1/2}$  for all  $x \in (\frac{1}{n}, n)$ , in such a way that:

$$\forall x > 0 \quad \lim_{n \rightarrow \infty} \phi_n(x) = \frac{1}{\sqrt{x}}. \quad (3.18)$$

### 3.2 Regularised problem

We now solve in Theorem 3.6 a regularised version of (3.2) with the operator  $\tilde{\mathcal{Q}}_{3,n}$  defined in (8.14)–(8.16). The solution  $h_n$  is obtained as a mild solution to the equation

$$\frac{\partial h_n}{\partial \tau}(\tau, x) = J_{3,n}(h_n(\tau))(x), \quad (3.19)$$

where  $J_{3,n}$  is defined in (8.17)–(8.20), and corresponds to a regularised version of the term  $J_3$  defined in (1.36). Namely,  $J_{3,n}(h) = J_3(h\phi_n)$ , where  $\phi_n$  is as in Cutoff 3.5.

**Theorem 3.6.** *For any  $n \in \mathbb{N}$  and any nonnegative function  $h_0 \in C_c([0, \infty))$ , there exists a unique nonnegative function  $h_n \in C([0, \infty), L^\infty(\mathbb{R}_+) \cap L^1_x(\mathbb{R}_+))$  such that for all  $\tau \in [0, \infty)$  and all  $\varphi \in L^1_{loc}(\mathbb{R}_+)$ :*

$$\tau \mapsto \int_{[0, \infty)} \varphi(x) h_n(\tau, x) dx \in W^{1, \infty}_{loc}([0, \infty)) \quad (3.20)$$

$$\frac{d}{d\tau} \int_0^\infty \varphi(x) h_n(\tau, x) dx = \tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\tau)). \quad (3.21)$$

$$h_n(0, x) = h_0(x) \quad (3.22)$$

Moreover, if we denote by  $N = M_0(h_0)$  and  $E = M_1(h_0)$ , then for every  $\tau \in [0, \infty)$  and  $\alpha \geq 3$ :

$$M_0(h_n(\tau)) \leq \left( \frac{E}{2} \tau + \sqrt{N} \right)^2, \quad (3.23)$$

$$M_1(h_n(\tau)) = E, \quad (3.24)$$

$$M_\alpha(h_n(\tau)) \leq \left( M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau \right)^{\frac{\alpha-1}{2}}. \quad (3.25)$$

Furthermore, there exist two positive constants  $C_{1,n}$  and  $C_{2,n}$  depending on  $n$  and  $\|h_0\|_{L^\infty \cap L^1_x}$  such that for all  $\tau > 0$ :

$$\|h_n(\tau)\|_\infty \leq C_{1,n} e^{C_{2,n}(\tau^2 + \tau)}. \quad (3.26)$$

**Proof.** Using (8.17) we write equation (3.19) as

$$\frac{\partial h_n}{\partial \tau} + h_n A_n(h_n) = K_n(h_n) + L_n(h_n), \quad (3.27)$$

and the solution  $h_n$  is obtained as a fixed point of the operator:

$$\begin{aligned} R_n(h_n)(\tau, x) = & h_0(x) S_n(0, \tau; x) \\ & + \int_0^\tau S_n(\sigma, \tau; x) (K_n(h_n)(\sigma, x) + L_n(h_n)(\sigma, x)) d\sigma, \end{aligned} \quad (3.28)$$

$$S_n(\sigma, \tau; x) = e^{-\int_\sigma^\tau A_n(h_n)(\sigma, x) d\sigma} \quad (3.29)$$

on

$$B(T) := \left\{ h \in C([0, T], L^\infty(\mathbb{R}_+) \cap L_x^1(\mathbb{R}_+)) : h \geq 0 \quad \text{and} \right. \\ \left. \sup_{\tau \in [0, T]} \|h(\tau)\|_{L^\infty \cap L_x^1} \leq 2\|h_0\|_{L^\infty \cap L_x^1} \right\}. \quad (3.30)$$

Let us show first that  $R_n$  sends  $B(T)$  into itself. Let  $r_0 := \|h_0\|_{L^\infty \cap L_x^1}$  and for an arbitrary  $T > 0$ , let  $h \in B(T)$ . By Proposition 8.9 with  $\rho(x) = x$ ,

$$R_n(h)(\tau, x) \geq 0 \quad \forall \tau \in [0, T], \forall x \in \mathbb{R}_+, \\ R_n(h) \in C([0, T], L^\infty(\mathbb{R}_+) \cap L_x^1(\mathbb{R}_+)).$$

Moreover, using (8.28) and (8.29):

$$\sup_{\tau \in [0, T]} \|R_n(h)(\tau)\|_{L^\infty \cap L_x^1} \leq r_0 + T C(n)(4r_0^2 + 2r_0).$$

If  $T$  satisfies:

$$T \leq \frac{1}{C(n)(4r_0 + 2)} \quad (3.31)$$

then  $R_n(h) \in B(T)$ .

To prove that  $R_n$  is a contraction, let  $h_1 \in B(T)$ ,  $h_2 \in B(T)$  and write:

$$\begin{aligned} |R_n(h_1)(\tau, x) - R_n(h_2)(\tau, x)| &\leq h_0(x) |S_1(0, \tau; x) - S_2(0, \tau; x)| + \\ &+ \int_0^\tau |S_1(\sigma, \tau; x) - S_2(\sigma, \tau; x)| (K_n(h_1)(\sigma, x) + L_n(h_1)(\sigma, x)) d\sigma \\ &+ \int_0^\tau |K_n(h_1)(\sigma, x) - K_n(h_2)(\sigma, x)| d\sigma \\ &+ \int_0^\tau |L_n(h_1)(\sigma, x) - L_n(h_2)(\sigma, x)| d\sigma. \end{aligned}$$

By (8.31), for all  $\sigma \geq 0$  and  $\tau \geq 0$

$$\begin{aligned} |S_1(\sigma, \tau; x) - S_2(\sigma, \tau; x)| &\leq \int_0^\tau |A_n(h_1)(\sigma, x) - A_n(h_2)(\sigma, x)| d\sigma \\ &\leq C(n) \tau \sup_{\tau \in [0, T]} \|h_1(\tau) - h_2(\tau)\|_\infty. \end{aligned} \quad (3.32)$$

Using now (3.32) and (8.28)–(8.31), we deduce:

$$\begin{aligned} \|R_n(h_1)(\tau) - R_n(h_2)(\tau)\|_{L^\infty \cap L_x^1} &\leq C_1 \sup_{\tau \in [0, T]} \|h_1(\tau) - h_2(\tau)\|_\infty, \\ C_1 &\equiv C_1(n, T, r_0) = C(n)T(1 + 3r_0 + 2Tr_0(1 + 2r_0)). \end{aligned}$$

If (3.31) holds and

$$C(n)T(1 + 3r_0 + 2Tr_0(1 + 2r_0)) < 1,$$

$R_n$  will be a contraction from  $B(T)$  into itself. This is achieved, for example, as soon as:

$$T < \min \left\{ \frac{1}{2r_0(1 + 2r_0)}, \frac{1}{2C(n)(1 + 2r_0)} \right\} = \kappa_{r_0}.$$

The fixed point  $h_n$  of  $R_n$  in  $B(T)$  is then a mild solution of (3.19), that can be extended to a maximal interval of existence  $[0, T_{n,\max})$ .

We claim now that  $h_n$  satisfies (3.20), (3.21). Since  $h_n$  is a mild solution of (3.19):

$$h_n(\tau, x) = h_0(x)S_n(0, \tau; x) + \int_0^\tau S_n(\sigma, \tau; x)(K_n(h_n)(\sigma, x) + L_n(h_n)(\sigma, x))d\sigma \quad (3.33)$$

We multiply this equation by  $\varphi \in L^1_{loc}(\mathbb{R}_+)$  and integrate on  $(0, \infty)$ :

$$\begin{aligned} \int_0^\infty h_n(\tau, x)\varphi(x)dx &= \int_0^\infty h_0(x)S_n(0, \tau; x)\varphi(x)dx + \\ &+ \int_0^\tau \int_0^\infty S_n(\sigma, \tau; x)(K_n(h_n)(\sigma, x) + L_n(h_n)(\sigma, x))\varphi(x)dx d\sigma. \end{aligned}$$

Using Lemma 8.9 and  $h_0 \in C_c([0, \infty))$ , it follows that the integrals above are well define. It also follows from Lemma 8.9 and (3.29) that  $\tau \mapsto \int_0^\infty h_n(\tau, x)\varphi(x)dx$  is locally Lipschitz on  $(0, T_{n,\max})$ , and:

$$\begin{aligned} \frac{d}{dt} \int_0^\infty h_n(\tau, x)\varphi(x)dx &= \int_0^\infty h_0(x)(S_n(0, \tau; x))_\tau \varphi(x)dx + \\ &+ \int_0^\infty (K_n(h_n)(\tau, x) + L_n(h_n)(\tau, x))\varphi(x)dx + \\ &+ \int_0^\tau \int_0^\infty (S_n(\sigma, \tau; x))_\tau (K_n(h_n)(\sigma, x) + L_n(h_n)(\sigma, x))\varphi(x)dx d\sigma. \end{aligned}$$

We use now that  $(S_n(\sigma, \tau; x))_\tau = -A_n(h_n)(\tau, x)S_n(\sigma, \tau; x)$  and the identity (3.33) to deduce:

$$\begin{aligned} \frac{d}{dt} \int_0^\infty h_n(\tau, x)\varphi(x)dx &= \int_0^\infty (K_n(h_n)(\tau, x) + L_n(h_n)(\tau, x))\varphi(x)dx - \\ &- \int_0^\infty A_n(h_n)h_n(\tau, x)\varphi(\tau, x)dx, \end{aligned}$$

that is (3.21).

Suppose now that  $T_{n,\max} < \infty$  and

$$\sup_{\tau \in [0, T_{n,\max})} \|h_n(\tau)\|_{L^\infty \cap L^1_x} < \infty.$$

Then there is an increasing sequence  $\tau_j \rightarrow T_{n,\max}$  as  $j \rightarrow \infty$  and  $L > 0$  such that

$$\sup_j \|h_n(\tau_j)\|_{L^\infty \cap L_x^1} \leq L < \infty.$$

Fix  $\delta > 0$  such that  $\delta < \kappa_{r_0+1}$ . Starting with the initial value  $h(\tau_j)$  we have a mild solution  $h_j$  defined on  $[0, \delta]$ . Gluing together  $h$  with  $h_j$  we obtain a mild solution on  $[0, t_j + \delta]$ . For  $j$  large enough,  $t_j + \delta > T_{n,\max}$ , and this is a contradiction. Therefore, either  $T_{n,\max} = \infty$  or, if  $T_{n,\max} = \infty$ , then  $\limsup \|h_n(\tau)\|_{L^\infty \cap L_x^1} = \infty$  as  $\tau \rightarrow T_{n,\max}$ .

Let us prove now the estimates (3.23), (3.24) and (3.26), first for all  $\tau \in (0, T_{n,\max})$ . Then, the property  $T_{n,\max} = \infty$  will follow. We start proving (3.24). To this end we use (3.21) with  $\varphi = x$ . Since in that case  $\Lambda(\varphi)(x, y) = 0$  and  $\mathcal{L}(\varphi)(x) = 0$ , (3.24) is immediate. To prove (3.23), we use (3.21) with  $\varphi = 1$ . Then,  $\Lambda(\varphi)(x, y) = 0$  and  $\mathcal{L}(\varphi)(x) = -x$  and then, using  $\phi_n \leq x^{-1/2}$ , Hölder inequality and (3.24):

$$\frac{d}{d\tau} \left( \int_0^\infty h_n(\tau, x) dx \right)^{1/2} \leq \frac{\sqrt{E}}{2},$$

from where (3.23) follows.

In order to prove (3.26) we use (3.23):

$$\begin{aligned} \|K_n(h_n)(\sigma)\|_\infty &\leq \|\phi_n\|_\infty^2 \|h_n(\sigma)\|_1 \|h_n(\sigma)\|_\infty \\ &\leq \|\phi_n\|_\infty^2 \left( \frac{\sqrt{E}}{2} \sigma + \sqrt{N} \right)^2 \|h_n(\sigma)\|_\infty, \end{aligned}$$

which combined with the estimate  $\|L_n(h_n)(\sigma)\|_\infty \leq 2\|\phi_n\|_1 \|h_n(\sigma)\|_\infty$ , gives

$$\begin{aligned} \|h_n(\tau)\|_\infty &\leq \|h_0\|_\infty + \int_0^\tau (\|K_n(h_n)(\sigma)\|_\infty + \|L_n(h_n)(\sigma)\|_\infty) d\sigma \\ &\leq \|h_0\|_\infty + C(n, h_0) \int_0^\tau (\sigma^2 + 1) \|h_n(\sigma)\|_\infty d\sigma. \end{aligned}$$

where

$$C(n, h_0) = \max \left\{ \|\phi_n\|_1 \|\phi_n\|_\infty^2 \|h_0\|_1, \frac{\|\phi_n\|_\infty^2}{4} \|h_0\|_{L_x^1} \right\}.$$

Then (3.26) follows from Gronwall's inequality.

For the proof of (3.25) we use (3.21) with  $\varphi(x) = x^\alpha$  for  $\alpha \geq 3$ :

$$\frac{d}{d\tau} M_\alpha(h_n(\tau)) = \tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\tau)). \quad (3.34)$$

Since:

$$\mathcal{L}(\varphi)(x) = \left( \frac{\alpha-1}{\alpha+1} \right) x^{\alpha+1} \geq 0, \quad (3.35)$$

we have,

$$\frac{d}{d\tau} M_\alpha(h_n(\tau)) \leq 2 \int_0^\infty \int_0^x \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) h_n(\tau, x) h_n(\tau, y) dy dx.$$

Then, we write  $\Lambda(\varphi)(x, y) = x^\alpha ((1+z)^\alpha + (1-z)^\alpha - 2)$ , where  $z = y/x$ , and by Taylor's expansion around  $z = 0$ :

$$u(z) \leq \frac{\|u''\|_\infty}{2} z^2 \leq \alpha(\alpha-1) 2^{\alpha-3} z^2.$$

Hence for all  $0 \leq y \leq x$ :

$$\Lambda(\varphi)(x, y) \leq C_\alpha x^{\alpha-2} y^2, \quad \text{where} \quad C_\alpha = \alpha(\alpha-1) 2^{\alpha-3}, \quad (3.36)$$

and then, using  $\phi_n(x) \phi_n(y) \leq y^{-1}$  and (3.24),

$$\frac{d}{d\tau} M_\alpha(h_n(\tau)) \leq 2C_\alpha M_{\alpha-2}(h_n(\tau)) E.$$

Since by Holder's inequality and (3.24)

$$M_{\alpha-2}(h_n(\tau)) \leq E^{\frac{2}{\alpha-1}} M_\alpha(h_n(\tau))^{\frac{\alpha-3}{\alpha-1}},$$

we deduce

$$\frac{d}{d\tau} \left( M_\alpha(h_n(\tau))^{\frac{2}{\alpha-1}} \right) \leq \frac{4C_\alpha}{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}},$$

and (3.25) follows.  $\square$

### 3.3 Proof of Theorem 3.4.

The solution  $h$  whose existence is claimed in Theorem 3.4 is obtained as the limit of a subsequence of solutions  $(h_n)_{n \in \mathbb{N}}$  to the regularized problems obtained in Theorem 3.6. We first prove the following Lemma.

**Lemma 3.7.** *Let  $h_0 \in C_c([0, \infty))$  be nonnegative with  $N = M_0(h_0) > 0$  and  $E = M_1(h_0) > 0$ , and consider  $(h_n)_{n \in \mathbb{N}}$  the sequence of functions given by Theorem 3.6. Then for every  $\tau \in [0, \infty)$  there exists a subsequence, still denoted  $(h_n(\tau))_{n \in \mathbb{N}}$ , and a measure  $h(\tau) \in \mathcal{M}_+^1([0, \infty))$  such that, as  $n \rightarrow \infty$ ,  $h_n(\tau)$  converges to  $h(\tau)$  in the following sense:*

$$\forall \varphi \in C([0, \infty)); \exists \theta \in [0, 1) : \sup_{x \geq 0} \frac{\varphi(x)}{1+x^\theta} < \infty, \quad (3.37)$$

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} \varphi(x) h_n(\tau, x) dx = \int_{[0, \infty)} \varphi(x) h(\tau, x) dx. \quad (3.38)$$

Moreover, for every  $\tau \in [0, \infty)$ :

$$M_0(h(\tau)) \leq \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^2, \quad (3.39)$$

$$M_1(h(\tau)) \leq E. \quad (3.40)$$

**Proof.** Let us prove first the convergence for a subsequence of  $(h_n(\tau))_{n \in \mathbb{N}}$ . For every  $\tau \geq 0$  we have by (3.23) that

$$\sup_{n \in \mathbb{N}} \int_0^\infty h_n(\tau, x) dx \leq \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^2.$$

Therefore, there exists a subsequence, still denoted  $(h_n(\tau))_{n \in \mathbb{N}}$ , and a measure  $h(\tau)$  such that  $(h_n(\tau))_{n \in \mathbb{N}}$  converges to  $h(\tau)$  in the weak\* topology of  $\mathcal{M}([0, \infty))$ , as  $n \rightarrow \infty$ :

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} \varphi(x) h_n(\tau, x) dx = \int_{[0, \infty)} \varphi(x) h(\tau, x) dx, \quad \forall \varphi \in C_0([0, \infty)). \quad (3.41)$$

Since for all  $n \in \mathbb{N}$ ,  $h_n(\tau)$  is nonnegative, then  $h(\tau)$  is a positive measure. Also by weak\* convergence and (3.23) we deduce that  $h(\tau)$  is a finite measure:

$$\int_{[0, \infty)} h(\tau, x) dx \leq \liminf_{n \rightarrow \infty} \int_0^\infty h_n(\tau, x) dx \leq \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^2. \quad (3.42)$$

Moreover, by (3.24) we also have that the sequence  $(h_n(\tau))_{n \in \mathbb{N}}$  is bounded in  $L_x^1(\mathbb{R}_+)$ . Hence there exists a subsequence (not relabelled) that converges to a measure  $\nu(\tau)$  in the weak\* topology of  $\mathcal{M}([0, \infty))$ , i.e., such that

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi(x) x h_n(\tau, x) dx = \int_{[0, \infty)} \varphi(x) \nu(\tau, x) dx, \quad \forall \varphi \in C_0([0, \infty)). \quad (3.43)$$

Again, since  $h_n(\tau)$  is nonnegative for all  $n \in \mathbb{N}$  then  $\nu(\tau)$  is a positive measure. Also by weak\* convergence and (3.24) we have

$$\int_{[0, \infty)} \nu(\tau, x) dx \leq \liminf_{n \rightarrow \infty} \int_0^\infty x h_n(\tau, x) dx = E. \quad (3.44)$$

Let us show now that  $\nu(\tau) = x h(\tau)$ . This will follow from

$$\forall \varphi \in C_0([0, \infty)) : \int_{[0, \infty)} \varphi(x) \nu(\tau, x) dx = \int_{[0, \infty)} \varphi(x) x h(\tau, x) dx \quad (3.45)$$

In a first step we show that (3.45) holds for  $\varphi \in C_c([0, \infty))$  and then we use a density argument. Let  $\varepsilon > 0$  and  $\varphi \in C_c([0, \infty))$ . Using (3.43) with test function  $\varphi$ , and (3.41) with test function  $x\varphi(x)$ , we deduce that

$$\begin{aligned} & \left| \int_{[0, \infty)} \varphi(x) \nu(\tau, x) dx - \int_{[0, \infty)} \varphi(x) x h(\tau, x) dx \right| \\ & \leq \left| \int_0^\infty \varphi(x) \nu(\tau, x) dx - \int_{[0, \infty)} \varphi(x) x h_n(\tau, x) dx \right| \\ & \quad + \left| \int_0^\infty \varphi(x) x h_n(\tau, x) dx - \int_{[0, \infty)} \varphi(x) x h(\tau, x) dx \right| < \varepsilon \end{aligned}$$

for  $n$  large enough. Hence (3.45) holds for all  $\varphi \in C_c([0, \infty))$ . Now let  $\varphi \in C_0([0, \infty))$  and consider a sequence  $(\varphi_k)_{k \in \mathbb{N}} \subset C_c([0, \infty))$  such that  $\|\varphi_k - \varphi\|_\infty \rightarrow 0$  as  $k \rightarrow \infty$ . Using (3.45) with  $\varphi_k$  and the bounds (3.42) and (3.44), we deduce that

$$\begin{aligned} & \left| \int_{[0, \infty)} \varphi(x) \nu(\tau, x) dx - \int_{[0, \infty)} \varphi(x) x h(\tau, x) dx \right| \\ & \leq \int_{[0, \infty)} |\varphi(x) - \varphi_k(x)| \nu(\tau, x) dx \\ & + \left| \int_{[0, \infty)} \varphi_k(x) \nu(\tau, x) dx - \int_{[0, \infty)} \varphi_k(x) x h(\tau, x) dx \right| \\ & + \int_{[0, \infty)} |\varphi_k(x) - \varphi(x)| x h(\tau, x) dx < \varepsilon \end{aligned}$$

for  $k$  large enough. Therefore (3.45) holds for all  $\varphi \in C_0([0, \infty))$ , i.e.,  $\nu(\tau) = x h(\tau)$ . Hence we rewrite (3.43) as

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi(x) x h_n(\tau, x) dx = \int_{[0, \infty)} \varphi(x) x h(\tau, x) dx, \quad \forall \varphi \in C_0([0, \infty)). \quad (3.46)$$

Let us show now (3.37), (3.38). Let then  $\varphi \in C([0, \infty))$  be any nonnegative test function that satisfies (3.37). We denote  $(\zeta_j)_{j \in \mathbb{N}}$  a sequence of nonnegative and nonincreasing functions of  $C_c^\infty([0, \infty))$  such that:

$$\zeta_j(x) = 1 \text{ if } x \in [0, j), \quad \zeta_j(x) = 0 \text{ if } x > j + 1,$$

and define  $\varphi_j = \varphi \zeta_j$ . Then for every  $n, j \in \mathbb{N}$ :

$$\begin{aligned} & \left| \int_0^\infty \varphi(x) h_n(\tau, x) dx - \int_{[0, \infty)} \varphi(x) h(\tau, x) dx \right| \\ & \leq \int_0^\infty |\varphi(x) - \varphi_j(x)| h_n(\tau, x) dx \\ & + \left| \int_0^\infty \varphi_j(x) h_n(\tau, x) dx - \int_{[0, \infty)} \varphi_j(x) h(\tau, x) dx \right| \\ & + \int_{[0, \infty)} |\varphi_j(x) - \varphi(x)| h(\tau, x) dx \end{aligned} \quad (3.47)$$

Since  $\varphi_j \in C_0([0, \infty))$ , using (3.43), the second term in the right hand side of (3.47) converges to zero as  $n \rightarrow \infty$  for every  $j \in \mathbb{N}$ . The first and the third term in the right hand side of (3.47) are treated in the same way, using



that  $\varphi_j(x) = \varphi(x)$  for all  $x \in [0, j]$ . For instance, in the first term:

$$\begin{aligned} \int_0^\infty |\varphi(x) - \varphi_j(x)| h_n(\tau, x) dx &= \int_j^\infty |\varphi(x) - \varphi_j(x)| h_n(\tau, x) dx \\ &\leq 2 \int_j^\infty |\varphi(x)| h_n(\tau, x) dx \leq 2C \int_j^\infty (1 + x^\theta) h_n(\tau, x) dx \\ &\leq 2C \left( \frac{1 + j^\theta}{j} \right) \int_j^\infty x h_n(\tau, x) dx \leq 2C \left( \frac{1 + j^\theta}{j} \right) E. \end{aligned}$$

Therefore this term is small provided  $j$  is large enough. In conclusion, the difference in (3.47) is less than  $\varepsilon$  for  $n$  sufficiently large, i.e., (3.38) holds.  $\square$

**Remark 3.8.** The so-called narrow topology  $\sigma(\mathcal{M}([0, \infty)), C_b([0, \infty)))$  on  $\mathcal{M}_+([0, \infty))$  is generated by the metric  $d(\mu, \nu) = \|\mu - \nu\|_0$ , where

$$\|\mu\|_0 = \sup \left\{ \int_{[0, \infty)} \varphi d\mu : \varphi \in \text{Lip}_1([0, \infty)), \|\varphi\|_\infty \leq 1 \right\},$$

(cf. [4] Theorem 8.3.2).

Using this Remark, Lemma 3.7 and the Arzelà-Ascoli's Theorem we prove now the following:

**Proposition 3.9.** *Let  $h_0$  and  $(h_n)_{n \in \mathbb{N}}$  be as in Lemma 3.7. Then there exist a subsequence (not relabelled) and  $h \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  such that*

$$h_n \xrightarrow{n \rightarrow \infty} h \quad \text{in } C([0, \infty), \mathcal{M}_+([0, \infty))). \quad (3.48)$$

Moreover, if we denote by  $N = M_0(h_0)$  and  $E = M_1(h_0)$ , then for all  $\tau \geq 0$

$$M_0(h(\tau)) \leq \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^2, \quad (3.49)$$

$$M_1(h(\tau)) \leq E, \quad (3.50)$$

and for all  $\varphi \in C([0, \infty))$  satisfying the growth condition (3.37):

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi(x) h_n(\tau, x) dx = \int_{[0, \infty)} \varphi(x) h(\tau, x) dx. \quad (3.51)$$

**Proof of Proposition 3.9.** By Lemma 3.7 the sequence  $(h_n(\tau))_{n \in \mathbb{N}}$  is relatively compact in  $\mathcal{M}([0, \infty))$  for every  $\tau \in [0, \infty)$ . Let us show now that  $(h_n)_{n \in \mathbb{N}}$  is also equicontinuous. To this end let  $\tau_2 \geq \tau_1 \geq 0$ , and consider  $\varphi$  as in Remark 3.8, i.e.,  $\varphi \in \text{Lip}([0, \infty))$  with Lipschitz constant  $\text{Lip}(\varphi) \leq 1$ ,

and  $\|\varphi\|_\infty \leq 1$ . Then, using  $\phi_n(x) \leq x^{-1/2}$ , (8.2) and (8.4) in Lemma 8.3, we have

$$\begin{aligned} & \left| \int_0^\infty \varphi(x) h_n(\tau_1, x) dx - \int_0^\infty \varphi(x) h_n(\tau_2, x) dx \right| \\ & \leq \int_{\tau_1}^{\tau_2} |\tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\sigma))| d\sigma \leq 2 \int_{\tau_1}^{\tau_2} \left( \int_0^\infty h_n(\sigma, x) dx \right)^2 d\sigma \\ & + 4 \int_{\tau_1}^{\tau_2} \int_0^\infty \sqrt{x} h_n(\sigma, x) dx d\sigma. \end{aligned} \quad (3.52)$$

Using Hölder's inequality and the estimates (3.23) and (3.24) in (3.52), it follows that

$$\begin{aligned} & \left| \int_0^\infty \varphi(x) h_n(\tau_1, x) dx - \int_0^\infty \varphi(x) h_n(\tau_2, x) dx \right| \\ & \leq 2 \int_{\tau_1}^{\tau_2} \left( \frac{\sqrt{E}}{2} \sigma + \sqrt{N} \right)^4 d\sigma + 4\sqrt{E} \int_{\tau_1}^{\tau_2} \left( \frac{\sqrt{E}}{2} \sigma + \sqrt{N} \right) d\sigma \quad \forall n \in \mathbb{N}. \end{aligned}$$

We then deduce using Remark 3.8 that  $(h_n)_{n \in \mathbb{N}}$  is equicontinuous. It then follows from Arzelà-Ascoli's Theorem (cf. for example [24]) that there exists  $h \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  such that  $h_n \rightarrow h$  in  $C([0, T], \mathcal{M}_+([0, \infty)))$ , for every  $T > 0$ , as  $n \rightarrow \infty$ .

The estimates (3.49), (3.50) and the convergence (3.51) are deduced in the same way as in the Proof of Lemma 3.7.  $\square$

**Proof of Theorem 3.4.** By Corollary 8.6, there exists a sequence of non-negative function  $(h_{0,n})_{n \in \mathbb{N}} \in C_c([0, \infty))$  that approximate  $h_0$  in the weak\* topology of the space  $C_b([0, \infty))^*$ . Let then  $(h_n)_{n \in \mathbb{N}} \subset C([0, \infty), \mathcal{M}_+([0, \infty)))$  be the sequence of solutions to (3.20), (3.21) obtained by Theorem 3.6 with the initial data  $h_{0,n}$ . By Proposition 3.9 there exists a subsequence, still denoted  $(h_n)_{n \in \mathbb{N}}$ , and  $h \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  such that  $h_n$  converges to  $h$  in the topology of  $C([0, \infty), \mathcal{M}_+([0, \infty)))$ .

By (3.21) and (3.22), for all  $\varphi \in C_b^1([0, \infty))$  and  $\tau > 0$ :

$$\int_0^\infty \varphi(x) h_n(\tau, x) dx - \int_0^\infty \varphi(x) h_{0,n}(x) dx = \int_0^\tau \tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\sigma)) d\sigma. \quad (3.53)$$

By construction, for every  $\varphi \in C_b^1([0, \infty))$  and every  $\tau \in [0, \infty)$ :

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi(x) h_n(\tau, x) dx = \int_{[0, \infty)} \varphi(x) h(\tau, x) dx. \quad (3.54)$$

We prove now the convergence of the linear term: for all  $\varphi \in C_b^1([0, \infty))$  and  $\tau \in [0, \infty)$

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}_{3,n}^{(1)}(\varphi, h_n(\tau)) = \tilde{\mathcal{Q}}_3^{(1)}(\varphi, h_n(\tau)). \quad (3.55)$$

By definition:

$$\begin{aligned}
& \left| \tilde{\mathcal{Q}}_3^{(1)}(\varphi, h(\tau)) - \tilde{\mathcal{Q}}_{3,n}^{(1)}(\varphi, h_n(\tau)) \right| \\
& \leq \left| \int_0^\infty \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} h(\tau, x) dx - \int_0^\infty \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} h_n(\tau, x) dx \right| \\
& + \int_0^\infty \left| \mathcal{L}(\varphi)(x) \phi_n(x) - \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} \right| h_n(\tau, x) dx. \tag{3.56}
\end{aligned}$$

From Lemma 8.3 (iii) and (3.51):

$$\lim_{n \rightarrow \infty} \left| \int_0^\infty \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} h(\tau, x) dx - \int_0^\infty \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} h_n(\tau, x) dx \right| = 0 \tag{3.57}$$

For the second term in the right hand side of (3.56) we split the integral  $\int_0^\infty$  in two:  $\int_0^R$  and  $\int_R^\infty$  for  $R > 0$ , and apply (8.4). We obtain:

$$\begin{aligned}
& \int_0^\infty \left| \mathcal{L}(\varphi)(x) \phi_n(x) - \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} \right| h_n(\tau, x) dx \\
& \leq \left\| \mathcal{L}(\varphi)(x) \phi_n(x) - \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} \right\|_{C([0, R])} \int_0^R h_n(\tau, x) dx + \\
& + 4 \|\varphi\|_\infty \int_R^\infty \sqrt{x} h_n(\tau, x) dx. \tag{3.58}
\end{aligned}$$

By (3.24), for any  $\varepsilon > 0$  and  $R > (E/\varepsilon)^2$ :

$$\int_R^\infty \sqrt{x} h_n(\tau, x) dx \leq \frac{E}{\sqrt{R}} < \varepsilon \quad \forall n \in \mathbb{N}.$$

Then by Lemma 8.4 and (3.23), the part on  $[0, R]$  converges to zero as  $n \rightarrow \infty$ . Since  $R > 0$  is arbitrary we finally deduce that (3.58) converges to zero as  $n \rightarrow \infty$ . Therefore (3.55) holds.

Let us prove now the convergence of the quadratic term: for all  $\varphi \in C_b^1([0, \infty))$  and all  $\tau \in [0, \infty)$ :

$$\lim_{n \rightarrow \infty} \mathcal{Q}_{3,n}^{(2)}(\varphi, h_n(\tau)) = \mathcal{Q}_3^{(2)}(\varphi, h_n(\tau)). \tag{3.59}$$

As before

$$\begin{aligned}
& \left| \mathcal{Q}_3^{(2)}(\varphi, h(\tau)) - \mathcal{Q}_{3,n}^{(2)}(\varphi, h_n(\tau)) \right| \\
& \leq \left| \mathcal{Q}_3^{(2)}(\varphi, h(\tau)) - \int_0^\infty \int_0^\infty \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} h_n(\tau, x) h_n(\tau, y) dx dy \right| \\
& + \int_0^\infty \int_0^\infty \left| \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) - \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \right| h_n(\tau, x) h_n(\tau, y) dx dy. \tag{3.60}
\end{aligned}$$

It follows from Lemma 8.3 (ii) and (3.51) that the first term in the right hand side above converges to zero as  $n \rightarrow \infty$ . For the second term we proceed as before. For any  $R > 0$  we split the double integral:

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \left| \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) - \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \right| h_n(\tau, x) h_n(\tau, y) dx dy \\
& \leq \left\| \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) - \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \right\|_{C([0, R]^2)} \left( \int_0^R h_n(\tau, x) dx \right)^2 \\
& + \int \int_{(0, \infty)^2 \setminus (0, R)^2} \left| \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) - \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \right| h_n(\tau, x) h_n(\tau, y) dx dy \\
& = I_1 + I_2.
\end{aligned}$$

By Lemma 8.4 and (3.23),  $I_1$  converges to zero as  $n \rightarrow \infty$ . For the term  $I_2$  we use (8.2) in Lemma 8.3 and the estimates (3.24) and (3.23):

$$\begin{aligned}
& \int_R^\infty \int_R^\infty \left| \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) - \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \right| h_n(\tau, x) h_n(\tau, y) dx dy \\
& \leq 4 \|\varphi'\|_\infty \left( \int_R^\infty h_n(\tau, x) dx \right)^2 \leq \frac{4 \|\varphi'\|_\infty E^2}{R^2} \quad \forall n \in \mathbb{N}
\end{aligned}$$

and

$$\begin{aligned}
& 2 \int_R^\infty \int_0^R \left| \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) - \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \right| h_n(\tau, x) h_n(\tau, y) dx dy \\
& \leq 4 \|\varphi'\|_\infty \int_R^\infty \int_0^R h_n(\tau, x) h_n(\tau, y) dx dy \leq \frac{4 \|\varphi'\|_\infty E}{R} \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^2.
\end{aligned}$$

Since  $R > 0$  is arbitrary we deduce that  $I_2$  also converges to zero as  $n \rightarrow \infty$ . We then conclude that (3.59) holds.

Combining (3.55) and (3.59) it follows that for all  $\varphi \in C_b^1([0, \infty))$  and all  $\tau \in [0, \infty)$ :

$$\lim_{n \rightarrow \infty} \tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\tau)) = \tilde{\mathcal{Q}}_3(\varphi, h(\tau)). \quad (3.61)$$

Moreover, using  $\phi_n(x) \leq x^{-1/2}$ , (8.2) and (8.4) in Lemma 8.3, and the estimates (3.23) and (3.24), we have for all  $\varphi \in C_b^1([0, \infty))$ , all  $\tau \in [0, \infty)$  and all  $n \in \mathbb{N}$ :

$$\begin{aligned}
& \left| \tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\tau)) \right| \leq \\
& \leq 2 \|\varphi'\|_\infty \left( \int_0^\infty h_n(\tau, x) dx \right)^2 + 4 \|\varphi\|_\infty \int_0^\infty \sqrt{x} h_n(\tau, x) dx \\
& \leq 2 \|\varphi'\|_\infty \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^4 + 4 \|\varphi\|_\infty \sqrt{E} \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right).
\end{aligned}$$

By (3.61) and the dominated convergence Theorem:

$$\lim_{n \rightarrow \infty} \int_0^\tau \tilde{\mathcal{Q}}_{3,n}(\varphi, h_n(\sigma)) d\sigma = \int_0^\tau \tilde{\mathcal{Q}}_3(\varphi, h(\sigma)) d\sigma. \quad (3.62)$$

Using now (3.54) and (3.62), we may pass to the limit as  $n \rightarrow \infty$  in (3.53) for all  $\varphi \in C_b^1([0, \infty))$  and all  $\tau \in [0, \infty)$  to obtain:

$$\int_{[0, \infty)} \varphi(x) h(\tau, x) dx = \int_{[0, \infty)} \varphi(x) h_0(x) dx + \int_0^\tau \tilde{\mathcal{Q}}_3(\varphi, h(\sigma)) d\sigma. \quad (3.63)$$

The map  $\tau \mapsto \int_{[0, \infty)} \varphi(x) h(\tau, x) dx$  is then locally Lipschitz on  $[0, \infty)$ , and  $h$  satisfies (3.1), (3.2) for all  $\varphi \in C_b^1([0, \infty))$  and for a.e.  $\tau \in [0, \infty)$ . It also follows from (3.63) that  $h(0) = h_0$  in  $\mathcal{M}_+$ .

The property (3.4) follows from (3.49). The conservation of energy (3.5) is obtained as follows. We already know by (3.50) that  $M_1(h(\tau)) \leq E$ . On the other hand, let  $\varphi_k \in C_b^1([0, \infty))$  be a concave test function such that  $\varphi_k(x) = x$  for  $x \in [0, k)$  and  $\varphi_k(x) = k + 1$  for  $x \geq k + 2$ . Notice that there exists a positive constant  $C$  such that

$$\sup_{k \in \mathbb{N}} \|\varphi_k'\|_\infty \leq C. \quad (3.64)$$

By Remark 8.2,  $\tilde{\mathcal{Q}}_3^{(1)}(\varphi_k, h) \leq 0$  and  $\mathcal{Q}_3^{(2)}(\varphi_k, h) \leq 0$  for all  $k \in \mathbb{N}$ , and then, from (3.63):

$$\int_{[0, \infty)} \varphi_k(x) h(\tau, x) dx \geq \int_{[0, \infty)} \varphi_k(x) h_0(x) dx + \int_0^\tau \mathcal{Q}_3^{(2)}(\varphi_k, h(\sigma)) d\sigma. \quad (3.65)$$

We now prove that for all  $\tau \in [0, \infty)$ :

$$\lim_{k \rightarrow \infty} \int_0^\tau \mathcal{Q}_3^{(2)}(\varphi_k, h(\sigma)) d\sigma = 0. \quad (3.66)$$

Notice that  $\Lambda(\varphi_k)(x, y) \rightarrow 0$  as  $k \rightarrow \infty$ , since  $\varphi_k(x) \rightarrow x$ . Then, using (8.2) in Lemma 8.3, (3.64) and (3.23), we deduce for all  $\tau \in [0, \infty)$  and  $\sigma \in (0, \tau)$ :

$$\lim_{k \rightarrow \infty} \mathcal{Q}_3^{(2)}(\varphi_k, h(\sigma)) = 0 \quad (3.67)$$

$$\left| \mathcal{Q}_3^{(2)}(\varphi_k, h(\tau)) \right| \leq 2C \left( \frac{\sqrt{E}}{2} \tau + \sqrt{N} \right)^4 \quad \forall k \in \mathbb{N}. \quad (3.68)$$

and (3.66) follows from the dominated convergence Theorem. We take now limits in (3.65) as  $k \rightarrow \infty$ . By (3.66) and the monotone convergence Theorem we obtain that  $M_1(h(\tau)) \geq E$  and then  $M_1(h(\tau)) = E$  for all  $\tau > 0$ .

We assume now that  $M_\alpha(h_0) < \infty$  for some  $\alpha \geq 3$  and prove (3.6). By (3.25) and (8.10) in Corollary 8.6:

$$M_\alpha(h(\tau)) \leq \liminf_{n \rightarrow \infty} \left( M_\alpha(h_{0,n})^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} M_1(h_{0,n})^{\frac{\alpha+1}{\alpha-1}} \tau \right)^{\frac{\alpha-1}{2}} \quad (3.69)$$

$$\leq \left( M_\alpha(h_0)^{\frac{2}{\alpha-1}} + \alpha 2^{\alpha-1} E^{\frac{\alpha+1}{\alpha-1}} \tau \right)^{\frac{\alpha-1}{2}}. \quad (3.70)$$

□

### 3.4 Proof of Theorem 3.1

**Proof of Theorem 3.1.** Consider again the sequence of initial data  $h_{0,n}$  used in the proof of Theorem 3.4 and the sequence of solutions  $h_n$  obtained by Theorem 3.4. Using (3.25) we know that  $M_\alpha(h_n(\tau)) < \infty$  for  $\tau > 0$  and  $n \in \mathbb{N}$ .

Our first step is to prove that (3.2) holds also true for  $\varphi(x) = x^\alpha$ . Notice that  $h_n$  solves now the equation (3.2), with the operator  $\tilde{\mathcal{Q}}_3$  in the right-hand side, whose kernel is not compactly supported and the argument in the proof of (3.25) must be slightly modified.

In order to use (3.2) we consider a sequence  $(\varphi_k) \subset C_b^1([0, \infty))$  such that:

$$\varphi_k \rightarrow \varphi \quad \text{as } k \rightarrow \infty \quad (3.71)$$

$$\varphi_k \leq \varphi_{k+1} \leq \varphi \quad (3.72)$$

$$\varphi' \geq \varphi'_k \geq 0. \quad (3.73)$$

Let us prove by the dominated convergence Theorem that for all  $\tau \geq 0$ :

$$(i) \quad \tilde{\mathcal{Q}}_3(\varphi, h_n) \in L^1(0, \tau), \quad (3.74)$$

$$(ii) \quad \lim_{k \rightarrow \infty} \int_0^\tau \tilde{\mathcal{Q}}_3(\varphi_k, h_n(\sigma)) d\sigma = \int_0^\tau \tilde{\mathcal{Q}}_3(\varphi, h_n(\sigma)) d\sigma. \quad (3.75)$$

To this end we first observe that, for  $x \geq y > 0$ :

$$\lim_{k \rightarrow \infty} \Lambda(\varphi_k)(x, y) = \Lambda(\varphi) \quad \text{and} \quad \lim_{n \rightarrow \infty} \mathcal{L}(\varphi_k) = \mathcal{L}(\varphi)(x) \quad (3.76)$$

and, by the mean value Theorem:

$$\frac{\Lambda(\varphi_k)(x, y)}{\sqrt{xy}} \leq \varphi'_k(\xi_1) - \varphi'_k(\xi_2)$$

for some  $\xi_1 \in (x, x+y)$  and  $\xi_2 \in (x-y, x)$ . Using then (3.73):

$$\frac{|\Lambda(\varphi_k)(x, y)|}{\sqrt{xy}} \leq \alpha(2^{\alpha-1} + 1)x^{\alpha-1} \quad \forall k \in \mathbb{N}, \quad (3.77)$$

and by (3.72):

$$\frac{|\mathcal{L}(\varphi_k)(x)|}{\sqrt{x}} \leq \left( \frac{\alpha + 3}{\alpha + 1} \right) x^{\alpha + \frac{1}{2}} \quad \forall k \in \mathbb{N}.$$

Since by Theorem 3.4:  $M_{\alpha-1}(h_n(\tau)) < \infty$  and  $M_{\alpha+1/2}(h_n(\tau)) < \infty$ , for every fixed  $n$  we may apply the Lebesgue's convergence Theorem to the sequences  $\left\{ \frac{\Lambda(\varphi_k)(x,y)}{\sqrt{xy}} h_n(\sigma, x) h_n(\sigma, y) \right\}_{k \in \mathbb{N}}$  and  $\left\{ \frac{\mathcal{L}(\varphi_k)(x)}{\sqrt{x}} h_n(\sigma, x) \right\}_{k \in \mathbb{N}}$  and obtain (3.74), (3.75).

We use now  $\varphi_k$  in (3.2) and take the limit  $k \rightarrow \infty$ . We obtain from (3.71), (3.72), (3.75) and monotone convergence:

$$M_\alpha(h_n(\tau)) = M_\alpha(h_{0,n}) + \int_0^\tau \tilde{\mathcal{Q}}_3(\varphi, h_n(\sigma)) d\sigma \quad \forall \tau \geq 0, \quad (3.78)$$

and then, using (3.74):

$$\frac{d}{d\tau} M_\alpha(h_n(\tau)) = \tilde{\mathcal{Q}}_3(\varphi, h_n(\tau)) \quad a.e. \tau > 0. \quad (3.79)$$

If we use (3.35) and (3.36) in the right hand side of (3.79), we obtain

$$\frac{d}{d\tau} M_\alpha(h_n) \leq 2^{\alpha-2} \alpha(\alpha-1) E_n M_{\alpha-2}(h_n) - \left( \frac{\alpha-1}{\alpha+1} \right) M_{\alpha+\frac{1}{2}}(h_n),$$

where  $E_n = M_1(h_{0,n})$ . Now by Hölder's inequality:

$$\begin{aligned} M_{\alpha-2}(h_n) &\leq E_n^{2/(\alpha-1)} M_\alpha(h_n)^{(\alpha-3)/(\alpha-1)} \\ M_\alpha(h_n) &\leq E_n^{1/(2\alpha-1)} M_{\alpha+\frac{1}{2}}(h_n)^{2(\alpha-1)/(2\alpha-1)}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \frac{d}{d\tau} M_\alpha(h_n) &\leq 2^{\alpha-2} \alpha(\alpha-1) E_n^{1+2/(\alpha-1)} M_\alpha(h_n)^{(\alpha-3)/(\alpha-1)} \\ &\quad - \left( \frac{\alpha-1}{\alpha+1} \right) E_n^{-1/(2(\alpha-1))} M_\alpha(h_n)^{(2\alpha-1)/(2(\alpha-1))}. \end{aligned}$$

Since  $(\alpha-3)/(\alpha-1) \in [0, 1)$  then

$$M_\alpha(h_n)^{(\alpha-3)/(\alpha-1)} \leq 1 + M_\alpha(h_n),$$

and :

$$\begin{aligned} \frac{d}{d\tau} M_\alpha(h_n) &\leq 2^{\alpha-2} \alpha(\alpha-1) E_n^{1+2/(\alpha-1)} (1 + M_\alpha(h_n)) \\ &\quad - \left( \frac{\alpha-1}{\alpha+1} \right) E_n^{-1/(2(\alpha-1))} M_\alpha(h_n)^{(2\alpha-1)/(2(\alpha-1))}, \end{aligned} \quad (3.80)$$

where  $(2\alpha - 1)/(2(\alpha - 1)) > 1$ . If we define:

$$\begin{aligned} u(\sigma) &= M_\alpha(h_n(\tau)), \quad \sigma = C_1\tau, \quad q = 2(\alpha - 1), \\ C_1 &= 2^{\alpha-2}\alpha(\alpha - 1)E^{1+2/(\alpha-1)} \\ C_2 &= \left(\frac{\alpha - 1}{\alpha + 1}\right) E^{-1/(2(\alpha-1))}, \quad C = \frac{C_2}{C_1}. \end{aligned}$$

We deduce from (3.80) that

$$u' \leq 1 + u - Cu^{1+1/q}, \quad (3.81)$$

and then by Lemma 6.3 in [2], for every  $n \in \mathbb{N}$ :

$$M_\alpha(h_n(\tau)) \leq C(\alpha, E_n) \left( \frac{1}{1 - e^{-\gamma(\alpha, E_n)\tau}} \right)^{2(\alpha-1)}, \quad (3.82)$$

where the constants  $C(\alpha, E_n)$  and  $\gamma(\alpha, E_n)$  are defined as in Theorem 3.1. We may argue now as in the proof of Theorem 3.4 and pass to the limit along a subsequence to obtain a limit  $h \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  satisfying (3.1)–(3.5) and (3.7). Using (3.7) and  $h \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  we deduce as in the proof of Theorem 3.4 that  $h \in C((0, \infty), \mathcal{M}_+^\alpha([0, \infty)))$  for all  $\alpha \geq 1$ .  $\square$

**Proof of Corollary 3.2.** We first observe that the map  $\tau \mapsto M_{1/2}(h(\tau))$  is locally bounded. Indeed by Hölder's inequality, (3.4) and (3.5):

$$M_{1/2}(h(\tau)) \leq \sqrt{M_1(h(\tau))M_0(h(\tau))} \leq \sqrt{E} \left( \frac{\sqrt{E}}{2}\tau + \sqrt{N} \right).$$

Then by (3.1) it follows (3.11). Now for all  $\varphi \in C_b^1([0, \infty))$  and for a.e.  $\tau > 0$ , we deduce from (3.2):

$$\begin{aligned} \frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) H(\tau, x) dx &= \tilde{\mathcal{Q}}_3(\varphi, h(\tau)) - \varphi(0)M_{1/2}(h(\tau)) \\ &= \mathcal{Q}_3(\varphi, h(\tau)). \end{aligned}$$

Since  $H = h$  on  $(0, \infty)$  then  $\mathcal{Q}_3(\varphi, H) \equiv \mathcal{Q}_3(\varphi, h)$ , and therefore (3.12) holds.

Now for the initial data:  $H(0) = h(0) = h_0$ . The conservation of mass (3.14) follows from (3.12) for  $\varphi = 1$ , since  $\Lambda(\varphi) = 0 = \mathcal{L}_0(\varphi)$ . The conservation of energy (3.15) follows directly from (3.5) since  $H = h$  on  $(0, \infty)$ .  $\square$

## 4 Properties of $h(\tau, \{0\})$ .

In all this Section we denote

$$m(\tau) = h(\tau, \{0\}). \quad (4.1)$$

The main result of this Section is the following.



**Theorem 4.1.** Suppose that  $h \in C([0, \infty); \mathcal{M}_+^1([0, \infty)))$  is a solution of (1.67) with  $h(0) = h_0 \in \mathcal{M}_+^1([0, \infty))$ ,  $N = M_0(h_0) > 0$  and  $E = M_1(h_0) > 0$ . Then  $m$  is right continuous, a.e. differentiable and strictly increasing on  $[0, \infty)$ .

We begin with the following properties of the function  $m$  defined in (4.1).

**Lemma 4.2.** The function  $m$  is nondecreasing, a.e. differentiable and right continuous on  $[0, \infty)$ .

**Proof.** Given any  $\varphi_\varepsilon$  as in Remark 1.6, then for all  $\tau \geq 0$

$$m(\tau) = \lim_{\varepsilon \rightarrow 0} \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx, \quad (4.2)$$

and by (1.67)-(1.40)

$$\frac{d}{d\tau} \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx = \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h(\tau)) - \tilde{\mathcal{Q}}_3^{(1)}(\varphi_\varepsilon, h(\tau)). \quad (4.3)$$

Since  $\varphi_\varepsilon$  is convex, nonnegative and decreasing, it follows from Lemma 8.1 that  $\mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h) \geq 0$  and  $\tilde{\mathcal{Q}}_3^{(1)}(\varphi_\varepsilon, h) \leq 0$  for all  $\varepsilon > 0$ . Then by (4.3)

$$\int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau_2, x) dx \geq \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau_1, x) dx \quad \forall \tau_2 \geq \tau_1 \geq 0.$$

Letting  $\varepsilon \rightarrow 0$  it follows from (4.2) that  $m$  is nondecreasing on  $[0, \infty)$  and, for any  $\tau \geq 0$  and  $\delta > 0$ ,

$$\liminf_{\delta \rightarrow 0^+} m(\tau + \delta) \geq m(\tau). \quad (4.4)$$

Using Lebesgue's Theorem (cf. [24]),  $m$  is a.e. differentiable on  $[0, \infty)$ . On the other hand, if in (4.3) the term  $\tilde{\mathcal{Q}}_3^{(1)}(\varphi_\varepsilon, h)$  is dropped,

$$\int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau + \delta, x) dx \leq \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx + \int_\tau^{\tau + \delta} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h(\sigma)) d\sigma.$$

Using  $\mathbb{1}_{\{0\}} \leq \varphi_\varepsilon$  for all  $\varepsilon > 0$ , and the bound (8.2), we deduce

$$m(\tau + \delta) \leq \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx + \frac{2\delta}{\varepsilon} (M_0(h(\tau)))^2.$$

If we take now superior limits as  $\delta \rightarrow 0^+$  at  $\varepsilon > 0$  fixed,

$$\limsup_{\delta \rightarrow 0^+} m(\tau + \delta) \leq \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx \quad \forall \varepsilon > 0.$$

We may pass now to the limit as  $\varepsilon \rightarrow 0$  in the right hand side above and use (4.2) to get,

$$\limsup_{\delta \rightarrow 0^+} m(\tau + \delta) \leq m(\tau). \quad (4.5)$$

The right continuity then follows from (4.4) and (4.5).  $\square$

**Corollary 4.3.** *The map  $\tau \mapsto H(\tau, \{0\})$ , defined for all  $\tau \geq 0$ , is right continuous on  $[0, \infty)$  and*

$$\limsup_{\delta \rightarrow 0^+} H(\tau - \delta, \{0\}) \leq H(\tau, \{0\}) \quad \forall \tau > 0. \quad (4.6)$$

**Proof.** By construction (cf.(1.68)) it follows

$$H(\tau, \{0\}) = m(\tau) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma.$$

Since  $M_{1/2}(h) \in L^1_{loc}(\mathbb{R}_+)$  then  $\tau \mapsto \int_0^\tau M_{1/2}(h(\sigma))d\sigma$  is absolutely continuous, and since  $m$  is right continuous by Lemma 4.2, it follows that  $\tau \mapsto H(\tau, \{0\})$  is also right continuous. To prove (4.6) we use the continuity of  $\tau \mapsto \int_0^\tau M_{1/2}(h(\sigma))d\sigma$  and the monotonicity of  $h(\tau, \{0\})$ : for all  $\tau > 0$  and  $\delta \in (0, \tau)$ ,

$$\begin{aligned} \limsup_{\delta \rightarrow 0^+} H(\tau - \delta, \{0\}) &= \limsup_{\delta \rightarrow 0^+} m(\tau - \delta) - \int_0^{\tau - \delta} M_{1/2}(h(\sigma))d\sigma \\ &\leq m(\tau) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma = H(\tau, \{0\}). \end{aligned}$$

□

**Remark 4.4.** We do not know if the map  $\tau \mapsto H(\tau, \{0\})$  is continuous. By property (4.6) however,  $H(\tau, \{0\})$  does not decrease through the possible discontinuities.

The proof of Theorem 4.1 closely follows the proof of Proposition 1.21 in [14] (see also [13], Ch. 3), where the authors proved the same result for the equation without the linear term  $\tilde{\mathcal{Q}}_3^{(1)}$ . The main arguments in the proof are, on the one hand, the invariance of the problem (1.67) with respect to time translation and under a suitable scaling transformation. On the other hand, the fact that  $\Lambda(\varphi) \geq 0$  on  $\mathbb{R}_+^2$  for convex test functions  $\varphi$ , and that the map  $\tau \mapsto \mathcal{Q}_3^{(2)}(\varphi, h(\tau))$  is locally integrable on  $[0, T)$ . When the linear term  $\tilde{\mathcal{Q}}_3^{(1)}$  is added, a slight modification of these argument still leads to the proof. Since by Lemma 8.1, for all nonnegative, convex decreasing test function  $\varphi \in C_b^1([0, \infty))$ , we have  $\tilde{\mathcal{Q}}_3^{(1)}(\varphi, h) \leq 0$ , then solutions  $h$  to (1.67) are also super solutions (cf. Definition 1.13).

**Proposition 4.5.** *Let  $h$  be a super solution of (1.67). Then for any  $R > 0$  and  $\theta \in (0, 1)$*

$$\int_{[0, R]} h(\tau, x)dx \geq (1 - \theta) \int_{[0, \theta R]} h(\tau_0, x)dx \quad \forall \tau \geq \tau_0 \geq 0. \quad (4.7)$$

**Proof.** Chose  $\varphi_R(x) = (1 - x/R)_+$  for  $R > 0$ , and consider a sequence  $(\varphi_{R,n})_{n \in \mathbb{N}} \subset C_b^1([0, \infty))$  such that  $\varphi_{R,n} \rightarrow \varphi_R$ ,  $\varphi_{R,n} \leq \varphi_R$  and  $\varphi_{R,n}(0) = 1$  for all  $n \in \mathbb{N}$ . Since by convexity  $\mathcal{Q}_3^{(2)}(\varphi_{R,n}, h) \geq 0$ , then for all  $\tau$  and  $\tau_0$  with  $\tau \geq \tau_0 \geq 0$ ,

$$\begin{aligned} \int_{[0, \infty)} \varphi_{R,n}(x) h(\tau, x) dx &\geq \int_{[0, \infty)} \varphi_{R,n}(x) h(\tau_0, x) dx \\ &\geq \int_{[0, \theta R]} \varphi_{R,n}(x) h(\tau_0, x) dx \geq \varphi_{R,n}(\theta R) \int_{[0, \theta R]} h(\tau_0, x) dx, \end{aligned}$$

and (4.7) follows since, if we let  $n \rightarrow \infty$ ,

$$\int_{[0, R]} h(\tau, x) dx \geq \int_{[0, \infty)} \varphi_R(x) h(\tau, x) dx \geq \varphi_R(\theta R) \int_{[0, \theta R]} h(\tau_0, x) dx.$$

□

**Lemma 4.6.** *Let  $h$  be a super solution of (1.67). Let  $R > 0$  and consider a sequence  $R := a_0 < a_1 < a_2 < \dots < a_n < \dots$  such that  $|a_i - a_{i-1}| \leq \frac{R}{2}$  for all  $i \in \{1, 2, 3, \dots\}$ . Then for all  $\tau \geq \tau_0 \geq 0$  there holds*

$$\int_{[0, R]} h(\tau, x) dx \geq \sum_{i=1}^{\infty} \frac{1}{2a_i} \int_{\tau_0}^{\tau} \left( \int_{(a_{i-1}, a_i]} h(\sigma, x) dx \right)^2 d\sigma. \quad (4.8)$$

**Proof.** We chose  $\varphi_R$  and  $\varphi_{R,n}$  as in the proof of Proposition 4.5 above. Since  $h$  is a super solution of (1.67), then for all  $n \in \mathbb{N}$ ,

$$\frac{d}{d\tau} \int_{[0, \infty)} h(\tau, x) \varphi_{R,n}(x) dx \geq \mathcal{Q}_3^{(2)}(\varphi_{R,n}, h(\tau)).$$

We have now:

$$\begin{aligned} \mathcal{Q}_3^{(2)}(\varphi_{R,n}, h(\tau)) &\geq \iint_{(R, \infty)^2} h(\tau, x) h(\tau, y) \frac{\varphi_{R,n}(|x - y|)}{\sqrt{xy}} dx dy \\ &\geq \sum_{i=1}^{\infty} \frac{\varphi_{R,n}(R/2)}{a_i} \iint_{(a_{i-1}, a_i]^2} h(\tau, x) h(\tau, y) dx dy \\ &= \sum_{i=1}^{\infty} \frac{\varphi_{R,n}(R/2)}{a_i} \left( \int_{(a_{i-1}, a_i]} h(\tau, x) dx \right)^2. \end{aligned}$$

Estimate (4.8) follows in the limit  $n \rightarrow \infty$ , since  $\varphi_{R,n}(R/2) \rightarrow 1/2$ . □

**Proposition 4.7.** *Let  $h$  be a super solution of (1.67) with initial data  $h_0 \in \mathcal{M}_+^1([0, \infty))$ , and denote  $N = M_0(h_0)$  and  $E = M_1(h_0)$ . Then for all  $R > 0$ ,  $\alpha \in (-\frac{1}{2}, \infty)$ , and  $\tau_1$  and  $\tau_2$  with  $0 \leq \tau_1 \leq \tau_2$ :*

$$\int_{\tau_1}^{\tau_2} \int_{(0, R]} x^\alpha h(\tau, x) dx d\tau \leq \frac{2R^{\frac{1}{2}+\alpha} \sqrt{\tau_2 - \tau_1}}{1 - (\frac{2}{3})^{\frac{1}{2}+\alpha}} \left( \frac{\sqrt{E}}{2} \tau_2 + \sqrt{N} \right). \quad (4.9)$$

**Proof.** Since  $h$  is a super solution of (1.67), if we chose  $\varphi(x) = (1 - x/r)_+^2$  for  $r > 0$ , then

$$\int_{[0, \infty)} \varphi(x) h(\tau_2, x) dx \geq \int_{\tau_1}^{\tau_2} \mathcal{Q}_3^{(2)}(\varphi, h(\tau)) d\tau. \quad (4.10)$$

Since  $\text{supp } \Lambda(\varphi) = \{(x, y) \in [0, \infty)^2 : |x - y| \leq r\}$  and  $\Lambda(\varphi)(x, y) = \varphi(|x - y|)$  for all  $(x, y) \in [r, \infty)^2$ , then for all  $\tau \geq 0$ :

$$\begin{aligned} \mathcal{Q}_3^{(2)}(\varphi, h(\tau)) &\geq \iint_{(r, \frac{3r}{2}]^2} \frac{\varphi(|x - y|)}{\sqrt{xy}} h(\tau, x) h(\tau, y) dx dy \\ &\geq \frac{1}{4} \left( \int_{(r, \frac{3r}{2}]} \frac{h(\tau, x)}{\sqrt{x}} dx \right)^2. \end{aligned}$$

If we use that  $\varphi \leq 1$  in the left hand side of (4.10), and the estimate above in the right hand side, then

$$\int_{\tau_1}^{\tau_2} \left( \int_{(r, \frac{3r}{2}]} \frac{h(\tau, x)}{\sqrt{x}} dx \right)^2 d\tau \leq 4M_0(h(\tau_2)).$$

Since for any  $\alpha \in (-1/2, \infty)$

$$\int_{(r, \frac{3r}{2}]} \frac{h(\tau, x)}{\sqrt{x}} dx \geq \left( \frac{3r}{2} \right)^{-\alpha - \frac{1}{2}} \int_{(r, \frac{3r}{2}]} x^\alpha h(\tau, x) dx,$$

we then obtain

$$\int_{\tau_1}^{\tau_2} \left( \int_{(r, \frac{3r}{2}]} x^\alpha h(\tau, x) dx \right)^2 d\tau \leq 4M_0(h(\tau_2)) \left( \frac{3r}{2} \right)^{1+2\alpha}. \quad (4.11)$$

For any given  $R > 0$ , using the decomposition

$$(0, R] = \bigcup_{k=0}^{\infty} (a_{k+1}, a_k], \quad a_k = \left( \frac{2}{3} \right)^k R,$$

and Cauchy-Schwarz inequality we obtain

$$\int_{\tau_1}^{\tau_2} \int_{(0, R]} x^\alpha h(\tau, x) dx d\tau \leq \sqrt{\tau_2 - \tau_1} \sum_{k=0}^{\infty} \left( \int_{\tau_1}^{\tau_2} \left( \int_{(a_{k+1}, a_k]} x^\alpha h(\tau, x) dx \right)^2 d\tau \right)^{\frac{1}{2}}.$$

If we chose  $r = a_{k+1}$  so that  $(a_{k+1}, a_k] = (r, (3/2)r]$  for every  $k \in \mathbb{N}$ , then by (4.11) we deduce

$$\int_{\tau_1}^{\tau_2} \int_{(0, R]} x^\alpha h(\tau, x) dx d\tau \leq 2\sqrt{(\tau_2 - \tau_1)M_0(h(\tau_2))} \sum_{k=0}^{\infty} a_k^{\frac{1}{2} + \alpha}.$$

Using the estimate (3.4) for  $M_0(h(\tau_2))$  and

$$\sum_{k=0}^{\infty} a_k^{\frac{1}{2}+\alpha} = \frac{R^{\frac{1}{2}+\alpha}}{1 - \left(\frac{2}{3}\right)^{\frac{1}{2}+\alpha}},$$

we finally obtain (4.9).  $\square$

**Lemma 4.8.** *Let  $h$  be a super solution of (1.67). Then for all  $r > 0$ ,  $\tau \geq \tau_0 \geq 0$  and  $n \in \mathbb{N}$ :*

$$\int_{[0,r]} h(\tau, x) dx \geq \frac{1}{4^{n+1}r} \int_{\tau_0}^{\tau} \left( \int_{(r, r2^n]} h(\sigma, x) dx \right)^2 d\sigma. \quad (4.12)$$

**Proof.** Consider the decomposition

$$(r, 2^n r] = \bigcup_{i=3}^{2^{n+1}} \left( \frac{r}{2}(i-1), \frac{r}{2}i \right].$$

Then by Lemma 4.6, and Lemma 3.12 in [14], we have

$$\begin{aligned} \int_{[0,r]} h(\tau, x) dx &\geq \int_{\tau_0}^{\tau} \sum_{i=3}^{2^{n+1}} \frac{1}{ri} \left( \int_{(\frac{r}{2}(i-1), \frac{r}{2}i]} h(\sigma, x) dx \right)^2 d\sigma \\ &\geq \int_{\tau_0}^{\tau} \frac{1}{r} \left( \sum_{i=3}^{2^{n+1}} i \right)^{-1} \left( \int_{(r, r2^n]} h(\sigma, x) dx \right)^2 d\sigma \\ &\geq \frac{1}{(2^n - 1)(2^{n+1} + 3)r} \int_{\tau_0}^{\tau} \left( \int_{(r, r2^n]} h(\sigma, x) dx \right)^2 d\sigma. \end{aligned}$$

Notice that  $(2^n - 1)(2^{n+1} + 3) \leq 4^{n+1}$ .  $\square$

The next Lemma takes into account the linear term  $\tilde{\mathcal{Q}}_3^{(1)}$ .

**Lemma 4.9.** *Let  $h$  be a solution of (1.67) with initial data  $h_0 \in \mathcal{M}_+^1([0, \infty))$  satisfying*

$$m_0 = \int_{(0, \infty)} h_0(x) dx > 0. \quad (4.13)$$

*Then, for any  $\tau_0 \geq 0$  there exist  $R_1 > 0$ ,  $C_1 > 0$  such that*

$$\int_{[0,r]} h(\tau, x) dx \geq C_1 r \quad \forall r \in [0, R_1], \quad \forall \tau \geq \tau_0. \quad (4.14)$$

**Proof.** By (4.13), there exist  $0 < a \leq b < \infty$  such that

$$\int_{(a,b]} h_0(x) dx > \frac{m_0}{2}. \quad (4.15)$$

We prove now

$$\exists T' > 0; \forall \tau \in [0, T'] : \int_{(\frac{a}{2}, 2b]} h(\tau, x) dx \geq \frac{m_0}{4}. \quad (4.16)$$

To this end we use (1.67) with a test function  $\varphi \in C_c^1([0, \infty))$  such that  $0 \leq \varphi \leq 1$ ,  $\varphi = 1$  on  $(a, b]$  and  $\varphi = 0$  on  $[0, \infty) \setminus (\frac{a}{2}, 2b]$  and (4.15) to obtain:

$$\int_{(\frac{a}{2}, 2b]} h(\tau, x) dx \geq \frac{m_0}{2} + \int_0^\tau \tilde{\mathcal{Q}}_3(\varphi, h(\sigma)) d\sigma. \quad (4.17)$$

Now using (8.2) and (3.4) we deduce

$$\left| \mathcal{Q}_3^{(2)}(\varphi, h(\sigma)) \right| \leq 2\|\varphi'\|_\infty \left( \frac{\sqrt{M_1(h_0)}}{2} \sigma + \sqrt{M_0(h_0)} \right)^4.$$

Using now  $\frac{|\mathcal{L}(\varphi)(x)|}{\sqrt{x}} \leq 3\|\varphi\|_\infty \sqrt{x}$  and  $M_{1/2}(h) \leq \sqrt{M_0(h)M_1(h)}$ , we have by the conservation of energy and the mass inequality

$$\left| \tilde{\mathcal{Q}}_3^{(1)}(\varphi, h(\sigma)) \right| \leq 2\|\varphi\|_\infty \sqrt{M_1(h_0)} \left( \frac{M_1(h_0)}{2} \sigma + \sqrt{M_0(h_0)} \right).$$

It follows that  $\tilde{\mathcal{Q}}_3(\varphi, h) \in L_{loc}^1(\mathbb{R}_+)$  and we deduce (4.16) from (4.17).

By Lemma 4.8 and (4.16), for any  $r \in (0, \frac{a}{2}]$  and  $n \in \mathbb{N}$  such that  $r2^n \in (2b, 3b]$  we have

$$\begin{aligned} \int_{[0, r]} h(\tau, x) dx &\geq \int_0^\tau \frac{1}{4^{n+1}r} \left( \int_{(\frac{a}{2}, 2b]} h(\sigma, x) dx \right)^2 d\sigma \\ &\geq \frac{\tau}{4^{n+1}r} \left( \frac{m_0}{4} \right)^2 \geq \frac{m_0^2}{4^3(3b)^2} \tau r \quad \forall \tau \in [0, T']. \end{aligned} \quad (4.18)$$

where  $(\frac{a}{2}, 2b] \subset (r, r2^n]$  has been used.

For any given  $\tau_0 \geq 0$  define  $\tau' = \min\{\tau_0, T'\}$ . Then by (4.7) in Proposition 4.5 with  $\theta = \frac{1}{2}$  and  $R = 2r$ , we deduce from (4.18):

$$\int_{[0, 2r]} h(\tau, x) dx \geq \frac{C\tau'}{2} r \quad \forall \tau \geq \tau'. \quad (4.19)$$

and this proves the Lemma, where  $R_1 = a/2$  and  $C_1 = C\tau'/4$ .  $\square$

**Proposition 4.10.** *Let  $h$  and  $h_0$  be as in Lemma 4.9. For all  $L > 0$  and every  $\tau_1 > 0$  there exists  $R_0 = R_0(h, L, \tau_1) > 0$  such that*

$$\int_{[0, R_0]} h(\tau, x) dx \geq LR_0 \quad \forall \tau \geq \tau_1. \quad (4.20)$$

**Proof.** By Lemma 4.9 for  $\tau_0 = \frac{\tau_1}{2}$

$$\exists C_1 > 0, \exists R_1 > 0; \int_{[0,r]} h(\tau, x) dx \geq C_1 r, \quad \forall r \in [0, R_1], \quad \forall \tau \geq \frac{\tau_1}{2}. \quad (4.21)$$

Now fix an integer  $p \geq 2$  such that  $C_1 p \geq 8L$ . We divide the proof in two parts. Assume first :

$$\exists r' \in (0, R_1], \exists \tau' \in \left[\frac{\tau_1}{2}, \tau_1\right] : \int_{[0, \frac{r'}{p}]} h(\tau', x) dx \geq \frac{C_1 r'}{2}. \quad (4.22)$$

It follows from lemma 4.5 with  $\theta = \frac{1}{2}$  and  $R = \frac{2r'}{p}$  that

$$\int_{[0, \frac{2r'}{p}]} h(\tau, x) dx \geq \frac{C_1 r'}{4} \quad \forall \tau \geq \tau',$$

If we take  $R_0 := \frac{2r'}{p}$ , we have, by our choice of  $p$ ,

$$\int_{[0, R_0]} h(\tau, x) dx \geq \frac{C_1 p}{8} R_0 \geq L R_0 \quad \forall \tau \geq \tau',$$

so (4.20) holds.

Assume now that (4.22) does not hold, then, by (4.21):

$$\int_{\left(\frac{r}{p}, r\right]} h(\tau, x) dx \geq \frac{C_1 r}{2} \quad \forall r \in (0, R_1], \quad \forall \tau \in \left[\frac{\tau_1}{2}, \tau_1\right]. \quad (4.23)$$

Take now any  $r \in \left(0, \frac{R_1}{p}\right]$ , let  $n \in \mathbb{N}$  be the largest integer such that  $rp^n \in \left(\frac{R_1}{p}, R_1\right]$ , and consider now the following decomposition

$$(r, rp^n] = \bigcup_{i=p+1}^{p^{n+1}} \left(\frac{r}{p}(i-1), \frac{r}{p}i\right] = \bigcup_{k=1}^n \bigcup_{i=p^k+1}^{p^{k+1}} \left(\frac{r}{p}(i-1), \frac{r}{p}i\right].$$

By lemma 4.6 on  $(\tau_1/2, \tau_1)$  with  $a_i = ri/p$ ,  $i = p+1, \dots, p^{n+1}$ :

$$\begin{aligned} \int_{[0,r]} h(\tau_1, x) dx &\geq \int_{\frac{\tau_1}{2}}^{\tau_1} \left[ \frac{p}{2r} \sum_{i=p+1}^{p^{n+1}} \frac{1}{i} \left( \int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x) dx \right)^2 \right] d\sigma \\ &= \int_{\frac{\tau_1}{2}}^{\tau_1} \left[ \frac{p}{2r} \sum_{k=1}^n \sum_{i=p^k+1}^{p^{k+1}} \frac{1}{i} \left( \int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x) dx \right)^2 \right] d\sigma \\ &\geq \int_{\frac{\tau_1}{2}}^{\tau_1} \left[ \frac{1}{2r} \sum_{k=1}^n \frac{1}{p^k} \sum_{i=p^k+1}^{p^{k+1}} \left( \int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x) dx \right)^2 \right] d\sigma. \end{aligned} \quad (4.24)$$

We use now Lemma 3.12 in [14]

$$\begin{aligned} \sum_{i=p^k+1}^{p^{k+1}} \left( \int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x) dx \right)^2 &\geq \frac{1}{p^k(p-1)} \times \\ \times \left( \sum_{i=p^k+1}^{p^{k+1}} \int_{\left(\frac{r}{p}(i-1), \frac{r}{p}i\right]} h(\sigma, x) dx \right)^2 &\geq \frac{1}{p^{k+1}} \left( \int_{(rp^{k-1}, rp^k]} h(\sigma, x) dx \right)^2 \end{aligned}$$

and deduce

$$\int_{[0, r]} h(\tau_1, x) dx \geq \int_{\frac{\tau_1}{2}}^{\tau_1} \left[ \frac{1}{2r} \sum_{k=1}^n \frac{1}{p^{2k+1}} \left( \int_{(rp^{k-1}, rp^k]} h(\sigma, x) dx \right)^2 \right] d\sigma.$$

Due to the choice of the integer  $n$ ,  $rp^k \in (0, R_1]$  for all  $k = 1, \dots, n$ , and we can use (4.23) on each interval  $(rp^{k-1}, rp^k]$  to obtain:

$$\int_{[0, r]} h(\tau_1, x) dx \geq \int_{\frac{\tau_1}{2}}^{\tau_1} \left[ \frac{1}{2r} \sum_{k=1}^n \frac{1}{p^{2k+1}} \left( \frac{C_1 rp^k}{2} \right)^2 \right] d\sigma = \frac{\tau_1 C_1^2 n}{16p} r.$$

It then follows from lemma 4.5 with  $\theta = \frac{1}{2}$  and  $R = 2r$  that

$$\int_{[0, 2r]} h(\tau, x) dx \geq \frac{\tau_1 C_1^2 n}{32p} r \quad \forall \tau \geq \tau_1. \quad (4.25)$$

Since  $rp^n \in \left(\frac{R_1}{p}, R_1\right]$ , then  $n \geq \frac{\log\left(\frac{R_1}{rp}\right)}{\log(p)}$ , and we chose  $r > 0$  small enough in order to have  $r \in (0, R_1/p)$  and

$$\frac{\tau_1 C_1^2}{64p} \frac{\log\left(\frac{R_1}{rp}\right)}{\log p} \geq L;$$

and set  $R_0 := 2r$ . The result then follows from (4.25).  $\square$

**Lemma 4.11.** *Let  $h$  be a solution of (1.67) and, for any  $\kappa > 0$  and  $\lambda > 0$ , consider the rescaled measure  $h_{\kappa, \lambda}$  defined as:*

$$\int_{[0, \infty)} h_{\kappa, \lambda}(\tau, x) \varphi(x) dx = \kappa \int_{[0, \infty)} h(\kappa \lambda \tau, x) \varphi\left(\frac{x}{\lambda}\right) dx, \quad \forall \varphi \in C_b([0, \infty)). \quad (4.26)$$

*Then  $h_{\kappa, \lambda}$  is a super solution of (1.67).*

**Proof.** Let  $\varphi \in C_b^1([0, \infty))$  be nonnegative, convex and decreasing,  $\psi(x) = \varphi(x/\lambda)$ , and  $\eta = \kappa \lambda \tau$ . By Lemma 8.1,  $\tilde{\mathcal{Q}}_3^{(1)}(\psi, h) \leq 0$ , and by (1.67)

$$\frac{d}{d\eta} \int_{[0, \infty)} \psi(x) h(\eta, x) dx \geq \mathcal{Q}_3^{(2)}(\psi, h(\eta)).$$



Since  $\mathcal{Q}_3^{(2)}(\psi, h(\eta)) = \kappa^{-2} \lambda^{-1} \mathcal{Q}_3^{(2)}(\varphi, h_{\kappa, \lambda}(\tau))$ , then

$$\frac{d}{d\tau} \int_{[0, \infty)} \varphi(x) h_{\kappa, \lambda}(\tau, x) dx = \kappa^2 \lambda \frac{d}{d\eta} \int_{[0, \infty)} \psi(x) h(\eta, x) dx \geq \mathcal{Q}_3^{(2)}(\varphi, h_{\kappa, \lambda}(\tau)).$$

□

**Lemma 4.12.** *Let  $h$  be a super solution of (1.67). Suppose that there exists  $\tau' > 0$  such that*

$$\int_{[0, 1]} h(\tau, x) dx \geq 1 \quad \forall \tau \geq \tau'. \quad (4.27)$$

*Then for any given  $\delta > 0$  there exist  $\tau_0$  such that*

$$\tau' \leq \tau_0 \leq \tau' + T_0(\delta), \quad T_0(\delta) = \frac{64}{\delta^3} \left(1 - \frac{\delta}{2}\right) \quad (4.28)$$

$$\text{and } \int_{[0, \frac{\delta}{4}]} h(\tau_0, x) dx \geq 1 - \frac{\delta}{2}. \quad (4.29)$$

**Proof.** The statement of the Lemma is equivalent to show that the following set

$$A := \left\{ \tau \in [\tau', \tau' + T_0(\delta)] : \int_{[0, \frac{\delta}{4}]} h(\tau, x) dx \geq 1 - \frac{\delta}{2} \right\}.$$

is non empty, where  $T_0(\delta)$  is defined in (4.28). To this end we first apply Lemma 4.6 with  $a_0 = \frac{\delta}{4}$ ,  $a_i = \frac{\delta}{4} \left(1 + \frac{i}{2}\right)$  for  $i \in \{1, \dots, n-1\}$  and  $a_n = 1$ . The number  $n$  is chosen to be the largest integer such that  $a_{n-1} < 1$ , which implies

$$\frac{1}{n+1} > \frac{\delta}{8}. \quad (4.30)$$

Then, using  $a_i^{-1} \geq 1$  for all  $i \in \{1, \dots, n\}$ :

$$\int_{[0, \frac{\delta}{4}]} h(\tau, x) dx \geq \frac{1}{2} \int_{\tau'}^{\tau} \sum_{i=1}^n \left( \int_{(a_{i-1}, a_i]} h(\sigma, x) dx \right)^2 d\sigma, \quad \forall \tau > \tau'.$$

Since by Lemma 3.12 in [14] and (4.30):

$$\sum_{i=1}^n \left( \int_{(a_{i-1}, a_i]} h(\sigma, x) dx \right)^2 \geq \frac{\delta}{8} \left( \int_{(\frac{\delta}{4}, 1]} h(\sigma, x) dx \right)^2,$$

we obtain, for all  $\tau > \tau'$

$$\int_{[0, \frac{\delta}{4}]} h(\tau, x) dx \geq \frac{\delta}{16} \int_{\tau'}^{\tau} \left( \int_{(\frac{\delta}{4}, 1]} h(\sigma, x) dx \right)^2 d\sigma. \quad (4.31)$$

Arguing by contradiction suppose that  $A = \emptyset$ :

$$\int_{(0, \frac{\delta}{4}]} h(\tau, x) dx < 1 - \frac{\delta}{2} \quad \forall \tau \in [\tau', \tau' + T_0(\delta)]$$

and by (4.27):

$$\int_{(\frac{\delta}{4}, 1]} h(\tau, x) dx \geq \frac{\delta}{2} \quad \forall \tau \in [\tau', \tau' + T_0(\delta)].$$

It follows from (4.31) that  $1 - \frac{\delta}{2} > \frac{\delta^3}{64}(\tau - \tau')$  for all  $\tau \in [\tau', \tau' + T_0(\delta)]$  which is a contradiction for  $\tau = \tau' + T_0(\delta)$ .  $\square$

**Proposition 4.13.** *Let  $h$  be a solution of (1.67). Suppose that there exist  $m, R > 0$  such that*

$$\int_{[0, R]} h(\tau, x) dx \geq m \quad \forall \tau \in [0, \infty). \quad (4.32)$$

*Then given any  $\alpha \in (0, 1)$  there exists  $T_* = T_*(\alpha) > 0$  such that*

$$\int_{[0, r]} h(\tau, x) dx \geq \frac{m}{(2R)^\alpha} r^\alpha \quad \forall r \in [0, R], \quad \forall \tau \in \left[ \frac{RT_*}{m}, \infty \right). \quad (4.33)$$

**Proof.** We argue by induction and define first the scaled measure  $h_1 = h_{\kappa_1, \lambda_1}$ , defined as in (4.26), that satisfies condition (4.27) for  $\kappa_1 = \frac{1}{m}$ ,  $\lambda_1 = R$ . From Lemma 4.11, and Lemma 4.12 with  $\tau' = 0$ , we deduce that for all  $\delta \in (0, 1)$  there exists  $\tau_1 > 0$  such that:

$$0 \leq \tau_1 \leq T_0(\delta), \quad \int_{[0, \frac{\delta}{4}]} h_1(\tau_1, x) dx \geq 1 - \frac{\delta}{2}.$$

Then from Lemma 4.11, and Proposition 4.5 with  $\theta = \delta/2$  and  $R = 1/2$ ,

$$\begin{aligned} \int_{[0, \frac{1}{2}]} h_1(\tau, x) dx &\geq \left(1 - \frac{\delta}{2}\right)^2, \quad \forall \tau \geq T_0(\delta), \\ \int_{[0, \frac{R}{2}]} h(\tau, x) dx &\geq m(1 - \delta), \quad \forall \tau \geq \frac{R}{m} T_0(\delta). \end{aligned} \quad (4.34)$$

Exactly as before we now define  $h_2 = h_{\kappa_2, \lambda_2}$  as in (4.26), that satisfies condition (4.27) for  $\kappa_2 = \frac{1}{m(1-\delta)^2}$ ,  $\lambda_2 = \frac{R}{2}$ ,  $\tau' = 2(1 - \delta)T_0(\delta)$ . The same argument gives then:

$$\int_{[0, \frac{R}{4}]} h(\tau, x) dx \geq m(1 - \delta)^2, \quad \forall \tau \geq \frac{RT_0(\delta)}{m} \left(1 + \frac{1}{2(1 - \delta)}\right). \quad (4.35)$$

We deduce after  $n$  iterations

$$\int_{[0, \frac{R}{2^n}]} h(\tau, x) dx \geq m(1 - \delta)^n, \quad \forall \tau \geq \frac{RT_0(\delta)}{m} \sum_{k=0}^{n-1} \frac{1}{2^k(1 - \delta)^k} \quad (4.36)$$

If we chose  $\delta = 1 - 2^{-\alpha}$ , for any  $0 < \alpha < 1$ , we may define

$$T_* = T_0(\delta) \sum_{k=0}^{\infty} 2^{-(1-\alpha)k} = \frac{T_0(\delta)}{1 - 2^{-(1-\alpha)}}. \quad (4.37)$$

Since for any  $r \in (0, R)$  there exists  $n \in \mathbb{N}$  such that  $r \in (\frac{R}{2^n}, \frac{R}{2^{n-1}}]$ ,

$$\int_{[0, r]} h(\tau, x) dx \geq m2^{-\alpha n}, \quad \forall \tau > \frac{RT_*}{m}$$

and using  $2^{-n} > r/2R$ , (4.33) follows.  $\square$

**Proposition 4.14.** *Let  $h$  be a solution of (1.67). Then, for all  $\tau_0 > 0$  and for any  $\alpha \in (0, 1)$  there exists  $R_* = R_*(h, \tau_0, \alpha) > 0$  such that*

$$\int_{[0, r]} h(\tau, x) dx \geq C r^\alpha \quad \forall r \in [0, R_*] \quad \forall \tau \in [\tau_0, \infty), \quad (4.38)$$

where  $C = \frac{T_*(\alpha)}{\tau_0} (2R_*)^{1-\alpha}$ , and  $T_*(\alpha)$  is given by Proposition 4.13.

**Proof.** By Proposition 4.10 with  $L > 0$  and for  $\tau_1 = \tau_0/2$ , there exists  $R_0(h, L, \tau_1) > 0$  such that

$$\int_{[0, R_0]} h(\tau, x) dx \geq LR_0 \quad \forall \tau \geq \frac{\tau_0}{2}.$$

Then by Proposition 4.13, with  $m = LR_0$  and  $R = R_0$ , we obtain that for any given  $\alpha \in (0, 1)$  there exists  $T_* = T_*(\alpha) > 0$  such that

$$\int_{[0, r]} h(\tau, x) dx \geq \frac{LR_0}{(2R_0)^\alpha} r^\alpha \quad \forall r \in [0, R_0], \quad \forall \tau \in \left[ \frac{\tau_0}{2} + \frac{T_*}{L}, \infty \right).$$

If we chose  $L = 2T_*/\tau_0$ , then the Proposition follows with  $R_* = R_0$ .  $\square$

**Proof of Theorem 4.1.** By Lemma 4.2 the map  $\tau \mapsto h(\tau, \{0\})$  is right continuous, nondecreasing and a.e. differentiable on  $[0, \infty)$ . It remains to prove that it is actually strictly increasing. We first suppose that  $h_0$  is such that

$$\int_{\{0\}} h_0(x) dx = 0, \quad \int_{(0, \infty)} h_0(x) dx > 0, \quad (4.39)$$

and prove

$$h(\tau, \{0\}) > 0 \quad \forall \tau > 0. \quad (4.40)$$

Arguing by contradiction, if we suppose that there exists  $\tau_0 > 0$  such that  $h(\tau_0, \{0\}) = 0$ , by monotonicity  $h(\tau, \{0\}) = 0$  for all  $\tau \in [0, \tau_0]$ . In particular

$$\int_{\frac{\tau_0}{2}}^{\tau_0} \int_{[0,r]} h(\sigma, x) dx d\sigma = \int_{\frac{\tau_0}{2}}^{\tau_0} \int_{(0,r]} h(\sigma, x) dx d\sigma \quad (4.41)$$

for all  $r > 0$ . Now using Proposition 4.7 with  $\alpha = 0$ , and Proposition 4.14, we deduce that, for any  $\alpha \in (0, 1/2)$ , there exists  $R_* = R_*(h, \tau_0/2, \alpha)$  such that

$$C_2 r^\alpha \leq \int_{\frac{\tau_0}{2}}^{\tau_0} \int_{(0,r]} h(\sigma, x) dx d\sigma \leq C_1 \sqrt{r}, \quad \forall r \in [0, R_*];$$

$$C_1 = 8 \sqrt{\frac{\tau_0}{2}} \left( \frac{\sqrt{M_1(h_0)}}{2} \tau_0 + \sqrt{M_0(h_0)} \right), \quad C_2 = \frac{T_*(\alpha)}{2} (2R_*)^{1-\alpha},$$

and that leads to a contradiction for  $r$  small enough.

Consider now a general initial data  $h_0$  such that  $\int_{\{0\}} h_0(x) dx > 0$ . Let  $h$  be a solution of (1.67) with initial data  $h_0$  and define

$$\tilde{h}(\tau) = h(\tau) - h_0(\{0\})\delta_0.$$

Then, on the one hand, the initial data of  $\tilde{h}$  satisfies  $\tilde{h}(0, \{0\}) = 0$ . On the other hand we claim that  $\tilde{h}$  is still a solution of (1.67). Notice indeed that  $\tilde{h}_\tau \equiv h_\tau$  and, moreover,  $\tilde{\mathcal{Q}}_3(\varphi, h(\tau)) = \tilde{\mathcal{Q}}_3(\varphi, \tilde{h}(\tau))$ . Using the previous case

$$\int_{\{0\}} \tilde{h}(\tau, x) dx > 0, \quad \forall \tau > 0,$$

and then

$$\int_{\{0\}} h(\tau, x) dx > \int_{\{0\}} h_0(x) dx, \quad \forall \tau > 0.$$

The Theorem follows using now the time translation invariance of the equation.  $\square$

The last result of this section describes the relation between the Lebesgue-Stieltjes measure associated to the (right continuous and strictly increasing) function  $m(\tau) = h(\tau, \{0\})$ , and the equation for  $h$  (1.67).

**Proposition 4.15.** *Let  $h$  be a solution of (1.67) for a initial data  $h_0 \in \mathcal{M}_+^1([0, \infty))$  with  $N = M_0(h_0) > 0$  and  $E = M_1(h_0) > 0$ . If we denote  $m(\tau) = h(\tau, \{0\})$  and  $\lambda$  is the Lebesgue-Stieltjes measure associated to  $m$ , then for all  $\varphi_\varepsilon$  as in Remark 1.6 and for all  $\tau_1$  and  $\tau_2$  with  $0 \leq \tau_1 < \tau_2$ :*

$$m(\tau_2) - m(\tau_1) = \lambda((\tau_1, \tau_2]), \quad (4.42)$$

$$\lambda((\tau_1, \tau_2]) = \lim_{\varepsilon \rightarrow 0} \int_{\tau_1}^{\tau_2} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h(\tau)) d\tau, \quad (4.43)$$

and 
$$0 < \lambda((\tau_1, \tau_2]) < \infty. \quad (4.44)$$

Furthermore, for all  $\varphi_\varepsilon$  as in Remark 1.6

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h) \in \mathcal{D}'(0, \infty), \quad (4.45)$$

and if we denote  $m'$  the derivative in the sense of Distributions of  $m$ , then

$$m' = \lambda = \lim_{\varepsilon \rightarrow 0} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h) \quad \text{in } \mathcal{D}'(0, \infty). \quad (4.46)$$

**Proof.** By Lemma 4.2,  $m$  is right continuous and nondecreasing on  $[0, \infty)$ . Then it has a Lebesgue-Stieltjes measure associated to it,  $\lambda$ , that satisfies (4.42) (c.f. [9] Ch.1).

On the other hand, since  $h$  is a solution of (1.67), using  $\varphi_\varepsilon$  as in Remark 1.6 and taking the limit  $\varepsilon \rightarrow 0$ , it follows from (8.25) in Lemma 8.8 that for all  $\tau_1$  and  $\tau_2$  with  $0 \leq \tau_1 < \tau_2$ :

$$m(\tau_2) - m(\tau_1) = \lim_{\varepsilon \rightarrow 0} \int_{\tau_1}^{\tau_2} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h(\sigma)) d\sigma, \quad (4.47)$$

and then (4.43) follows from (4.42). Moreover, since by Theorem 4.1  $m$  is strictly increasing, then (4.44) holds.

Notice that the limit in (4.47) is independent of the choice of the test function  $\varphi_\varepsilon$ . Indeed, if  $\psi_\varepsilon$  is another test function as in Remark 1.6, since for all  $\tau \geq 0$

$$\lim_{\varepsilon \rightarrow 0} \int_{[0, \infty)} \psi_\varepsilon(x) h(\tau, x) dx = m(\tau) = \lim_{\varepsilon \rightarrow 0} \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx,$$

it follows from (4.47) that for all  $0 \leq \tau_1 \leq \tau_2$

$$\lim_{\varepsilon \rightarrow 0} \int_{\tau_1}^{\tau_2} \mathcal{Q}_3^{(2)}(\psi_\varepsilon, h(\sigma)) d\sigma = \lim_{\varepsilon \rightarrow 0} \int_{\tau_1}^{\tau_2} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h(\sigma)) d\sigma.$$

Now, for all  $\varphi_\varepsilon$  as in Remark 1.6, consider the absolutely continuous function

$$\theta_\varepsilon(\tau) = \int_{[0, \infty)} \varphi_\varepsilon(x) h(\tau, x) dx.$$

Then the equation in (3.2) reads  $\theta'_\varepsilon(\tau) = \tilde{\mathcal{Q}}_3(\varphi_\varepsilon, h(\tau))$ . Using integration by parts we deduce that for all  $\varepsilon > 0$ :

$$-\int_0^\infty \phi'(\tau) \theta_\varepsilon(\tau) d\tau = \int_0^\infty \phi(\tau) \tilde{\mathcal{Q}}_3(\varphi_\varepsilon, h(\tau)) d\tau \quad \forall \phi \in C_c^\infty(0, \infty).$$

Taking the limit  $\varepsilon \rightarrow 0$  it then follows from Lemma 8.8 that

$$-\int_0^\infty \phi'(\tau) m(\tau) d\tau = \lim_{\varepsilon \rightarrow 0} \int_0^\infty \phi(\tau) \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h(\tau)) d\tau,$$

hence,  $m' = \lim_{\varepsilon \rightarrow 0} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, h)$ . On the other hand, by Fubini's theorem

$$\int_0^\infty \phi(\tau) d\lambda(\tau) = \int_0^\infty \int_0^\tau \phi'(\sigma) d\sigma d\lambda(\tau) = - \int_0^\infty \phi'(\sigma) m(\sigma) d\sigma$$

for all  $\phi \in C_c^\infty(0, \infty)$  (cf. [25], Example 6.14), thus  $m' = \lambda$ .  $\square$

## 5 Existence of solutions $G$ , proof of Theorem 1.3.

Given a initial data  $G_0 \in \mathcal{M}_+^1$  as in Theorem 1.3, let  $h \in C([0, \infty), \mathcal{M}_+([0, \infty)))$  satisfy (3.1)–(3.5), (3.7), given by (3.2) and  $H$  defined by (3.10) and satisfying (3.11)–(3.15), (3.17) by Corollary 3.2. It is natural, in view of the change of variables (1.65) to define now,

$$G(t) = H(\tau), \quad \tau = \int_0^t G(s, \{0\}) ds. \quad (5.1)$$

Notice nevertheless that since  $G(s, \{0\})$  is still unknown, (5.1) does not define  $G(t)$  actually. What we know is rather, given  $\tau > 0$ , what would be the value of  $t$  such that

$$t = \int_0^\tau \frac{d\sigma}{H(\sigma, \{0\})}, \quad (5.2)$$

since we expect to have  $G(s, \{0\}) = H(\sigma, \{0\})$  for  $s$  and  $\sigma$  such that

$$\sigma = \int_0^s G(r, \{0\}) dr, \quad \text{or} \quad s = \int_0^\sigma \frac{d\rho}{H(\rho, \{0\})}.$$

If  $G$  is going to be defined in that way it is then necessary first to check that the range of values taken by the variable  $t$  in (5.2) is all of  $[0, \infty)$ . By definition (3.10),

$$H(\tau, \{0\}) = h(\tau, \{0\}) - \int_0^\tau M_{1/2}(h(\sigma)) d\sigma. \quad (5.3)$$

Since both terms in the right hand side are nonnegative,  $H(\tau, \{0\})$  has no a priori definite sign. We must then consider that question in some detail. Our first step is to prove the following

**Lemma 5.1.** *If  $G_0(\{0\}) > 0$ , then*

$$\tau_* = \inf\{\tau > 0 : H(\tau, \{0\}) = 0\} > 0, \quad (5.4)$$

$$H(\tau_*, \{0\}) = 0, \quad (5.5)$$

$$H(\tau, \{0\}) > 0 \quad \forall \tau \in [0, \tau_*). \quad (5.6)$$

**Proof.**  $H(0) = G_0$  by (3.13), and then, using  $\varphi_\varepsilon$  as in Remark 1.6, we deduce  $H(0, \{0\}) = G_0(\{0\})$ , which is strictly positive by hypothesis. Then (5.4) follows from the right continuity of  $H(\tau, \{0\})$  (cf. Corollary 4.3).

In order to prove (5.5) we use a minimizing sequence  $(\tau_n)_{n \in \mathbb{N}}$ , i.e.,  $\tau_n \geq \tau_*$ ,  $H(\tau_n, \{0\}) = 0$  for every  $n \in \mathbb{N}$ , and  $\tau_n \rightarrow \tau_*$  as  $n \rightarrow \infty$ . Then from the right continuity (5.5) holds.

Let us prove now (5.6). If  $H(\tau_0, \{0\}) < 0$  for some  $\tau_0 \in (0, \tau_*)$ , then  $\tau_0$  must be a left discontinuity point of  $H(\tau, \{0\})$  and

$$\limsup_{\delta \rightarrow 0^+} H(\tau_0 - \delta, \{0\}) > H(\tau_0, \{0\}),$$

and this would contradict (4.6). That proves (5.6).  $\square$

It follows from Lemma 5.1 that the function:

$$t = \xi(\tau) = \int_0^\tau \frac{d\sigma}{H(\sigma, \{0\})} \quad (5.7)$$

introduced in (5.2) is well defined, monotone nondecreasing and continuous on the interval  $[0, \tau_*)$ . We then define,

$$\forall t \in [0, \xi(\tau_*)) : \quad G(t) = H(\xi^{-1}(t)). \quad (5.8)$$

By (5.8) and (5.3), if  $G(t) = G(t, \{0\})\delta_0 + g(t)$  and  $H(\tau) = H(\tau, \{0\})\delta_0 + \tilde{h}(\tau)$ , then

$$G(t, \{0\}) = H(\tau, \{0\}), \quad (5.9)$$

$$\tilde{h}(\tau) = h(\tau) - h(\tau, \{0\})\delta_0, \quad (5.10)$$

$$g(t) = \tilde{h}(\tau). \quad (5.11)$$

**Remark 5.2.** Formula (5.8) defines the function  $G$  at time  $t \in (0, \xi(\tau_*))$  from the knowledge of the function  $H(\tau)$  for  $\tau > 0$  such that  $\tau = \xi^{-1}(t)$ . Moreover,

$$\forall t \in (0, \xi(\tau_*)) : \quad \xi^{-1}(t) = \int_0^t G(s, \{0\})ds. \quad (5.12)$$

We have now,

**Proposition 5.3.** *The function  $G$  defined by (5.8) is such that*

$$G \in C([0, \xi(\tau_*)), \mathcal{M}_+^1([0, \infty))), \quad G(0) = G_0 \quad (5.13)$$

*and satisfies (1.45), (1.46), (1.48) and (1.49) on the time interval  $[0, \xi(\tau_*))$ .*

**Proof.** We first prove that  $G(t)$  is a positive measure for all  $t \in [0, \xi(\tau_*))$ . By (5.6) and (5.9) we have  $G(t, \{0\}) > 0$  for all  $t \in [0, \xi(\tau_*))$ . Then, since  $h(\tau)$  is a positive measure for all  $\tau \in [0, \infty)$ , we deduce from (5.11) and

(5.10) that  $g(t)$  is a positive measure for all  $t \in [0, \xi(\tau_*)]$ . Hence  $G(t) = G(t, \{0\})\delta_0 + g(t)$  is also positive.

All the properties of  $G(t)$  at  $t \in [0, \xi(\tau_*)]$  fixed follow from the corresponding property of  $H(\tau)$  with  $t = \xi(\tau)$ . The only property where  $t$  is not fixed are (1.44) and (1.45). Since

$$\left| \frac{\partial G(t)}{\partial t} \right| = \left| \frac{\partial \tau}{\partial t} \frac{\partial H(\tau)}{\partial \tau} \right| \leq |H(\tau, \{0\})| \left| \frac{\partial H(\tau)}{\partial \tau} \right|$$

By definition,

$$|H(\tau, \{0\})| \leq |h(\tau, \{0\})| + \int_0^\tau M_{1/2}(h(\sigma))d\sigma.$$

Since  $h \in C([0, \infty), \mathcal{M}_+^1)$  it follows using also (3.4), (3.5) and Hölder inequality that  $H(\tau, \{0\}) \in L_{loc}^\infty([0, \infty))$ . Then, by (3.1),  $G(t)$  is locally Lipschitz on  $[0, \xi(\tau_*)]$  and satisfies (1.45). Since  $H$  satisfies (3.2) the change of variables ensures that  $G$  satisfies (1.46).  $\square$

We prove now the following property of the function  $G$  defined in (5.8).

**Proposition 5.4.** *Let  $G$  be the function defined in (5.8) for  $t \in (0, \xi(\tau_*))$ . Then the map  $t \mapsto G(t, \{0\})$  is right continuous and differentiable for almost every  $t \in [0, \xi(\tau_*)]$  and, for all  $t_0 \in (0, \xi(\tau_*))$*

$$G(t, \{0\}) \geq G(t_0, \{0\})e^{-\int_{t_0}^t M_{1/2}(g(s))ds} \quad \forall t \in (t_0, \xi(\tau_*)). \quad (5.14)$$

*In particular, if  $G(0, \{0\}) > 0$ , then  $G(t, \{0\}) > 0$  for all  $t \in (0, \xi(\tau_*))$ .*

**Proof.** Using (1.46) and (1.43) with  $\varphi_\varepsilon$  as in Remark 1.6, we have

$$\frac{d}{dt} \int_{[0, \infty)} \varphi_\varepsilon(x) G(t, x) dx + G(t, \{0\}) M_{1/2}(G(t)) = G(t, \{0\}) \tilde{\mathcal{Q}}_3(\varphi_\varepsilon, G(t)). \quad (5.15)$$

We use now that for all  $\varepsilon > 0$ :

$$G(t, \{0\}) \leq \int_{[0, \infty)} \varphi_\varepsilon(x) G(t, x) dx, \quad (5.16)$$

and we deduce from (5.15), using  $J(t) = \exp\left(\int_0^t M_{1/2}(G(s))ds\right)$ ,

$$\frac{d}{dt} \left( J(t) \int_{[0, \infty)} \varphi_\varepsilon(x) G(t, x) dx \right) \geq G(t, \{0\}) J(t) \tilde{\mathcal{Q}}_3(\varphi_\varepsilon, G(t)). \quad (5.17)$$

By Lemma 8.1 the right hand side of (5.17) is nonnegative, and we deduce

$$J(t) \int_{[0, \infty)} \varphi_\varepsilon(x) G(t, x) dx \geq J(t_0) \int_{[0, \infty)} \varphi_\varepsilon(x) G(t_0, x) dx$$



for all  $t \in (t_0, \xi(\tau_*))$  and all  $\varepsilon > 0$ . If we pass now to the limit as  $\varepsilon \rightarrow 0$ :

$$J(t)G(t, \{0\}) \geq J(t_0)G(t_0, \{0\}), \quad (5.18)$$

and this proves the estimate (5.14). It also follows from Lebesgue's Theorem that  $J(t)G(t, \{0\})$  is differentiable for almost every  $t \in (0, \xi(\tau_*))$  (cf. [24], Theorem 2). On the other hand, since  $J(t)$  is a.e differentiable and  $J(t) > 0$  for all  $t \in [0, \xi(\tau_*))$ , we deduce that  $G(t, \{0\})$  is also differentiable for almost every  $t \in [0, \xi(\tau_*))$ .

We prove now the right continuity of  $G(t, \{0\})$ . It follows from (5.18),

$$J(t + \delta)G(t + \delta, \{0\}) \geq J(t)G(t, \{0\}), \quad \forall \delta > 0 \quad \forall t > 0.$$

If we take inferior limits and use that  $J$  is continuous and strictly positive we obtain,

$$\liminf_{\delta \rightarrow 0} G(t + \delta, \{0\}) \geq G(t, \{0\}), \quad \forall t > 0. \quad (5.19)$$

Since  $\mathcal{L}_0(\varphi_\varepsilon) \geq 0$  by convexity (cf. Lemma 8.1), we deduce

$$\frac{d}{dt} \int_{[0, \infty)} \varphi_\varepsilon(x) G(t, x) dx \leq G(t, \{0\}) \iint_{(0, \infty)^2} \frac{\Lambda(\varphi_\varepsilon)(x, y)}{\sqrt{xy}} G(t, x) G(t, y) dx dy,$$

and the argument follows now as in the proof of the right continuity of  $H$ . From the inequality (5.16), the bound (8.2) and the conservation of mass, we deduce for all  $t \in [0, \xi(\tau_*))$  fixed and  $\delta \in [0, \xi(\tau_*) - t)$ ,

$$G(t + \delta, \{0\}) \leq \int_{[0, \infty)} \varphi_\varepsilon(x) G(t, x) dx + \frac{2N^2\delta}{\varepsilon} \int_0^t G(s, \{0\}) ds.$$

If we take superior limits as  $\delta \rightarrow 0$ , and then let  $\varepsilon \rightarrow 0$  we obtain, using (4.2) with  $G$  instead of  $H$ :

$$\limsup_{\delta \rightarrow 0} G(t + \delta, \{0\}) \leq G(t, \{0\}).$$

and this combined with (5.19) proves that  $G(t, \{0\})$  is right continuous on  $[0, \xi(\tau_*))$ .  $\square$

In the next Lemma we prove that the function  $G$  defined by (5.8) is actually well defined for all  $t > 0$ .

**Lemma 5.5.**

$$\lim_{\tau \rightarrow \tau_*^-} \xi(\tau) = \infty. \quad (5.20)$$

**Proof.** Since the function  $\xi(\tau)$  is monotone nondecreasing and continuous on  $[0, \tau_*)$ , its limit as  $\tau \rightarrow \tau_*^-$  exists in  $\overline{\mathbb{R}}_+$ . Let us call it  $\ell$  and suppose

$\ell \in \mathbb{R}_+$ . Now, from (5.14) and the fact that  $G$  satisfies :  $0 \leq M_{1/2}(G(s)) \leq \sqrt{NE}$ , we deduce

$$\limsup_{t \rightarrow \ell^-} G(t, \{0\}) \geq e^{-\sqrt{NE}\ell} G(0, \{0\}) > 0, \quad (5.21)$$

and by (4.6)

$$H(\tau_*, \{0\}) \geq \limsup_{\tau \rightarrow \tau_*^-} H(\tau, \{0\}) = \limsup_{t \rightarrow \ell^-} G(t, \{0\}) > 0,$$

and this contradicts (5.5). This proves that  $\ell = \infty$ .  $\square$

**Proof of Theorem 1.3.** By Lemma 5.5 the function  $G$  is defined for all  $t > 0$ . As we have seen in the proof of Lemma 5.5,  $G(t) \in \mathcal{M}_+([0, \infty))$  for all  $t > 0$ . It then follows from Proposition 5.3 that  $G$  satisfies now all the conditions (1.44)–(1.46) and (1.47)–(1.49). Property (1.50) follows from the corresponding estimate (3.6) for  $h$ . Similarly, property (1.52) follows from the property (3.7) of  $h$ . We prove now the point (iv). Suppose then  $\alpha \in (1, 3]$  and condition (1.53). For  $\varphi(x) = x^\alpha$  we have,

$$\mathcal{Q}_3^{(1)}(\varphi, G(t)) = \left( \frac{\alpha - 1}{\alpha + 1} \right) M_{\alpha + \frac{1}{2}}(G(t)).$$

On the other hand, for  $0 \leq y \leq x$ ,

$$\Lambda(\varphi)(x, y) = x^\alpha \left( (1 + z)^\alpha + (1 - z)^\alpha - 2 \right), \quad z = \frac{y}{x} \in [0, 1],$$

If  $\alpha \in (1, 2]$ , for all  $x \geq y > 0$ ,

$$\frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \leq (2^\alpha - 2) x^{\alpha - \frac{3}{2}} y^{\frac{1}{2}} \leq (2^\alpha - 2) (xy)^{\frac{\alpha - 1}{2}}.$$

We deduce

$$\mathcal{Q}_3^{(2)}(\varphi, G(t)) \leq (2^\alpha - 2) \left( M_{\frac{\alpha - 1}{2}}(G(t)) \right)^2.$$

and obtain

$$\frac{d}{dt} M_\alpha(G(t)) \leq G(t, \{0\}) \left[ C_{1,1} \left( M_{\frac{\alpha - 1}{2}}(G(t)) \right)^2 - C_2 M_{\alpha + \frac{1}{2}}(G(t)) \right],$$

where  $C_{1,1} = 2^\alpha - 2$  and  $C_2 = (\alpha - 1)/(\alpha + 1)$ . Using Hölder's inequality

$$\frac{d}{dt} M_\alpha(G(t)) \leq G(t, \{0\}) \left[ C_{1,1} N^{3 - \alpha} E^{\alpha - 1} - C_2 E^{(2\alpha + 1)/2} N^{(1 - 2\alpha)/2} \right]. \quad (5.22)$$

By (1.53), the right hand side of (5.22) is negative, and then  $M_\alpha(G(t))$  is decreasing on  $(0, \infty)$ .

For  $\alpha \in [2, 3]$  we use the estimate (3.36) with  $C_{1,2} = \alpha(\alpha - 1)$  instead of  $C_\alpha$ . Then we proceed as in the previous case to obtain

$$\frac{d}{dt} M_\alpha(G(t)) \leq G(t, \{0\}) \left[ C_{1,2} N^{3-\alpha} E^{\alpha-1} - C_2 E^{(2\alpha+1)/2} N^{(1-2\alpha)/2} \right]. \quad (5.23)$$

As before, (1.53) implies that the right hand side of (5.23) is negative, and then  $M_\alpha(G(t))$  is decreasing.  $\square$

**Proof of Theorem 1.4.** By construction

$$G(t) = H(\tau) = h(\tau) - \left( \int_0^\tau M_{1/2}(h(\sigma)) d\sigma \right) \delta_0,$$

where  $\tau$  and  $t$  are related by

$$t = \xi(\tau) = \int_0^\tau \frac{d\sigma}{H(\sigma, \{0\})}; \quad \tau = \xi^{-1}(t) = \int_0^t G(s, \{0\}) ds. \quad (5.24)$$

Therefore  $G(t, x) = h(\tau, x)$  for  $x \in (0, \infty)$ , and

$$\int_0^T G(t, \{0\}) \int_{(0, \infty)} x^\alpha G(t, x) dx dt = \int_0^{\xi^{-1}(T)} \int_{(0, \infty)} x^\alpha h(\tau, x) dx d\tau.$$

The result then follows from Proposition 4.7.  $\square$

**Remark 5.6.** One could try to directly solve the system (1.34), (1.35), written in  $(g, n)$  variables. First, to obtain a sequence of solutions  $(g_k, n_k)$  of an approximated system where the factor  $x^{-1/2}$  is modified by truncation and regularization, and then pass to the limit. However, the limit obtained in that way, say  $(g, n)$  is not a solution of (1.34), (1.35). The reason is that all the solutions  $g_k$  of the approximated system will be functions with a bounded moment of order  $-1/2$ . Then, the right hand side of the equation (1.37) is equal to  $M_{1/2}(g_k)$  and by passage to the limit the equation for  $n$  will be  $n'(t) = -n(t)M_{1/2}(g(t))$ , and the total mass will not be conserved.

## 6 Proofs of Theorems 1.7, 1.8 and 1.9.

We first prove Theorem 1.7.

**Proof of Theorem 1.7 .** We already know by Proposition 5.4 and Lemma 5.5 that  $n$  is right continuous and a.e. differentiable on  $[0, \infty)$ . Then, by construction

$$G(t) = H(\tau) = h(\tau) - \left( \int_0^\tau M_{1/2}(h(\sigma)) d\sigma \right) \delta_0,$$

where  $\tau \in [0, \tau^*)$  and  $t \in [0, \infty)$  are related by (5.24). Hence

$$n(t) = m(\tau) - \int_0^\tau M_{1/2}(h(\sigma))d\sigma = m(\tau) - \int_0^t n(s)M_{1/2}(g(s))ds. \quad (6.1)$$

Since  $n(0) = m(0)$ , it then follows from Proposition 4.15 that for all  $t > 0$ :

$$n(t) - n(0) + \int_0^t n(s)M_{1/2}(g(s))ds = \lambda((0, \tau]), \quad (6.2)$$

and using (5.24)

$$\lambda((0, \tau]) = \lim_{\varepsilon \rightarrow 0} \int_0^t n(s) \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, g(s))ds. \quad (6.3)$$

If we denote  $\mu = \xi_{\#}\lambda$  (c.f. [1], Ch. 5), i.e., the push-forward of  $\lambda$  through the function  $\xi : [0, \tau^*) \rightarrow [0, \infty)$  in (5.24), then from the definition of  $\mu$  we obtain

$$\mu((0, t]) = \lambda((0, \tau]) \quad \forall t > 0. \quad (6.4)$$

Then (1.58) and (1.57) follows from (6.2), (6.3) and (6.4). Moreover, (1.59) follows from (4.44) in Proposition 4.15.  $\square$

The following properties of  $n(t)$  follows by the same arguments used in the proofs of properties (4.45) and (4.46) of Proposition 4.15

**Proposition 6.1.** *Let  $G$ ,  $g$ , and  $n(t)$  be as in Theorem 1.7. Then, for all  $\varphi_\varepsilon$  as in Remark 1.6, the following limit exists in  $\mathcal{D}'(0, \infty)$ :*

$$\lim_{\varepsilon \rightarrow 0} n \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, g) = T(G), \quad (6.5)$$

$$\text{and} \quad n' + nM_{1/2}(g) = T(G) \quad \text{in} \quad \mathcal{D}'(0, \infty). \quad (6.6)$$

**Proof.** Consider, for all  $\varphi_\varepsilon$  as in Remark 1.6, the absolutely continuous functions

$$\eta_\varepsilon(t) = \int_{[0, \infty)} \varphi_\varepsilon(x)G(t, x)dx. \quad (6.7)$$

Then equation (1.46) becomes  $\eta'_\varepsilon = n \mathcal{Q}_3(\varphi_\varepsilon, g)$ . Using integration by parts,

$$-\int_0^\infty \phi'(t)\eta_\varepsilon(t)dt = \int_0^\infty \phi(t)n(t) \mathcal{Q}_3(\varphi_\varepsilon, g(t))dt \quad \forall \phi \in C_c^\infty(0, \infty).$$

Taking the limit  $\varepsilon \rightarrow 0$  we deduce, using Lemma 8.8, that

$$\begin{aligned} -\int_0^\infty \phi'(t)n(t)dt &= \lim_{\varepsilon \rightarrow 0} \int_0^\infty \phi(t)n(t) \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, g(t))dt \\ &\quad - \int_0^\infty \phi(t)n(t)M_{1/2}(g(t))dt, \end{aligned}$$

and then (6.5), (6.6) follows.  $\square$

**Remark 6.2.** If we take distributional derivatives in both sides of (1.58) we obtain:

$$n' + nM_{1/2}(g) = \mu \quad \text{in } \mathcal{D}'(0, \infty),$$

and by (6.6),  $\mu = T(G)$ .

**Proof of Theorem 1.8.** The statement of the Theorem follows from (4.44) in Proposition 4.15 and (6.4).  $\square$

**Proof of Theorem 1.9.** Proof of part (i). By Theorem 1.7,  $n$  is given by (1.58) and (1.57). On the other hand, since  $G$  satisfies (1.46), and for all  $\varphi \in C_b^1([0, \infty))$  such that  $\varphi(0) = 0$ :

$$\int_{[0, \infty)} \varphi(x) G(t, x) dx = \int_{[0, \infty)} \varphi(x) g(t, x) dx, \quad (6.8)$$

then  $g$  satisfies (1.60). In order to prove part (ii) we first show the existence of the limit in (1.57). To this end we write  $\varphi_\varepsilon = (1 - \psi_\varepsilon)$ , where  $\psi_\varepsilon$  is as in Remark 1.6. Then  $\varphi_\varepsilon(0) = 0$ , and by (1.60) and (1.43), using that  $\mathcal{Q}_3(1 - \psi_\varepsilon, g) = \mathcal{Q}_3(1, g) - \mathcal{Q}_3(\psi_\varepsilon, g)$ , and  $\mathcal{Q}_3(1, g) = 0$ , we deduce

$$\begin{aligned} \int_0^t n(s) \tilde{\mathcal{Q}}_3(\psi_\varepsilon, g(s)) ds &= \int_{(0, \infty)} \varphi_\varepsilon(x) (g(0, x) - g(t, x)) dx \\ &\quad + \int_0^t n(s) M_{1/2}(g(s)) ds. \end{aligned} \quad (6.9)$$

The existence of the limit in (1.57) follows and, if we pass to the limit,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^t n(s) \mathcal{Q}_3^{(2)}(\psi_\varepsilon, g(s)) ds &= \int_{(0, \infty)} (g(0, x) - g(t, x)) dx \\ &\quad + \int_0^t n(s) M_{1/2}(g(s)) ds. \end{aligned} \quad (6.10)$$

We now check that, if  $n$  satisfies the equation (1.61) then  $G$  satisfies equation (1.46) for a.e.  $t > 0$  and for every  $\varphi \in C_b^1([0, \infty))$ . If  $\varphi(0) = 0$  this follows from (1.60) and (6.8).

For  $\varphi(0) \neq 0$  we may assume without loss of generality that  $\varphi(0) = 1$ , and write  $\varphi = (\varphi - \psi_\varepsilon) + \psi_\varepsilon$ , where  $\psi_\varepsilon$  is as in Remark 1.6. Since  $(\varphi - \psi_\varepsilon)(0) = 0$ , using (1.60) and (1.45)

$$\begin{aligned} \int_{[0, \infty)} (\varphi - \psi_\varepsilon)(x) g(t, x) dx &= \int_{[0, \infty)} (\varphi - \psi_\varepsilon)(x) g(0, x) dx \\ &\quad + \int_0^t n(s) \tilde{\mathcal{Q}}_3((\varphi - \psi_\varepsilon), g(s)) ds. \end{aligned} \quad (6.11)$$

In order to pass to the limit as  $\varepsilon \rightarrow 0$ , we first use  $\tilde{\mathcal{Q}}_3((\varphi - \psi_\varepsilon), g) = \tilde{\mathcal{Q}}_3(\varphi, g) - \tilde{\mathcal{Q}}_3(\psi_\varepsilon, g)$ . Then, since for all  $t \geq 0$

$$\lim_{\varepsilon \rightarrow 0} \int_{[0, \infty)} \psi_\varepsilon(x) g(t, x) dx = 0, \quad (6.12)$$

and  $n$  satisfies (1.61), we deduce from (6.11) and Lemma 8.8:

$$\begin{aligned} \int_{[0, \infty)} \varphi(x) g(t, x) dx &= \int_{[0, \infty)} \varphi(x) g(0, x) dx + \int_0^t n(s) \tilde{\mathcal{Q}}_3(\varphi, g(s)) ds \\ &\quad + n(0) - n(t) - \int_0^t n(s) M_{1/2}(g(s)) ds. \end{aligned}$$

Since  $\tilde{\mathcal{Q}}_3(\varphi, G) - M_{1/2}(g) = \mathcal{Q}_3(\varphi, G)$ , it follows that  $G$  satisfies

$$\int_{[0, \infty)} \varphi(x) G(t, x) dx = \int_{[0, \infty)} \varphi(x) G(0, x) dx + \int_0^t n(s) \mathcal{Q}_3(\varphi, g(s)) ds,$$

thus (1.46) holds for a.e.  $t > 0$ .

In order to check that  $G$  satisfies (1.44) we first use (1.46) with  $\varphi = 1 \in C_b^1([0, \infty))$ . For that choice of  $\varphi$  we have  $\Lambda(\varphi) = \mathcal{L}_0(\varphi) \equiv 0$  and then:

$$\int_{[0, \infty)} G(t, x) dx = \int_{[0, \infty)} G_0(x) dx.$$

Because:

$$\int_{[0, \infty)} x G(t, x) dx = \int_{[0, \infty)} x g(t, x) dx,$$

$G$  satisfies (1.44) since by hypothesis so does  $g$ .  $\square$

**Remark 6.3.** If  $G$  is a weak radial solution of (1.1), (1.2), we know by Theorem 1.9 that  $g$  satisfies (1.60). It is straightforward to check that it also satisfies,

$$\frac{d}{dt} \int_{(0, \infty)} \varphi(x) g(t, x) dx = n(t) \tilde{\mathcal{Q}}_3(\varphi, g(t)) - \varphi(0) \frac{d}{dt} \mu((0, t]),$$

where  $\mu$  is as in Theorem 1.7, and  $\tilde{\mathcal{Q}}_3$  is defined in (1.40)–(1.42).

**Proof of Corollary 1.10.** If we prove that  $n$  satisfies (1.61), the conclusion of the Corollary will follow from part (ii) of Theorem 1.9. By the hypothesis and part (ii) of Theorem 1.9, the limit in (1.57) exists, and (6.10) holds, that we write:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_0^t n(s) \mathcal{Q}_3^{(2)}(\psi_\varepsilon, g(s)) ds - \int_0^t n(s) M_{1/2}(g(s)) ds &= \\ &= \int_{[0, \infty)} (G(0, x) - G(t, x)) dx + n(t) - n(0). \end{aligned}$$

Using the conservation of mass (1.62) it follows that  $n$  satisfies equation (1.61).  $\square$

**Proposition 6.4.** *Let  $G \in \mathcal{M}_+([0, \infty))$ . If  $G$  has no atoms on  $(0, \infty)$  and  $\int_{(0, \infty)} \frac{G(x)}{\sqrt{x}} dx < \infty$ , then, for all  $\varphi_\varepsilon$  as in Remark 1.6,*

$$\mathcal{T}(G) = \lim_{\varepsilon \rightarrow 0} \mathcal{Q}_3^{(2)}(\varphi_\varepsilon, G) = 0.$$

**Proof.** By definition

$$\mathcal{T}(G) = \lim_{\varepsilon \rightarrow 0} \iint_{(0, \infty)^2} \frac{\Lambda(\varphi_\varepsilon)(x, y)}{\sqrt{xy}} G(x)G(y) dx dy,$$

Since  $\Lambda(\varphi_\varepsilon) \leq 1$  for all  $\varepsilon > 0$  and

$$\lim_{\varepsilon \rightarrow 0} \Lambda(\varphi_\varepsilon)(x, y) = \mathbb{1}_{\{x=y>0\}}(x, y) \quad \forall (x, y) \in (0, \infty)^2,$$

and  $\int_{(0, \infty)} \frac{G(x)}{\sqrt{x}} dx < \infty$ , then by dominated convergence

$$\mathcal{T}(G) = \iint_{\{x=y>0\}} \frac{G(x)G(y)}{\sqrt{xy}} dx dy.$$

Since  $G$  has no atoms on  $(0, \infty)$ , i.e.,  $G(\{x\}) = 0$  for all  $x > 0$ , by Fubini's theorem

$$\iint_{\{x=y>0\}} \frac{G(x)G(y)}{\sqrt{xy}} dx dy = \int_{(0, \infty)} \frac{G(x)}{x} G(\{x\}) dx = 0.$$

□

**Remark 6.5.** From Proposition 6.4, if  $M_{-1/2}(g) < \infty$  and  $g$  has no atoms, then  $\mu((0, t]) = 0$  for all  $t > 0$ . If  $g \in L^1(0, \infty)$  and  $x = 0$  is a Lebesgue point of  $g$  then  $\mathcal{T}(g) = 0$  (cf. [23]) and again  $\mu((0, t]) = 0$  for all  $t > 0$ . If  $g(x) = x^{-1/2}$ , then  $\mathcal{T}(g) = \pi^2/6$ , (cf. [20]), and a similar result holds if  $\lim_{x \rightarrow 0} \sqrt{x}g(x) = C > 0$  (cf. [27]). In that case,  $\mu((0, t]) = \pi^2/6 \int_0^t n(s) ds$ .

## 7 Proof of Theorem 1.11

**Proof.** By (5.22) and (5.23), we deduce that for all  $t > t_0 > 0$ :

$$\begin{aligned} \int_{t_0}^t G(s, \{0\}) ds &\leq (M_\alpha(G(t_0)) - M_\alpha(G(t))) C(N, E, \alpha) \\ C(N, E, \alpha) &= \left[ \left( \frac{\alpha - 1}{\alpha + 1} \right) E^{(2\alpha+1)/2} N^{(1-2\alpha)/2} - C_1 N^{3-\alpha} E^{\alpha-1} \right]^{-1}, \end{aligned} \quad (7.1)$$

where  $C_1 = 2^\alpha - 2$  for  $\alpha \in (1, 2]$  and  $C_1 = \alpha(\alpha - 1)$  for  $\alpha \in [2, 3]$ . Since by part (i),  $0 \leq M_\alpha(G(t_0)) - M_\alpha(G(t)) \leq M_\alpha(G(t_0))$  for every  $t > t_0$ , we immediately deduce (1.63).

We prove now (1.64). Since, as we have seen in (5.18), the function  $n(t)J(t)$  is monotone nondecreasing, from where, for all  $t > 0$  and  $s \in (0, t)$ :

$$n(t) \geq e^{-\int_s^t M_{1/2}(g(r))dr} n(s).$$

As we have  $M_{1/2}(g(r)) \leq \sqrt{NE}$  for all  $r \geq 0$ ,

$$n(t) \geq e^{-\sqrt{NE}(t-s)} n(s). \quad (7.2)$$

By (1.63) we already have a sequence of times  $\theta_k$  such that  $\theta_k \rightarrow \infty$  and  $n(\theta_k) \rightarrow 0$  as  $k \rightarrow \infty$ . Suppose that there exists, for some  $\rho > 0$ , an increasing sequence of times  $(s_k)_{k \in \mathbb{N}}$  such that  $s_k \rightarrow \infty$  as  $k \rightarrow \infty$  and :

$$\forall k, n(s_k) \geq \rho \quad \text{and} \quad s_{k+1} - s_k > \frac{\log 2}{\sqrt{NE}}.$$

Then, if we denote  $t_k = s_k + \frac{\log 2}{\sqrt{NE}}$ , we deduce from (7.2) that for all  $t \in (s_k, t_k)$ :

$$n(t) \geq e^{-\sqrt{NE}(t-s_k)} n(s_k) \geq e^{-\sqrt{NE}(t_k-s_k)} \rho = \frac{\rho}{2}.$$

This would imply

$$\int_0^\infty n(t)dt \geq \sum_{k=0}^\infty \int_{s_k}^{t_k} n(t)dt = \infty,$$

and this contradiction proves (1.64).  $\square$

## 8 Appendix

We have gathered in this Section several results that are important and useful, but not directly related to the main results. For the sake of clarity, we present them in two different Sub Sections. In the first one, we find results that are used all along the manuscript, perhaps several times. In the second, we present results that are needed in Section 2.

### 8.1 A1

**Lemma 8.1** (Convex-positivity). *Let  $\varphi \in C([0, \infty))$ . If  $\varphi$  is convex then  $\Lambda(\varphi)(x, y) \geq 0$  for all  $(x, y) \in [0, \infty)^2$  and  $\mathcal{L}_0(\varphi)(x) \geq 0$  for all  $x \in [0, \infty)$ . If  $\varphi$  is nonnegative and nonincreasing, then  $\mathcal{L}(\varphi)(x) \leq 0$  for all  $x \in [0, \infty)$ .*

**Proof.** Since  $\Lambda(\varphi)(x, y)$  is symmetric we may reduce the proof to the case  $0 \leq y \leq x$ . Putting  $x = \frac{x+y}{2} + \frac{x-y}{2}$ , then by the very definition of convexity

$$\varphi(x) \leq \frac{\varphi(x+y)}{2} + \frac{\varphi(x-y)}{2},$$



therefore  $\Lambda(\varphi)(x, y) \geq 0$ .

The positivity of  $\mathcal{L}_0(\varphi)$  is equivalent to prove

$$\frac{1}{x} \int_0^x \varphi(y) dy \leq \frac{\varphi(0) + \varphi(x)}{2} \quad \forall x \in [0, \infty). \quad (8.1)$$

Since for any  $0 \leq y \leq x$  we may trivially write  $y = (1 - \frac{y}{x}) 0 + \frac{y}{x} x$ , then by convexity  $\varphi(y) \leq (1 - \frac{y}{x}) \varphi(0) + \frac{y}{x} \varphi(x)$ , which implies (8.1).

If  $\varphi$  is nonnegative and nonincreasing, then  $\mathcal{L}(\varphi)(x) \leq -x \varphi(x) \leq 0$  for all  $x \in [0, \infty)$ .  $\square$

**Remark 8.2.** By linearity and Lemma 8.1, it follows that for all  $\varphi \in C([0, \infty))$  concave,  $\Lambda(\varphi)(x, y) \leq 0$  for all  $(x, y) \in [0, \infty)^2$  and  $\mathcal{L}_0(\varphi)(x) \leq 0$  for all  $x \in [0, \infty)$ .

**Lemma 8.3.** Consider the operators  $\Lambda(\cdot)$ ,  $\mathcal{L}_0(\cdot)$  and  $\mathcal{L}(\cdot)$  given in (1.30), (1.31) and (1.42) respectively. Then

(i) If  $\varphi \in \text{Lip}([0, \infty))$  with Lipschitz constant  $L$ , then

$$\frac{|\Lambda(\varphi)(x, y)|}{\sqrt{xy}} \leq 2L \quad \forall (x, y) \in [0, \infty)^2. \quad (8.2)$$

(ii) If  $\varphi \in C^1([0, \infty))$ , then the map  $(x, y) \mapsto \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}}$  belongs to  $C([0, \infty)^2)$  and

$$\frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} = 0 \quad \forall (x, y) \in \partial[0, \infty)^2. \quad (8.3)$$

(iii) If  $\varphi \in C([0, \infty))$  then the maps  $x \mapsto \frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}}$  and  $x \mapsto \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$  belong to  $C([0, \infty))$  and  $\frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}} = \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} = 0$  at  $x = 0$ . If in addition  $\varphi$  is bounded, then

$$\frac{|\mathcal{L}_0(\varphi)(x)|}{\sqrt{x}} \leq 4\|\varphi\|_\infty \sqrt{x} \quad \forall x \in [0, \infty), \quad (8.4)$$

$$\frac{|\mathcal{L}(\varphi)(x)|}{\sqrt{x}} \leq 3\|\varphi\|_\infty \sqrt{x} \quad \forall x \in [0, \infty). \quad (8.5)$$

**Proof.** (i) By the symmetry of  $\Lambda(\varphi)$  we can assume that  $0 \leq y \leq x$ , and directly from the Lipschitz continuity

$$|\Lambda(\varphi)(x, y)| \leq |\varphi(x+y) - \varphi(x)| + |\varphi(x-y) - \varphi(x)| \leq 2Ly,$$

which implies (8.2).

(ii) The only possible problem for the continuity is on the boundary of  $[0, \infty)^2$ . Again by the symmetry of  $\Lambda(\varphi)$  we can assume  $0 \leq y \leq x$ . Then

by the mean value theorem  $\Lambda(\varphi)(x, y) = y(\varphi'(\xi_1) - \varphi'(\xi_2))$  for some  $\xi_1 \in (x, x+y)$  and  $\xi_2 \in (x-y, x)$ . Hence

$$\frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} \leq \varphi'(\xi_1) - \varphi'(\xi_2),$$

and the continuity of  $\frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}}$  on  $[0, \infty)^2$  and (8.3) follow from the continuity of  $\varphi'$ .

(iii) The continuity of  $\frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}}$  and  $\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$  are clear for  $x > 0$ . Using that  $\frac{1}{x} \int_0^x \varphi(y) dy \rightarrow \varphi(0)$  as  $x \rightarrow 0$  by Lebesgue differentiation Theorem, it follows the continuity at  $x = 0$  and that  $\frac{\mathcal{L}_0(\varphi)(x)}{\sqrt{x}} = \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} = 0$  for  $x = 0$ . The bounds (8.4) and (8.5) are straightforward for  $\varphi \in C_b([0, \infty))$ .  $\square$

**Lemma 8.4.** *Consider the operators  $\Lambda(\cdot)$  and  $\mathcal{L}_0(\cdot)$  given in (1.30) and (1.31), and a sequence  $(\phi_n)_{n \in \mathbb{N}} \subset C_c([0, \infty))$  as in Cutoff 3.5.*

(i) *If  $\varphi \in C^1([0, \infty))$  then  $\Lambda(\varphi)(x, y)\phi_n(x)\phi_n(y) \xrightarrow{n \rightarrow \infty} \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}}$  uniformly on the compact sets of  $[0, \infty)^2$ .*

(ii) *If  $\varphi \in C([0, \infty))$  then  $\mathcal{L}(\varphi)(x)\phi_n(x) \xrightarrow{n \rightarrow \infty} \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$  uniformly on the compact sets of  $[0, \infty)$ .*

**Proof.** (i) The pointwise convergence on  $[0, \infty)^2$  is trivial since  $\phi_n(x) \rightarrow x^{-1/2}$  as  $n \rightarrow \infty$ . Then, let  $\varepsilon > 0$  and  $R > 0$ . For  $n \geq R$  there holds  $\phi_n(x) = x^{-1/2}$  for all  $x \in [1/n, R]$ , so we only need to show the uniform convergence on the regions  $(x, y) \in [0, R] \times [0, 1/n]$  and  $(x, y) \in [0, 1/n] \times [0, R]$ . By the symmetry of  $\Lambda(\varphi)$ , we may study only one region.

Using that  $\frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}}$  is continuous (hence uniformly continuous on compacts) and vanishes when  $(x, y) \in \partial[0, \infty)^2$  (c.f. Lemma 8.3), there holds for all  $(x, y) \in [0, R] \times [0, 1/n]$  that, for  $n$  large enough,

$$\left| \frac{\Lambda(\varphi)(x, y)}{\sqrt{xy}} - \Lambda(\varphi)(x, y)\phi_n(x)\phi_n(y) \right| \leq \frac{|\Lambda(\varphi)(x, y)|}{\sqrt{xy}} \leq \varepsilon$$

(ii) Let  $\varepsilon > 0$  and  $R > 0$ . Since for  $n \geq R$  there holds  $\phi_n(x) = x^{-1/2}$  for all  $x \in [1/n, R]$ , we only need to prove the uniform convergence on the region  $[0, 1/n]$ . Using that  $\frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}}$  is continuous (hence uniformly continuous on compacts) and vanishes when  $x \rightarrow 0$  (cf. Lemma 8.3), we have

$$\left| \frac{\mathcal{L}(\varphi)(x)}{\sqrt{x}} - \mathcal{L}(\varphi)(x)\phi_n(x) \right| \leq \frac{|\mathcal{L}(\varphi)(x)|}{\sqrt{x}} \leq \varepsilon \quad \forall x \in [0, 1/n]$$

for  $n$  large enough.  $\square$

The following Lemma is about the approximation of a measure by continuous functions. It is a simplified version of Lemma 4 in [18].

**Lemma 8.5.** *Let  $\nu \in \mathcal{M}_+^\alpha([0, \infty))$  for some  $\alpha \geq 0$ . Then, there exists a sequence of functions  $(\nu_n)_{n \in \mathbb{N}} \subset C([0, \infty)) \cap L^1(\mathbb{R}_+, (1 + x^\alpha)dx)$  such that*

$$\forall \varphi \in C([0, \infty)) : \sup_{x \geq 0} \frac{|\varphi(x)|}{1 + x^\alpha} < \infty, \quad (8.6)$$

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi(x) \nu_n(x) dx = \int_{[0, \infty)} \varphi(x) d\nu(x). \quad (8.7)$$

**Proof.** Let  $J(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$  for  $x \geq 0$  and define, for  $n \in \mathbb{N}$ ,  $x \geq 0$ ,

$$\nu_n(x) = e^n \int_{[0, \infty)} J(e^n |x - y(1 - e^{-n})|) d\nu(y).$$

In order to prove that  $\nu_n$  is a continuous function on  $[0, \infty)$ , let  $x \geq 0$  and  $(x_k)_{k \in \mathbb{N}} \subset [0, \infty)$  be such that  $x_k \rightarrow x$  as  $k \rightarrow \infty$ . Since  $J$  is a bounded continuous function on  $[0, \infty)$  and  $M_0(\nu) < \infty$ , it is easily deduced using dominated convergence theorem that, for all  $n \in \mathbb{N}$ ,  $\nu_n(x_k) \rightarrow \nu_n(x)$  as  $k \rightarrow \infty$ , and therefore  $\nu_n \in C([0, \infty))$ .

Let us prove now that  $\nu_n \in L^1(\mathbb{R}_+, (1 + x^\alpha)dx)$ . To this end, let  $F_n(x, y) = (1 + x^\alpha) e^n J(e^n |x - y(1 - e^{-n})|)$ . Using the change of variables  $z = e^n(y(1 - e^{-n}) - x)$  we deduce that for all  $y \geq 0$ ,  $n \in \mathbb{N}$ ,

$$\begin{aligned} \int_0^\infty |F_n(x, y)| dx &= \int_0^{y(e^n - 1)} (1 + [y(1 - e^{-n}) - e^{-n}z]^\alpha) J(z) dz \\ &\quad + \int_0^\infty (1 + [y(1 - e^{-n}) + e^{-n}z]^\alpha) J(z) dz. \end{aligned}$$

Since

$$\begin{aligned} 1 + [y(1 - e^{-n}) - e^{-n}z]^\alpha &\leq 1 + [y(1 - e^{-n}) + e^{-n}z]^\alpha \leq 1 + 2^\alpha(y^\alpha + z^\alpha) \\ &\leq 2^\alpha(1 + y^\alpha)(1 + z^\alpha), \end{aligned} \quad (8.8)$$

and  $\nu \in \mathcal{M}_+^\alpha([0, \infty))$ , then for all  $n \in \mathbb{N}$ ,

$$\int_{[0, \infty)} \int_0^\infty |F_n(x, y)| dx d\nu(y) \leq 2^{\alpha+1} \int_{[0, \infty)} (1 + y^\alpha) d\nu(y) \int_0^\infty (1 + z^\alpha) J(z) dz < \infty,$$

which implies, by Fubini's theorem, that  $\nu_n \in L^1(\mathbb{R}_+, (1 + x^\alpha)dx)$ .

Now, for any  $\varphi \in C([0, \infty))$  satisfying (8.6), using Fubini's theorem and the change of variables  $z = e^n(x - y(1 - e^{-n}))$ :

$$\begin{aligned} \int_0^\infty \varphi(x) \nu_n(x) dx &= \int_{[0, \infty)} I_n(\varphi)(y) d\nu(y), \\ I_n(\varphi)(y) &= \int_0^{y(e^n - 1)} \varphi(y(1 - e^{-n}) - ze^{-n}) J(z) dz \\ &\quad + \int_0^\infty \varphi(y(1 - e^{-n}) + ze^{-n}) J(z) dz. \end{aligned} \quad (8.9)$$

By a similar estimate as in (8.8), using (8.6) we obtain that for some constant  $C > 0$ ,

$$\max \{ |\varphi(y(1 - e^{-n}) - ze^{-n})|, |\varphi(y(1 - e^{-n}) + ze^{-n})| \} \leq C(1 + y^\alpha)(1 + z^\alpha),$$

and  $|I_n(\varphi)(y)| \leq C(1 + y^\alpha)$ . We then deduce, using dominated convergence, that

$$\lim_{n \rightarrow \infty} I_n(\varphi)(y) = 2\varphi(y) \int_0^\infty J(z)dz = \varphi(y), \quad \forall y \geq 0,$$

and

$$\lim_{n \rightarrow \infty} \int_{[0, \infty)} I_n(\varphi)(y) d\nu(y) = \int_{[0, \infty)} \varphi(y) d\nu(y),$$

which completes the proof, in view of (8.9).  $\square$

**Corollary 8.6.** *Let  $\nu \in \mathcal{M}_+^\alpha([0, \infty))$  for some  $\alpha \geq 1$ . Then, there exists a sequence of nonnegative functions  $(f_n)_{n \in \mathbb{N}} \subset C_c([0, \infty))$  such that*

$$\limsup_{n \rightarrow \infty} M_\alpha(f_n) \leq M_\alpha(\nu), \quad (8.10)$$

and for all  $\varphi \in C_b([0, \infty))$ ,

$$\lim_{n \rightarrow \infty} \int_0^\infty \varphi(x) f_n(x) dx = \int_{[0, \infty)} \varphi(x) d\nu(x). \quad (8.11)$$

**Proof.** We consider the sequence  $(\nu_n)_{n \in \mathbb{N}}$  given by Lemma 8.5 and a smooth cutoff  $\zeta_n \in C([0, \infty))$  such that  $0 \leq \zeta_n \leq 1$ ,  $\zeta_n(x) = 1$  for  $x \in [0, n]$  and  $\zeta_n(x) = 0$  for  $x \geq n + 1$ . Then we define for all  $n \in \mathbb{N}$ :

$$f_n(x) = \nu_n(x) \zeta_n(x). \quad (8.12)$$

It then follows that  $f_n$  is a nonnegative continuous function on  $[0, \infty)$  with compact support. Since  $f_n \leq \nu_n$ , the property (8.10) follows directly from (8.7) in Lemma 8.5. Now, let  $\varphi \in C_b([0, \infty))$ . Since  $\nu_n$  satisfies (8.7), in order to prove (8.11) it is sufficient to prove

$$\lim_{n \rightarrow \infty} \left| \int_0^\infty \varphi(x) f_n(x) dx - \int_0^\infty \varphi(x) \nu_n(x) dx \right| = 0, \quad (8.13)$$

and (8.13) follows from

$$\lim_{n \rightarrow \infty} \int_n^\infty \varphi(x) \nu_n(x) dx \leq \lim_{n \rightarrow \infty} \frac{\|\varphi\|_\infty M_1(\nu_n)}{n} = 0,$$

where we have used that  $M_1(\nu_n) \rightarrow M_1(\nu) < \infty$  as  $n \rightarrow \infty$  by (8.7) in Lemma 8.5.  $\square$

**Definition 8.7.** Let  $h$ ,  $\phi_n$  and  $\varphi$  be real-valued functions with domain  $\mathbb{R}_+$ . Then, let

$$\tilde{\mathcal{Q}}_{3,n}(\varphi, h) = \mathcal{Q}_{3,n}^{(2)}(\varphi, h) - \tilde{\mathcal{Q}}_{3,n}^{(1)}(\varphi, h), \quad (8.14)$$

where

$$\mathcal{Q}_{3,n}^{(2)}(\varphi, h) = \int_0^\infty \int_0^\infty \Lambda(\varphi)(x, y) \phi_n(x) \phi_n(y) h(x) h(y) dx dy, \quad (8.15)$$

$$\tilde{\mathcal{Q}}_{3,n}^{(1)}(\varphi, h) = \int_0^\infty \mathcal{L}(\varphi)(x) \phi_n(x) h(x) dx, \quad (8.16)$$

and let, for  $x \in \mathbb{R}_+$ :

$$J_{3,n}(h)(x) = K_n(h)(x) + L_n(h)(x) - h(x)A_n(h)(x), \quad (8.17)$$

where

$$\begin{aligned} K_n(h)(x) &= \int_0^x h(x-y)h(y)\phi_n(x-y)\phi_n(y)dy \\ &\quad + 2 \int_x^\infty h(y)h(y-x)\phi_n(y)\phi_n(y-x)dy, \end{aligned} \quad (8.18)$$

$$L_n(h)(x) = 2 \int_x^\infty h(y)\phi_n(y)dy, \quad (8.19)$$

$$A_n(h)(x) = \phi_n(x) \left( x + 4 \int_0^x h(y)\phi_n(y)dy \right). \quad (8.20)$$

**Lemma 8.8.** Let  $G \in \mathcal{M}_+([0, \infty))$ ,  $\varphi_\varepsilon$  as in Remark 1.6, and  $\phi_n$  as in Cutoff 3.5. Then

$$G(\{0\}) = \lim_{\varepsilon \rightarrow 0} \int_{[0, \infty)} \varphi_\varepsilon(x) G(x) dx, \quad (8.21)$$

$$\lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{Q}}_{3,n}^{(1)}(\varphi_\varepsilon, G) = 0 \quad \forall n \in \mathbb{N}. \quad (8.22)$$

If in addition  $G$  has no singular part in  $(0, \infty)$ , then

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}_{3,n}^{(2)}(\varphi_\varepsilon, G) = 0 \quad \forall n \in \mathbb{N}. \quad (8.23)$$

Furthermore, if  $G \in \mathcal{M}_+^{1/2}([0, \infty))$ , then

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}_3^{(1)}(\varphi_\varepsilon, G) = M_{1/2}(G), \quad (8.24)$$

$$\lim_{\varepsilon \rightarrow 0} \tilde{\mathcal{Q}}_3^{(1)}(\varphi_\varepsilon, G) = 0, \quad (8.25)$$

where  $\mathcal{Q}_3^{(1)}$  and  $\tilde{\mathcal{Q}}_3^{(1)}$  are defined in (1.29) and (1.41) respectively.

**Proof.** The proof only uses dominated convergence. Since  $\varphi_\varepsilon \leq 1$  for all  $\varepsilon > 0$ , and  $M_0(G) < \infty$ , and  $\varphi_\varepsilon \rightarrow \mathbf{1}_{\{0\}}$  as  $\varepsilon \rightarrow 0$ , then (8.21) holds. Then, since for all  $x \in [0, \infty)$  it follows from dominated convergence that

$$\lim_{\varepsilon \rightarrow 0} \mathcal{L}_0(\varphi_\varepsilon)(x) = x \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \mathcal{L}(\varphi_\varepsilon)(x) = 0, \quad (8.26)$$

and  $\phi_n$  is compactly supported, then (8.22) follows. Also, since for all  $(x, y) \in [0, \infty)^2$ ,  $\Lambda(\varphi_\varepsilon)(x, y) \leq 1$  for all  $\varepsilon > 0$ , and

$$\lim_{\varepsilon \rightarrow 0} \Lambda(\varphi_\varepsilon)(x, y) = \mathbf{1}_{\{x=y>0\}}(x, y),$$

then

$$\lim_{\varepsilon \rightarrow 0} \mathcal{Q}_{3,n}^{(2)}(\varphi_\varepsilon, G) = \iint_{\{x=y>0\}} \phi_n(x)\phi_n(y)G(x)G(y)dxdy,$$

Using that  $G$  has no singular part on  $(0, \infty)$ , (8.23) follows.

Lastly, since

$$\tilde{\mathcal{Q}}_3^{(1)}(\varphi_\varepsilon, G) \leq \mathcal{Q}_3^{(1)}(\varphi_\varepsilon, G) = \int_{(0,\infty)} \frac{\mathcal{L}_0(\varphi_\varepsilon)(x)}{\sqrt{x}} G(x)dx, \quad (8.27)$$

and by (8.4)

$$\int_{(0,\infty)} \frac{|\mathcal{L}_0(\varphi_\varepsilon)(x)|}{\sqrt{x}} G(x)dx \leq 4M_{1/2}(G) \quad \forall \varepsilon > 0.$$

then (8.24) and (8.25) follows from (8.26) and dominated convergence.  $\square$

**Lemma 8.9.** *Consider  $n \in \mathbb{N}$ ,  $\phi_n \in C_c([0, \infty))$  nonnegative and  $\rho \in L_{loc}^1(\mathbb{R}_+)$  nonnegative. Then for every nonnegative functions  $h$ ,  $h_1$  and  $h_2$  in  $L^\infty(\mathbb{R}_+)$ , the functions  $K_n(h)$ ,  $L_n(h)$ ,  $A_n(h)$  and  $hA_n(h)$  are also nonnegative, belong to  $L^\infty(\mathbb{R}_+) \cap L_\rho^1(\mathbb{R}_+)$ , and there exists a positive constant  $C(n, \rho)$  such that:*

$$\|K_n(h_1) - K_n(h_2)\|_{L^\infty \cap L_\rho^1} \leq C(n, \rho) \|h_1\|_\infty \|h_1 - h_2\|_\infty \quad (8.28)$$

$$\|L_n(h)\|_{L^\infty \cap L_\rho^1} \leq C(n, \rho) \|h\|_\infty \quad (8.29)$$

$$\|A_n(h)\|_{L^\infty \cap L_\rho^1} \leq C(n, \rho) (1 + \|h\|_\infty) \quad (8.30)$$

$$\|A_n(h_1) - A_n(h_2)\|_{L^\infty \cap L_\rho^1} \leq C(n, \rho) \|h_1 - h_2\|_\infty. \quad (8.31)$$

$$\text{Moreover} \quad J_{3,n}(h) \in L^\infty(\mathbb{R}_+) \cap L_\rho^1(\mathbb{R}_+). \quad (8.32)$$

**Proof.** The positivity of the operators is clear from their definitions. Notice that since  $\phi_n$  is bounded and compactly supported on  $\mathbb{R}_+$  and  $\rho \in L^1_{loc}(\mathbb{R}_+)$ , there exist two positive constants  $C(n)$  and  $C(n, \rho)$  such that

$$\begin{aligned} \sup_{x \geq 0} \int_0^\infty \phi_n(|x-y|) \phi_n(y) dy &\leq C(n), \\ \int_0^\infty \int_0^\infty \rho(x) \phi_n(|x-y|) \phi_n(y) dy dx &\leq C(n, \rho). \end{aligned}$$

1. Estimates for  $K_n$ . For all  $x \geq 0$ :

$$K_n(h)(x) \leq 3 \|h\|_\infty^2 \int_0^\infty \phi_n(|x-y|) \phi_n(y) dy \leq 3 \|h\|_\infty^2 C(n),$$

and

$$\|K_n(h)\|_{L^1_\rho} \leq 3 \|h\|_\infty^2 \int_0^\infty \int_0^\infty \rho(x) \phi_n(|x-y|) \phi_n(y) dy dx \leq 3 \|h\|_\infty^2 C(n, \rho).$$

Then for all  $x \geq 0$ :

$$\begin{aligned} &|K_n(h_1)(x) - K_n(h_2)(x)| \\ &\leq 3 \int_0^\infty \phi_n(|x-y|) \phi_n(y) |h_1(|x-y|)h_1(y) - h_2(|x-y|)h_2(y)| dy. \end{aligned} \tag{8.33}$$

Without loss of generality we assume that  $\|h_1\|_\infty \geq \|h_2\|_\infty$ . Using

$$|h_1(|x-y|)h_1(y) - h_2(|x-y|)h_2(y)| \leq 2\|h_1\|_\infty \|h_1 - h_2\|_\infty$$

in (8.33) then (8.28) follows.

2. Estimates for  $L_n$ . Since  $\phi_n$  is bounded and compactly supported and  $\rho \in L^1_{loc}(\mathbb{R}_+)$ , there exist two positive constants  $C(n)$  and  $C(n, \rho)$  such that

$$\int_0^\infty \phi_n(x) dx \leq C(n) \quad \text{and} \quad \int_0^\infty \rho(x) \int_x^\infty \phi_n(y) dy dx \leq C(n, \rho)$$

and (8.29) follows.

3. Estimates for  $A_n$ . The estimate (8.30) follows from

$$\|A_n(h)\|_\infty \leq \|x \phi_n(x)\|_\infty + 4 \|\phi_n\|_\infty^2 \|h\|_\infty |\text{supp}(\phi_n)| \leq C(n)(1 + \|h\|_\infty),$$

and

$$\begin{aligned} \|A_n(h)\|_{L^1_\rho} &\leq \int_0^\infty \rho(x) x \phi_n(x) dx + 4 \|h\|_\infty \int_0^\infty \rho(x) \phi_n(x) \int_0^x \phi_n(y) dy dx \\ &\leq C(n, \rho)(1 + \|h\|_\infty). \end{aligned}$$

For all  $x \geq 0$ ,

$$\begin{aligned} |A_n(h_1)(x) - A_n(h_2)(x)| &\leq 4\|h_1 - h_2\|_\infty \phi_n(x) \int_0^x \phi_n(y) dy \\ &\leq C(n)\|h_1 - h_2\|_\infty. \end{aligned}$$

We also have,

$$\begin{aligned} \|A_n(h_1) - A_n(h_2)\|_{L_\rho^1} &\leq 4\|h_1 - h_2\|_\infty \int_0^\infty \rho(x) \phi_n(x) \int_0^x \phi_n(y) dy dx \\ &\leq C(n, \rho) \|h_1 - h_2\|_\infty, \end{aligned}$$

and then, (8.31) follows.

4. Since  $h \in L^\infty(\mathbb{R}_+)$  and  $A_n(h) \in L^\infty(\mathbb{R}_+) \cap L_\rho^1(\mathbb{R}_+)$ , then  $hA_n(h) \in L^\infty(\mathbb{R}_+) \cap L_\rho^1(\mathbb{R}_+)$ .

5. It also follows from points 1 to 4 that  $J_{3,n}(h)$  has the desired regularity.  $\square$

## 8.2 A2

**Lemma 8.10.** *Let  $\varphi \in C^{1,1}([0, \infty))$ . Then, for all  $(x_1, x_2, x_3) \in [0, \infty)^3$  such that  $x_1 + x_2 \geq x_3$ :*

$$\begin{aligned} \Delta\varphi(x_1, x_2, x_3) &= (x_1 - x_3)(x_2 - x_3) \times \\ &\quad \times \int_0^1 \int_0^1 \varphi''(x_3 + t(x_1 - x_3) + s(x_2 - x_3)) ds dt. \end{aligned}$$

Moreover, if  $\varphi \in C_b^{1,1}([0, \infty))$ , then for all  $(x_1, x_2, x_3) \in [0, \infty)^3$

$$|\Delta\varphi(x_1, x_2, x_3)| \leq \min\{A, B, C, D\}. \quad (8.34)$$

where  $A = 4\|\varphi\|_\infty$ ,  $B = 2\|\varphi'\|_\infty|x_1 - x_3|$ ,  $C = 2\|\varphi'\|_\infty|x_2 - x_3|$ ,  
 $D = \|\varphi''\|_\infty|x_1 - x_3||x_2 - x_3|$ .

**Proof.** Let  $(x_1, x_2, x_3) \in [0, \infty)^3$  be such that  $x_1 + x_2 \geq x_3$ . By the fundamental Theorem of calculus

$$\begin{aligned} \Delta\varphi(x_1, x_2, x_3) &= [\varphi(x_4) - \varphi(x_2)] - [\varphi(x_1) - \varphi(x_3)] \\ &= \int_0^1 \frac{d}{dt} \varphi(x_2 + t(x_1 - x_3)) dt - \int_0^1 \frac{d}{dt} \varphi(x_3 + t(x_1 - x_3)) dt \\ &= (x_1 - x_3) \int_0^1 [\varphi'(x_2 + t(x_1 - x_3)) - \varphi'(x_3 + t(x_1 - x_3))] dt \\ &= (x_1 - x_3) \int_0^1 \int_0^1 \frac{d}{ds} \varphi'(x_3 + t(x_1 - x_3) + s(x_2 - x_3)) ds dt \\ &= (x_1 - x_3)(x_2 - x_3) \int_0^1 \int_0^1 \varphi''(x_3 + t(x_1 - x_3) + s(x_2 - x_3)) ds dt. \end{aligned}$$



Assume now that  $\varphi \in C_b^{1,1}([0, \infty))$ . Using the first, the third, and the fifth line above, estimate (8.34) follows.  $\square$

We now consider the function  $w$  given in (1.24) and define

$$W(x_1, x_2, x_3) = \begin{cases} \frac{w(x_1, x_2, x_3)}{\sqrt{x_1 x_2 x_3}} & \text{if } (x_1, x_2, x_3) \in (0, \infty)^3 \\ \frac{1}{\sqrt{x_1 x_2}} & \text{if } x_3 = 0, (x_1, x_2) \in (0, \infty)^2 \\ \frac{1}{\sqrt{x_i x_3}} & \text{if } x_j = 0, x_i > x_3 > 0; \{i, j\} = \{1, 2\} \\ 0 & \text{otherwise.} \end{cases} \quad (8.35)$$

We then have:

**Lemma 8.11.** *Consider the function  $\Phi_\varphi = W\Delta\varphi$ , where  $\Delta\varphi$  and  $W$  are defined in (1.23) and (8.35) respectively.*

(i) *If  $\varphi \in C^{1,1}([0, \infty))$  then  $\Phi_\varphi \in C([0, \infty)^3)$ .*

(ii) *If  $\varphi \in C_b^{1,1}([0, \infty))$  then  $\Phi_\varphi \in C_0([0, \infty)^3)$ . In particular  $\Phi_\varphi$  is uniformly continuous on  $[0, \infty)^3$ .*

**Proof. Proof of (i).** By definition  $\Phi_\varphi \in C((0, \infty)^3)$ . Therefore it only remains to study the behaviour of  $\Phi_\varphi$  in a neighborhood of the boundary  $\partial[0, \infty)^3$  of  $[0, \infty)^3$ . First we show that  $\Phi_\varphi$  is continuous on  $\partial[0, \infty)^3$ .

Thanks to the symmetry of  $\Phi_\varphi$  in the  $x_1, x_2$  variables, we just need to prove: (i) for all  $(x_1, x_2) \in (0, \infty)^2$ ,

$$\Phi_\varphi(x_1, x_2, 0) = \frac{\Delta\varphi(x_1, x_2, 0)}{\sqrt{x_1 x_2}} \longrightarrow 0 \quad (8.36)$$

whenever  $x_1 \rightarrow 0$  or  $x_2 \rightarrow 0$  or  $(x_1, x_2) \rightarrow (0, 0)$ , and

(ii) for all  $x_1 > x_3 > 0$ ,

$$\Phi_\varphi(x_1, 0, x_3) = \frac{\Delta\varphi(x_1, 0, x_3)}{\sqrt{x_1 x_3}} \longrightarrow 0 \quad (8.37)$$

whenever  $x_1 \rightarrow x_3$  or  $x_3 \rightarrow 0$  or  $(x_1, x_3) \rightarrow (0, 0)$ .

By (8.34)  $|\Delta\varphi(x_1, x_2, 0)| \leq \|\varphi''\|_\infty x_1 x_2$  for all  $(x_1, x_2) \in (0, \infty)^2$ , which implies (8.36). Also  $|\Delta\varphi(x_1, 0, x_3)| \leq \|\varphi''\|_\infty x_3(x_1 - x_3)$  for all  $x_1 > x_3 > 0$ . Hence

$$\frac{|\Delta\varphi(x_1, 0, x_3)|}{\sqrt{x_1 x_3}} \leq \|\varphi''\|_\infty \sqrt{\frac{x_3}{x_1}} (x_1 - x_3) \leq \|\varphi''\|_\infty (x_1 - x_3),$$

which implies (8.37).

Then we prove that for any  $x \in \partial[0, \infty)^3$  and for any  $(x_n)_{n \in \mathbb{N}} \subset (0, \infty)^3$  such that  $x_n \rightarrow x$ , then  $\Phi_\varphi(x_n) \rightarrow \Phi_\varphi(x)$  as  $n \rightarrow \infty$ . Let us denote

$$\Omega = \{(x_1, x_2, x_3) \in (0, \infty)^3 : x_1 + x_2 \leq x_3\}.$$

Since  $x_4$  is defined as  $x_4 = (x_1 + x_2 - x_3)_+$ , then for all  $(x_1, x_2, x_3) \in (0, \infty)^3$ ,

$$(x_1, x_2, x_3) \in \Omega \quad \text{if and only if} \quad x_4 = 0.$$

It might happen that the sequence  $(x_n)_{n \in \mathbb{N}}$  “jumps” from  $\Omega$  to  $\Omega^c$ . If in every neighbourhood of  $x$  the sequence has points in both regions, then we may consider two subsequences, each one contained in one region only. For the sequel, the main estimate is the following: if we denote  $x_n = (x_1^n, x_2^n, x_3^n)$  and  $w(x_n) = \min \{ \sqrt{x_1^n}, \sqrt{x_2^n}, \sqrt{x_3^n}, \sqrt{x_4^n} \}$ , then by (8.34)

$$|\Phi_\varphi(x_n)| \leq \|\varphi''\|_\infty \frac{w(x_n)}{\sqrt{x_1^n x_2^n x_3^n}} |x_1^n - x_3^n| |x_2^n - x_3^n|. \quad (8.38)$$

We study case by case depending on where  $x$  lies.

Case  $x = (0, 0, 0)$ . If  $(x_n) \subset \Omega$  then  $x_4^n = 0$ ,  $w(x_n) = \sqrt{x_4^n} = 0$  and thus  $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$ .

If  $\{x_n\} \subset \Omega^c$  then  $x_4^n > 0$  and we study case by case depending on the relative order of  $x_1^n$ ,  $x_2^n$ , and  $x_3^n$ . Since  $\Phi_\varphi$  is symmetric in the  $x_1, x_2$  variables, we may assume without loss of generality that  $x_1^n \leq x_2^n$ . Note by (8.38) that we also may assume  $x_3^n \neq x_1^n$ ,  $x_3^n \neq x_2^n$ ; otherwise the result follows directly.

If  $x_1^n \leq x_2^n < x_3^n$ , then  $w(x_n) = \sqrt{x_4^n}$  and by (8.38)

$$\begin{aligned} |\Phi_\varphi(x_n)| &\leq \|\varphi''\|_\infty \frac{\sqrt{x_4^n}}{\sqrt{x_1^n x_2^n x_3^n}} (x_3^n - x_1^n)(x_3^n - x_2^n) \\ &\leq \|\varphi''\|_\infty \left( \frac{\sqrt{x_4^n} (x_3^n)^{3/2}}{\sqrt{x_1^n x_2^n}} + \frac{\sqrt{x_4^n x_1^n x_2^n}}{\sqrt{x_3^n}} \right) \\ &\leq \|\varphi''\|_\infty \left( \frac{(x_3^n)^{3/2}}{\sqrt{x_2^n}} + \sqrt{x_1^n x_2^n} \right). \end{aligned}$$

Since  $x_n \rightarrow x = 0$ , then  $\sqrt{x_1^n x_2^n} \rightarrow 0$ . Moreover, since  $x_n \in \Omega^c$  and  $x_1^n \leq x_2^n$ , then  $x_3^n < 2x_2^n$ , and so

$$\frac{(x_3^n)^{3/2}}{\sqrt{x_2^n}} \leq 2^{3/2} x_2^n \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

If  $x_1^n < x_3^n < x_2^n$ , then  $w(x_n) = \sqrt{x_1^n}$  and by (8.38)

$$\begin{aligned} |\Phi_\varphi(x_n)| &\leq \|\varphi''\|_\infty \frac{(x_2^n - x_3^n)(x_3^n - x_1^n)}{\sqrt{x_2^n x_3^n}} \\ &\leq \|\varphi''\|_\infty \left( \sqrt{x_2^n x_3^n} + \frac{x_1^n \sqrt{x_3^n}}{\sqrt{x_2^n}} \right) \\ &\leq \|\varphi''\|_\infty \left( \sqrt{x_2^n x_3^n} + \sqrt{x_1^n x_3^n} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Lastly, if  $x_3^n < x_1^n \leq x_2^n$ , then  $w(x_n) = \sqrt{x_3^n}$  and by (8.38)

$$\begin{aligned} |\Phi_\varphi(x_n)| &\leq \|\varphi''\|_\infty \frac{(x_1^n - x_3^n)(x_2^n - x_3^n)}{\sqrt{x_1^n x_2^n}} \\ &\leq \|\varphi''\|_\infty \left( \sqrt{x_1^n x_2^n} + \frac{(x_3^n)^2}{\sqrt{x_1^n x_2^n}} \right) \\ &\leq 2\|\varphi''\|_\infty \left( \sqrt{x_1^n x_2^n} + x_1 \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, in the three cases above  $\Phi_\varphi(x_n) \rightarrow 0 = \Phi_\varphi(x)$ .

Case  $x = (x_1, 0, 0)$  with  $x_1 > 0$ . Then  $w(x_n) = \min \{ \sqrt{x_2^n}, \sqrt{x_3^n} \}$  for  $n$  large enough. On the other hand

$$\begin{aligned} |x_2^n - x_3^n| &= (\sqrt{x_2^n} + \sqrt{x_3^n}) |\sqrt{x_2^n} - \sqrt{x_3^n}| \\ &\leq 2 \max \{ \sqrt{x_2^n}, \sqrt{x_3^n} \} |\sqrt{x_2^n} - \sqrt{x_3^n}|. \end{aligned}$$

Since  $\min \{ \sqrt{x_2^n}, \sqrt{x_3^n} \} \max \{ \sqrt{x_2^n}, \sqrt{x_3^n} \} = \sqrt{x_2^n x_3^n}$ , then by (8.38)

$$|\Phi_\varphi(x_n)| \leq 2\|\varphi''\|_\infty \frac{|x_1^n - x_3^n|}{\sqrt{x_1^n}} |\sqrt{x_2^n} - \sqrt{x_3^n}|$$

for  $n$  large enough. It then follows  $\Phi_\varphi(x_n) \rightarrow 0 = \Phi_\varphi(x)$  as  $n \rightarrow \infty$ .

The case  $x = (0, x_2, 0)$  with  $x_2 > 0$  is analogous to the previous one thanks to the symmetry of  $\Phi_\varphi$  in the  $x_1, x_2$  variables.

Case  $x = (0, 0, x_3)$  with  $x_3 > 0$ . Then  $x_n \in \Omega$  for  $n$  large enough,  $x_4^n = 0$  and  $w(x_n) = \sqrt{x_4^n} = 0$ . Thus  $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$  for  $n$  large enough.

Case  $x = (0, x_2, x_3)$  with  $x_2 > 0$  and  $x_3 > 0$ . If  $x_2 > x_3$  then  $w(x_n) = \sqrt{x_1^n}$  for  $n$  large enough and

$$|\Phi_\varphi(x_n) - \Phi_\varphi(x)| = \left| \frac{1}{\sqrt{x_2^n x_3^n}} \Delta\varphi(x_1^n, x_2^n, x_3^n) - \frac{1}{\sqrt{x_2 x_3}} \Delta\varphi(0, x_2, x_3) \right|,$$

which clearly goes to zero as  $n \rightarrow \infty$ . If  $x_2 < x_3$  then  $x_4^n = 0$  for  $n$  large enough and  $w(x_n) = \sqrt{x_4^n} = 0$ , thus  $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$ . If  $x_2 = x_3$  and  $(x_n) \subset \Omega$  for  $n$  large enough, then  $x_4^n = 0$ , thus  $\Phi_\varphi(x_n) = 0 = \Phi_\varphi(x)$ . If  $x_2 = x_3$  and  $(x_n) \subset \Omega^c$  for  $n$  large enough, then  $w(x_n) = \min \{ \sqrt{x_1^n}, \sqrt{x_4^n} \}$ , and by (8.38)

$$|\Phi_\varphi(x_n)| \leq \|\varphi''\|_\infty \frac{\min \{ \sqrt{x_1^n}, \sqrt{x_4^n} \}}{\sqrt{x_1^n x_2^n x_3^n}} |x_1^n - x_3^n| |x_2^n - x_3^n|.$$

On the one hand

$$|x_1^n - x_3^n| \leq 2 \max \{ \sqrt{x_1^n}, \sqrt{x_3^n} \} |\sqrt{x_1^n} - \sqrt{x_3^n}|.$$

On the other hand  $\min \{ \sqrt{x_1^n}, \sqrt{x_4^n} \} \leq \min \{ \sqrt{x_1^n}, \sqrt{x_3^n} \}$  for  $n$  large enough. Since  $\min \{ \sqrt{x_1^n}, \sqrt{x_3^n} \} \max \{ \sqrt{x_1^n}, \sqrt{x_3^n} \} = \sqrt{x_1^n x_3^n}$ , then

$$|\Phi_\varphi(x_n)| \leq 2\|\varphi''\|_\infty \frac{|x_2^n - x_3^n|}{\sqrt{x_2^n}} |\sqrt{x_1^n} - \sqrt{x_3^n}|,$$

which goes to zero as  $n \rightarrow \infty$  since  $x_2 = x_3$ . Thus  $\Phi_\varphi(x_n) \rightarrow 0 = \Phi_\varphi(x)$ .

The case  $x = (x_1, 0, x_3)$  with  $x_1 > 0$  and  $x_3 > 0$  is analogous to the previous one thanks to the symmetry of  $\Phi_\varphi$  in the  $x_1, x_2$  variables.

Case  $x = (x_1, x_2, 0)$  with  $(x_1, x_2) \in (0, \infty)^2$ . Then  $w(x_n) = \sqrt{x_3^n}$  for  $n$  large enough and

$$|\Phi_\varphi(x_n) - \Phi_\varphi(x)| = \left| \frac{1}{\sqrt{x_1^n x_2^n}} \Delta\varphi(x_1^n, x_2^n, x_3^n) - \frac{1}{\sqrt{x_1 x_2}} \Delta\varphi(x_1, x_2, 0) \right|,$$

which clearly goes to zero as  $n \rightarrow \infty$ .

**Proof of (ii).** By part (i)  $\Phi_\varphi \in C([0, \infty)^3)$ . Let us show now that for any given  $\varepsilon > 0$  there exists  $R(\varepsilon) > 0$  such that  $|\Phi_\varphi(x)| \leq \varepsilon$  for all  $x \in [0, \infty)^3 \setminus [0, R(\varepsilon)]^3$ .

Given  $R > 0$  and  $\alpha > 0$ , let  $(x_1, x_2, x_3) \in [0, \infty)^3 \setminus [0, R]^3$  and denote  $x_i = \min\{x_1, x_2, x_3\}$ ,  $x_k = \max\{x_1, x_2, x_3\}$  and  $x_j$  neither  $x_i$  nor  $x_k$ . Notice that  $x_k > R$  and the function  $W$  defined in (8.35) satisfies  $W(x_1, x_2, x_3) \leq \frac{1}{\sqrt{x_j x_k}}$ . If  $x_i > \alpha$  or  $x_j > \alpha$  then by (8.34)

$$|\Phi_\varphi(x_1, x_2, x_3)| \leq \frac{|\Delta\varphi(x_1, x_2, x_3)|}{\sqrt{x_j x_k}} \leq \frac{4\|\varphi\|_\infty}{\sqrt{\alpha R}} \leq \varepsilon,$$

provided  $R \geq \frac{16\|\varphi\|_\infty^2}{\alpha\varepsilon^2}$ . If  $x_i \leq \alpha$  and  $x_j \leq \alpha$  we study case by case depending on the relative position of  $x_1, x_2, x_3$ . Since  $\Phi_\varphi$  is symmetric in variables  $x_1$  and  $x_2$ , we may assume without loss of generality that  $x_2 \leq x_1$ . If  $x_k = x_1$ , using (8.34)

$$|\Phi_\varphi(x_1, x_2, x_3)| \leq \frac{2\|\varphi'\|_\infty(x_j - x_i)}{\sqrt{x_1 x_j}} \leq \frac{2\|\varphi'\|_\infty \sqrt{x_j}}{\sqrt{x_1}} \leq \frac{2\|\varphi'\|_\infty \sqrt{\alpha}}{\sqrt{R}} \leq \varepsilon,$$

provided  $R \geq \frac{4\|\varphi'\|_\infty^2 \alpha}{\varepsilon^2}$ . If  $x_k = x_3$  and  $x \in \Omega$  then  $x_4 = 0$  and  $\Phi_\varphi(x) = 0$ . If  $x_k = x_3$  and  $x \in \Omega^c$ , then  $x_1 \geq R/2$  and

$$|\Phi_\varphi(x_1, x_2, x_3)| \leq \frac{4\|\varphi\|_\infty}{\sqrt{x_1 x_3}} \leq \frac{4\sqrt{2}\|\varphi\|_\infty}{R} \leq \varepsilon,$$

provided  $R \geq \frac{4\sqrt{2}\|\varphi\|_\infty}{\varepsilon}$ .

Finally, if we chose  $R \geq \max \left\{ \frac{16\|\varphi\|_\infty^2}{\alpha\varepsilon^2}, \frac{4\|\varphi'\|_\infty^2 \alpha}{\varepsilon^2}, \frac{4\sqrt{2}\|\varphi\|_\infty}{\varepsilon} \right\}$  then  $\Phi_\varphi \in C_0([0, \infty)^3)$  and in particular,  $\Phi_\varphi$  is uniformly continuous in  $[0, \infty)^3$ .  $\square$

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