# Hypersingular integral equations over a disc: convergence of a spectral method and connection with Tranter's method

Leandro Farina[a], P. A. Martin[b], Victor Péron[c]

[a] *Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Porto Alegre, RS, Brazil (permanent address; email:* farina@mat.ufrgs.br*) and BCAM–Basque Center for Applied Mathematics, Bilbao, Basque Country, Spain*

[b] *Department of Applied Mathematics and Statistics, Colorado School of Mines, Golden, Colorado* 80401*, USA. Corresponding author. Email:* pamartin@mines.edu*. Phone: 1-303-273-3895*

[c] *Université de Pau et des Pays de l'Adour, MAGIQUE3D, INRIA-Bordeaux Sud Ouest, LMA,* 64013 *Pau cedex, France. Email:* victor.peron@inria.fr

## Abstract

Two-dimensional hypersingular equations over a disc are considered. A spectral method is developed, using Fourier series in the azimuthal direction and orthogonal polynomials in the radial direction. The method is proved to be convergent. Then, Tranter's method is discussed. This method was devised in the 1950s to solve certain pairs of dual integral equations. It is shown that this method is also convergent because it leads to the same algebraic system as the spectral method.

**Keywords**: hypersingular integral equations; spectral method; Galerkin method; Tranter's method; Jacobi polynomials; screen problems

## 1. Introduction

Two-dimensional boundary-value problems involving a Neumann-type boundary condition on a thin plate or crack can often be reduced to one-dimensional hypersingular integral equations. Examples are potential flow past a rigid plate, acoustic scattering by a hard strip, water-wave interaction with thin impermeable barriers [1], and stress fields around cracks [2]; for many additional references, see [3] and [4, §6.7.1]. The basic equation encountered takes the form

$$\times\!\!\!\!\!\!\int_{-1}^{1} \left\{ \frac{1}{(x-t)^2} + K(x,t) \right\} v(t)\, dt = f(x) \quad \text{for } -1 < x < 1, \tag{1}$$

supplemented by two boundary conditions, which we take to be $v(-1) = v(1) = 0$. Here, $v$ is the unknown function, $f$ is prescribed and the kernel $K$ is known. The cross on the integral sign indicates that it is to be interpreted as a two-sided finite-part integral of order two: if $g'$ is Hölder continuous ($g \in C^{1,\alpha}$),

$$\times\!\!\!\!\!\!\int_{a}^{b} \frac{g(t)}{(x-t)^2}\, dt = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{g(t)}{(x-t)^2}\, dt + \int_{x+\varepsilon}^{b} \frac{g(t)}{(x-t)^2}\, dt - \frac{2g(x)}{\varepsilon} \right\}. \tag{2}$$

Assuming that $f$ is sufficiently smooth, the solution $v$ has square-root zeros at the end-points. This suggests that we write

$$v(x) = w(x)\, u(x) \quad \text{with} \quad w(x) = \sqrt{1-x^2}.$$

Then, we expand $u$ using a set of orthogonal polynomials; a good choice is to use Chebyshev polynomials of the second kind, $U_n$, defined by [5, 18.5.2]

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad n = 0, 1, 2, \ldots.$$

To effect the integration, we use *Tranter's integral* [21, 23],

$$\frac{2^{1-\mu}\Gamma(n+m+1)}{\Gamma(n+1)\Gamma(m+\mu)}\int_0^1 x^{n+1}\mathcal{F}_m(n+\mu,n+1,x^2)\,\frac{J_n(\xi x)}{(1-x^2)^{1-\mu}}\,dx = \xi^{-\mu}J_{n+2m+\mu}(\xi), \tag{44}$$

with $n > -1$ and $\mu > 0$. Thus, (43) gives

$$\sum_{m=0}^{\infty}\alpha_m^{n\sigma}\int_0^{\infty}(1+h(\kappa))\kappa^{2-2\mu}J_{n+2m+\mu}(\kappa)J_{n+2l+\mu}(\kappa)\,d\kappa = E(n,l,\mu), \tag{45}$$

where the constants on the right-hand side are given by

$$E(n,l,\mu) = \frac{2^{1-\mu}(n+l)!}{n!\,\Gamma(l+\mu)}\int_0^1 r^{n+1}(1-r^2)^{\mu-1}\mathcal{F}_l(n+\mu,n+1,r^2)\,g_n^{\sigma}(r)\,dr.$$

Now, how should we select $\mu$? Tranter [21, p. 319] observes that the system (45) can be solved explicitly if the term $(1+h)\kappa^{2-2\mu}$ in the integrand is replaced by $\kappa^{-1}$, and then suggests that the difference between these two terms should be made 'fairly small', if possible, by the choice of $\mu$. (There is a similar suggestion in Tranter's book [23, p. 116] and in Duffy's book [24, p. 248].) If we interpret Tranter's prescription as meaning in the limit $\kappa \to \infty$, we find that $\mu = \frac{3}{2}$. With this choice, (45) reduces to

$$\begin{aligned}\sum_{m=0}^{\infty}\alpha_m^{n\sigma}\int_0^{\infty}(1+h(\kappa))j_{n+2m+1}(\kappa)j_{n+2l+1}(\kappa)\,d\kappa &= \frac{\pi}{2}E(n,l,3/2)\\ &= \frac{\pi}{2}\frac{2^{-1/2}(n+l)!}{n!\,\Gamma(l+\frac{3}{2})}\int_0^1 r^{n+1}\sqrt{1-r^2}\mathcal{F}_l(n+3/2,n+1,r^2)g_n^{\sigma}(r)\,dr\\ &= \frac{\pi}{2\sqrt{2}}\int_0^1 \Phi_l^n(r)g_n^{\sigma}(r)\,r\,dr,\end{aligned} \tag{46}$$

where $j_n$ is a spherical Bessel function, and (44) gives

$$\int_0^1 \Phi_m^n(x)\,J_n(\xi x)x\,dx = \frac{2}{\xi\sqrt{\pi}}\,j_{n+2m+1}(\xi). \tag{47}$$

Then, as the spherical Bessel functions are orthogonal in the following sense [5, 10.22.55],

$$\int_0^{\infty} j_{n+2m+1}(\kappa)j_{n+2l+1}(\kappa)\,d\kappa = \frac{\pi\delta_{lm}}{2(2n+4l+3)}, \tag{48}$$

the system (46) becomes

$$\frac{\alpha_l^{n\sigma}}{2n+4l+3} + \frac{2}{\pi}\sum_{m=0}^{\infty}\alpha_m^{n\sigma}\int_0^{\infty}h(\kappa)j_{n+2m+1}(\kappa)j_{n+2l+1}(\kappa)\,d\kappa = \frac{1}{\sqrt{2}}\int_0^1 \Phi_l^n(r)g_n^{\sigma}(r)\,dr.$$

This system is the same as (35). As the spectral method leading to (35) has been shown to be convergent, we infer that truncated forms of Tranter's method are convergent.

Notice the choice $\mu = \frac{3}{2}$ made with Tranter's method. This choice is not arbitrary. Indeed, with any particular application, the quantity $B$ can be related to a physical quantity, $v$, a quantity that has a known behaviour near the edge of the disc, $D$. This behaviour is enforced by the correct choice for $\mu$. Similar remarks can be made when Tranter's method is used for other boundary value problems, such as acoustic scattering by a sound-soft disc (Dirichlet condition).

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