A JACOBIAN MODULE FOR DISENTANGLEMENTS AND APPLICATIONS TO MOND'S CONJECTURE

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ABSTRACT. Let $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a germ whose image is given by $g = 0$. We define an $\mathcal{O}_{n+1}$-module $M(g)$ with the property that $\mathscr{A}_e$-codim$(f) \leq \dim_{\mathbb{C}} M(g)$, with equality if f is weighted homogeneous. We also define a relative version $M_y(G)$ for unfoldings F, in such a way that $M_y(G)$ specialises to $M(g)$ when G specialises to g. The main result is that if $(n, n+1)$ are nice dimensions, then $\dim_{\mathbb{C}} M(g) \geq \mu_I(f)$, with equality if and only if $M_y(G)$ is Cohen-Macaulay, for some stable unfolding F. Here, $\mu_I(f)$ denotes the image Milnor number of f, so that if $M_y(G)$ is Cohen-Macaulay, then Mond's conjecture holds for f; furthermore, if f is weighted homogeneous, Mond's conjecture for f is equivalent to the fact that $M_y(G)$ is Cohen-Macaulay. Finally, we observe that to prove Mond's conjecture, it is enough to prove it in a suitable family of examples.

1. Introduction

For any hypersurface with isolated singularity $(X, 0)$, we have $\tau(X, 0) \leq \mu(X, 0)$, with equality if $(X, 0)$ is weighted homogeneous. Here, $\tau(X, 0)$ is the Tjurina number, that is, the minimal number of parameters in a versal deformation of $(X, 0)$ and $\mu(X, 0)$ is the Milnor number, which is the number of spheres in the Milnor fibre of $(X, 0)$. If $g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ is a function such that $g = 0$ is a reduced equation of $(X, 0)$, then we can compute both numbers in terms of g:

$$\tau(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g) + \langle g \rangle}, \quad \mu(X, 0) = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1}}{J(g)},$$

where $\mathcal{O}_{n+1}$ is the local ring of holomorphic germs from $(\mathbb{C}^{n+1}, 0)$ to $\mathbb{C}$ and $J(g)$ is the Jacobian ideal generated by the partial derivatives of g. Thus, the initial statement about τ and μ becomes evident. The Jacobian algebra deforms in flat manner over the parameter space of any deformation g_t of g,

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it is known to encode crucial properties of the vanishing cohomology and its monodromy by its relation with the Brieskorn lattice and it is crucial in the construction of Frobenius manifold structures in the bases of versal unfoldings. See the works of Brieskorn, Varchenko, Steenbrink, Scherk, Hertling and others, and the books [2], [8] and [6]

Inspired by the previous inequality, D. Mond [13] tried to obtain a result of the same nature in the context of singularities of mappings. He considered a hypersurface $(X, 0)$ given by the image of a map germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, with $S \subset \mathbb{C}^n$ a finite set and which has isolated instability under the action of the Mather group $\mathscr{A}$ of biholomorphisms in the source and the target. The Tjurina number has to be substituted by the $\mathscr{A}_e$-codimension, which is equal to the minimal number of parameters in an $\mathscr{A}$-versal deformation of f. Instead of the Milnor fibre, one considers the disentanglement, that is, the image X_u of a stabilisation f_u of f. Then, X_u has the homotopy type of a wedge of spheres and Mond defined the image Milnor number $\mu_I(f)$ as the number of such spheres. Note that, away from Mather's nice dimensions, some germs do not admit a stabilisation. Then, he stated the following conjecture:

Conjecture 1.1. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be an $\mathscr{A}$-finite map germ, with $(n, n+1)$ nice dimensions. Then,*

$$\mathscr{A}_e\text{-}\operatorname{codim}(f) \leq \mu_I(f),$$

with equality if f is weighted homogeneous.

The conjecture is known to be true for $n = 1, 2$ (see [9, 13, 14]) but it remains open until now for $n \geq 3$. There is a related result for map germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ with $n \geq p$, where one considers Δ the discriminant of f instead of its image and defines the discriminant Milnor number $\mu_\Delta(f)$ in the same way. Damon and Mond showed in [4] that if (n, p) are nice dimensions, then $\mathscr{A}_e$-codim$(f) \leq \mu_\Delta(f)$ with equality if f is weighted homogeneous. There are many papers in the literature with related results, partial proofs and examples in which the conjecture has been checked. We refer to [15] for a recent account of these results.

For hypersurfaces $\{g = 0\}$ with non-isolated singularities the relation between the Jacobian algebra and the vanishing cohomology is not so clear. Moreover, it is apparent from easy examples that the Jacobian algebra does not deform in a flat manner in unfoldings. In fact the possibility of studying the vanishing cohomology via deformations that simplify the critical set (in the same way that Morsifications do for isolated singularities) does not exist in general. However, for restricted classes of singularities Siersma, Pellikaan, Zaharia, Nemethi, Marco-Buzunáriz and the first author have developed methods that allow to split the vanishing cohomology of a non-isolated singularity in two direct summands according with the geometric properties of a deformation g_u of g which plays the role of a Morsification (one may find a nice survey in [20]). The first is a vector space contributing to the middle dimension cohomology of the Milnor fibre with the number of Morse points that appear away from the zero set of g_u $(u \neq 0)$. The second is determined by the non-isolated singularities of the zero-set of g_u $(u \neq 0)$.

Given an $\mathscr{A}$-finite map germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, we consider a generic 1-parameter deformation f_u of it (a stabilisation). Let g_u be the equation defining the image of f_u. It turns out that the deformation g_u is suitable to split the vanishing cohomology of g in two direct summands, as explained in the paragraph above, and that the first summand corresponds with the cohomology of the image X_u, whose rank is the image Milnor number. The main novelty of this paper is the definition of an Artinian $\mathcal{O}_{n+1}$ -module $M(g)$, which satisfies

$$\dim_{\mathbb{C}} M(g) = \mathscr{A}_e\text{- codim}(f) + \dim_{\mathbb{C}}((g) + J(g)/J(g))$$

and, in the nice dimensions, this dimension upper bounds the image Milnor number. Moreover we define a relative version $M_y(G)$, where G is an equation of the image of an unfolding F of f. We prove that, if G specializes to g, then $M_y(G)$ specializes to the original $M(g)$.

The first main result of this paper is Theorem 6.1, which implies that the dimension of $M(g)$ equals the image Milnor number if and only if $M_y(G)$ is flat over the base of the unfolding. We also prove that this is equivalent to the flatness of the Jacobian algebra over the base of the unfolding. Thus, under the flatness condition, $M(g)$ is expected to play the role of the Milnor algebra for isolated singularities, in the sense of encoding the first direct summand of the vanishing cohomology, which is the only one present for isolated singularities. It is very interesting to investigate whether the relation of the vanishing cohomology of isolated singularities with the Jacobian algebra explained admits a generalization to a relation between the first direct summand of the vanishing cohomology of g and the module $M(g)$.

More importantly, Theorem 6.1 also states that the flatness of $M_y(G)$ implies Mond's conjecture for f, and it is equivalent to it if g is weighted homogeneous. As a consequence, in Theorems 7.1 and 7.2 we derive that in order to settle Mond's conjecture in complete generality it is enough to prove it for a series of examples of increasing multiplicity.

In Section 5 we obtain formulas to compute the modules $M(g)$ and $M_y(G)$ which are well suited for computer algebra programs and also lead to new formulas for the $\mathscr{A}_e$-codimension, see Corollary 5.7 and Remark 5.8. The formula for $M_y(G)$ when F is a stable unfolding and G is a good defining equation of its image implies that $M(g)$ coincides with Damon's normal space with respect to $\mathscr{K}_{G,e}$-equivalence (see [4]). The advantage of our definition of $M(g)$ is that it can be computed directly in terms of f, without taking a stable unfolding (see Corollary 5.6). Our version of $M(g)$ has been used recently by Sharland [18] to show that certain corank 3 map germs from $(\mathbb{C}^3, 0)$ to $(\mathbb{C}^4, 0)$ satisfy Mond's conjecture. As the referee has pointed out, our module should also coincide with de Jong and van Straten module $T^1(g, \Sigma)$, introduced in [9], and some of our results recover those proved in loc.cit., see Remarks 3.4 and 6.2.

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2. The $\mathscr{A}_e$-codimension and the image Milnor number

We recall the definition of codimension of a map germ with respect to the Mather $\mathscr{A}$ group. Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ be any holomorphic map multi-germ. We write $\mathcal{O}_n = \mathcal{O}_{\mathbb{C}^n,S}$ and $\mathcal{O}_p = \mathcal{O}_{\mathbb{C}^p,0}$ for the rings of holomorphic function germs in the source and the target, respectively. We also write $\theta_n = \theta_{\mathbb{C}^n,S}$ and $\theta_p = \theta_{\mathbb{C}^p,0}$ for the corresponding modules of germs of vector fields and $\theta(f)$ for the module of germs of vector fields along f (that is, the sections of the pullback by f of the tangent bundle of the target). There are associated morphisms: $tf\colon \theta_n \to \theta(f)$, given by $tf(\eta) = df \circ \eta$, and $\omega f\colon \theta_p \to \theta(f)$, given by $\omega f(\xi) = \xi \circ f$. The $\mathscr{A}_e$-codimension of f is defined to be

$$\mathscr{A}_e\text{-}\operatorname{codim}(f) = \dim_{\mathbb{C}} \frac{\theta(f)}{tf(\theta_n) + \omega f(\theta_p)}.$$

We say that f is $\mathscr{A}$-stable (resp. $\mathscr{A}$-finite) if its $\mathscr{A}_e$-codimension is zero (resp. finite).

By an r-parameter unfolding of a map multi-germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ we mean a multi-germ $F\colon (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) \to (\mathbb{C}^r \times \mathbb{C}^p, 0)$, given by $F(u, x) = (u, f_u(x))$ and satisfying $f_0 = f$. It was proved by Mather [10] that f is $\mathscr{A}$-stable if and only if any unfolding F of f is trivial. This means that there exist Φ and Ψ unfoldings of the identity in $(\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S)$ and $(\mathbb{C}^r \times \mathbb{C}^p, 0)$, respectively, such that $\Psi \circ F \circ \Phi^{-1}$ is the constant unfolding $\operatorname{id} \times f$.

We present now a result due to Mond which gives a way to compute the $\mathscr{A}_e$-codimension in the case $p = n + 1$ in terms of a defining equation of the image of f. We need to introduce some notation.

From now on, we assume that $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is finite and generically one-to-one and write its image as $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$. The restriction $\bar{f}\colon (\mathbb{C}^n, S) \to (X, 0)$ is the normalization map, hence the induced morphism $\bar{f}^*\colon \mathcal{O}_{X,0} \to \mathcal{O}_n$ is a monomorphism and we may regard $\mathcal{O}_{X,0}$ as a subring of $\mathcal{O}_n$. Thus, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{n+1} & \xrightarrow{f^*} & \mathcal{O}_n \\ & \searrow^{\pi} & \uparrow{\scriptstyle i} \\ & & \mathcal{O}_{X,0} \end{array}$$

where π is the epimorphism induced by the inclusion $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$. Here, we consider both $\mathcal{O}_{X,0}$ and $\mathcal{O}_n$ as $\mathcal{O}_{n+1}$-modules via the corresponding morphisms. Finally, let $g \in \mathcal{O}_{n+1}$ be such that $g = 0$ is a reduced equation of $(X, 0)$ and let $J(g) \subset \mathcal{O}_{n+1}$ be the Jacobian ideal of g.

Lemma 2.1. [13, Proposition 2.1] *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite, with $n \geq 2$. Then,*

$$\mathscr{A}_e\text{-}\operatorname{codim}(f) = \dim_{\mathbb{C}} \frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}}.$$

Note that this lemma is given in [13] only for monogerms, but a careful revision of the proof shows that it also works for multigerms. Note also that the lemma is not true for $n = 1$. In fact, in that case (see [14]):

$$\mathscr{A}_e\text{- codim}(f) = \dim_{\mathbb{C}} \frac{J(g)\cdot \mathcal{O}_1}{J(g)\cdot \mathcal{O}_{X,0}} + \dim_C \frac{\mathcal{O}_1}{\langle f_1', f_2' \rangle},$$

where f_i' is the derivative of f_i.

Next we recall the definition of image Milnor number. Consider any r-parameter unfolding $F(u,x) = (u, f_u(x))$ of $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$. Let $(\mathcal{X}, 0)$ be the image hypersurface of F in $(\mathbb{C}^r \times \mathbb{C}^{n+1}, 0)$, and X_u be the fibre of $\mathcal{X}$ over $u \in \mathbb{C}^r$. We fix a small enough representative

$$F\colon W \to T \times B_\epsilon,$$

where W, T, B_ϵ are open neighbourhoods of S and the origin in $\mathbb{C}^{r+n}, \mathbb{C}^r, \mathbb{C}^{n+1}$, respectively, such that

(1) F is finite (i.e., closed and finite-to-one),
(2) $F^{-1}(0) = \{0\} \times S$,
(3) B_ϵ is a Milnor ball for the hypersurface $X_0 \subset \mathbb{C}^{n+1}$,
(4) T is small enough so that the intersection $X_u \cap \partial B_\epsilon$ of the hypersurface with the Milnor sphere is topologically trivial over all $u \in T$.

In order to understand the topology of $X_u \cap B_\epsilon$, we use the following general result, due to Siersma:

Theorem 2.2. [19] *Let $g\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define a reduced hypersurface $(X_0, 0)$, not necessarily with isolated singularity, and let $G\colon (\mathbb{C}^r \times \mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a deformation of g such that, for all u,*

(1) $\{g_u = 0\}$ is topologically trivial over the Milnor sphere ∂B_ϵ, and
(2) all the critical points of g_u which are not in $X_u = g_u^{-1}(0) \cap B_\epsilon$ are isolated.

Then $X_u \cap B_\epsilon$ is homotopy equivalent to a wedge of n-spheres and the number of such n-spheres is equal to

$$\sum_{y \in B_\epsilon \setminus X_u} \mu(g_u; y),$$

where $\mu(g_u; y)$ denotes the Milnor number of the function g_u at the point y.

Definition 2.3. Assume $r = 1$. Given a representative $F\colon W \to T \times B_\epsilon$ as above, we say that F is a *stabilisation* if for any $u \in T \setminus \{0\}$ and any point $y \in X_u \cap B_\epsilon$ the multigerm of f_u at y is $\mathscr{A}$-stable.

It is well known that every map f admits a stabilisation if $(n, n+1)$ are nice dimensions in the sense of Mather [12]. As an application of Siersma's previous result, Mond proves the following theorem in [13]. Again, the original proof is given for monogerms, but it is easy to check that it also works for multigerms.

Theorem 2.4. [13] *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be $\mathscr{A}$-finite with $(n, n+1)$ nice dimensions and let F be a stabilisation of f. Then, for any $u \in T \setminus \{0\}$, the image of f_u has the homotopy type of a wedge of n-spheres. Moreover, the number of such n-spheres is independent of the parameter u and on the stabilisation F.*

Definition 2.5. The number of spheres in the above mentioned wedge is called the *image Milnor number* of f and is written as $\mu_I(f)$.

Remark 2.6. Instead of using stabilisations, an equivalent definition of the image Milnor number can be formulated in terms of stable unfoldings. Indeed, for any stable unfolding F of f and any generic point u in the space of parameters, the image of f_u has the homotopy type of a wedge of n-spheres, and the number of spheres is $\mu_I(f)$.

3. The module $M(g)$

Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be a finitely determined map germ. Denote its image by $(X, 0)$. Let g be an equation of $(X, 0)$. We denote by $C(f)$ the conductor ideal of $\mathcal{O}_{X,0}$ in $\mathcal{O}_n$ and by $\mathscr{C}(f)$ its inverse image through $\pi\colon \mathcal{O}_{n+1} \to \mathcal{O}_{X,0}$, that is,

$$C(f) := \{h \in \mathcal{O}_{X,0} :\ h \cdot \mathcal{O}_n \subset \mathcal{O}_{X,0}\}, \quad \mathscr{C}(f) := \pi^{-1}(C(f)).$$

The conductor has the property that it is the largest ideal of $\mathcal{O}_{X,0}$ which is also an ideal of $\mathcal{O}_n$. We can compute easily $C(f)$ by using the following result of Piene [17] (see also Bruce-Marar [3]).

Lemma 3.1. *There exists a unique $\lambda \in \mathcal{O}_n$, such that*

$$\frac{\partial g}{\partial y_i} \circ f = (-1)^i \lambda \det(df_1, \dots, df_{i-1}, df_{i+1}, \dots, df_{n+1}),\ 1 \le i \le n+1.$$

Moreover, the ideal $C(f)$ is generated by λ.

From Lemma 3.1 follows the inclusion $J(g) \cdot \mathcal{O}_n \subset C(f)$, which motivates the following definition.

Definition 3.2. We define $M(g)$ as the kernel of the following epimorphism of $\mathcal{O}_{n+1}$-modules, induced by f^*:

$$\frac{\mathscr{C}(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g) \cdot \mathcal{O}_n}.$$

Proposition 3.3. *The following sequence of $\mathcal{O}_{n+1}$-modules is exact:*

$$0 \longrightarrow K(g) \longrightarrow M(g) \longrightarrow \frac{J(g) \cdot \mathcal{O}_n}{J(g) \cdot \mathcal{O}_{X,0}} \longrightarrow 0,$$

where $K(g) := (\langle g \rangle + J(g))/J(g)$.

Remark 3.4. The second map from the previous exact sequence coincides with the natural map $T^1(\Sigma, g) \to T^1(\Sigma, X)$ of [9]. There, the authors define a relative version of these modules for unfoldings which, in the case of $T^1(\Sigma, g)$, also coincides with the relative version of $M(g)$ that we define in the next section. Using their definitions, they prove that flatness of the Jacobian algebra of the equation of a stabilisation of f implies Mond's conjecture for f. This is one of the implications of our Theorem 6.1. We thank the referee for noticing this fact and communicating it to us.

Here we go a bit further, by proving the equivalence of Mond's conjecture with the flatness of our relative module and of the jacobian algebra in the weighted homogeneous case. This allows us to prove the reduction to series of examples of the last section of this paper. An advantage of our approach is that the definition of the module $M(f)$ and its relative version is elementary, more explicit and computable. For instance, it leads to new formulas for the $\mathcal{A}_e$-codimension (see Corollary 5.7 and Remark 5.8). In fact, from these explicit computations one obtains that our module also coincides with Damon's normal space of the inclusion $j\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C}^r \times \mathbb{C}^{n+1},0)$ in diagram (1) with respect to $\mathscr{K}_{G,e}$-equivalence (see [4]), see Remark 5.5.

Proof of Proposition 3.3. Consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K(g) & \longrightarrow & M(g) & \longrightarrow & \frac{J(g)\cdot\mathcal{O}_n}{J(g)\cdot\mathcal{O}_{X,0}} & & \\
 & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_2} & & \downarrow{\scriptstyle \mu_3} & & \\
0 & \longrightarrow & K(g) & \longrightarrow & \frac{\mathcal{C}(f)}{J(g)} & \longrightarrow & \frac{C(f)}{J(g)\cdot\mathcal{O}_{X,0}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \lambda_1} & & \downarrow{\scriptstyle \lambda_2} & & \downarrow{\scriptstyle \lambda_3} & & \\
 & & 0 & \longrightarrow & \frac{C(f)}{J(g)\cdot\mathcal{O}_n} & \longrightarrow & \frac{C(f)}{J(g)\cdot\mathcal{O}_n} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where the λ_i and the μ_i are the natural maps. Observe that, since all columns and the second and third rows are exact, Snake Lemma gives the desired exact sequence. □

Corollary 3.5. *Let $f\colon (\mathbb{C}^n,S) \to (\mathbb{C}^{n+1},0)$ be $\mathscr{A}$-finite with $n \geq 2$. Then*

(1) $M(g) = 0$ *if and only if f is $\mathscr{A}$-stable and $g \in J(g)$,*
(2) $\dim_{\mathbb{C}} M(g) = \mathscr{A}_e\text{-}\operatorname{codim}(f) + \dim_{\mathbb{C}} K(g)$.

This corollary is important, because it gives a simple method to compute the $\mathscr{A}_e$-codimension of a map germ $f\colon (\mathbb{C}^n,S) \to (\mathbb{C}^{n+1},0)$, with $n \geq 2$, just by means of a reduced equation of the image. This will be explained in detail in Section 5.

A priori, the length of $M(g)$ may depend on the defining equation g. Since $\dim_{\mathbb{C}} K(g)$ is upper semi-continuous in the space of possible equations for the image of g, for *generic* equations the length of $K(g)$ is minimal and independent of g, and the same happens for the length of $M(g)$.

Definition 3.6. An equation g of the image of f is *adequate* if $\operatorname{length}(K(g))$ is minimal among all possible equations.

Proposition 3.7. *If f is $\mathscr{A}$-finite and weighted homogeneous, all equations of its image are adequate.*

Proof. If f is weighted homogeneous its image has a weighted homogeneous equation g of degree d_0 for weights $(w_1, ..., w_{n+1})$. Any other equation g' can be written as $g' = g(\lambda + g'')$ for $\lambda \in \mathbb{C}^*$ and g'' vanishing at the origin. Decompose $g'' = \sum_{d>0} g''_d$ in weighted homogeneous components. For $t \in \mathbb{C}^*$ define $\phi_t(y_1, ..., y_{n+1}) := (t^{w_1}y_1, ..., t^{w_{n+1}}y_{n+1})$ and $g'|_t := g' \circ \phi_t$. Since $K(g')$ is invariant by changes of coordinates we have length$(K(g')) = $ length$(K(g'|_t))$ for all $t \in \mathbb{C}^*$. But we can write $g'|_t = t^{d_0}(\lambda g + \sum_{d>0} t^d g''_d)$, and dividing by t^{d_0} we obtain a family $g'''|_t$ of equations which at $t = 0$ specializes to λg. On one hand, we have proven that length$(K(g'''|_t))$ is independent of t if $t \neq 0$. On the other hand, we have $K(g) = 0$ by weighted homogeneity. By upper semi-continuity, we deduce that $K(g'''|_t) = 0$ for all t. So $K(g') = 0$. □

We finish this section with a couple of interesting properties of the ideals $C(f)$ and $\mathscr{C}(f)$ which will be used later.

Remark 3.8. It follows from the proof of [16, Theorem 3.4] that $\mathscr{C}(f)$ coincides with the first Fitting ideal of $\mathcal{O}_n$ as an $\mathcal{O}_{n+1}$-module via $f^*\colon \mathcal{O}_{n+1} \to \mathcal{O}_n$ (that is, $\mathscr{C}(f)$ is the ideal generated by the sub-maximal minors of a matrix presentation of $\mathcal{O}_n$). Furthermore, the same theorem also states that $\mathcal{O}_{n+1}/\mathscr{C}(f)$ is a determinantal ring of dimension $n-1$. By [16, Proposition 1.5], the zero locus of $\mathscr{C}(f)$ is

$$V(\mathscr{C}(f)) = \left\{ y \in \mathbb{C}^{n+1} : \sum_{f(x)=y} \dim_{\mathbb{C}} \frac{\mathcal{O}_{\mathbb{C}^n,x}}{f^*\mathfrak{m}_{\mathbb{C}^{n+1},y}} > 1 \right\},$$

which is equal to the points $y \in \mathbb{C}^{n+1}$ such that either $y = f(x)$ and x is a non-immersive point of f or $y = f(x) = f(x')$ with $x \neq x'$. Hence, we deduce that $V(\mathscr{C}(f))$ is the singular locus of $(X, 0)$. This space is also known as the target double point space of f.

Remark 3.9. Another consequence of Lemma 3.1 is that multiplication by λ induces an isomorphism:

$$\frac{C(f)}{J(g)\cdot \mathcal{O}_n} \cong \frac{\mathcal{O}_n}{R(f)},$$

where $R(f) \subset \mathcal{O}_n$ is the ramification ideal, that is, the ideal generated by the maximal minors of the Jacobian matrix of f. If f is $\mathscr{A}$-finite and $n \geq 2$, then $\mathcal{O}_n/R(f)$ is a determinantal ring (of dimension $n-2$ in this case). The zero locus $V(R(f)) \subset (\mathbb{C}^n, S)$ is the set of non-immersive points of f.

4. The relative version for unfoldings

We are interested in the behavior of the module $M(g)$ under deformations. With this motivation, we define a relative version of this module for unfoldings. For any r-parameter unfolding F of $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, there is a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^r \times \mathbb{C}^n, \{0\} \times S) & \xrightarrow{F} & (\mathbb{C}^r \times \mathbb{C}^{n+1}, 0) \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ (\mathbb{C}^n, S) & \xrightarrow{f} & (\mathbb{C}^{n+1}, 0), \end{array} \tag{1}$$

where $i(x) = (0, x)$ and $j(y) = (0, y)$. This induces another commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{r+n+1} & \xrightarrow{F^*} & \mathcal{O}_{r+n} \\ {\scriptstyle j^*}\downarrow & & \downarrow{\scriptstyle i^*} \\ \mathcal{O}_{n+1} & \xrightarrow{f^*} & \mathcal{O}_n, \end{array} \tag{2}$$

whose columns are epimorphisms. The conductor ideal $C(f)$ and its inverse image $\mathscr{C}(f)$ behave well under deformations, meaning that

$$i^*(C(F)) = C(f), \quad j^*(\mathscr{C}(F)) = \mathscr{C}(f).$$

The claim for $C(f)$ follows immediately from Piene's Lemma 3.1 and the claim for $\mathscr{C}(f)$ is a consequence of the claim for $C(f)$ and the commutative diagram (2), since

$$\begin{aligned} j^*(\mathscr{C}(F)) &= j^*\left((F^*)^{-1}(C(F))\right) = (f^*)^{-1}\left(i^*(C(F))\right) \\ &= (f^*)^{-1}(C(f)) = \mathscr{C}(f). \end{aligned}$$

Now we need an ideal which gives a deformation of the Jacobian ideal $J(g)$. Let $G \in \mathcal{O}_{r+n+1}$ be such that $G = 0$ is a reduced equation of $(\mathcal{X}, 0)$ of F and such that $j^*(G) = g$. It is not true that $j^*(J(G)) = J(g)$ since $J(G)$ contains the additional partial derivatives with respect to the parameters u_i. Instead of this, we consider the relative Jacobian ideal $J_y(G)$, that is, the ideal in $\mathcal{O}_{r+n+1}$ generated by the partial derivatives of G with respect to the variables y_i, $i = 1, \dots, n+1$. This obviously satisfies

$$j^*(J_y(G)) = J(g).$$

Definition 4.1. We define $M_y(G)$ as the kernel of the following epimorphism of $\mathcal{O}_{r+n+1}$-modules, induced by F^*:

$$\frac{\mathscr{C}(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}}.$$

The main result of this section will be that the module $M_y(G)$ specialises to $M(g)$ when $u = 0$, that is,

$$M_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong M(g),$$

where $\mathfrak{m}_r$ is the maximal ideal of $\mathcal{O}_r$, and the isomorphism is induced by the epimorphism j^*.

Lemma 4.2. *For any r-parameter unfolding F of f, we have*

$$\mathscr{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \mathscr{C}(f).$$

Moreover, $I \cdot \mathscr{C}(F) = I \cap \mathscr{C}(F)$, where $I = \mathfrak{m}_r \cdot \mathcal{O}_{r+n+1}$.

Proof. Since I is the kernel of j^*, we have $(\mathscr{C}(F) + I/I) = j^*(\mathscr{C}(F)) = \mathscr{C}(f)$ and from this we deduce:

$$\frac{\mathcal{O}_{r+n+1}}{\mathscr{C}(F)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} = \frac{\mathcal{O}_{r+n+1}}{\mathscr{C}(F) + I} = \frac{\mathcal{O}_{r+n+1}/I}{(\mathscr{C}(F) + I)/I} \cong \frac{\mathcal{O}_{n+1}}{\mathscr{C}(f)}.$$

Take the exact sequence of $\mathcal{O}_r$-modules

$$0 \longrightarrow \mathscr{C}(F) \longrightarrow \mathcal{O}_{r+n+1} \longrightarrow \frac{\mathcal{O}_{r+n+1}}{\mathscr{C}(F)} \longrightarrow 0$$

and consider the induced long exact Tor-sequence:

$$\cdots \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_r}\left(\frac{\mathcal{O}_{r+n+1}}{\mathscr{C}(F)}, \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right) \longrightarrow \mathscr{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \longrightarrow \mathcal{O}_{n+1} \longrightarrow \frac{\mathcal{O}_{n+1}}{\mathscr{C}(f)} \longrightarrow 0.$$

By Remark 3.8, $\mathcal{O}_{r+n+1}/\mathscr{C}(F)$ is determinantal of dimension $r + n - 1$. In particular, it is Cohen-Macaulay and since the fibre $\mathcal{O}_{n+1}/\mathscr{C}(f)$ has dimension $n - 1$, it is $\mathcal{O}_r$-flat. Therefore,

$$\operatorname{Tor}_1^{\mathcal{O}_r}\left(\frac{\mathcal{O}_{r+n+1}}{\mathscr{C}(F)}, \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right) = 0,$$

and the above exact sequence implies

$$\mathscr{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \mathscr{C}(f).$$

To show the second part, on one hand we have

$$\mathscr{C}(f) \cong \frac{\mathscr{C}(F) + I}{I} = \frac{\mathscr{C}(F)}{\mathscr{C}(F) \cap I}.$$

On the other hand, we have

$$\mathscr{C}(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} = \frac{\mathscr{C}(F)}{\mathfrak{m}_r \cdot \mathscr{C}(F)} = \frac{\mathscr{C}(F)}{I \cdot \mathscr{C}(F)},$$

and the result follows from the first part of the lemma. □

Lemma 4.3. *For any r-parameter unfolding F of f, we have*

$$C(F) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong C(f).$$

Moreover, $L \cdot C(F) = L \cap C(F)$, where $L = \mathfrak{m}_r \cdot \mathcal{O}_{r+n}$.

Proof. Let $L = \mathfrak{m}_r \cdot \mathcal{O}_{n+r}$. From $C(f) = i^*(C(F))$, it follows that $C(f) \cong (C(F) + L)/L$. Now one proceeds exactly as in the proof of Lemma 4.2, with $C(f)$, $\mathcal{O}_{n+r}$ and L playing the respective roles of $\mathscr{C}(f)$, $\mathcal{O}_{n+r+1}$ and I above. The argument that $\mathcal{O}_{n+r+1}/\mathscr{C}(F)$ is determinantal –and hence Cohen Macaulay– with $(n-1)$-dimensional fibre $V(\mathscr{C}(f))$ is replaced by the argument that $V(C(F))$ is a hypersurface –and hence Cohen Macaulay– with $(n-1)$-dimensional fibre $V(C(f))$. □

Proposition 4.4. *For any r-parameter unfolding F of f, the following hold:*

(1) $\dfrac{\mathscr{C}(F)}{J_y(G)} \otimes \dfrac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \dfrac{\mathscr{C}(f)}{J(g)}$,

(2) $\dfrac{C(F)}{J_y(G) \cdot \mathcal{O}_{n+r}} \otimes \dfrac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \dfrac{C(f)}{J(g) \cdot \mathcal{O}_n}$.

Proof. In order to simplify the notation for the first item, we write $\mathscr{C} := \mathscr{C}(F)$ and $J := J_y(G)$. From Lemma 4.2, we obtain that

$$\begin{aligned}\frac{\mathscr{C}}{J}\otimes\frac{\mathcal{O}_r}{\mathfrak{m}_r} &\cong \frac{\mathscr{C}/J}{I\cdot(\mathscr{C}/J)} = \frac{\mathscr{C}/J}{(I\cdot\mathscr{C}+J)/J}\\ &= \frac{\mathscr{C}}{I\cap\mathscr{C}+J} = \frac{\mathscr{C}/I\cap\mathscr{C}}{(I\cap\mathscr{C}+J)/I\cap\mathscr{C}}\\ &= \frac{\mathscr{C}/I\cap\mathscr{C}}{J/I\cap\mathscr{C}\cap J} = \frac{\mathscr{C}/I\cap\mathscr{C}}{J/I\cap J}\\ &= \frac{(\mathscr{C}+I)/I}{(J+I)/I} \cong \frac{\mathscr{C}(f)}{J(g)}.\end{aligned}$$

The proof of the second item is analogous to that of the first, just follow the same sequence of isomorphisms, replacing the modules $\mathscr{C}(F)$, $J_y(G)$ and I by the modules $C(F)$, $J_y(G)\cdot\mathcal{O}_{n+r}$ and $\mathfrak{m}_r\cdot\mathcal{O}_{n+r}$ and use Lemma 4.3 in place of Lemma 4.2. □

Next lemma shows that, over the source ring $\mathcal{O}_{r+n}$, the Jacobian ideals $J(G)$ and $J_y(G)$ coincide.

Lemma 4.5. *For any r-parameter unfolding F of f, we have*

$$J_y(G)\cdot\mathcal{O}_{r+n} = J(G)\cdot\mathcal{O}_{r+n}.$$

Proof. Let us write $F(u,x) = (u, f_u(x))$, then the Jacobian matrix of F has the following format:

$$dF = \begin{pmatrix} I_r & 0\\ * & df_u\end{pmatrix},$$

where df_u is the Jacobian matrix of f_u, but considered with entries in $\mathcal{O}_{r+n}$. Denote by $M_1,\dots,M_r,M'_1,\dots,M'_{n+1}$ the $r+n$-minors of dF in such a way that $M'_1,\dots,M'_{n+1}$ are the n-minors of df_u. Then $M_1,\dots,M_r$ can be generated from the other minors $M'_1,\dots,M'_{n+1}$. That is, we can put

$$M_i = \sum_j a_{ij}M'_j,$$

for some $a_{ij}\in\mathcal{O}_{r+n}$. Now, by Piene's Lemma 3.1:

$$\frac{\partial G}{\partial u_i}\circ F = \Lambda M_i,\quad \frac{\partial G}{\partial y_j}\circ F = \Lambda M'_j,$$

where Λ is the generator of the conductor ideal $C(F)$. We have:

$$\frac{\partial G}{\partial u_i}\circ F = \sum a_{ij}\frac{\partial G}{\partial y_j}\circ F. \quad \square$$

Now we are ready to show the main result of this section.

Theorem 4.6. *If F is any r-parameter unfolding of $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$, with $n\geq 2$, then:*

$$M_y(G)\otimes\frac{\mathcal{O}_r}{\mathfrak{m}_r}\cong M(g).$$

Proof. We have a short exact sequence coming from the definition of $M_y(G)$:

$$0 \longrightarrow M_y(G) \longrightarrow \frac{\mathscr{C}(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}} \longrightarrow 0.$$

By tensoring with $\mathcal{O}_r/\mathfrak{m}_r$, from the results of Proposition 4.4 we obtain the associated long exact Tor-sequence

$$\dots \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_r}\left(\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}}, \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right) \longrightarrow M_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \longrightarrow \frac{\mathscr{C}(f)}{J(g)} \longrightarrow \frac{C(f)}{J(g) \cdot \mathcal{O}_n} \longrightarrow 0.$$

We claim that $C(F)/J_y(G) \cdot \mathcal{O}_{r+n}$ is $\mathcal{O}_r$-flat. In fact, by Lemma 4.5,

$$\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}} = \frac{C(F)}{J(G) \cdot \mathcal{O}_{r+n}} \cong \frac{\mathcal{O}_{r+n}}{R(F)},$$

where $R(F)$ is the ramification ideal and the isomorphism is induced by multiplication of the generator of $C(F)$ (see Remark 3.9). Since $\mathcal{O}_{r+n}/R(F)$ is determinantal of dimension $r+n-2$, it is Cohen-Macaulay and, since the fibre $\mathcal{O}_n/R(f)$ has dimension $n-2$, it follows that $\mathcal{O}_{r+n}/R(F)$ is $\mathcal{O}_r$-flat. Here is where we use the hypothesis $n \geq 2$, since in this case we know that, for finitely determined mappings, the ramification locus is of pure codimension 2. Note that this is false for $n = 1$.

Therefore,

$$\operatorname{Tor}_1^{\mathcal{O}_r}\left(\frac{C(F)}{J_y(G) \cdot \mathcal{O}_{r+n}}, \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right) = 0,$$

and, from the above exact sequence, we obtain

$$M_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong M(g). \quad \square$$

Remark 4.7. The above statement does not hold for $n = 1$. An unfolding F of a germ $f\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ satisfying $M_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \ncong M(g)$ can be found in Example 5.9.

We finish this section with the next proposition, which gives the relative version of the short exact sequence of Proposition 3.3 for unfoldings. The proof is based on a commutative diagram analogous to that in the proof of 3.3, but using relative versions of the modules involved.

Proposition 4.8. *Let F be an r-parameter unfolding of f. We have the following exact sequence of $\mathcal{O}_{r+n+1}$-modules:*

$$0 \longrightarrow K_y(G) \longrightarrow M_y(G) \longrightarrow \frac{J_y(G) \cdot \mathcal{O}_{r+n}}{J_y(G) \cdot \mathcal{O}_{\mathcal{X},0}} \longrightarrow 0$$

where $K_y(G) := (\langle G \rangle + J_y(G))/J_y(G)$.

Remark 4.9. By analogous arguments to those of Theorem 4.6, it is not difficult to prove that the module on the right hand side of the above exact sequence specialises to the module which controls the $\mathscr{A}_e$-codimension, for $n \geq 2$. To be precise, there is an isomorphism

$$\frac{J_y(G)\cdot \mathcal{O}_{r+n}}{J_y(G)\cdot \mathcal{O}_{\mathcal{X},0}} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong \frac{J(g)\cdot \mathcal{O}_n}{J(g)\cdot \mathcal{O}_{X,0}}.$$

The proof follows easily by using the short exact sequence

$$0 \longrightarrow \frac{J_y(G)\cdot \mathcal{O}_{r+n}}{J_y(G)\cdot \mathcal{O}_{\mathcal{X},0}} \longrightarrow \frac{C(F)}{J_y(G)\cdot \mathcal{O}_{\mathcal{X},0}} \longrightarrow \frac{C(F)}{J_y(G)\cdot \mathcal{O}_{r+n}} \longrightarrow 0$$

The desired result is obtained after tensoring with $\mathcal{O}_r/\mathfrak{m}_r$, taking into account that the module on the right hand side is $\mathcal{O}_r$-flat.

It is not true in general that the module $K_y(G)$ specialises to $K(g)$. That is, we may have that

$$K_y(G) \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r} \ncong K(g).$$

In fact, by using the short exact sequence of Proposition 4.8, if $K_y(G)\otimes\frac{\mathcal{O}_r}{\mathfrak{m}_r} \cong K(g)$, this would imply that $\frac{J_y(G)\cdot\mathcal{O}_{r+n}}{J_y(G)\cdot\mathcal{O}_{\mathcal{X},0}}$ is $\mathcal{O}_r$-flat. But it is obvious that this module is not flat when f is $\mathscr{A}$-finite and F is a stabilisation of f, since it is supported only at the origin.

5. An equivalent description of the module $M(g)$

In this section we show a description of the modules $M(g)$ and $M_y(G)$, better suited for applications. Proposition 5.1 allows us to compute $M(g)$ easily using a computer algebra system, such as Singular [5]. Since $K(g)$ can be computed as well, from Corollary 3.5 we obtain an expression for the $\mathscr{A}_e$-codimension of any map germ $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$, with $n \geq 2$.

Proposition 5.1. *Let $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ be any map germ and F any r-parameter unfolding of f. Let g be an equation of the image of f and G an equation of the image of F that specializes to g. Then*

$$M(g) = \frac{(f^*)^{-1}(J(g)\cdot \mathcal{O}_n)}{J(g)},$$

$$M_y(G) = \frac{(F^*)^{-1}(J(G)\cdot \mathcal{O}_{r+n})}{J_y(G)}.$$

Proof. By construction, $M(g)$ is given by

$$M(g) = \frac{(f^*)^{-1}(J(g)\cdot \mathcal{O}_n) \cap \mathscr{C}(f)}{J(g)}.$$

But Lemma 3.1 implies the inclusion $J(g)\cdot \mathcal{O}_n \subset C(f)$, hence we have the inclusion

$$(f^*)^{-1}(J(g)\cdot \mathcal{O}_n) \subset (f^*)^{-1}(C(f)) = \mathscr{C}(f).$$

The proof for $M_y(G)$ is analogous, taking into account that $J_y(G)\cdot \mathcal{O}_{r+n}$ equals $J(G)\cdot \mathcal{O}_{r+n}$ by Lemma 4.5. □

Corollary 5.2. *Let F be a stable unfolding of f. Let g be an equation of the image of f and G an equation of the image of F that specializes to g. Then,*

$$M_y(G) = \frac{J(G) + \langle G\rangle}{J_y(G)}.$$

Proof. Since F is stable, $M(F) = K(G)$ by Proposition 3.3. The first part Proposition 5.1 implies that $(F^*)^{-1}(J(G) \cdot \mathcal{O}_{r+n})) = J(G) + \langle G \rangle$ and the result follows from the second part of Proposition 5.1. □

Definition 5.3. Let F be an unfolding of f. We say that G is a *good defining equation* for F if $G = 0$ is a reduced equation of the image of F and moreover $G \in J(G)$.

Note that there is always a stable unfolding F which admits a good defining equation. In fact, if $F(u,x) = (u, f_u(x))$ is any r-parameter stable unfolding, then let F' be the trivial 1-parameter unfolding of F, that is, $F'(t,u,x) = (t,u,f_u(x))$. Let $G = 0$ be a reduced equation of the image of F and take $G'(t,u,y) = e^t G(u,y)$. Then, $G' = 0$ is a reduced equation of the image of F' and $\partial G'/\partial t = G'$, hence $G' \in J(G')$.

Corollary 5.4. *Let F be a stable unfolding of f and G a good defining equation for F. Then,*

$$M_y(G) = \frac{J(G)}{J_y(G)}.$$

Remark 5.5. A consequence of this corollary is that $M(g)$ coincides with Damon's normal space of the inclusion $j\colon (\mathbb{C}^{n+1},0) \to (\mathbb{C}^r \times \mathbb{C}^{n+1},0)$ in diagram (1) with respect to $\mathscr{K}_{G,e}$-equivalence (see [4]).

Corollary 5.6. *Let F be a stable unfolding of f and G a good defining equation for F. Then, the evaluation map $ev\colon \theta_{r+n+1} \to J(G)$ given by $ev(\xi) = \xi(G)$, induces an isomorphism*

$$M(g) \cong \frac{\theta(j)}{tj(\theta_{n+1}) + j^*(\operatorname{Derlog}(G))},$$

where $\operatorname{Derlog}(G) = \{\xi \in \theta_{r+n+1} : \ \xi(G) = 0\}$.

Proof. The evaluation map is an epimorphism with kernel $\operatorname{Derlog}(G)$. Thus, it induces an isomorphism $\theta_{r+n+1}/\operatorname{Derlog}(G) \cong J(G)$ and hence an isomorphism

$$\frac{\theta_{r+n+1}}{\mathcal{O}_{r+n+1}\left\{\frac{\partial}{\partial y_1},\dots,\frac{\partial}{\partial y_{n+1}}\right\} + \operatorname{Derlog}(G)} \cong \frac{J(G)}{J_y(G)} = M_y(G).$$

After tensoring with $\mathcal{O}_r/\mathfrak{m}_r$, from Theorem 4.6 we get the desired isomorphism

$$\frac{\theta(j)}{tj(\theta_{n+1}) + j^*(\operatorname{Derlog}(G))} \cong M(g). \quad \square$$

Corollary 5.7. *Let $f\colon (\mathbb{C}^n,S) \to (\mathbb{C}^{n+1},0)$ be $\mathscr{A}$-finite, with $n \geq 2$ and $(n,n+1)$ nice dimensions. Let F be a stable unfolding of f, then*

$$\mathscr{A}_e\text{-}\operatorname{codim}(f) = \dim_{\mathbb{C}}\left(\frac{J(G)+\langle G\rangle}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right) - \dim_{\mathbb{C}}\left(\frac{J(g)+\langle g\rangle}{J(g)}\right).$$

Proof. It follows immediately by putting together Theorem 4.6, and Corollary 5.2. □

Remark 5.8. Observe that if G is a good defining equation for F, then we have

$$\mathscr{A}_e\text{- codim}(f) = \dim_{\mathbb{C}}\left(\frac{J(G)}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right) - \dim_{\mathbb{C}}\left(\frac{J(g)+\langle g\rangle}{J(g)}\right).$$

If, moreover, f is weighted homogeneous, then

$$\mathscr{A}_e\text{- codim}(f) = \dim_{\mathbb{C}}\left(\frac{J(G)}{J_y(G)} \otimes \frac{\mathcal{O}_r}{\mathfrak{m}_r}\right).$$

Example 5.9. Let $f\colon (\mathbb{C},0) \to (\mathbb{C}^2,0)$, given by $x \mapsto (x^3, x^4)$, be the parametrization of the plane curve E_6. The equation of E_6 is $g(y) = y_1^4 - y_2^3 = 0$. We have

$$J(g) = \langle y_1^3, y_2^2\rangle,\ J(g)\cdot\mathcal{O}_1 = \langle x^8\rangle \text{ and } (f^*)^{-1}(J(g)\cdot\mathcal{O}_1) = \langle y_1^3, y_2^2y_2, y_2^2\rangle,$$

hence $\dim_{\mathbb{C}} M(g) = 1$ by Proposition 5.1.

Let $G(a,b,c,y) = 0$ be the equation of the image of the versal unfolding

$$F(a,b,c,x) = (a,b,c,x^3+ax, x^4+bx^2+cx).$$

Since F is stable and weighted homogeneous, it follows from Corollary 5.4 that $M_y(G) = J(G)/J_y(G)$. Therefore, a presentation matrix R of $M_y(G)$ is obtained by deleting the two rows corresponding to the generators $\frac{\partial G}{\partial y_1}$ and $\frac{\partial G}{\partial y_2}$ in a presentation matrix of $J(G)$. A computation with Singular yields the matrix R, whose transpose is

$$\begin{pmatrix} 2a & 4a-2b & -c-4y_1 \\ 0 & c+y_1 & -y_2+a^2-ab \\ 6c-3y_1 & -9y_1 & -3y_2-9a^2+11ab-4b^2 \\ 9y_1 & 9y_1 & 3y_2+a^2+ab \\ 2a^2 & 3y_2+3a^2-ab & -3by_1 \\ 3y_2-3a^2+3ab & -4a^2+6ab-2b^2 & 0 \end{pmatrix}.$$

A presentation matrix of $M_y(G)\otimes\frac{\mathcal{O}_3}{\mathfrak{m}_3}$ is obtained by substituting $a = b = c = 0$ in R, that is,

$$\begin{pmatrix} 0 & 0 & -3y_1 & 9y_1 & 0 & 3y_2 \\ 0 & y_1 & -9y_1 & 9y_1 & 3y_2 & 0 \\ -4y_1 & -y_2 & -3y_2 & 3y_2 & 0 & 0 \end{pmatrix}.$$

We get $\dim_{\mathbb{C}} M_y(G)\otimes\frac{\mathcal{O}_3}{\mathfrak{m}_3} = 3$, which equals the $\mathscr{A}_e$-codimension of f. The discrepance between $\dim_{\mathbb{C}} M_y(G)\otimes\frac{\mathcal{O}_3}{\mathfrak{m}_3}$ and $\dim_{\mathbb{C}} M(g)$ is due to the contribution of the ramification ideal (see the discussion after Lemma 2.1).

6. Flatness of the Jacobian Module and the Mond Conjecture

In this section, we assume that $f : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ is a germ such that $\dim_{\mathbb{C}} M(g) < \infty$ and F is an r-parameter unfolding of f. By Theorem 4.6 and the Preparation Theorem, $M_y(G)$ is finite over $\mathcal{O}_r$. We consider a small enough representative $F\colon W \to T\times B_\epsilon$ with the properties required in Section 2 and such that the restriction of the projection onto the parameter space

$$\pi\colon \operatorname{supp}\mathcal{M}_y(G) \to T$$

is finite and $\pi^{-1}(0)=\{0\}$. Here $\mathcal{M}_y(G)$ is the coherent sheaf of modules on $T\times B_\epsilon$ whose stalk at the origin is $M_y(G)$. We also denote by $M_y(G)_{(u,p)}$ the stalk of $\mathcal{M}_y(G)$ at $(u,p)\in T\times B_\epsilon$. The main result of the section is the following:

Theorem 6.1. *Let $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ be $\mathscr{A}$-finite, with $(n,n+1)$ nice dimensions and $n\geq 2$. Let $F\colon\mathbb{C}^n\times T\to\mathbb{C}^{n+1}\times T$ be either an stable unfolding or an stabilization. Let g be an equation of the image of f and G be an equation of the image of F which specializes to g. The following statements are equivalent and imply Mond's conjecture for f:*

1. *$\dim_{\mathbb{C}} M(g)=\mu_I(f)$;*
2. *$M_y(G)$ is Cohen-Macaulay;*
3. *$\mathcal{O}_{\mathbb{C}^{n+1}\times T}/J_y(G)$ is $\mathcal{O}_T$-flat.*

Moreover, if f is weighted homogeneous and satisfies Mond's conjecture then the above assertions hold.

Remark 6.2. In [9] it is proved that the flatness of $\mathcal{O}_{\mathbb{C}^{n+1}\times T}/J_y(G)$ implies Mond's conjecture for f. See Remark 3.4.

Proof of Theorem 6.1. We claim that, for any $u\in T$, we have the inequality

$$\dim_{\mathbb{C}} M(g)\geq\sum_{p\in X_u\cap B_\epsilon}\dim_{\mathbb{C}} M(g_u)_p+b_n(X_u\cap B_\epsilon), \tag{3}$$

where $b_n(X_u\cap B_\epsilon)$ denotes the n-th Betti number of $X_u\cap B_\epsilon$. Moreover, the equality holds for all $u\in T$ if and only if the module $M_y(G)$ is Cohen-Macaulay of dimension r (equivalently, if it is $\mathcal{O}_r$-flat).

To prove the claim, we define

$$\Theta(u):=\sum_{p\in B_\epsilon}\dim_{\mathbb{C}}\left(M_y(G)_{(u,p)}\otimes\frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}}\right),$$

where $\mathfrak{m}_{T,u}$ denotes the maximal ideal of $\mathcal{O}_{T,u}$. By upper semi-continuity, we obtain the inequality

$$\Theta(0)\geq\Theta(u). \tag{4}$$

Well known facts from analytic geometry show that $M_y(G)$ is Cohen-Macaulay if and only if the equality $\Theta(0)=\Theta(u)$ holds for all $u\in T$. Let us identify both sides of the inequality (4). The left hand side is equal to $\dim_{\mathbb{C}} M_y(G)\otimes(\mathcal{O}_r/\mathfrak{m}_r)=\dim_{\mathbb{C}} M(g)$ by Theorem 4.6. For the same reason, if $p\in X_u\cap B_\epsilon$, then

$$M_y(G)_{(u,p)}\otimes\frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}}=M(g_u)_p.$$

We split the sum $\Theta(u)$ as

$$\Theta(u)=\sum_{p\in X_u\cap B_\epsilon}\dim_{\mathbb{C}} M(g_u)_p+\sum_{p\in B_\epsilon\setminus X_u}\dim_{\mathbb{C}}\left(M_y(G)_{(u,p)}\otimes\frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}}\right).$$

The first summand coincides with the first summand of the right hand side of the desired inequality. For the second summand we use the short exact sequence of Proposition 4.8. If $p \in B_\epsilon \setminus X_u$, then

$$M_y(G)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = K_y(G)_{(u,p)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = \frac{\mathcal{O}_{T\times B_\epsilon,(u,p)}}{J_y(G)} \otimes \frac{\mathcal{O}_{T,u}}{\mathfrak{m}_{T,u}} = \frac{\mathcal{O}_{B_\epsilon,p}}{J(g_u)},$$

which is the Jacobian algebra of g_u at p. Thus, the second summand equals $\sum_{p\in B_\epsilon\setminus X_u} \mu(g_u;p)$, where $\mu(g_u;p)$ is the Milnor number of g_u at p. By Siersma's Theorem 2.2, the sum of all Milnor numbers $\mu(g_u;p)$, with $p \notin X_u$, is equal to the Betti number $b_n(X_u \cap B_\epsilon)$. This proves the claim.

If F is either a stabilisation or a stable unfolding, the right hand side is identified with $\mu_I(f)$ for u generic. This proves the equivalence of the first two assertions. The third assertion is also equivalent because of the fact that $M_y(G)$ is Cohen Macaulay if and only if it is flat over T, and this happens if and only if $\mathcal{O}_{\mathbb{C}^{n+1}\times T}/J_y(G)$ is flat over T. In fact, we consider the exact sequence defining $M_y(G)$:

$$0 \longrightarrow M_y(G) \longrightarrow \frac{\mathscr{C}(F)}{J_y(G)} \longrightarrow \frac{C(F)}{J_y(G)\cdot \mathcal{O}_{\mathbb{C}^n\times T}} \longrightarrow 0.$$

Since the last module is $\mathcal{O}_T$-flat by Lemma 4.5 and Remark 3.9 the first module is $\mathcal{O}_r$-flat if and only if the second is. Now consider the exact sequence:

$$0 \longrightarrow \frac{\mathscr{C}(F)}{J_y(G)} \longrightarrow \frac{\mathcal{O}_{\mathbb{C}^{n+1}\times T}}{J_y(G)} \longrightarrow \frac{\mathcal{O}_{\mathbb{C}^{n+1}\times T}}{\mathscr{C}(F)} \longrightarrow 0.$$

The last module of the sequence is $\mathcal{O}_T$-flat by Remark 3.8, hence the $\mathcal{O}_T$-flatness is equivalent for the first two modules of the sequence.

Suppose that we have the equality $\dim_\mathbb{C} M(g) = \mu_I(f)$. Mond's conjecture for f follows now immediately from Corollary 3.5.

Suppose that f is a weighted homogeneous germ satisfying Mond's conjecture, and that g is also weighted homogeneous. Then $\dim_\mathbb{C} M(g) = \mu_I(f)$ by Corollary 3.5 and the fact that $K(g)$ vanishes. If g is not weighted homogeneous, use Proposition 3.7. □

Remark 6.3. If g is not an adequate equation for X, then the three equivalent conditions of the previous theorem fail necessarily. Indeed, if g' is an adequate equation we have the strict inequality $\dim_\mathbb{C} M(g) > \dim_\mathbb{C} M(g')$, and $\dim_\mathbb{C} M(g')$ is greater than or equal to $\mu_I(f)$, as follows from Inequality (3).

Remark 6.4. For germs $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ of corank one, Theorem 6.1 applies without the hypothesis that $(n, n+1)$ are nice dimensions. This is due to the well known facts that all corank one germs admit stabilisations and all stable corank one germs are weighted-homogeneous.

Example 6.5. Now we show that any map germ $H\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^4, 0)$ which is an unfolding of $f(x) = (x^3, x^4)$ (see Example 5.9) satisfies Mond's conjecture. After coordinate changes, H is a pullback of the versal unfolding F, that is,

$$H(u, x) = (u, x^3 + \alpha(u)x, x^4 + \beta(u)x^2 + \gamma(u)x),$$

for some functions $\alpha, \beta, \gamma \in \mathcal{O}_2$. A stable unfolding of H is the sum $\mathcal{F}$ of the unfoldings F and H, given by

$$\mathcal{F}(a,b,c,u,x) = (a,b,c,u,x^3 + (a+\alpha(u))x, x^4 + (b+\beta(u))x^2 + (c+\gamma(u))x).$$

The equation of the image of $\mathcal{F}$ is $\mathcal{G} = G(a+\alpha(u), b+\beta(u), c+\gamma(u), y) = 0$. By the chain rule, the partial derivatives of $\mathcal{G}$ with respect to u_i are

$$\frac{\partial \mathcal{G}}{\partial u_i} = \frac{\partial \alpha}{\partial u_i}\frac{\partial G}{\partial a} + \frac{\partial \beta}{\partial u_i}\frac{\partial G}{\partial b} + \frac{\partial \gamma}{\partial u_i}\frac{\partial G}{\partial c},$$

and the other partial derivatives are equal to those of G. Let S be the matrix whose transpose is

$$\begin{pmatrix} 2a & 4a-2b & -c-4y_1 \\ 0 & c+y_1 & -y_2+a^2-ab \\ 6c-3y_1 & -9y_1 & -3y_2-9a^2+11ab-4b^2 \\ 9y_1 & 9y_1 & 3y_2+a^2+ab \\ 2a^2 & 3y_2+3a^2-ab & -3by_1 \\ 3y_2-3a^2+3ab & -4a^2+6ab-2b^2 & 0 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}.$$

A presentation matrix of $M_{u,y}(\mathcal{G})$ is obtained from S by substituting a, b and c by $a+\alpha(u), b+\beta(u)$ and $c+\gamma(u)$ and A_i, B_i and C_i by $\frac{\partial \alpha}{\partial u_i}, \frac{\partial \beta}{\partial u_i}$ and $\frac{\partial \gamma}{\partial u_i}$, respectively. We use Singular to check that $\operatorname{coker} S$ is a Cohen-Macaulay $\mathcal{O}_{11}$-module of codimension 4. Being $M_{u,y}(\mathcal{G})$ a pullback of $\operatorname{coker} S$ of the same codimension, $M_{u,y}(\mathcal{G})$ is Cohen-Macaulay. By Theorem 6.1, the map-germ H satisfies Mond's conjecture.

Example 6.6. It is stated in [7] that the map germ P_2, given by

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_2x_3 + x_3^5, x_1x_3 + x_3^3),$$

satisfies Mond's conjecture (in fact, the whole list of [7] is checked by Altıntaş Sharland in [1]). Here we show this by using a simpler argument based on the flatness of the relative Jacobian ideal. Take the one-parameter stable unfolding $F\colon (\mathbb{C}^4, 0) \to (\mathbb{C}^5, 0)$ of P_2, given by

$$(u, x) \mapsto (u, x_1, x_2, x_2x_3 + x_3^5 + ux_3^2, x_1x_3 + x_3^3 + ux_3).$$

Let $J_y(G)$ be the relative Jacobian ideal of the equation G defining the image of F. A computation with Singular shows that $J_y(G) = J_y(G) : \langle u \rangle$, and hence u is not a zero divisor in $\mathcal{O}_5/J_y(G)$. In other words, $\mathcal{O}_5/J_y(G)$ is $\mathcal{O}_1$-flat and the claim follows by Theorem 6.1.

We do not know a more theoretical reason why this module is Cohen-Macaulay. In fact, we believe that finding such reason would imply a fundamental advance towards the conjecture.

7. Reduction of Mond's conjecture to families of examples

We exploit the results from the previous section to reduce the general validity of Mond's conjecture for map germs to its validity in suitable families of examples.

We define the *multiplicity of a map germ* $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ *to be* the minimum of the multiplicities of the components $(f_1,\dots,f_{n+1})$ of f at all points in S.

Theorem 7.1. *Let $(n,n{+}1)$ be nice dimensions. Suppose that, for any natural number N, there exists a weighted homogenous $\mathscr{A}$-finite germ $h\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$, of multiplicity at least N, for which Mond's conjecture holds. Then Mond's conjecture holds for any $\mathscr{A}$-finite germ $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$.*

Proof. Let $f\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$ be $\mathscr{A}$-finite. By finite determinacy we may assume, up to right-left equivalence, that f is N-determined for a certain natural number N. Let h be the germ predicted by hypothesis. By N-determinacy, if we consider the 1-parameter family of germs $h_t:=h+tf\colon(\mathbb{C}^n,S)\to(\mathbb{C}^{n+1},0)$, then h_t is equivalent to f for any $t\neq0$. Let H be a stable and versal unfolding of h parametrised by a base T. Let G be an equation of the image of H. Since Mond's conjecture holds for h and h is weighted homogeneous, by Theorem 6.1, the module $M_y(G)$ is T-flat. By versality, and because h_t is equivalent to f for any $t\neq0$ we have that H is also a versal unfolding of f. Thus, applying again Theorem 6.1, we obtain Mond's conjecture for f. □

This result can be adapted to study maps of fixed corank. For simplicity, we state the result only for mono-germs, although it can be easily adapted to multi-germs. An $\mathscr{A}$-finite map-germ $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$ of corank r is right-left equivalent to a map germ whose components admit the normal form

$$(x_1,\dots,x_{n-r},f_{n-r+1},\dots,f_{n+1}).$$

We call the *corank-r multiplicity* of f the minimum of the multiplicities of $f_{n-r+1},\dots,f_{n+1}$.

Theorem 7.2. *Let $(n,n{+}1)$ be nice dimensions. Suppose that, for any natural number N, there exists a weighted homogenous $\mathscr{A}$-finite germ $h\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$, of corank r and of corank-r multiplicity at least N, for which Mond's conjecture holds. Then Mond's conjecture holds for any $\mathscr{A}$-finite germ of corank r. $f\colon(\mathbb{C}^n,0)\to(\mathbb{C}^{n+1},0)$.*

Proof. The proof is entirely analogous to the proof of the previous theorem. The only modification is that one needs to put f and h in normal form before constructing the family h_t. □

Remark 7.3. In order to obtain examples of increasing corank-1 multiplicity, one may consider unfoldings of the plane curve parametrization (x^N,x^{N+1}). An $\mathscr{A}$-versal deformation of the parametrization (x^N,x^{N+1}) has the following form:

$$F:\mathbb{C}^{\frac{(N-1)N}{2}}\times\mathbb{C}\to\mathbb{C}^{\frac{(N-1)N}{2}}\times\mathbb{C}^2,$$

where

$$F(a_i,b_j,x)=\Big(x^N+\sum_{i=1}^{N-2}a_ix^i,x^{N+1}+\sum_{j=1}^{N-1}b_jx^j+\sum_{k=2}^{N-2}\Big(\sum_{j=(k-1)N+k+1}^{kN-1}b_jx^j\Big)\Big).$$

Replacing a_i and b_j by generic polynomials in two variables of multiplicity at least $N-i$ and $N-j$, respectively, one obtains a finitely determined map-germ from $\mathbb{C}^3$ to $\mathbb{C}^4$ of corank-1 multiplicity equal to N. This is the first dimension in which the conjecture is open.

It may be difficult to compute the image Milnor number and the $\mathscr{A}$-codimension for these examples. The difficulty comes from the fact that generic examples are hard to compute and that it is hard to find explicit, sufficiently simple finitely-determined examples. Producing other series of examples, where computations become easier, is a subject of current research.

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