RIGHT UNIMODAL AND BIMODAL SINGULARITIES IN POSITIVE CHARACTERISTIC

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Abstract. The problem of classification of real and complex singularities was initiated by Arnold in the sixties who classified simple, unimodal and bimodal singularities w.r.t. right equivalence. The classification of simple singularities positive characteristic was achieved by Greuel and the author in 2014. In the present paper we classify right unimodal and bimodal singularities in positive characteristic by giving explicit normal forms. It is surprising that in positive characteristic, there are no infinite series of unimodal and bimodal singularities. Moreover, the Milnor number of simple, unimodal and bimodal singularity satisfies \( \mu(f) \leq 4p \). As an application we prove that, for singularities of right modality at most 2, the \( \mu \)-constant stratum is smooth and its dimension is equal to the right modality.

1. Introduction

We classify hypersurface singularities \( f \in K[[x_1, \ldots, x_n]] \) which are unimodal and bimodal w.r.t. right equivalence, where \( K \) is an algebraically closed field of positive characteristic. That is, the singularities have modality 1 resp. 2 up to the change of coordinates (or right equivalence, see Section 2.1). The notion of modality was introduced by Arnold in the seventies into the singularity theory for real and complex singularities. He classified simple, unimodal and bimodal hypersurface singularities w.r.t. right equivalence (2,5,6). He showed that the simple singularities are the ADE-singularities, i.e. the two infinite series \( A_k, k \geq 1 \), \( D_k, k \geq 4 \), and the three exceptional singularities \( E_6, E_7, E_8 \). The unimodal singularities include three families of parabolic singularities, a three-index family of hyperbolic singularities and 14 exceptional singularities. There are 8 infinite series and 14 exceptional bimodal singularities. The unimodal and bimodal singularities are also classified by degenerations of elliptic curves (11), which were studied by Kodaira. In positive characteristic, the right simple singularities were recently classified by Greuel and the author in [14].

The main result of the present paper is the classification of right unimodal and bimodal singularities in positive characteristic with tables of normal forms. Our lists of normal forms for unimodal and bimodal singularities are given in [3]. A surprising fact of our classification is that in positive characteristic, there are no infinite series of unimodal and bimodal singularities. Moreover, the Milnor number of simple, unimodal and bimodal singularity satisfies \( \mu(f) \leq 4p \) (Corollary 3.7). Another surprising fact is that, some A- or D-singularities occurs in the lists of unimodal and bimodal singularities. On the other hand, if the characteristic is 2 or 3, then all simple, unimodal and bimodal singularities must be of type \( A, D \) or \( E \). The adjacency diagrams of simple, unimodal and bimodal singularities are also determined (Theorem 3.5). As an application of the classification, we show that the \( \mu \)-constant stratum of simple, unimodal and bimodal singularities is smooth and its dimension coincides with the right modality (Corollary 3.8, 3.9).

We organize our proof in the form of a singularity determinator (§ 4), finding for every given singularities its place in the list of § 3. This method was used by Arnold in [6] to study local normal
forms of complex functions. He formed a singularity determinant to recover the classification of simple, unimodal and bimodal singularities, to obtain classifications of all the singularities with Milnor number \( \mu \leq 16 \), all the singularities of corank 2 with nonzero 4-jet, all the singularities of corank 3 with a 3-jet, which determine an irreducible cubic, and some other singularities. However, in positive characteristic, we obtain only the classification of simple, unimodal and bimodal singularities; the other classifications are still unknown. It seems that the classification problems are very different in positive characteristic even for the singularities of corank 1 \([22]\). The main difficulty is to find for each series \( A_k, D_k, J_{2,q}, \ldots \) in \( \mathbb{N} \) the exact bound of its index. This also shows that, there are only finitely many series of simple, unimodal and bimodal singularities.

Note that, for contact equivalence and for \( K = \mathbb{C} \), it was proved by Giusti in \([11]\) that ADE-singularities are contact simple. The classification of contact unimodal singularities was achieved by Wall in \([24]\). Greuel and Kröning showed in \([12]\) that the contact simple singularities over a field of positive characteristic are again exactly the ADE-singularities or the rational double points of Artin’s list \([7]\).

2. Modality

Modality was introduced by Arnol’d in connection with the classification of singularities of functions under right equivalence. It has been generalized to arbitrary actions of algebraic groups by Vinberg \([18]\). Wall \([20]\) described two possible generalizations for use in other classification problems in singularity theory. Both generalizations are developed in detail by Greuel and the author \([14]\) for any characteristic and it was proved that they coincide.

2.1. Right modality. Consider an action of algebraic group \( G \) on a variety \( X \) (over a given algebraically closed field \( K \)) and a Rosenlicht stratification \( \{(X_i, p_i), i = 1, \ldots, s\} \) of \( X \) w.r.t. \( G \). That is, a stratification \( X = \bigcup_{i=1}^{s} X_i \), where the stratum \( X_i \) is a locally closed \( G \)-invariant subvariety of \( X \) such that the projection \( p_i : X_i \to X_i/G \) is a geometric quotient. For each open subset \( U \subset X \) the modality of \( U \), \( G\text{-mod}(U) \), is the maximal dimension of the images of \( U \cap X_i \) in \( X_i/G \). The modality \( G\text{-mod}(x) \) of a point \( x \in X \) is the minimum of \( G\text{-mod}(U) \) over all open neighbourhoods \( U \) of \( x \).

Let \( K[[x]] = K[[x_1, \ldots, x_n]] \) the formal power series ring and let the right group, \( R := \text{Aut}(K[[x]]) \), act on \( K[[x]] \) by \( (\Phi, f) \to \Phi(f) \). Two elements \( f, g \in K[[x]] \) are called right equivalent, \( f \sim_r g \), if they belong to the same \( R \)-orbit, or equivalently, there exists a coordinate change \( \Phi \in \text{Aut}(K[[x]]) \) such that \( g = \Phi(f) \).

Recall that for \( f \in m \subset K[[x]] \), \( \mu(f) := \dim K[[x]]/j(f), \quad j(f) = (j_x f, \ldots, j_{x^n} f) \), denotes the Milnor number of \( f \) and that \( f \) is isolated if \( \mu(f) < \infty \). The \( k \)-jet of \( f \), \( j^k(f) \), is the image of \( f \) in the jet space \( J_k := m/m^{k+1} \). We call \( f \) to be right \( k \)-determined if each singularity having the same \( k \)-jet with \( f \), is right equivalent to \( f \). A number \( k \) is called right sufficiently large for \( f \), if there exists a neighbourhood \( U \) of the \( j^k f \) in \( J_k \) such that every \( g \in K[[x]] \) with \( j^k g \in U \) is right \( k \)-determined. The right modality of \( f \), \( R\text{-mod}(f) \), is defined to be the \( R_k \)-modality of \( j^k f \) in \( J_k \) with \( k \) right sufficiently large for \( f \), where \( R_k \) the \( k \)-jet of \( R \). A singularity \( f \in K[[x]] \) is called (right) simple, uni-modal, bi-modal and \( r \)-modal if its (right) modality is equal to 0,1,2 and \( r \) respectively. These notions are independent of the right sufficiently large \( k \) \([14]\), Def. 2.3).

The second description is in relation with versal or complete deformation. Let \( T \) be an affine variety with its algebra of global section \( \mathcal{O}(T) \). Then a family \( f_t(x) := F(x, t) \in \mathcal{O}(T)[[x]] \) is called an unfolding (deformation with trivial section) of \( f \) over a pointed space \( T, t_0 \) if \( F(x, t_0) = f \) and \( f_t \in m \) for all \( t \in T \). For a basis \( \{x^{\alpha_1}, \ldots, x^{\alpha_N}\} \) of \( m/m \cdot j(f) \), the semiuniversal unfolding is represented by \( F(x, \lambda) := f(x) + \sum_{i=1}^{N} \lambda_i x^{\alpha_i} \) with \( \lambda = (\lambda_1, \ldots, \lambda_N) \) the coordinate of \( \lambda \in \mathbb{A}^N \). Note that from the exact sequence

\[ 0 \to j(f)/m \cdot j(f) \to m/m \cdot j(f) \to m/j(f) \to 0 \]

we get \( N = \mu + n - 1 \). Since \( m \cdot j(f) \) is the tangent space of the orbit of the right group \( R \) at \( f \) \([9]\), \( N \) is the codimension of the orbit in \( m \).

An unfolding \( F(x, t) \) over \( T, t_0 \) is called right complete if any unfolding \( H(x, s) \) over \( S, s_0 \) is isomorphic to a pullback of \( F \) after passing to some étale neighbourhood of \( S, s_0 \), see \([14]\). An important
property of the complete unfoldings is that they are sufficient to determine the modality, i.e. if $F$ is a complete unfolding of $f$, then modality of $f$ w.r.t. $F$ [13 Def. 2.5] equals to modality of $f$, see [14 Prop. 2.12(ii)]. The semiuniversal unfolding of an isolated hypersurface singularity is right complete (see [15] for the analytic case and [14] for the general case). Consequently, we may define modality of $f$ as follows: “Let $f_\lambda$ be the semiuniversal unfolding of $f$ over $\mathbb{A}^N,0$. If the set of singularities $f_\lambda \in K[[x]]$ ($\lambda$ in some Zariski neighbourhood of $0 \in \mathbb{A}^N$) falls into finitely many families of right classes, each depending on $r$ parameters (at most) then $f$ is right (resp. contact) $r$-modal (at most).”

**Remark 2.1.** (1) For convergent power series over the complex numbers it does not make any difference whether we consider the semiuniversal deformation (without section) given by the Milnor algebra $\mathbb{C}\{x_1, \ldots, x_n\}/j(f)$ or the semiuniversal deformation with section given by $m/m \cdot j(f)$. However, in positive characteristic we have to consider the latter (cf. [14] and 2.2).

(2) The difference between the classical versal and our complete deformation is twofold. First, we consider deformations over algebraic varieties and not just of the spectrum of a complete local ring (as in positive characteristic we have to consider the latter (cf. [14] and 2.2).

2.2. **Proper modality.** In [10] Gabrielov showed in the complex analytic case that the right modality is equal to the dimension of the $\mu$-constant stratum in a semi-universal deformation of $f$. This is not true in positive characteristic since $f = x^2 + y^4 \in A_3 \subset K[[x,y]]$ with $\text{char}(K) = 3$, is unimodal, but the dimension of the stratum $\mu = 3$ into the semiuniversal deformation $f + a_0 + a_1y + a_2y^2$, is equal to 0. In positive characteristic we need to consider deformations with section. Let $f_\lambda(x) := F(x,\lambda)$ be the semiuniversal unfolding of $f$ with trivial section over affine variety $\mathbb{A}^N,0$ with $N = \mu + n - 1$ as above. We define the proper modality of $f$, denoted by $\text{pmod}(f)$, to be the dimension at 0 of the $\mu$-constant stratum in $\mathbb{A}^N$:

$$\Delta_\mu := \{ \lambda \in \mathbb{A}^N \mid \mu(f_\lambda) = \mu \}.$$  

**Conjecture 2.2.** $\text{pmod}(f) = \text{rmod}(f)$.

See Corollary 3.9 for a partial result of the conjecture. Namely, if $\text{rmod}(f) \leq 2$ then $\text{pmod}(f) = \text{rmod}(f)$.

3. **Right unimodal, bimodal singularities and adjacency diagrams**

In this section we present the result of our classification by explicit lists of normal forms, the adjacency diagrams of simple, unimodal and bimodal singularities, and their applications. Recall that a normal form is a modular family $F(x,t) \in \mathcal{O}(T)[[x]]$, i.e. for each $t \in T$ there are only finitely many $t' \in T$ such that $f_{t'} \sim f_t$. Notice that, if $F(x,t)$ is a normal form, then the right modality of $F(x,t)$ for all $t \in T$, is not less than the dimension of $T$.

3.1. **Right unimodal singularities.**

**Theorem 3.1.** Let $p = \text{char}(K) > 2$. A hypersurface singularity $f \in m^2$ is right unimodal if and only if it is right equivalent to one of the following forms:

I. $n = 1$ ($f \in K[[x]]$). The classification is given in Table 1

II. $n = 2$ ($f \in K[[x,y]]$). The classification is given in Table 2

III. $n = 3$ ($f \in K[[x,y,z]]$). The classification is given in Table 3

IV. $n > 3$. The classification is given in Table 4

**Theorem 3.2.** Let $p = \text{char}(K) = 2$. A hypersurface singularity $f \in m^2$ is right unimodal if and only if $n$ is odd and $f$ is right equivalent to one of singularities in the Table 5.
Let Theorem 3.3.

3.2. Right bimodal singularities.

**Theorem 3.3.** Let $p = \text{char}(K) > 2$. A hypersurface singularity $f \in m^2$ is right bimodal if and only if it is right equivalent to one of the following forms

I. $n = 1$ ($f \in K[[x]]$). The list is given in Table 7

II. $n = 2$ ($f \in K[[x, y]]$). The list is given in Table 8

III. $n = 3$ ($f \in K[[x, y, z]]$). The list is given in Table 8

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form</th>
<th>Conditions</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_k$</td>
<td>$x^p + ax^{k+1}$</td>
<td>$p \leq k \leq 2p - 2$</td>
<td>$k$</td>
</tr>
<tr>
<td>$D_p$</td>
<td>$x^2y + y^{p-1}$</td>
<td>$3 &lt; p$</td>
<td>$p$</td>
</tr>
<tr>
<td>$D_k$</td>
<td>$x^2y + ay^p + y^{k-1}$</td>
<td>$3 \leq p &lt; k - 1 \leq 2p - 2$</td>
<td>$k$</td>
</tr>
<tr>
<td>$E_{12}$</td>
<td>$x^3 + y^7 + axy^5$</td>
<td>$7 &lt; p$</td>
<td>12</td>
</tr>
<tr>
<td>$E_{13}$</td>
<td>$x^3 + xy^5 + ay^8$</td>
<td>$7 &lt; p$</td>
<td>13</td>
</tr>
<tr>
<td>$E_{14}$</td>
<td>$x^3 + y^8 + axy^6$</td>
<td>$7 &lt; p$</td>
<td>14</td>
</tr>
<tr>
<td>$J_{10} = J_{2.0} = T_{2,3,6}$</td>
<td>$x^3 + y^6 + ax^2y^2$</td>
<td>$5 &lt; p$</td>
<td>10</td>
</tr>
<tr>
<td>$J_{2,q}$</td>
<td>$x^3 + ax^2y^2 + y^{6+q}$</td>
<td>$6 &lt; 6 + q &lt; p$</td>
<td>$q + 10$</td>
</tr>
<tr>
<td>$W_{12}$</td>
<td>$x^4 + y^5 + ax^2y^3$</td>
<td>$p &gt; 5$</td>
<td>12</td>
</tr>
<tr>
<td>$W_{13}$</td>
<td>$x^4 + xy^4 + axy^6$</td>
<td>$p &gt; 5$</td>
<td>12</td>
</tr>
<tr>
<td>$X_9 = X_{1,0} = T_{2,4,4}$</td>
<td>$x^4 + y^4 + ax^2y^2$</td>
<td>$3 &lt; p$</td>
<td>9</td>
</tr>
<tr>
<td>$X_{1,q} = T_{2,4,4+q}$</td>
<td>$x^4 + x^2y^2 + ay^{4+q}$</td>
<td>$4 &lt; 4 + q &lt; p$</td>
<td>$q + 9$</td>
</tr>
<tr>
<td>$Y_{r,s} = T_{2,4+1+r+s}$</td>
<td>$x^{4+r} + ax^2y^2 + y^{4+s}$</td>
<td>$4 &lt; 4 + r \leq 4 + s &lt; p$</td>
<td>$9 + r + s$</td>
</tr>
<tr>
<td>$Z_{11}$</td>
<td>$x^3y + y^3 + axy^4$</td>
<td>$5 &lt; p$</td>
<td>11</td>
</tr>
<tr>
<td>$Z_{12}$</td>
<td>$x^3y + xy^4 + ax^2y^3$</td>
<td>$5 &lt; p$</td>
<td>12</td>
</tr>
<tr>
<td>$Z_{13}$</td>
<td>$x^3y + y^6 + axy^5$</td>
<td>$5 &lt; p$</td>
<td>13</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form</th>
<th>Conditions</th>
<th>$\mu(g)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_8 = T_{3,3,3}$</td>
<td>$x^3 + y^3 + z^3 + axyz$</td>
<td>$3 &lt; p$</td>
<td>8</td>
</tr>
<tr>
<td>$Q_{10}$</td>
<td>$x^3 + y^3 + yz^2 + axy^3$</td>
<td>$3 &lt; p$</td>
<td>10</td>
</tr>
<tr>
<td>$Q_{11}$</td>
<td>$x^3 + y^2z + xz^2 + axz^5$</td>
<td>$3 &lt; p$</td>
<td>11</td>
</tr>
<tr>
<td>$Q_{12}$</td>
<td>$x^3 + y^2z + yz^2 + axy^4$</td>
<td>$5 &lt; p$</td>
<td>12</td>
</tr>
<tr>
<td>$S_{11}$</td>
<td>$x^4 + y^2z + axz^2 + axz^3$</td>
<td>$3 &lt; p$</td>
<td>11</td>
</tr>
<tr>
<td>$S_{12}$</td>
<td>$x^2y + y^2z + xz^2 + ax^2z^5$</td>
<td>$3 &lt; p$</td>
<td>12</td>
</tr>
<tr>
<td>$T_{q,r,s}$</td>
<td>$x^3 + y^r + z^s + axyz$</td>
<td>$3 \leq q \leq r \leq s &lt; p, \frac{1}{q} + \frac{1}{r} + \frac{1}{s} &lt; 1, q + r + s - 1$</td>
<td>$q + r + s - 1$</td>
</tr>
<tr>
<td>$U_{12}$</td>
<td>$x^3 + y^3 + z^4 + axyz^2$</td>
<td>$3 &lt; p$</td>
<td>12</td>
</tr>
</tbody>
</table>

| Normal form | $g(x_1, x_2, x_3) + x_1^4 + \ldots + x_n^4$ | $g$ is one of the singularities in Table 4 |

3.2. Right bimodal singularities.

**Theorem 3.3.** Let $p = \text{char}(K) > 2$. A hypersurface singularity $f \in m^2$ is right bimodal if and only if it is right equivalent to one of the following forms

I. $n = 1$ ($f \in K[[x]]$). The list is given in Table 7

II. $n = 2$ ($f \in K[[x, y]]$). The list is given in Table 8

III. $n = 3$ ($f \in K[[x, y, z]]$). The list is given in Table 8

**Table 1.** Unimodal singularities ($p > 2, n = 1$)

**Table 2.** Unimodal singularities ($p > 2, n = 2$)

**Table 3.** Unimodal singularities ($p > 2, n = 3$)

**Table 4.** Unimodal singularities ($p > 2, n > 3$)
IV. $n > 3$. The list is given in Table 7.

Theorem 3.4. Let $p = \text{char}(K) = 2$. A hypersurface singularity $f \in \mathfrak{m}^2$ is right bimodal if and only if it is right equivalent to one of the following forms

I. $n$ odd: The list is given in the Table 10.

II. $n$ even: The list is given in the Table 11.

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form</th>
<th>Conditions</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$</td>
<td>$ax_1^2 + x_1^3 + x_2x_3 + \ldots + x_{n-1}x_n \ a \in K$</td>
<td>$2$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5. Unimodal singularities ($p = 2, n \geq 1$)

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form</th>
<th>Conditions</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_k$</td>
<td>$a_1x^p + a_2x^{2p} + x^{k+1}$</td>
<td>$2p \leq k \leq 3p - 2$</td>
<td>$k$</td>
</tr>
</tbody>
</table>

Table 6. Bimodal singularities ($p > 2, n = 1$)

<table>
<thead>
<tr>
<th>Name</th>
<th>Normal form ($a = a_0 + a_1y$)</th>
<th>Conditions</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{10} = J_{2,0} = T_{2,3,6}$</td>
<td>$x^3 + bxy^2 + y^5 + ay^5$</td>
<td>$4b^3 + 27 \neq 0, p = 5$</td>
<td>$10$</td>
</tr>
<tr>
<td>$J_{2,q} = T_{2,3,6+q}$</td>
<td>$x^3 + x^2y^2 + ayq + by^q$</td>
<td>$p \leq 6 + q &lt; 2p, b \neq 0, p \geq 5$</td>
<td>$q + 10$</td>
</tr>
<tr>
<td>$J_{3,0}$</td>
<td>$x^3 + 2xy^3 + cyx^q + y^9$</td>
<td>$4b^3 + 27 \neq 0, q \leq 7 &lt; p$</td>
<td>$16$</td>
</tr>
<tr>
<td>$J_{3,q}$</td>
<td>$x^3 + x^2y^q + ay^{3+q}$</td>
<td>$a_0 \neq 0, 9 &lt; q &lt; p$</td>
<td>$q + 16$</td>
</tr>
<tr>
<td>$W_{17}$</td>
<td>$x^4 + xy^5 + ay^7$</td>
<td>$7 &lt; p$</td>
<td>$17$</td>
</tr>
<tr>
<td>$W_{18}$</td>
<td>$x^4 + ay^7 + ax^2y^q$</td>
<td>$7 &lt; p$</td>
<td>$18$</td>
</tr>
<tr>
<td>$W_{1,0}$</td>
<td>$x^4 + ax^2y^3 + y^6$</td>
<td>$a_0^2 \neq 4.5 &lt; p$</td>
<td>$15$</td>
</tr>
<tr>
<td>$W_{1,q}$</td>
<td>$x^4 + ax^2y^3 + ay^{6+q}$</td>
<td>$a_0 \neq 0, 7 \leq 6 + q &lt; p$</td>
<td>$q + 15$</td>
</tr>
<tr>
<td>$W_{1,2q-1}$</td>
<td>$(x^2 + y^5)^2 + axy^{4+q}$</td>
<td>$a_0 \neq 0, 5 \leq 4 + q &lt; p$</td>
<td>$2q + 14$</td>
</tr>
<tr>
<td>$W_{1,2q}$</td>
<td>$(x^2 + y^5)^2 + ax^2y^{3+q}$</td>
<td>$a_0 \neq 0, 4 \leq 3 + q &lt; p &gt; 5$</td>
<td>$2q + 15$</td>
</tr>
<tr>
<td>$Z_{12}$</td>
<td>$x^3y + xy^4 + ay^5 + bx^2y^3$</td>
<td>$p = 5$</td>
<td>$12$</td>
</tr>
<tr>
<td>$Z_{13}$</td>
<td>$x^3y + y^6 + ay^5 + bx^2y^3$</td>
<td>$p = 5$</td>
<td>$13$</td>
</tr>
<tr>
<td>$Z_{17}$</td>
<td>$x^3y + ax^6 + y^8$</td>
<td>$7 &lt; p$</td>
<td>$17$</td>
</tr>
<tr>
<td>$Z_{18}$</td>
<td>$x^3y + xy^6 + ay^9$</td>
<td>$7 &lt; p$</td>
<td>$18$</td>
</tr>
<tr>
<td>$Z_{19}$</td>
<td>$x^3y + y^9 + axy^7$</td>
<td>$7 &lt; p$</td>
<td>$19$</td>
</tr>
<tr>
<td>$Z_{1,0}$</td>
<td>$x^3y + bx^2y^3 + cxy^6 + y^7$</td>
<td>$4b^3 + 27 \neq 0, 7 &lt; p$</td>
<td>$15$</td>
</tr>
<tr>
<td>$Z_{1,q}$</td>
<td>$x^3y + x^2y^3 + ay^{7+q}$</td>
<td>$a_0 \neq 0, 7 &lt; 7 + q &lt; p$</td>
<td>$q + 15$</td>
</tr>
</tbody>
</table>

Table 7. Bimodal singularities ($p > 2, n = 2$)
3.3. Adjacencies of simple, unimodal and bimodal singularities. In the following we give diagrams of adjacencies for all class of simple singularities and singularities in Tables 1–11. Moreover a singularity in these tables deforms only into classes listed in the diagrams. Recall that a class \( \mathcal{D} \) of singularities is adjacent to class \( \mathcal{C} \), \( \mathcal{C} \leftarrow \mathcal{D} \), if every \( f \in \mathcal{D} \) can be deformed into an element in \( \mathcal{C} \) by a deformation. That is, there exists an unfolding \( f_t \) of \( f = f_{t_0} \) over an affine variety \( T, t_0 \) and a Zariski open subset \( V \subset T \) such that \( f_t \in \mathcal{C} \) for all \( t \in V \).

**Theorem 3.5.** Any singularity in Tables 1–11 deforms only into singularities given in the following adjacency diagrams (\( T_{2, 3, 6+q} = J_{2, q}, T_{2, 4, 4+q} = X_{1, q}, T_{2, 4+r, 4+s} = Y_{r, s} \)).
3.4. Milnor number, \( \mu \)-constant stratum and proper modality. In this section we give several applications of the classification of unimodal and bimodal singularities. The first two corollaries below follow from the classification of right simple, unimodal and bimodal singularities (\cite{14}, and Theorems 3.1,3.3). The remainings will be proved in Section 5.5.

\begin{itemize}
  \item \( A_{k-1} \leftarrow D_k \leftarrow E_{k+1} \leftarrow A_p \leftarrow D_p \leftarrow J_{2,k-1} \leftarrow J_{3,k-4} \)
  \item \( E_k \leftarrow J_{2,0}; E_{14} \leftarrow J_{3,0}; J_{6,s} \leftarrow E_{6s+k-1}; s = 2, 3; k = 1, 2, 3. \)
  \item \( T_{q', r', s'} \leftarrow T_{q, r, s} \) if \( (q', r', s') \leq (q, r, s) \), i.e. \( q' \leq q, r' \leq r, s' \leq s \).
  \item \( Q_{12} \leftarrow Q_{2,0} \leftarrow Q_{2,1} \leftarrow Q_{2,2} \leftarrow Q_{2,3} \leftarrow \cdots \)
  \item \( S_{1,1} \leftarrow S_{1,2} \leftarrow \cdots \)
  \item \( S_{1,0} \leftarrow S_{16} \leftarrow S_{17} \)
  \item \( S_{1,3} \leftarrow S_{1,2} \leftarrow \cdots \)
  \item \( T_{3,3,3} \leftarrow T_{3,3,4} \leftarrow Q_{10} \leftarrow Q_{11} \leftarrow Q_{12} \)
  \item \( T_{3,4,4} \leftarrow S_{11} \leftarrow S_{12} \leftarrow S_{1,0} \leftarrow U_{16} \)
  \item \( T_{4,4,4} \leftarrow U_{12} \leftarrow U_{1,0} \leftarrow U_{1,1} \leftarrow \cdots \)
  \item \( Z_{13} \leftarrow Z_{1,0} \leftarrow Z_{1,1} \leftarrow Z_{1,2} \leftarrow Z_{1,3} \leftarrow \cdots \)
  \item \( E_7 \leftarrow T_{2,4,4} \leftarrow T_{2,4,5} \leftarrow Z_{11} \leftarrow Z_{12} \leftarrow Z_{13} \)
  \item \( T_{2,5,5} \leftarrow W_{12} \leftarrow W_{13} \leftarrow W_{1,0} \)
\end{itemize}

\section{Milnor number, \( \mu \)-constant stratum and proper modality}

In this section we give several applications of the classification of unimodal and bimodal singularities. The first two corollaries below follow from the classification of right simple, unimodal and bimodal singularities (\cite{14}, and Theorems 3.1,3.3). The remainings will be proved in Section 5.5.

\begin{itemize}
  \item \textbf{Corollary 3.6.} If \( p \leq 3 \), then \( f \) is of type \( A, D \) or \( E \).
  \item Let \( f \in K[[x]] \), with \( p = \text{char}(K) > 0 \) such that \( \text{rmod}(f) \leq 2 \). Then
  \item \textbf{Corollary 3.7.} We have \( \mu(f) \leq 4p \).
  \item \textbf{Corollary 3.8.} The \( \mu \)-constant stratum of \( f \) is a linear space, and hence smooth.
\end{itemize}

Note that this is not true in general as shown by Luengo in \cite{17}.
Corollary 3.9. We have \( \text{rmod}(f) = \text{pmod}(f) \).

4. Singularity Determinator

In [9] Arnol’d supplied lists of normal forms which contain all the singularities with the modality number \( \text{mod} = 0, 1, 2 \), all the singularities with Milnor number \( \mu \leq 16 \), all the singularities of corank 2 with nonzero 4-jet, all the singularities of corank 3 with a 3-jet, which determine an irreducible cubic, and some other singularities. The proof of Arnol’d is organized as a determinator consisting of 105 theorems. We follow this scheme and organize our proof a singularity determinator of 152 theorems. This gives an algorithm finding for every given singularity its place in the list of §3. However we obtain only the classification of simple, unimodal and bimodal singularities. For the other classifications a new method should be needed.

Notations:
\[ \Rightarrow \quad \text{“implies”}. \]
\[ \rightarrow \quad \text{“see”}. \]
\[ \text{crk} \quad \text{the corank of the Hessian of } f \text{ at the origin, which is used to reduce the number of variables, see } \]
\[ \Delta \quad \text{discriminant, in Theorems 71, 72: } \Delta = 4(a^3 + b^3) + 27 - a^2b^2 - 18ab. \]
\[ j_{\{x^\alpha\}}(f(x)) \quad \text{quasijet of } f \text{ determined by } \{x^\alpha\}, \text{ defined as follows.} \]

Here \( \{\alpha_i\} \) is a system of \( n \) points defining an affine hyperplane \( H \) in \( \mathbb{R}^n \). Let \( v: \mathbb{R}^n \rightarrow \mathbb{R} \) be the linear form defining \( H \) with \( v(\alpha_i) = 1 \) for all \( i \). Then \( j_{\{x^\alpha\}}(f) \) is the image of \( f \) in \( K[[x]] \) modulo the ideal generated by \( x^\alpha \), \( v(\alpha) > 1 \).

4.1. Singularity determinator in characteristic \( \geq 5 \).

1. \( \mu(f) < \infty \Rightarrow \) one of the four possibilities holds:
   \[ \text{crk}(f) \leq 1 \implies 2, \]
   \[ = 2 \implies 4 - 73, \]
   \[ = 3 \implies 74 - 119, \]
   \[ > 3 \implies 120. \]
2. \( \text{crk}(f) \leq 1, \mu < 3p \Rightarrow \text{rmod}(f) = |\mu/p| \text{ and } f \in A_\mu. \]
3. \( \text{crk}(f) \leq 1, \mu \geq 3p \Rightarrow \text{rmod}(f) \geq 3. \)

Corank 2 Singularities

Through theorems 4–73, \( f \in K[[x, y]]. \)

4. \( j^3(f) = 0 \Rightarrow \) one of the four possibilities holds:
   \[ j^3f \sim_r x^2y + y^3 \implies 5, \]
   \[ \sim_r x^2y \implies 6, \]
   \[ \sim_r x^3 \implies 9–30, \]
   \[ = 0 \implies 31–73. \]
5. \( j^3(f) = x^2y + y^3 \Rightarrow f \in D_4. \)
6. \( j^3(f) = x^2y \Rightarrow f \sim_r x^2y + \alpha(y), j^3(\alpha) = 0 \Rightarrow 7–8. \)
7. \( f = x^2y + \alpha(y), j^3(\alpha) = 0, k := \mu(\alpha) \leq 3p - 1 \Rightarrow f \in D_{k+2}. \)
8. \( f = x^2y + \alpha(y), j^3(\alpha) = 0, \mu(\alpha) \geq 3p \Rightarrow \text{rmod}(f) \geq 3. \)

Through theorems 9–12, \( k = 1, 2, 3 \) for \( p > 7, k = 1, 2 \) for \( p = 7, k = 1 \) for \( p = 5. \)

9. \( j_{x^3,y^{3k}}f(x, y) = x^3 \Rightarrow \) one of the four possibilities holds:
   \[ j_{x^3,y^{3k+1}}f(x, y) \sim_r x^3 + y^{3k+1} \implies 10, \]
   \[ j_{x^3,y^{3k+1}}f(x, y) \sim_r x^3 + xy^{2k+1} \implies 11, \]
   \[ j_{x^3,y^{3k+2}}f(x, y) \sim_r x^3 + y^{3k+2} \implies 12, 13, \]
   \[ j_{x^3,y^{3k+2}}f(x, y) = x^3 \implies 13, 26. \]
10. \( j_{x^3,y^{3k+1}}f(x, y) = x^3 + y^{3k+1} \text{ and } 3k + 1 < p \Rightarrow f \in E_{6k}. \)
11. \( j_{x^3,y^{2k+1}}f(x, y) = x^3 + xy^{2k+1} \text{ and } 3k + 1 < p \Rightarrow f \in E_{6k+1}. \)
12. \( j_{x^3,y^{3k+2}}f(x, y) = x^3 + y^{3k+2} \text{ and } 3k + 2 < p \Rightarrow f \in E_{6k+2}. \)
13. \( p = 5 \) and \( j_{x^3,y^5} f(x,y) = x^3 + ay^5 \Rightarrow \) one of the three possibilities holds:
   \( j_{x^3,y^5} f(x,y) \sim_r x^3 + bx^2y^2 + y^6 + ay^5, \ 4b^3 + 27 = \neq 0 \Rightarrow 14 \),
   \( j_{x^3,y^5} f(x,y) \sim_r x^3 + x^2y^2 + ay^5 \Rightarrow 15,16 \),
   \( j_{x^3,y^5} f(x,y) \sim_r x^3 + ay^5 \Rightarrow 17 \).
14. \( p = 5 \) and \( j_{x^3,y^5} f = x^3 + bx^2y^2 + y^6 + ay^5, \ 4b^3 + 27 = \neq 0 \Rightarrow f \in J_{2,9} \).
15. \( p = 5, j_{x^3,y^5} f = x^3 + x^2y^2 + ay^5 \) and \( \mu < 14 \Rightarrow f \in J_{2,9} \) with \( q = \mu - 10 > 0 \).
16. \( p = 5, j_{x^3,y^5} f = x^3 + x^2y^2 + ay^5 \) and \( \mu \geq 14 \Rightarrow \text{rmod}(f) \geq 3 \).
17. \( p = 5, j_{x^3,y^5} f = x^3 + ay^5 \Rightarrow \) one of the two possibilities holds:
   \( j_{x^3,y^5} f(x,y) \sim_r x^3 + ay^5 + y^7 \Rightarrow 18 \),
   \( j_{x^3,y^5} f(x,y) = x^3 + ay^5 \Rightarrow 19 \).
18. \( p = 7 \) and \( j_{x^3,y^5} f = x^3 + ay^5 + y^7 \Rightarrow f \in E_{12} \).
19. \( p = 5, j_{x^3,y^5} f = x^3 + ay^5 \Rightarrow \text{rmod}(f) \geq 3 \).
20. \( p = 7 \) and \( j_{x^3,y^5} f = x^3 + ay^7 \Rightarrow \) one of the three possibilities holds:
   \( j_{x^3,y^5} f(x,y) \sim_r x^3 + xy^5 + ay^7 \Rightarrow 21 \),
   \( j_{x^3,y^5} f(x,y) \sim_r x^3 + y^8 + ay^7 \Rightarrow 22 \),
   \( j_{x^3,y^5} f(x,y) = x^3 + ay^7 \Rightarrow 23 \).
21. \( p = 7 \) and \( j_{x^3,y^5} f = x^3 + xy^5 + ay^7 \Rightarrow f \in E_{13} \).
22. \( p = 7 \) and \( j_{x^3,y^5} f = x^3 + y^8 + ay^7 \Rightarrow f \in E_{14} \).
23. \( p = 7 \) and \( j_{x^3,y^5} f = x^3 + ay^7 \Rightarrow \text{rmod}(f) \geq 3 \).
24. \( p = 11 \) and \( j_{x^3,y^{11}} f = x^3 + y^{11} \Rightarrow \text{rmod}(f) \geq 3 \).
25. \( j_{x^3,y^{11}} f(x,y) = x^3 \Rightarrow f \in \langle x, y^4 \rangle \Rightarrow \text{rmod}(f) \geq 3 \).

Through theorems 26–29, \( k = 2,3 \).
26. \( j_{x^3,y^k} f(x,y) = x^3 \Rightarrow \) one of the three possibilities holds:
   \( j_{x^3,y^k} f(x,y) \sim_r x^3 + ax^2y^k + y^{3k}, \ 4a^3 + 27 = \neq 0 \Rightarrow 27 \),
   \( j_{x^3,y^k} f(x,y) \sim_r x^3 + x^2y^k \Rightarrow 28,29 \),
   \( j_{x^3,y^k} f(x,y) \sim_r x^3 \Rightarrow 9,30 \).
27. \( j_{x^3,y^k} f(x,y) = x^3 + ax^2y^k + y^{3k}, \ 4a^3 + 27 = \neq 0 \) and \( 3k < p \Rightarrow f \in J_{k,0} \).
28. \( j_{x^3,y^k} f(x,y) = x^3 + x^2y^k, \ 3k < p \) and \( \mu - 3k + 2 < 2p \Rightarrow f \in J_{k,q} \) with \( q = \mu - 6k + 2 \).
29. \( j_{x^3,y^k} f(x,y) = x^3 + x^2y^k, \ 3k < p \) and \( \mu - 3k + 2 \geq 2p \Rightarrow \text{rmod}(f) \geq 3 \).
30. \( p = 7 \) and \( j_{x^3,y^k} f(x,y) = x^3 + x^2y^2 \) and \( \mu < 18 \Rightarrow f \in J_{2,9} \) with \( q = \mu - 10 > 0 \).

Series X
31. \( j^3 f = 0 \Rightarrow \) one of the six possibilities holds:
   \( j^3 f \sim_r x^4 + ax^2y^2 + y^4, \ a^2 + 4 \neq 0 \Rightarrow 32 \),
   \( j^3 f \sim_r x^4 + x^2y^2 \Rightarrow 33,34 \),
   \( j^3 f \sim_r x^2y^2 \Rightarrow 35–38 \),
   \( j^3 f \sim_r x^3y \Rightarrow 39 \),
   \( j^3 f \sim_r x^4 \Rightarrow 54 \),
   \( j^3 f = 0 \Rightarrow 71 \).
32. \( j^4(f) = x^4 + ax^2y^2 + y^4, \ a^2 + 4 \neq 0 \Rightarrow f \in X_9 \).
33. \( j^4(f) = x^4 + x^2y^2 \) and \( \mu(f) < 2p + 5 \Rightarrow f \in X_{1,q} \).
34. \( j^4(f) = x^4 + x^2y^2 \) and \( \mu(f) \geq 2p + 5 \Rightarrow \text{rmod}(f) \geq 3 \).
35. \( j^4(f) = x^4y^2 \Rightarrow f = f_1 \cdot f_2 \) with \( \text{int}(f_1) = \text{int}(f_2) = 2 \) and \( 2 \leq \mu(f_1) < \mu(f_2) \Rightarrow 36 \).

Through theorems 36–39, \( 1 \leq r := \mu(f_1) - 1 \leq s := \mu(f_2) - 1 \).
36. \( \mu(f_1) \geq p \) or \( \mu(f_2) \geq 2p \Rightarrow \text{rmod}(f) \geq 3 \).
37. \( \mu(f_2) < p \Rightarrow \text{rmod}(f) = 1 \) and \( f \in Y_{r,s} \).
38. \( \mu(f_1) < p \) and \( p < \mu(f_2) < 2p \Rightarrow \text{rmod}(f) = 2 \) and \( f \in Y_{r,s} \).
39. \( j^4(f) = x^4y \Rightarrow j_{x^3,y^4} f = x^3y \Rightarrow 40,44 \).

Series Z
30. \( j_{x^3,y^k} f \sim_x = \) one of the four possibilities holds:

   \( j_{x^3,y^k} f \sim x^3 \Rightarrow 40–43 \), \( 5 < p \) and \( q = 1,2 \).
\[
\begin{align*}
\text{j}_{x^3y, y^{q+2}} f & \sim_r x^3y + y^{3q+2} \implies 41, \\
\text{j}_{x^3y, xy^{q+2}} f & \sim_r x^3y + xy^{2q+2} \implies 42, \\
\text{j}_{x^3y, y^{q+3}} f & \sim_r x^3y + y^{3q+3} \implies 43, \\
\text{j}_{x^3y, y^{q+3}} f & = x^3y \implies 49-53.
\end{align*}
\]
\begin{align*}
41. \text{j}_{x^3y, y^{q+2}} f = x^3y + y^{3q+2}, 3q + 2 < p \implies f \in Z_{6q+5}. \\
42. \text{j}_{x^3y, xy^{q+2}} f = x^3y + xy^{2q+2}, 3q + 3 < p \implies f \in Z_{6q+6}. \\
43. \text{j}_{x^3y, y^{q+3}} f = x^3y + y^{3q+3}, 3q + 3 < p \implies f \in Z_{6q+7}.
\end{align*}

44. \( p = 5 \) and \( \text{j}_{x^3y, y^4} f = x^3y \) imply one of the three possibilities holds:
\[
\begin{align*}
\text{j}_{x^3y, xy} f & \sim_r x^3y + xy^4 + ay^5 \implies 45, \\
\text{j}_{x^3y, y^6} f & \sim_r x^3y + y^6 + ay^5 \implies 46, \\
\text{j}_{x^3y, y^6} f & = x^3y + ay^5 \implies 47.
\end{align*}
\]

45. \( p = 5 \) and \( \text{j}_{x^3y, xy^4} f = x^3y + xy^4 + ay^5 \implies f \in Z_{12}. \\
46. \( p = 5 \) and \( \text{j}_{x^3y, xy^4} f = x^3y + y^6 + ay^5 \implies f \in Z_{13}. \\
47. \( p = 5 \) and \( \text{j}_{x^3y, xy^4} f = x^3y + ay^5 \implies \text{mod}(f) \geq 3. \\
48. \( p = 7 \) and \( \text{j}_{x^3y, xy^4} f = x^3y \implies \text{mod}(f) \geq 3.
\]

49. \( \text{j}_{x^3y, y^6} f = x^3y \) imply one of the three possibilities holds:
\[
\begin{align*}
\text{j}_{x^3y, y^6} f & = y(x^3 + bx^2y^2 + y^3), 4b^3 + 27 \neq 0 \implies 50 \\
& = x^3y \implies 53.
\end{align*}
\]

50. \( \text{j}_{x^3y, y^7} f = y(x^3 + bx^2y^2 + y^3), 4b^3 + 27 \neq 0 \implies f \in Z_{1,0}. \\
51. \( \text{j}_{x^3y, y^8} f = y(x^3 + x^2y^2) \) and \( \mu - 8 < p \implies f \in Z_{1,r} \) with \( r = \mu - 15 > 0. \\
52. \( \text{j}_{x^3y, y^9} f = y(x^3 + x^2y^2) \) and \( \mu - 8 \geq p \implies \text{mod}(f) \geq 3. \\
53. \( \text{j}_{x^3y, y^9} f = x^3y \implies f \in \langle y \rangle \cdot \langle x, y^3 \rangle \implies \text{mod}(f) \geq 3. \\
\]

Series W

Through theorems 54–56, \( 5 < p. \)

54. \( j^4 f = x^4 \implies \text{j}_{x^4, y^4} f = x^4 \implies \) one of the three possibilities holds:
\[
\begin{align*}
\text{j}_{x^4, y^4} f & \sim_r x^4 + y^5 \implies 55, \\
\text{j}_{x^4, xy^4} f & \sim_r x^4 + xy^4 \implies 56, \\
\text{j}_{x^4, xy^4} f & = x^4 \implies 57, 60.
\end{align*}
\]

55. \( \text{j}_{x^4, y^6} f = x^4 + y^5 \implies f \in W_{12}. \\
56. \( \text{j}_{x^4, xy^4} f = x^4 + xy^4 \implies f \in W_{13}. \\
57. \( p = 5 \) and \( \text{j}_{x^4, y^4} f = x^4 \implies \) one of the two possibilities holds:
\[
\begin{align*}
\text{j}_{x^4, xy^4} f & \sim_r x^4 + xy^4 + ay^5 \implies 58, \\
\text{j}_{x^4, xy^4} f & = x^4 + ay^5 \implies 59.
\end{align*}
\]

58. \( p = 5 \) and \( \text{j}_{x^4, xy^4} f = x^4 + xy^4 + ay^5 \implies f \in W_{13}. \\
59. \( p = 5 \) and \( \text{j}_{x^4, xy^4} f = x^4 + ay^5 \implies \text{mod}(f) \geq 3.
\]

Through theorems 60–70, \( 5 < p. \)

60. \( \text{j}_{x^4, xy^4} f = x^4 \implies \) one of the four possibilities holds:
\[
\begin{align*}
\text{j}_{x^4, y^6} f & \sim_r x^4 + bx^2y^3 + y^6, b^2 \neq 4 \implies 61, \\
& \sim_r x^4 + x^2y^3 \implies 62, 63, \\
& \sim_r (x^2 + y^3)^2 \implies 64, 65 \\
& = x^4 \implies 66.
\end{align*}
\]

61. \( \text{j}_{x^4, y^6} f = x^4 + bx^2y^3 + y^6, b^2 \neq 4 \implies f \in W_{1,0}. \\
62. \( \text{j}_{x^4, y^6} f = x^4 + x^2y^3 \) and \( \mu - 8 < p \implies f \in W_{1,q} (q = \mu - 15 > 0). \\
63. \( \text{j}_{x^4, y^6} f = x^4 + x^2y^3 \) and \( \mu - 8 \geq p \implies \text{mod}(f) \geq 3. \\
64. \( \text{j}_{x^4, y^6} f = (x^2 + y^3)^2 \) and \( \mu - 8 < p \implies f \in W_{k,q} (q = \mu - 15 > 0). \\
65. \( \text{j}_{x^4, y^6} f = (x^2 + y^3)^2 \) and \( \mu - 8 \geq p \implies \text{mod}(f) \geq 3. \\
66. \( \text{j}_{x^4, y^6} f = x^4 \implies \) one of the three possibilities holds:
Corank 3 Singularities

Through theorems 74–120, \( f \in K[[x, y, z]] \).

74. \( j^2 f(x, y, z) = 0 \Rightarrow \) one of the ten possibilities holds:

\[
\begin{align*}
\text{j}^3 f &\sim_r x^3 + y^3 + z^3 + axyz, a^3 + 27 \neq 0 \quad \Rightarrow \quad 75, \\
&\sim_r x^3 + y^3 + xyz \\ &\sim_r x^3 + xyz \\ &\sim_r x^3 + yz \\ &\sim_r x^2 + yz^2 \\ &\sim_r x^2 + xz^2 \\ &\sim_r x^2 y \\ &\sim_r x^3. \\
&= 0 \\
&\Rightarrow \quad 120.
\end{align*}
\]

Series T

75. \( j^3 f(x, y, z) = x^3 + y^3 + z^3 + axyz, a^3 + 27 \neq 0 \Rightarrow f \in P_8. \)

76. \( j^3 f(x, y, z) = x^3 + y^3 + xyz \Rightarrow f \sim_r x^3 + y^3 + xyz + o(z), j^3 \alpha = 0 \Rightarrow 77, 78. \)

77. \( f = x^3 + y^3 + xy + o(z), j^3 \alpha = 0, q := \mu(\alpha) + 1 < 2p \Rightarrow f \in P_{q+5} = T_{3,3,q}(q > 3). \)

78. \( f = x^3 + y^3 + xy + o(z), j^3 \alpha = 0, \mu(\alpha) + 1 \geq 2p \Rightarrow \text{rmod}(f) \geq 3. \)

79. \( j^3 f(x, y, z) = x^3 + xy + \Rightarrow f = x^3 + xy + o(y) + \beta(z), j^3(\alpha, \beta) = 0 \quad \text{and} \quad q := \mu(\alpha) + 1 < r := \mu(\beta) + 1 \Rightarrow \) one of the three possibilities holds:

\[
\begin{align*}
(i) \quad r < p &\quad \Rightarrow \quad \text{rmod}(f) = 1 \quad \text{and} \quad f \in T_{3, q, r}, \\
(ii) q < p \leq r < 2p &\quad \Rightarrow \quad \text{rmod}(f) = 2 \quad \text{and} \quad f \in T_{3, q, r}, \\
(iii) \quad \text{otherwise} &\quad \Rightarrow \quad \text{rmod}(f) \geq 3.
\end{align*}
\]

80. \( j^3 f(x, y, z) = xyz \Rightarrow f \sim_r xyz + o(x) + \beta(y) + \gamma(z), j^3(\alpha, \beta, \gamma) = 0 \quad \text{and} \quad q := \mu(\alpha) + 1 \leq r := \mu(\beta) + 1 \leq s := \mu(\gamma) + 1 \Rightarrow \) one of the three possibilities holds:

\[
\begin{align*}
(i) \quad s < p &\quad \Rightarrow \quad \text{rmod}(f) = 1 \quad \text{and} \quad f \in T_{q, r, s}, \\
(ii) r < p \leq s < 2p &\quad \Rightarrow \quad \text{rmod}(f) = 2 \quad \text{and} \quad f \in T_{q, r, s}, \\
(iii) \quad \text{otherwise} &\quad \Rightarrow \quad \text{rmod}(f) \geq 3.
\end{align*}
\]

Series Q

Through theorems 81–91, \( \varphi = x^3 + yz^2, j^3_{\varphi} = j_{\varphi z^2, x^3, \lambda}, (\lambda \text{ is a polynomial}). \)

81. \( j^3 f = \varphi \Rightarrow f \sim_r \varphi + o(y) + \beta(y), j^3(\alpha, x, \beta) = 0 \Rightarrow 82. \)

Through theorems 82–85, \( k = 1, 2. \)

82. \( f = \varphi + o(y) + \beta(y), j^3_{\varphi + y} f = \varphi \Rightarrow \) one of the four possibilities holds:
Through theorems 92–106, \( \varphi = x^2z + yz^2 \), \( j_*^\varphi = j_{x2yz^2 \lambda} \), (\( \lambda \) is a polynomial).

92. \( j_*^\psi \) = \( \varphi \Rightarrow f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y), j_*^\psi f = \varphi \Rightarrow \) one of the three possibilities holds:

\[
j_*^\psi f \sim_r \varphi + x^2y^2, f \not\equiv 4 \Rightarrow 94.
\]

93. \( f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y), j_*^\psi f = \varphi \Rightarrow \) one of the three possibilities holds:

\[
j_*^\psi f \sim_r \varphi + y^4, f \not\equiv 4 \Rightarrow 95.
\]

94. \( j_*^\psi f = \varphi + y^4 \Rightarrow f \in S_{11} \).

Series S

Through theorems 96–105, \( p > 5 \).

96. \( f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y), j_*^\psi f = \varphi \Rightarrow \) one of the four possibilities holds:

\[
j_*^\psi f \sim_r \varphi + y^5 + bxy^3, b^2 \not\equiv 4 \Rightarrow 97.
\]

97. \( j_*^\psi f = \varphi + y^5 + bxy^3, b^2 \not\equiv 4 \Rightarrow f \in S_{10} \).

98. \( j_*^\psi f = \varphi + x^2y^2 \) and \( \mu - 9 < p \Rightarrow f \in S_{10(q = \mu - 14 > 0)}. \)

99. \( j_*^\psi f = \varphi + x^2y^2 \) and \( \mu - 9 \geq p \Rightarrow \text{rmod}(f) \geq 3 \).

100. \( j_*^\psi f = \varphi + yz^3 \) and \( \mu - 9 < p \Rightarrow f \in S_{10(q = \mu - 14 > 0)}. \)

101. \( j_*^\psi f = \varphi + yz^3 \) and \( \mu - 9 \geq p \Rightarrow \text{rmod}(f) \geq 3 \).

Series U

Through theorems 107–117, \( \varphi = x^3 + xz^2 \), \( j_*^\lambda = j_{x^3 \lambda} \), (\( \lambda \) is a polynomial).

107. \( j_*^3 f = \varphi \Rightarrow f \sim_r \varphi + \alpha(y) + x\beta(y) + z\gamma(y), j_*^3 f = \varphi \) and \( p = 5 \Rightarrow \text{rmod}(f) \geq 3 \).
4.2. \( f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y), j_{y^4}^*f = \varphi \Rightarrow \) one of the two possibilities holds:

\[ j_{y^4}^*f \sim_{t} \varphi + y^4 \quad \mapsto \quad 109, \]

\[ = \varphi \quad \mapsto \quad 110. \]

109. \( j_{y^4}^*f = \varphi + y^4 \Rightarrow f \in U_{12}. \)

110. \( f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y), j_{xy^3}^*f = \varphi \Rightarrow \) one of the three possibilities holds:

\[ j_{xy^3}^*f \sim_{t} \varphi + xy^3 + cy^3, c(c^2 + 1) \neq 0 \quad \mapsto \quad 111, \]

\[ = \varphi + xy^3 \quad \mapsto \quad 112, 113, \]

\[ = \varphi \quad \mapsto \quad 114. \]

111. \( j_{xy^3}^*f = \varphi + xy^3 + cy^3, c(c^2 + 1) \neq 0 \Rightarrow f \in U_{1,0}. \)

112. \( j_{xy^3}^*f = \varphi + xy^3 \) and \( \mu - 13 < p \Rightarrow f \in U_{1,0}(q := \mu - 14 \geq 0). \)

113. \( j_{xy^3}^*f = \varphi + xy^3 \) and \( \mu - 13 \geq p \Rightarrow \text{rmod}(f) \geq 3. \)

114. \( f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y), j_{xy^3}^*f = \varphi \Rightarrow \) one of the two possibilities holds:

\[ j_{y^5}^*f \sim_{t} \varphi + y^5 \quad \mapsto \quad 115, 116, \]

\[ = \varphi \quad \mapsto \quad 116, 117. \]

115. \( j_{y^5}^*f = \varphi + y^5 \) and \( p > 5 \Rightarrow f \in U_{16}. \)

116. \( p = 5, f = \varphi + \alpha(y) + x\beta(y) + z\gamma(y) + x^2\delta(y) \) and \( j_{xy^3}^*f = \varphi \Rightarrow \text{rmod}(f) \geq 3. \)

117. \( j_{xy^3}^*f = \varphi \Rightarrow \text{rmod}(f) \geq 3. \)

118. \( j^3f = x^2y \Rightarrow f \sim x^2y + \alpha(y, z) + x\beta(z) \) and then \( \text{rmod}(f) \geq 3. \)

119. \( j^3f = x^3 \Rightarrow \text{rmod}(f) \geq 4. \)

120. \( j^3f = 0 \Rightarrow \text{rmod}(f) \geq 6. \)

**Corank > 3 Singularities**

121. \( \text{crk}(f) > 3 \Rightarrow \text{rmod}(f) \geq 4. \)

4.2. Singularity determinator in characteristic 2.

122. \( \mu(f) < \infty \Rightarrow \) one of the three possibilities holds:

\[ \text{crk}(f) \leq 1 \quad \mapsto \quad 123, \]

\[ = \quad 2 \quad \mapsto \quad 124, \]

\[ \geq \quad 3 \quad \mapsto \quad 136. \]

123. \( \text{crk}(f) \leq 1 \Rightarrow f \in A_k (1 \leq k \leq 5). \)

Through theorems, \( f \in K[x, y]. \)

124. \( \text{crk}(f) = 2 \Rightarrow \) one of the four possibilities holds:

\[ j^3f \sim_{t} ax^2 + by^2 + x^3 + y^3 \quad \mapsto \quad 125, \]

\[ = ax^2 + by^2 + x^2y \quad \mapsto \quad 126, \]

\[ = ax^2 + by^2 + x^3 \quad \mapsto \quad 131, \]

\[ = ax^2 + by^2 \quad \mapsto \quad 134, 135. \]

125. \( j^3f = ax^2 + by^2 + x^3 + y^3 \Rightarrow f \in D_4. \)

126. \( j^3f = ax^2 + by^2 + x^2y \Rightarrow \) one of the two possibilities holds:

\[ j^4f \sim_{t} ax^2 + by^2 + x^2y + xy^3 \quad \mapsto \quad 127, \]

\[ = ax^2 + by^2 + x^2y \quad \mapsto \quad 128, 130. \]

127. \( j^4f = ax^2 + by^2 + x^2y + xy^3 \Rightarrow f \in D_6. \)

128. \( j^4f = ax^2 + by^2 + x^2y \Rightarrow \) one of the two possibilities holds:

\[ j^5f \sim_{t} ax^2 + by^2 + x^2y + xy^4 \quad \mapsto \quad 129, \]

\[ = ax^2 + by^2 + x^2y \quad \mapsto \quad 130. \]

129. \( j^5f = ax^2 + by^2 + x^2y + xy^4 \Rightarrow \) one of the two possibilities holds:

\[ j^6f \sim_{t} ax^2 + by^2 + x^2y + xy^4 + cxy^5 \quad \mapsto \quad 130, \]

\[ = ax^2 + by^2 + x^2y \quad \mapsto \quad 130. \]

130. \( j^6f = ax^2 + by^2 + x^2y \Rightarrow \text{rmod}(f) \geq 3, \mu(f) \geq 8. \)

131. \( j^3f = ax^2 + by^2 + x^3 \Rightarrow \) one of the two possibilities holds:

\[ j^4f \sim_{t} ax^2 + by^2 + x^3 \quad \mapsto \quad 132, \]

\[ = ax^2 + by^2 + x^3 \quad \mapsto \quad 133. \]
132. \( j^4 f = ax^2 + by^2 + x^3 + xy^3 \Rightarrow f \in E_7. \)
133. \( j^4 f = ax^2 + by^2 + x^3 \Rightarrow \text{rmod}(f) \geq 3. \)
134. \( j^3 f = ax^2 + by^2, (a, b) \neq (0, 0) \Rightarrow \) one of the two possibilities holds:
   \[ j^4 f \sim_r ax^2 + by^2 + x^3 y \Rightarrow 135. \]
   \[ = ax^2 + by^2 \Rightarrow 135. \]
135. \( j^3 f = ax^2 + by^2 \Rightarrow \text{rmod}(f) \geq 4, \mu(f) \geq 10. \)
136. \( \text{crk}(f) \geq 3 \Rightarrow \text{rmod}(f) \geq 4, \mu(f) \geq 8. \)

4.3. **Singularity determinator in characteristic 3.**

137. \( \mu(f) < \infty \Rightarrow \) one of the four possibilities holds:
   \[ \text{crk}(f) \leq 1 \Rightarrow 138. \]
   \[ = 2 \Rightarrow 139–146, \]
   \[ = 3 \Rightarrow 147–151, \]
   \[ > 3 \Rightarrow 152. \]
138. \( \text{crk}(f) \leq 1 \Rightarrow f \in A_k (1 \leq k \leq 8). \)

**Corank 2 Singularities**

Through theorems 139–146, \( f \in K[[x, y]]. \)

139. \( j^2(f) = 0 \Rightarrow \) one of the three possibilities holds:
   \[ j^3 f \sim_r x^2 y + \epsilon y^3, \epsilon \in \{0, 1\} \Rightarrow 140, \]
   \[ \sim_r x^3 \Rightarrow 145, \]
   \[ = 0 \Rightarrow 146. \]
140. \( j^3(f) = x^2 y + \epsilon y^3, \epsilon \in \{0, 1\} \Rightarrow f \sim_r x^2 y + g(y), j^2 g = 0 \Rightarrow 141. \)
141. \( j^4 f = x^2 y + g(y), j^2 g = 0 \Rightarrow \) one of the three possibilities holds:
   \[ 2 < \mu(g) < 5 \Rightarrow 142, \]
   \[ 5 < \mu(g) < 8 \Rightarrow 143, \]
   \[ 8 < \mu(g) \Rightarrow 144. \]
142. \( 2 < \mu(g) < 5 \Rightarrow f \in D_5, D_6. \)
143. \( 5 < \mu(g) < 8 \Rightarrow f \in D_5, D_6. \)
144. \( 8 < \mu(g) \Rightarrow \text{rmod}(f) \geq 3, \mu(f) \geq 11. \)
145. \( j^3(f) = x^3 \Rightarrow \text{rmod}(f) \geq 3, \mu(f) \geq 9. \)
146. \( j^3(f) = 0 \Rightarrow \text{rmod}(f) \geq 3, \mu(f) \geq 9. \)

**Corank 3 Singularities**

Through theorems 147–151, \( f \in K[[x, y, z]]. \)

147. \( j^2(f; x, y, z) = 0 \Rightarrow \) one of the eleven possibilities holds:
   \[ j^3 f \sim_r x^3 + ax^2 z + z^3 + y^2 z, a \neq 0 \Rightarrow 148, \]
   \[ \sim_r x^3 + ax z^2 + z^3 + y^2 z, a \neq 0 \Rightarrow 149, \]
   \[ \sim_r x^3 + y^3 + x y z \Rightarrow 150, \]
   \[ \sim_r x^3 + x y z \Rightarrow 150, \]
   \[ \sim_r x y z \Rightarrow 150, \]
   \[ \sim_r x^3 + y z^2 \Rightarrow 150, \]
   \[ \sim_r x^2 z + y z^2 \Rightarrow 150, \]
   \[ \sim_r x^3 + x z^2 \Rightarrow 150, \]
   \[ \sim_r x^2 y \Rightarrow 150, \]
   \[ \sim_r x^3 \Rightarrow 150, \]
   \[ = 0 \Rightarrow 151. \]
148. \( j^3(f) = x^3 + ax^2 z + z^3 + y^2 z, a \neq 0 \Rightarrow \text{rmod}(f) \geq 4, \mu(f) \geq 11. \)
149. \( j^3(f) = x^3 + ax^2 z + z^3 + y^2 z, a \neq 0 \Rightarrow \text{rmod}(f) \geq 4, \mu(f) \geq 11. \)
150. \( j^3(f) \text{ is degenerate} \Rightarrow \text{rmod}(f) \geq 4, \mu(f) \geq 11. \)
151. \( j^3 f = 0 \Rightarrow \text{rmod}(f) \geq 6. \)

**Corank > 3 Singularities**

152. \( \text{crk}(f) > 3 \Rightarrow \text{rmod}(f) \geq 4. \)
5. Proof of the main results

We first use the splitting lemma to reduce the number of variables. Namely, if \( f \in m^2 \subset K[[x]] \) has \( \text{crank}(f) = k \geq 0 \), then

\[
f \sim_r g(x_1, \ldots, x_k) + Q(x_{k+1}, \ldots, x_n)
\]

with \( g \in m^3 \) and \( Q \) is a nondegenerate quadratic singularity (cf. [14] Lemma 3.9, 3.12). One has moreover that \( \text{rmod}(f) \) in \( K[[x]] \) is equal to \( \text{rmod}(g) \) in \( K[[x_1, \ldots, x_k]] \), cf. [14] Lemma 3.11, 3.13.

Theorems 1, 92, 122, 137 and 141 are obvious. Theorems 9, 17, 20, 25, 39, 40, 44, 54, 57, 66, 82, 93, 102, 108, 114 are proved by the Newton method [20] of a moving ruler (line, plane). This method reduces the proof to the counting of the integer points in triangles resp. polyhedrons on the exponent plane (resp. in the space).

Theorems concerning the geometrical classification problems: The proofs of theorems 1, 13, 26, 31, 49, 60, 72, 74, 96, 110, 124, 139, 147 can be reduced to the classifications of orbits of the actions of some quasihomogenous diffeomorphism groups on the spaces of quasihomogenous polynomials, see Section 5.1 for a proof of Theorem 147.

Theorems on normal forms: Theorems 2, 123, 138 follow from [22 Thm 2.11]. The proofs of theorems 5, 6, 7, 10, 11, 12, 14, 15, 18, 21, 22, 27, 28, 30, 32, 33, 37, 38, 41, 42, 43, 45, 46, 50, 51, 55, 56, 58, 61, 62, 64, 67, 68, 72, 75, 76, 77, 79, 80, 81, 83, 84, 85, 88, 89, 94, 95, 97, 98, 100, 103, 104, 107, 109, 111, 112, 115, 125, 127, 132, 142, 143 are based on the techniques introduced in [2] and generalized in [9], see Section 5.2 for a proof of Theorem 14.

Theorems on low bound of modality: Theorems 121, 151, 152 are consequences of [14 Prop. 2.18]. Theorems 3, 8, 16, 19, 23, 24, 25, 29, 34, 36, 47, 48, 52, 53, 59, 63, 65, 69, 70, 72, 73, 78, 79 (iii), 80 (iii), 86, 90, 99, 105, 106, 113, 116–120, 130, 133, 135, 136, 144, 145, 146, 148, 149, 150 are proved by using the theory in [14] ([21]), see Section 5.3 for a proof of 25 and 70.


Classification of unimodal and bimodal singularities (Theorems 3.1, 7.4): Applying Theorems 1-153 and the splitting lemma (cf. [14]) we obtain the list of families of singularities in Tables 1–11. The modularity of these families follows from simple calculations. To prove these singularities are unimodal resp. bimodal we use the theory of modality in [14]. See Section 5.4 for a proof that \( E_{12} \) with \( p > 7 \), is a class of unimodal singularities.

Smoothness of \( \mu \)-constant stratum and proper modality (Corollaries 3.8, 3.9): are proved by using the theory of modality diagrams (Theorem 3.5). See Section 5.4 for a proof that if \( f \) is of type \( T_{q,r,s} \) as in Table 8 then \( \mu \)-constant stratum \( \Delta_{\mu} \) of \( f \) is isomorphic to \( K^2 \). This also show that \( \text{rmod}(f) = \text{pmod}(f) = 2 \).

5.1. Proof of Theorem 147. The theorem is obtained by combining the following lemmas [5.1 5.2 5.3]. Let \( 0 \neq f \in K[x,y,z] \), with \( \text{char}(K) = 3 \), be a homogeneous polynomial of degree 3.

**Lemma 5.1.** If \( f \) is nonsingular, then \( f \) is right equivalent to one of the following forms \((a \neq 0)\)

\[
x^3 + ax^2z + z^3 + y^2z, \quad x^3 + axz^2 + z^3 + y^2z.
\]

*Proof.* cf. [18 Chap. II, Prop.1.2] \( \square \)

**Lemma 5.2.** If \( f \) is singular in \( \mathbb{P}^2_K \) and irreducible, then it is right equivalent to one of the following forms

\[
x^3 + y^3 + yz, \quad x^3 + y^2z.
\]

*Proof.* Let \( C \) be the curve in \( \mathbb{P}^2 \) defined by \( f \). Take \( P \in \text{Sing}(C) \) and \( P \neq Q \in C \). Let \( L \) be the line in \( \mathbb{P}^2 \) connecting \( P, Q \). Applying Bézout theorem we obtain that

\[
3 = \deg(C) \cdot \deg(L) \geq \text{mt}_P(C) + \text{mt}_Q(C).
\]
Lemma 5.3. If $f$ is reducible, then it is right equivalent to one of the following forms

$$x^3, x^2y, x^2z + yz^2, x^3 + xyz, x^3 + xz^2, xyz.$$  

Proof. Let $f = g_1 \cdot g_2$ with $\text{mt}(g_1) = 1$, $\text{mt}(g_2) = 2$. By the splitting lemma (Cf. [14])

$$g_2 \sim_\sim ax^2 + byz$$

with $a, b \in \{0, 1\}$. That is $f \sim_\sim g_1 \cdot (ax^2 + byz)$. Consider the following cases:

- $a = 1, b = 0$: Then $f$ is right equivalent to $x^3$ or $x^2y$.
- $a = 1, b = 1$: Then $f \sim_\sim g_1 \cdot (x^2 + yz)$. Without loss of generality we may assume moreover that
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is right equivalent to one of the following forms: $y^2z$, $xyz$, $(y + z)yz$, or, one of the forms: $x^2y$, $xyz$, $x^3 + xz^2$.

5.2. Proof of Theorem 14. Let $f \in K[[x, y]]$ with $p = \text{char}(K) = 5$ and $j_{x^3, y^6} f = x^3 + bx^2y^2 + y^6 + ay^5$. We will show that $f$ is right equivalent to $f_0 := x^3 + bx^2y^2 + y^6 + ay^5$, i.e. $f$ is of type $J_{2.0}$.

In fact, put $g := f - ay^5$, then $j_{x^3, y^6} g = x^3 + bx^2y^2 + y^6$. Applying [8] Thm. 4.4 we obtain that $g \sim_\sim x^3 + bx^2y^2 + y^6$. We can see moreover that there exists a coordinate change of the form

$$x \mapsto x + \varphi_1(x, y), y \mapsto y + \varphi_2(x, y)$$

with $\text{mt}(\varphi_i) \geq 2$ such that

$$g(x + \varphi_1, y + \varphi_2) = x^3 + bx^2y^2 + y^6.$$  

It yields

$$f_1 := f(x + \varphi_1, y + \varphi_2) = x^3 + bx^2y^2 + y^6 + a(y + \varphi_2)^5 = x^3 + bx^2y^2 + y^6 + ay^5 + a\varphi_2^5.$$  

It is easy to see that $m^7 \subset m^2 \cdot j(f_1)$. By [21] Thm. 2.1, $f_1$ is right 9-determined and hence $f_1 \sim_\sim f_0$ since $\text{mt}(\varphi_2^5) \geq 10$. This completes the proof.

5.3. Proof of theorems on lower bound of modality. For the proof of these theorems we need the following lemma which is deduced from Corollaries A.4, A.9, A.10 of [14] (see [21] Prop. 3.2.4, Cor. 3.3.4 and Cor. 3.3.6 for more details).

Lemma 5.4. Let the algebraic groups $G$ resp. $G'$ act on the varieties $X$ resp. $X'$. Let $h : Y \to X$ a morphism of varieties and let $h' : Y \to X'$ an open morphism such that

$$h^{-1}(G \cdot h(y)) \subset h'^{-1}(G' \cdot h'(y)), \forall y \in Y.$$  

Then for all $y \in Y$ we have

$$G \cdot \text{mod}(h(y)) \geq G' \cdot \text{mod}(h'(y)) \geq \dim X' - \dim G'.$$

Lemma 5.5. Let $f \in K[[x, y]]$ with $\text{char}(K) > 3$. Then $\text{rmod}(f) \geq 2 + l$ with $l \geq 0$, if either

(i) $f \in \langle x, y^{3+l} \rangle^3$; or
(ii) \( f \in \langle x^2, y^{3+l} \rangle^2 \).

**Proof.** We prove only for (i) since the proof for (ii) is similar. Let \( k \) be sufficiently large for \( f \), i.e. \( \text{rmod}(f) = \mathcal{R}_k \cdot \text{mod}(f) \). We denote

\[
\Delta := \{(3; s), (2; 3 + s), (1; 6 + s), (0; 9 + s) \mid 0 \leq s \leq l + 1\} \subset \mathbb{N}^2,
\]

\[
\Delta_1 := \{(1; s), (0; 3 + l + s) \mid 0 \leq s \leq l + 1\} \text{ and } \Delta_2 := \{(0; 1 + s) \mid 0 \leq s \leq l + 1\} \subset \mathbb{N}^2
\]

and define

\[
X := \left\{ \sum_{(i,j) \in \Delta} a_{i,j}x^iy^j \in K[[x, y]] \mid a_{i,j} \in K \right\} \cong \mathbb{A}^{4(l+2)},
\]

\[
G := X_1 \times X_2 \cong \mathbb{A}^{3(l+2)},
\]

where

\[
X_1 := \left\{ \sum_{(i,j) \in \Delta_1} a_{i,j}x^iy^j \in K[[x, y]] \mid a_{i,j} \in K, a_{10} \neq 0 \right\},
\]

\[
X_2 := \left\{ \sum_{(i,j) \in \Delta_2} b_{i,j}x^iy^j \in K[[x, y]] \mid b_{i,j} \in K, b_{01} \neq 0 \right\}.
\]

Using the projections

\[
\pi_1 : J_k \to X_1, \sum_{(i,j) \in \Delta_1} a_{i,j}x^iy^j \mapsto \sum_{(i,j) \in \Delta_1} a_{i,j}x^iy^j,
\]

\[
\pi_2 : J_k \to X_2, \sum_{(i,j) \in \Delta_2} a_{i,j}x^iy^j \mapsto \sum_{(i,j) \in \Delta_2} a_{i,j}x^iy^j,
\]

\[
\pi : J_k \to X, \sum_{(i,j) \in \Delta} a_{i,j}x^iy^j \mapsto \sum_{(i,j) \in \Delta} a_{i,j}x^iy^j
\]

and

\[
\bar{\pi} : \mathcal{R}_k \to G = X_1 \times X_2
\]

\[
\Phi = (\Phi_1, \Phi_2) \mapsto (\pi_1(\Phi_1), \pi_2(\Phi_2))
\]

we may define a multiplication on \( G \), resp. an action map of \( G \) on \( X \) as follows

\[
\bullet : G \times G \to G
\]

\[
(\phi, \phi') \mapsto \bar{\pi}(\phi \circ \phi'),
\]

resp.

\[
G \times X \to X
\]

\[
(\phi, g) \mapsto \pi(\phi(g)).
\]

By a simple calculation we can verify that the morphisms \( \iota : Y := \langle x, y^{3+l} \rangle^3 / \mathfrak{m}^{k+1} \to J_k \) and \( \pi : Y \to X \) satisfy

\[
\iota^{-1}(\mathcal{R}_k \cdot \iota(g)) \subset \pi^{-1}(G \cdot \pi(g)), \forall g \in Y.
\]

Hence applying Lemma[5,4] we obtain that

\[
\text{rmod}(f) = \mathcal{R}_k \cdot \text{mod}(i(f)) \geq \dim X - \dim G = 2 + l.
\]

\( \Box \)
5.4. Computing the modality of $E_{12}$. We shall show that $E_{12}$ is a class of unimodal singularities. To compute the modality of a singularity we use the general argument in [14], in particular, the following lemma.

**Lemma 5.6.** Assume that $f \in K[[x]]$ deforms only into finitely many families $h_i^{(i)}(x)$ over varieties $T^{(i)}, i \in I$. Then

$$\text{rmod}(f) \leq \max_{i \in I} \dim T^{(i)}.$$  

Assume further that the families $h_i^{(i)}(x)$ are all modular. Then

$$\text{rmod}(f) = \max_{i \in I} \dim T^{(i)}.$$ 

**Proof.** cf. [14], Prop. 2.15. □

**Proof for $E_{12}$.** Assume that $f = x^3 + y^7 + axy^6 \in K[[x, y]]$ with $p = \text{char}(K) > 7$ and $a \in K$, is of type $E_{12}$. We will show that

$$\text{rmod}(f) = 1.$$ 

In fact, by Theorem 3.5 (or, Theorems 1–9, 26, 27, 28), $f$ deforms only into the following modular families

$$E_{12}, A_k(k \leq 6), D_k(k \leq 8), E_6, E_7, E_8, J_{2,0}, J_{2,1}.$$ 

Hence it follows from Lemma 5.6 that $f$ is right unimodal singularities. □

5.5. Smoothness of $\mu$-constant stratum and proper modality. Let $f = x^q + y^r + z^s + axyz + bz^p$ be of type $T_{q,r,s}$ with $3 \leq q \leq r < p \leq s < 2p$ as in Table 8. Then

$$m/m \cdot j(f) = \{x, \ldots, x^{q-1}, y, \ldots, y^{r-1}, z, \ldots, z^{s-1}, xy, yz, zx, xyz\}$$ 

and the semiuniversal unfolding $f_\lambda$ of $f$ over $A^{q+r+s+1}, 0$ is given by

$$f_\lambda = f + \sum_{i=1}^{q-1} a_i x^i + \sum_{j=1}^{r-1} b_j y^j + \sum_{l=1}^{s-1} c_l z^l + d_1 xy + d_2 yz + d_3 zx + d_4 xyz$$ 

with $\lambda = (a_1, \ldots, a_{q-1}, b_1, \ldots, b_{r-1}, c_1, \ldots, c_{s-1}, d_1, d_2, d_3, d_4)$ the coordinate of $\lambda \in A^{q+r+s+1}$.  

Consider the $\mu$-constant stratum $\Delta_\mu$ of $f$, and assume that $\lambda \in \Delta_\mu$. It follows Theorem 3.5 that

$$a_1 = \cdots = a_{q-1} = b_1 = \cdots = b_{r-1} = c_1 = \cdots = c_{s-1} = d_1 = d_2 = d_3 = 0.$$ 

Moreover Theorems 76, 77, 79, 80 yield that, $c_p, d_4$ can be arbitrary, and hence $\Delta_\mu \cong A^2$. This implies that $\text{rmod}(f) = p\text{mod}(f) = 2$.

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**References**


