

HARDY-TYPE INEQUALITIES FOR FRACTIONAL POWERS OF THE DUNKL–HERMITE OPERATOR

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ABSTRACT. We prove Hardy-type inequalities for a fractional Dunkl–Hermite operator which incidentally give Hardy inequalities for the fractional harmonic oscillator as well. The idea is to use h -harmonic expansions to reduce the problem in the Dunkl–Hermite context to the Laguerre setting. Then, we push forward a technique based on a non-local ground representation, initially developed by R. L. Frank, E. H. Lieb and R. Seiringer in the Euclidean setting, to get a Hardy inequality for the fractional-type Laguerre operator. The above-mentioned method is shown to be adaptable to an abstract setting, whenever there is a “good” spectral theorem and an integral representation for the fractional operators involved.

1. INTRODUCTION AND MAIN RESULTS

The original Hardy’s inequality for the Laplacian on \mathbb{R}^d says that

$$\frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx, \quad d \geq 3,$$

where ∇ stands for the gradient. The statement of Hardy’s inequality can be generalized to other contexts and operators. Let $(X, d\eta)$ be a measure space, where $d\eta$ is a positive measure on X . Given $0 < \sigma < 1$, let us denote by L^σ the fractional powers of a non-negative, self-adjoint operator L on $L^2(X)$. We are interested in Hardy-type inequalities of the form

$$\int_X \frac{|f(x)|^2}{(1+|x|^2)^\sigma} d\eta(x) \leq B_\sigma \langle L^\sigma f, f \rangle,$$

for certain constant B_σ , or Hardy-type inequalities where the potential involved is homogeneous. Namely, for $0 < \sigma < 1$, $f \in C_0^\infty(X)$ we are also concerned with inequalities of the type

$$(1.1) \quad \int_X \frac{|f(x)|^2}{|x|^{2\sigma}} d\eta(x) \leq C_{d,\sigma} \langle L^\sigma f, f \rangle.$$

When $(X, d\eta) = (\mathbb{R}^d, dx)$ and $L = -\Delta$, the sharp constant $C_{d,\sigma}$ in (1.1) is already known [2, 17, 31]. Nevertheless, R. L. Frank, E. H. Lieb, and R. Seiringer [11] found a different proof of the inequality (1.1) in the Euclidean setting when $0 < \sigma < \min\{1, d/2\}$ by using a non-local version of the *ground state representation*. Such representation improved the previous results in the sense that it provided quantitative information on the error by delivering a remainder term, see e.g. Theorem 2.2 in the general setting. Hardy inequalities and their generalizations are of relevant importance in mathematical analysis. They may be applied, among others, to problems concerning mathematical physics, partial differential equations, spectral theory, harmonic analysis, and potential theory.

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The aim of this paper is two-fold. First, we revisit the technique developed by Frank–Lieb–Seiringer [11] (see also [12]). We present an exposition of such procedure in a systematic and unified way so that the method can be applied in more general settings. This description will be developed in Section 2. Secondly, we will apply this method to get Hardy inequalities in several settings. In particular, we will prove Hardy-type inequalities for certain fractional Laguerre operators $L_{\alpha,\sigma}$, and from these we will deduce the Hardy inequalities for the corresponding fractional Dunkl–Hermite operator $\mathbf{H}_{\kappa,\sigma}$. The definitions of these operators can be found, respectively, in (3.5) and Subsection 4.4 below. As an immediate consequence, we will also obtain Hardy inequalities for the fractional harmonic oscillator. Observe that in [24] the adapted method was already applied to obtain Hardy inequalities on the Heisenberg group.

We notice that a sharp Pitt’s inequality for the fractional powers of the Dunkl operator in $L^2(\mathbb{R}^d)$ was recently proved by D. V. Gorbachev, V. I. Ivanov, and S. Yu. Tikhonov in [13]. Such Pitt’s inequality, which is a weighted norm inequality for the Dunkl transform on \mathbb{R}^d , can be rewritten as a Hardy type inequality for fractional powers of the Dunkl-Laplacian Δ_κ . Thus the inequality takes the form

$$\int_{\mathbb{R}^d} \frac{|f(x)|^2}{|x|^{2\sigma}} h_\kappa(x) d(x) \leq B_\sigma \langle (-\Delta_\kappa)^\sigma f, f \rangle$$

with a sharp constant B_σ . Our guess is that an expression for the error in such an inequality could be accomplished by using the ground state representation.

We will not work with the pure fractional powers of the operators under consideration in this paper, but with their *conformally invariant* fractional powers, which will suit better for our purposes. As a consequence of the Hardy inequalities obtained for these operators, we will be able to deduce Hardy inequalities for the corresponding pure fractional powers. We borrow the terminology “conformally invariant fractional powers” from the context of sublaplacian on Heisenberg groups, see [24]. As this group arises as the boundary of Siegel’s upper half space, conformally invariant operators make perfect sense, see [5, 15] for the exact definition of this invariance. The spectral theory of sublaplacian \mathcal{L} on the Heisenberg group \mathbb{H}^d is closely related to Laguerre operators of type $(d-1)$. More precisely, the action of \mathcal{L} on functions of the form $e^{it}f(|z|)$ is given by the action of the Laguerre operator L_{d-1} on f . Consequently, when we consider the conformally invariant fractional power \mathcal{L}_σ acting on such functions, we obtain certain fractional powers of the Laguerre operator which we call conformally invariant. The analogues of such operators for any $\alpha > -\frac{1}{2}$ will be defined via spectral theorem and they will have a superficial resemblance with conformally invariant fractional powers of the sublaplacian on the Heisenberg group.

As it will be explained in Section 2, some of the main features of the general way to proceed consist of having at hand both a “good” spectral theorem and an integral representation for the ground state representation. In the case of Dunkl–Hermite setting we do not have a convolution structure (neither we have an explicit formula for the fundamental solution, which seems to be playing an important role). So our idea is to use h -harmonics to reduce the Dunkl–Hermite setting to the Laguerre case, where we have a convolution structure, viz., the Laguerre convolution is at our disposal. Moreover, in the latter setting we have “good” spectral theorem (in our case this means to have analogous results to those ones proved by M. Cowling and U. Haagerup in [7, Section 3], and slightly generalized by the second and third authors in [24, Section 3]) and explicit fundamental solutions. This is why we will focus on the Laguerre differential operator and as an application we will deduce the results for the Dunkl–Hermite operator (which includes the ordinary Hermite operator as a particular case).

Now we state the Hardy inequalities for our operators. For $\alpha > -1/2$, the Laguerre differential operator defined by

$$L_\alpha = -\frac{d^2}{dr^2} + r^2 - \frac{2\alpha + 1}{r} \frac{d}{dr}$$

is symmetric on $L^2(0, \infty)$ taken with respect to the measure $d\mu_\alpha(r) = r^{2\alpha+1}dr$. For $0 < \sigma < 1$ we use spectral theorem to define the conformally invariant fractional powers $L_{\alpha,\sigma}$, see Section 3 for the precise definition. For $\delta > 0$ and $\alpha > -1/2$, we let

$$(1.2) \quad w_{\alpha,\sigma}^\delta(r) := c_{\alpha,\sigma}(\delta + r^2)^{-(\alpha+1+\sigma)/2} K_{(\alpha+1+\sigma)/2}((\delta + r^2)/2),$$

where K_ν is the Macdonald's function of order ν (see ([19, Chapter 5, Section 5.7] for the definition of such function), and $c_{\alpha,\sigma}$ to be the constant

$$(1.3) \quad c_{\alpha,\sigma} := \frac{\sqrt{\pi}2^{1-\sigma}}{\Gamma[(\alpha + 2 + \sigma)/2]}.$$

The first Hardy inequality concerns the fractional Laguerre operators $L_{\alpha,\sigma}$. Let us define the constant $B_{\alpha,\sigma}^\delta$ by

$$(1.4) \quad B_{\alpha,\sigma}^\delta := \delta^\sigma \frac{\Gamma(\frac{\alpha+2+\sigma}{2})}{\Gamma(\frac{\alpha+2-\sigma}{2})}.$$

Theorem 1.1 (Hardy inequality for the Laguerre operator). *Let $0 < \sigma < 1$, $\delta > 0$, and $\alpha > -1/2$. Then*

$$B_{\alpha,\sigma}^\delta \int_0^\infty \frac{|f(r)|^2}{(\delta + r^2)^\sigma} d\mu_\alpha(r) \leq \frac{4^\sigma}{\delta^\sigma} (B_{\alpha,\sigma}^\delta)^2 \int_0^\infty |f(r)|^2 \frac{w_{\alpha,\sigma}^\delta(r)}{w_{\alpha,-\sigma}^\delta(r)} d\mu_\alpha(r) \leq \langle L_{\alpha,\sigma} f, f \rangle_{d\mu_\alpha}$$

for all functions $f \in C_0^\infty(0, \infty)$.

The constant just after the first inequality above turns out to be sharp. Actually, the function $w_{\alpha,\sigma}^\delta$ is an optimizer, in view of Theorem 3.11.

We use the results of the above theorem in order to prove the following Hardy inequality for fractional powers of the Dunkl-Hermite operators $\mathbf{H}_\kappa = -\Delta_\kappa + |x|^2$ on \mathbb{R}^d . Here κ is the multiplicity function defined on the Coxeter group associated to a given root system and Δ_κ is the Dunkl Laplacian on \mathbb{R}^d . We refer to Section 4 for the precise definition of these operators and their (conformally invariant) fractional powers $\mathbf{H}_{\kappa,\sigma}$. For a given Coxeter group W and a non-negative multiplicity function κ let $\gamma = \sum_{w \in W} \kappa(w)$ and define the following constant

$$(1.5) \quad \lambda := \frac{d}{2} + \gamma - 1.$$

Details about these notations and definitions for the theorem below can be found in Section 4.

Theorem 1.2 (Hardy inequality for the Dunkl-Hermite operator). *Let $0 < \sigma < 1$, $\delta > 0$, and λ be as in (1.5). Then*

$$B_{\lambda,\sigma}^\delta \int_{\mathbb{R}^d} \frac{|f(x)|^2}{(\delta + |x|^2)^\sigma} h_\kappa(x) dx \leq \langle \mathbf{H}_{\kappa,\sigma} f, f \rangle_{L^2(\mathbb{R}^d, h_\kappa)}$$

for all functions $f \in C_0^\infty(\mathbb{R}^d)$, where $B_{\lambda,\sigma}^\delta$ is as in (1.4).

When $\kappa = 0$ the Dunkl-Hermite operator \mathbf{H}_κ reduces to the standard Hermite operator and hence from Theorem 1.2, we immediately obtain the following result for H_σ , the conformally invariant fractional power of the Hermite operator (or harmonic oscillator) H .

Corollary 1.3 (Hardy inequality for the harmonic oscillator). *Let $0 < \sigma < 1$ and $\delta > 0$. Then*

$$B_{(d/2-1),\sigma}^\delta \int_{\mathbb{R}^d} \frac{|f(x)|^2}{(\delta + |x|^2)^\sigma} dx \leq \langle H_\sigma f, f \rangle_{L^2(\mathbb{R}^d)}$$

for all functions $f \in C_0^\infty(\mathbb{R}^d)$, where $B_{(d/2-1),\sigma}^\delta$ is as in (1.4).

From the results just stated, we can deduce Hardy inequalities for pure fractional powers of the underlying operators. Define

$$(1.6) \quad S_n^{\alpha,\sigma} := \frac{\Gamma\left(\frac{4n+2\alpha+2}{4} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{4n+2\alpha+2}{4} + \frac{1-\sigma}{2}\right)}.$$

With the same notation as in the previous theorems and corollary, let U_σ be the operator defined as $U_\sigma := L_{\alpha,\sigma} L_\alpha^{-\sigma}$. It can be shown that this operator is bounded, and the operator norm $\|U_\sigma\|$ can be expressed explicitly. Indeed, it is easy to see that

$$\|U_\sigma\| = \sup_{n \geq 0} \left((4n + 2\alpha + 2)^{-\sigma} 4^\sigma S_n^{\alpha,\sigma} \right).$$

Further, by using Stirling's formula for the Gamma function, one can check that $\|U_\sigma\| \sim 1$, for α large enough. Then we have a Hardy inequality for L_α^σ in the following corollary.

Corollary 1.4. *Let $0 < \sigma < 1$, $\delta > 0$, and $\alpha > -1/2$. Then*

$$B_{\alpha,\sigma}^\delta \int_0^\infty \frac{|f(r)|^2}{(\delta + r^2)^\sigma} d\mu_\alpha(r) \leq \frac{4^\sigma}{\delta^\sigma} (B_{\alpha,\sigma}^\delta)^2 \int_0^\infty |f(r)|^2 \frac{w_{\alpha,\sigma}^\delta(r)}{w_{\alpha,-\sigma}^\delta(r)} d\mu_\alpha(r) \leq \|U_\sigma\| \langle L_\alpha^\sigma f, f \rangle_{d\mu_\alpha}$$

for all functions $f \in C_0^\infty(0, \infty)$, where $B_{\alpha,\sigma}^\delta$ is as in (1.4).

Analogously, if we consider the operator $V_{\kappa,\sigma} := \mathbf{H}_{\kappa,\sigma} \mathbf{H}_\kappa^{-\sigma}$, we get the corresponding result for the fractional powers of the Hermite–Dunkl operator

Corollary 1.5. *Let $0 < \sigma < 1$, $\delta > 0$, and λ be as in (1.5). Then*

$$B_{\lambda,\sigma}^\delta \int_{\mathbb{R}^d} \frac{|f(x)|^2}{(\delta + |x|^2)^\sigma} h_\kappa(x) dx \leq \|V_{\kappa,\sigma}\| \langle \mathbf{H}_\kappa^\sigma f, f \rangle_{L^2(\mathbb{R}^d, h_\kappa)}$$

for all functions $f \in C_0^\infty(\mathbb{R}^d)$, where $B_{\lambda,\sigma}^\delta$ is as in (1.4).

From the above, we also obtain the result for the fractional powers of the harmonic oscillator.

Corollary 1.6. *Let $0 < \sigma < 1$ and $\delta > 0$. Then*

$$B_{(d/2-1),\sigma}^\delta \int_{\mathbb{R}^d} \frac{|f(x)|^2}{(\delta + |x|^2)^\sigma} dx \leq \|V_{0,\sigma}\| \langle H^\sigma f, f \rangle_{L^2(\mathbb{R}^d, dx)}$$

for all functions $f \in C_0^\infty(\mathbb{R}^d)$, where $B_{(d/2-1),\sigma}^\delta$ is as in (1.4).

Remark 1.7. Observe that all the results above are stated for smooth functions with compact support in the corresponding spaces. However, they could be extended to suitable Sobolev spaces, as in [11] or [24]. Since the procedure is quite standard we leave details to the interested reader.

Remark 1.8. Uncertainty principles could also be deduced from the Hardy inequalities as in [24, Corollary 1.7].

Remark 1.9. In [24] the second and third authors have also obtained Hardy-type inequalities where the weight functions are homogeneous. We can therefore follow the same method in obtaining such inequalities for Laguerre and Dunkl–Hermite operators. However, as the resulting inequalities are not known to be sharp even in the case of the Heisenberg group we do not pursue them here. Definitively, a quick inspection shows that a Hardy-type inequality with homogeneous weight cannot be obtained simply by letting δ go to zero in the Hardy inequalities stated above. On the other hand, the Laguerre setting can be viewed as arising from the Heisenberg setting when the functions considered are radial in the first variable. In this way, the weight $w_{\alpha,\sigma}^\delta$ appears in connection with certain function $u_{\sigma,\delta}$ (see Subsection 3.4) which is closely related to the function $u_{\lambda,\delta}(v, z)$ in [7, p.530]. When δ goes to zero, $u_{\lambda,\delta}$ tends (in a distributional sense) to the kernel of the intertwining operator of A. W. Knap and E. M. Stein [18]. In its turn, the non-homogeneous

function $u_{\lambda,1}$ is connected with a Poisson-type kernel when considering groups of type H which include the nilpotent components of the Iwasawa decomposition, see [7, Section 3] (also [16, Ch. IX Theorem 3.8]).

The outline of the paper is as follows. In Section 2 we describe the general method to get a Hardy-type inequality for fractional powers of operators in an abstract setting. Such a procedure is then applied to Laguerre operators (see the proof of Theorem 1.1 at the end of Section 3). We will then use the Hardy inequality in the Laguerre context to obtain a Hardy inequality for the corresponding fractional-type Dunkl–Hermite operator in Section 4. As explained earlier, the idea is to use expansions in h -harmonics. Moreover, we will prove a Hecke–Bochner identity that allows us to write the Dunkl–Hermite projections in terms of Laguerre convolutions which plays a crucial role in the proof. Such identity is of independent interest. Once the reduction is achieved, Theorem 1.2 will be proved at the end of Section 4.

2. THE GENERAL METHOD

In this section we consider a measure space $(X, d\eta)$, where $d\eta$ is a positive measure on a smooth Riemannian manifold X . The function spaces $L^p(X)$ are understood to be taken with respect to $d\eta$. Let A be a non-negative, self-adjoint operator on $L^2(X)$. Then, there is a unique resolution E of the identity, supported on the spectrum of A (which is a subset of $[0, \infty)$), so that the spectral resolution of A is given by

$$A = \int_0^\infty \lambda dE(\lambda),$$

or, equivalently,

$$\langle Af, g \rangle = \int_0^\infty \lambda dE_{f,g}(\lambda), \quad f \in \text{Dom } A, \quad g \in L^2(X).$$

Here, $dE_{f,g}(\lambda)$ is a regular complex Borel measure of bounded variation concentrated on the spectrum of A , and we use the notation $\langle f, g \rangle = \int_X f(x)\overline{g(x)} d\eta(x)$. The role of Plancherel formula will be played by

$$\langle Af, Af \rangle = \int_0^\infty \lambda^2 dE_{f,f}(\lambda).$$

Let $0 < \sigma < 1$. In the most general formulation of the method, we will denote by Λ_σ a fractional-type operator related to A . Our aim is to prove a Hardy-type inequality of the form

$$(2.1) \quad \langle \Lambda_\sigma f, f \rangle \geq C_\sigma \int_X h_\sigma(x) |f(x)|^2 d\eta(x),$$

where $h_\sigma(x)$ is an appropriate positive function, and we look for the explicit positive constant C_σ to be sharp.

There are two steps to be furnished in the method we present to obtain the Hardy-type inequality:

- (1) To get an integral representation for $\Lambda_\sigma f$ and an explicit form for $\langle \Lambda_\sigma f, f \rangle$.
- (2) To write the ground state representation and use the expression for $\langle \Lambda_\sigma f, f \rangle$ obtained previously.

2.1. Integral representation and the expression for $\langle \Lambda_\sigma f, f \rangle$. Let us assume that we have an integral representation of the following form for the operator Λ_σ , which is valid for all $f \in C_0^\infty(X)$:

$$(2.2) \quad \Lambda_\sigma f(x) = a_\sigma \int_X (f(x) - f(y)) K_\sigma(x, y) d\eta(y) + f(x) B_\sigma(x),$$

where $K_\sigma(x, y)$ is a symmetric (in the sense that $K_\sigma(x, y) = K_\sigma(y, x)$), positive kernel not necessarily known explicitly and a_σ is a positive constant, that depends on the kernel and the dimension of the underlying manifold X . Furthermore, $B_\sigma(x)$ is a continuous non-negative bounded function.

The next expression follows from the integral representation. We will also assume that the kernel $K_\sigma(x, y)$ is measurable and $\int_X \int_X |x - y|^2 K_\sigma(x, y) dx dy < \infty$ to allow the interchange of the order of integration in the following computations.

Lemma 2.1. *Let $0 < \sigma < 1$ and assume that the representation (2.2) is valid. Then, for all $f \in C_0^\infty(X)$*

$$(2.3) \quad \langle \Lambda_\sigma f, f \rangle = \frac{a_\sigma}{2} \int_X \int_X |f(x) - f(y)|^2 K_\sigma(x, y) d\eta(x) d\eta(y) + B_\sigma(x) \langle f, f \rangle.$$

Proof. From the integral representation (2.2) and Fubini we have

$$\langle \Lambda_\sigma f, f \rangle = \int_X \Lambda_\sigma f(x) \overline{f(x)} d\eta(x) = a_\sigma \int_X \int_X (f(x) - f(y)) \overline{f(x)} K_\sigma(x, y) d\eta(x) d\eta(y) + B_\sigma(x) \langle f, f \rangle.$$

Since the kernel $K_\sigma(x, y)$ is symmetric, the above also equals to

$$\langle \Lambda_\sigma f, f \rangle = \int_X \Lambda_\sigma f(x) \overline{f(x)} d\eta(x) = a_\sigma \int_X \int_X (f(y) - f(x)) \overline{f(y)} K_\sigma(x, y) d\eta(x) d\eta(y) + B_\sigma(x) \langle f, f \rangle.$$

Adding them up we get the announced statement. \square

Even though we have proved the above lemma under the assumption that $f \in C_0^\infty(X)$, in order to make use of the representation (2.3) it is necessary to assume its validity for a suitable Sobolev space which depends on the operator A and σ . Let us denote such a Sobolev space by $W_A^\sigma(X)$ and assume that (2.2) and hence the result of Lemma 2.1 are valid for all $f \in W_A^\sigma(X)$.

2.2. The ground state representation and a Hardy-type inequality. Let $0 < \sigma < 1$. For a suitable positive function $w_\sigma(x)$ to be specified later, let us set

$$\mathcal{H}_\sigma[f] := \langle \Lambda_\sigma f, f \rangle - C_\sigma \int_X \frac{\tilde{w}_\sigma(x)}{w_\sigma(x)} |f(x)|^2 d\eta(x),$$

where C_σ is a positive constant and $\tilde{w}_\sigma(x) := C_\sigma^{-1} \Lambda_\sigma w_\sigma(x)$. If we can show that $\mathcal{H}_\sigma[f]$ is nonnegative, then we get a version of Hardy's inequality for the operator Λ_σ .

Theorem 2.2. *Let $0 < \sigma < 1$ and assume the validity of Lemma 2.1 for all $f \in W_A^\sigma(X)$. Given $f \in C_0^\infty(X)$ and a positive function $w_\sigma \in W_A^\sigma(X)$ set $g(x) = f(x)(w_\sigma(x))^{-1}$. Then*

$$\mathcal{H}_\sigma[f] = \frac{a_\sigma}{2} \int_X \int_X |g(x) - g(y)|^2 K_\sigma(x, y) w_\sigma(x) w_\sigma(y) d\eta(x) d\eta(y),$$

where a_σ is the positive constant in Lemma 2.1.

Proof. By polarizing the expression in Lemma 2.1 we get, for any $F, G \in C_0^\infty(X)$,

$$(2.4) \quad \langle \Lambda_\sigma F, G \rangle = \frac{a_\sigma}{2} \int_X \int_X (F(x) - F(y)) \overline{(G(x) - G(y))} K_\sigma(x, y) d\eta(x) d\eta(y) + B_\sigma(x) \langle F, G \rangle.$$

Now, we take $G(x) = w_\sigma(x)$ and $F(x) = |f(x)|^2 w_\sigma(x)^{-1}$. After simplification, the right hand side of (2.4) becomes

$$\frac{a_\sigma}{2} \int_X \int_X \left(|f(x) - f(y)|^2 - \left| \frac{f(x)}{w_\sigma(x)} - \frac{f(y)}{w_\sigma(y)} \right|^2 w_\sigma(x) w_\sigma(y) \right) K_\sigma(x, y) d\eta(x) d\eta(y) + B_\sigma(x) \langle f, f \rangle.$$

On the other hand, using the fact that Λ_σ is self-adjoint and recalling the definition of \tilde{w}_σ we could write the left hand side of (2.4) as

$$C_\sigma \int_X \frac{\tilde{w}_\sigma(x)}{w_\sigma(x)} |f(x)|^2 d\eta(x).$$

Thus we have

$$C_\sigma \int_X \frac{\tilde{w}_\sigma(x)}{w_\sigma(x)} |f(x)|^2 d\eta(x) = \frac{a_\sigma}{2} \int_X \int_X |f(x) - f(y)|^2 K_\sigma(x, y) d\eta(x) d\eta(y) + B_\sigma(x) \langle f, f \rangle - \frac{a_\sigma}{2} \int_X \int_X |g(x) - g(y)|^2 K_\sigma(x, y) w_\sigma(x) w_\sigma(y) d\eta(x) d\eta(y).$$

The latter identity and Lemma 2.1 proves the theorem. □

Consequently, Theorem 2.2 leads to the following Hardy-type inequality.

Corollary 2.3. *Let $0 < \sigma < 1$. Then*

$$\langle \Lambda_\sigma f, f \rangle \geq C_\sigma \int_X |f(x)|^2 \frac{\tilde{w}_\sigma(x)}{w_\sigma(x)} d\eta(x),$$

for all functions $f \in C_0^\infty(X)$.

Remark 2.4. In the proof of Theorem 2.2 we have assumed that the action of the operator Λ_σ on the weight w_σ can be calculated. In several cases, this can be done via the spectral theorem but it is important that the resulting function \tilde{w}_σ is explicit. Though this can be done in several particular cases, we do not have a general result guaranteeing such a simplification. Actually, this step is crucial, and so the choice of such weight a function is dictated by the requirement that \tilde{w}_σ can be computed explicitly. In most cases, we use a function w_σ related to the fundamental solution of the operator Λ_σ .

Remark 2.5. Corollary 2.3 is not necessarily a sharp Hardy inequality, but it is an intermediate step towards a sharp Hardy inequality of the type (2.1). The task boils down to minimizing the function $a_\sigma \frac{\tilde{w}_\sigma(x)}{w_\sigma(x)}$, keeping a careful track of the constants involved. In several specific cases this function can be explicitly computed, so the sharp Hardy inequality, modulo verifications on the convergence of the integrals, is directly obtained from Theorem 2.2 (see [11, Proposition 4.1] for the Euclidean setting or [24, Theorems 5.2 and 5.4] for the Heisenberg group).

The definitions and the context above are rather general. We remind that in the following sections we will develop the procedure in the particular cases of Laguerre and Dunkl–Hermite settings. We will deal with the conformally invariant fractional σ -th powers of the corresponding operators A , which we denote by Λ_σ . Such operators will be defined by using spectral decomposition.

We have just outlined a general technique to get Hardy inequalities. This technique was first developed in the Euclidean context in [11], and improved and generalized, always related to the Euclidean Laplacian, in [12]. Such a method was for the first time adapted to get Hardy inequalities in the Heisenberg group in [24]. Our main purpose in this paper is to show the robustness of this technique by proving Hardy inequalities related to operators which are not translation invariant. In the subsequent sections we will apply this procedure to get Hardy inequalities for the Laguerre operator, the harmonic oscillator, and the Dunkl–Hermite operator.

3. A HARDY INEQUALITY FOR THE FRACTIONAL LAGUERRE OPERATOR

3.1. The fractional powers of the Laguerre operator. We will introduce first some facts related to Laguerre functions. From now on, let $\alpha > -1/2$. For $r, s > 0$, we define the *Laguerre translation* $\mathcal{T}_r^\alpha f$ of a function f on $(0, \infty)$ by

$$\mathcal{T}_r^\alpha f(s) = \frac{\Gamma(\alpha + 1)2^{\alpha}}{\sqrt{2\pi}} \int_0^\pi f((r^2 + s^2 + 2rs \cos \theta)^{1/2}) J_{\alpha-1/2}(rs \sin \theta) (rs \sin \theta)^{-(\alpha-1/2)} (\sin \theta)^{2\alpha} d\theta,$$

where J_ν is the Bessel function of order ν . If f and g are functions defined on $(0, \infty)$, the *Laguerre convolution* $f *_\alpha g$ is given by

$$(3.1) \quad f *_\alpha g = \int_0^\infty \mathcal{T}_r^\alpha f(s) g(s) s^{2\alpha+1} ds,$$

see [29, Chapter 6] and references therein for the definitions and properties of Laguerre translation and Laguerre convolution.

For $r \in (0, \infty)$, we define the Laguerre functions of type α as

$$\varphi_n^\alpha(r) := L_n^\alpha(r^2)e^{-r^2/2}, \quad n = 0, 1, \dots,$$

where L_n^α are the Laguerre polynomials of order α . The functions $\{\varphi_n^\alpha\}_{n=0}^\infty$ form an orthogonal basis for $L^2((0, \infty), d\mu_\alpha)$, where $d\mu_\alpha(r) = r^{2\alpha+1}$. When we apply the Laguerre translation to Laguerre functions we have the identity (see [29, (6.1.28)])

$$(3.2) \quad \mathcal{T}_r^\alpha \varphi_n^\alpha(s) = \frac{n!}{(\alpha+1)_n} \varphi_n^\alpha(r) \varphi_n^\alpha(s), \quad \alpha > -1/2,$$

with the notation $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ standing for the Pochhammer symbol.

As stated in the introduction, we will deal with the Laguerre differential operator given by

$$(3.3) \quad L_\alpha = -\frac{d^2}{dr^2} + r^2 - \frac{2\alpha+1}{r} \frac{d}{dr},$$

which is symmetric on $L^2((0, \infty), d\mu_\alpha)$ taken with respect to the measure $d\mu_\alpha$. The functions φ_n^α are eigenfunctions of the differential operator (3.3). Indeed, for $n = 0, 1, 2, \dots$

$$L_\alpha \varphi_n^\alpha = (4n + 2\alpha + 2) \varphi_n^\alpha.$$

The family of functions ψ_n^α , given by

$$\psi_n^\alpha(r) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right)^{1/2} \varphi_n^\alpha(r), \quad r > 0,$$

forms an orthonormal basis for $L^2((0, \infty), d\mu_\alpha)$. The Laguerre expansion of a function $f \in L^2((0, \infty), d\mu_\alpha)$, namely the expansion

$$f = \sum_{n=0}^{\infty} \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)} \right) (f, \varphi_n^\alpha) \varphi_n^\alpha$$

can be written in a compact form in terms of Laguerre convolution.

Lemma 3.1. *For a function $f \in L^2((0, \infty), d\mu_\alpha)$ we have*

$$f = \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} f *_\alpha \varphi_n^\alpha$$

where the series converges in norm. In particular,

$$(3.4) \quad \frac{2}{\Gamma(\alpha+1)} \varphi_n^\alpha *_\alpha \varphi_j^\alpha = \delta_{nj} \varphi_n^\alpha.$$

Proof. Let us define P_n^α to be the projection onto the n th eigenspace

$$P_n^\alpha f(r) = \psi_n^\alpha(r) \int_0^\infty f(s) \psi_n^\alpha(s) d\mu_\alpha(s).$$

Therefore, by (3.2),

$$\begin{aligned} f(r) &= \sum_{n=0}^{\infty} P_n^\alpha f(r) = 2 \sum_{n=0}^{\infty} \int_0^\infty f(s) \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \varphi_n^\alpha(r) \varphi_n^\alpha(s) d\mu_\alpha(s) \\ &= \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} \int_0^\infty f(s) \mathcal{T}_r^\alpha \varphi_n^\alpha(s) d\mu_\alpha(s) = \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} f *_\alpha \varphi_n^\alpha(r), \end{aligned}$$

where we used [29, Proposition 6.1.1] in the fourth equality above. Finally, by computing $P_n^\alpha \varphi_j^\alpha$, and using that $P_n^\alpha f(r) = \frac{2}{\Gamma(\alpha+1)} f *_\alpha \varphi_n^\alpha(r)$ (which is easily deduced from the equalities above), we obtain (3.4). \square

Thus the spectral decomposition of the Laguerre operator is given by

$$L_\alpha f(r) = \frac{2}{\Gamma(\alpha + 1)} \sum_{n=0}^{\infty} (4n + 2\alpha + 2) f *_\alpha \varphi_n^\alpha(r).$$

Therefore, a natural way to define fractional powers of the Laguerre operator is via the spectral decomposition:

$$L_\alpha^\sigma f(r) = \frac{2}{\Gamma(\alpha + 1)} \sum_{n=0}^{\infty} (4n + 2\alpha + 2)^\sigma f *_\alpha \varphi_n^\alpha(r).$$

Nevertheless, it will be more convenient to work with the following modified fractional powers $L_{\alpha,\sigma}$. For $0 \leq \sigma \leq 1$ we define $L_{\alpha,\sigma}$ by

$$(3.5) \quad L_{\alpha,\sigma} f(r) = \frac{2}{\Gamma(\alpha + 1)} \sum_{n=0}^{\infty} 4^\sigma S_n^{\alpha,\sigma} f *_\alpha \varphi_n^\alpha(r),$$

where $S_n^{\alpha,\sigma}$ is as in (1.6). In short, the above means that $L_{\alpha,\sigma}$ is the operator

$$L_{\alpha,\sigma} := 4^\sigma \frac{\Gamma\left(\frac{L_\alpha}{4} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{L_\alpha}{4} + \frac{1-\sigma}{2}\right)}$$

corresponding to the spectral multiplier $4^\sigma S_n^{\alpha,\sigma}$ with $S_n^{\alpha,\sigma}$ defined in (1.6). The motivation for this definition goes back, for instance, to [4, (1.33)]. Observe that $L_{\alpha,1} = L_\alpha$. This operator has an explicit fundamental solution, see Subsection 3.4, and this fact makes it more suitable than L_α^σ , whose fundamental solution cannot be written down explicitly. Moreover, by using Stirling's formula for the Gamma function, one can readily see that $L_{\alpha,\sigma} = U_{\alpha,\sigma} L_\alpha^\sigma$, where $U_{\alpha,\sigma}$ is a bounded operator on $L^2((0, \infty), d\mu_\alpha)$.

3.2. The Laguerre heat semigroup. We will use the language of semigroups to get a suitable integral representation for $L_{\alpha,\sigma}$. The heat semigroup related to L_α is defined on $L^2((0, \infty), d\mu_\alpha)$ by

$$(3.6) \quad e^{-tL_\alpha} f := T_{\alpha,t} f = \frac{2}{\Gamma(\alpha + 1)} \sum_{n=0}^{\infty} e^{-t(4n+2\alpha+2)} f *_\alpha \varphi_n^\alpha, \quad t > 0.$$

If we define

$$q_{t,\alpha}(r) = \frac{2}{\Gamma(\alpha + 1)} \sum_{n=0}^{\infty} e^{-(4n+2\alpha+2)t} \varphi_n^\alpha(r),$$

then the generating function identity for φ_n^α (see [30, (1.4.24)]) immediately gives

$$(3.7) \quad q_{t,\alpha}(r) = \frac{1}{2^\alpha \Gamma(\alpha + 1)} (\sinh 2t)^{-\alpha-1} e^{-(\coth 2t)r^2/2},$$

and we can write

$$e^{-tL_\alpha} f = f *_\alpha q_{t,\alpha}.$$

Let us denote by I_α the modified Bessel function of the first kind and order α , see [19, Chapter 5, Section 5.7]. Then, from the definition of $q_{t,\alpha}(r)$, the identity (3.2), and another generating function identity for $\varphi_n^\alpha(r)$ (see [29, p. 83])

$$(3.8) \quad \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \varphi_n^\alpha(r) \varphi_n^\alpha(s) w^{2n} = (1-w^2)^{-1} (rsw)^{-\alpha} \exp\left\{-\frac{1}{2}\left(\frac{1+w^2}{1-w^2}\right)(r^2+s^2)\right\} I_\alpha\left(\frac{2wrs}{1-w^2}\right),$$

we obtain readily the following lemma.

Lemma 3.2. *Let $\alpha > -1/2$, we have that*

$$\mathcal{T}_r^\alpha q_{t,\alpha}(s) = \frac{e^{-\frac{\cosh 2t}{2}(r^2+s^2)}}{(rs)^\alpha \sinh 2t} I_\alpha \left(\frac{rs}{\sinh 2t} \right).$$

The Laguerre heat semigroup is not conservative, i.e. $T_{\alpha,t}1 \neq 1$. However, it is easy to calculate and estimate $T_{\alpha,t}1$ which is the content of the following lemma.

Lemma 3.3. *Let $\alpha > -1/2$, we have that*

$$T_{\alpha,t}1(r) = \frac{1}{(\cosh 2t)^{\alpha+1}} e^{-\frac{\tanh 2t}{2}r^2} \leq 1.$$

Proof. By (3.1) and the previous lemma

$$\begin{aligned} T_{\alpha,t}1(r) &= 1 *_{\alpha} q_{t,\alpha}(r) = \int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) s^{2\alpha+1} ds \\ &= \frac{e^{-\frac{\cosh 2t}{2}r^2}}{r^\alpha \sinh 2t} \int_0^\infty e^{-\frac{\cosh 2t}{2}s^2} I_\alpha \left(\frac{rs}{\sinh 2t} \right) s^{\alpha+1} ds. \end{aligned}$$

Now, the result is a consequence of the identity (see [23, (2.15.5.4)])

$$\int_0^\infty e^{-px^2} I_\alpha(cx) x^{\alpha+1} dx = \frac{c^\alpha e^{c^2/(4p)}}{(2p)^{\alpha+1}}, \quad p > 0, \quad \alpha > -1.$$

□

3.3. Integral representation for $L_{\alpha,\sigma}$. In order to obtain an integral representation for the operator $L_{\alpha,\sigma}$ we first prove a numerical identity. We will use the identity (see [14, p. 382, 3.541.1])

$$(3.9) \quad \int_0^\infty e^{-\mu t} \sinh^\nu \beta t dt = \frac{1}{2^{\nu+1}} \frac{\Gamma(\frac{\mu}{2\beta} - \frac{\nu}{2}) \Gamma(\nu+1)}{\Gamma(\frac{\mu}{2\beta} + \frac{\nu}{2} + 1)},$$

which is valid for $\operatorname{Re} \beta > 0$, $\operatorname{Re} \nu > -1$, $\operatorname{Re} \mu > \operatorname{Re} \beta \nu$.

Lemma 3.4. *Let $0 < \sigma < 1$, and $\lambda \in \mathbb{R}$ such that $\lambda + 2\sigma > -2$. Then,*

$$2^\sigma |\Gamma(-\sigma)| \frac{\Gamma(\frac{\lambda}{4} + \frac{1+\sigma}{2})}{\Gamma(\frac{\lambda}{4} + \frac{1-\sigma}{2})} = \int_0^\infty (\cosh 2t - 1) (\sinh 2t)^{-\sigma-1} dt + \int_0^\infty (1 - e^{-t\lambda}) (\sinh 2t)^{-\sigma-1} dt.$$

Proof. We take $\nu = -\sigma$, $\beta = 2$, and $\mu = \lambda + 2$ in (3.9). Thus

$$\int_0^\infty e^{-(\lambda+2)t} (\sinh 2t)^{-\sigma} dt = \frac{2^{\sigma-1} \Gamma(\frac{\lambda+2}{4} + \frac{\sigma}{2}) \Gamma(1-\sigma)}{\Gamma(\frac{\lambda}{4} + \frac{1-\sigma}{2} + 1)}$$

or, equivalently,

$$(3.10) \quad \left(\frac{\lambda}{2} + 1 - \sigma\right) \int_0^\infty e^{-(\lambda+2)t} (\sinh 2t)^{-\sigma} dt = \frac{2^\sigma \Gamma(\frac{\lambda+2}{4} + \frac{1+\sigma}{2}) \Gamma(1-\sigma)}{\Gamma(\frac{\lambda}{4} + \frac{1-\sigma}{2})}.$$

Moreover, an integration by parts yields

$$\begin{aligned} (\lambda+2) \int_0^\infty e^{-(\lambda+2)t} (\sinh 2t)^{-\sigma} dt &= \int_0^\infty \frac{d}{dt} (1 - e^{-(\lambda+2)t}) (\sinh 2t)^{-\sigma} dt \\ &= 2\sigma \int_0^\infty (1 - e^{-(\lambda+2)t}) (\sinh 2t)^{-\sigma-1} \cosh 2t dt. \end{aligned}$$

Combining this with (3.10), and noticing that $\frac{\Gamma(1-\sigma)}{\sigma} = |\Gamma(-\sigma)|$, we deduce

$$\begin{aligned} \frac{2^\sigma \Gamma\left(\frac{\lambda}{4} + \frac{1+\sigma}{4}\right) |\Gamma(-\sigma)|}{\Gamma\left(\frac{\lambda}{4} + \frac{1-\sigma}{2}\right)} &= \int_0^\infty \left((1 - e^{-(\lambda+2)t}) (\sinh 2t)^{-\sigma-1} \cosh 2t - e^{-(\lambda+2)t} (\sinh 2t)^{-\sigma} \right) dt \\ &= \int_0^\infty \left(\cosh 2t - e^{-(\lambda+2)t} (\cosh 2t + \sinh 2t) \right) (\sinh 2t)^{-\sigma-1} dt \\ &= \int_0^\infty \left(\cosh 2t - e^{-\lambda t} \right) (\sinh 2t)^{-\sigma-1} dt \\ &= \int_0^\infty (\cosh 2t - 1) (\sinh 2t)^{-\sigma-1} dt + \int_0^\infty (1 - e^{-t\lambda}) (\sinh 2t)^{-\sigma-1} dt. \end{aligned}$$

The proof is complete. \square

Define the functions

$$\mathcal{K}_{\alpha,\sigma}(r, s) := \int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) \frac{dt}{(\sinh 2t)^{\sigma+1}}, \quad F_{\alpha,\sigma}(r) := \int_0^\infty (1 - T_{\alpha,t} 1(r)) \frac{dt}{(\sinh 2t)^{\sigma+1}}.$$

The integral defining the function $F_{\alpha,\sigma}(r)$ has to be taken in the sense of Bochner, i.e. as the integral of the $L^2((0, \infty), d\mu_\alpha)$ valued function $t \rightarrow (1 - 1 *_\alpha q_{t,\alpha})$. Observe that this function is non-negative, by Lemma 3.3. The expression in Lemma 3.2 allows us to deduce that $\mathcal{K}_{\alpha,\sigma}$ is a positive function. Let us call $E_\sigma := \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (\cosh 2t - 1) (\sinh 2t)^{-\sigma-1} dt$. The constant E_σ can be explicitly computed, see Remark 3.6. We are ready to prove the integral representation for $L_{\alpha,\sigma}$.

Proposition 3.5. *Let $0 < \sigma < 1$ and $f \in C_0^\infty(0, \infty)$. Then we have the following pointwise representation*

$$L_{\alpha,\sigma} f(r) = \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (f(r) - f(s)) \mathcal{K}_{\alpha,\sigma}(r, s) d\mu_\alpha(s) + f(r) \left(E_\sigma + \frac{2^\sigma}{|\Gamma(-\sigma)|} F_{\alpha,\sigma}(r) \right).$$

Proof. In the identity of Lemma 3.4 we take $\lambda = 4n + 2\alpha + 2$, multiply both sides by $4^\sigma \frac{2}{\Gamma(\alpha+1)} f *_\alpha \varphi_n^\alpha$ and then sum over n . Thus, with the constants $S_n^{\alpha,\sigma}$ as in (1.6), and taking into account (3.6),

$$\begin{aligned} L_{\alpha,\sigma} f(r) &= \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^\infty 4^\sigma S_n^{\alpha,\sigma} f *_\alpha \varphi_n^\alpha \\ &= E_\sigma f(r) + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (f(r) - f *_\alpha q_{t,\alpha}(r)) (\sinh 2t)^{-\sigma-1} dt, \end{aligned}$$

and this last integral has also to be interpreted as the Bochner integral of the $L^2((0, \infty), d\mu_\alpha)$ valued function $t \rightarrow f - f *_\alpha q_{t,\alpha}$. We have

$$\begin{aligned} f(r) - f *_\alpha q_{t,\alpha}(r) &= f(r) - \int_0^\infty \mathcal{T}_r^\alpha f(s) q_{t,\alpha}(s) d\mu_\alpha(s) = f(r) - \int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) f(s) d\mu_\alpha(s) \\ &= f(r) - f(r) 1 *_\alpha q_{t,\alpha}(r) + f(r) \int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) d\mu_\alpha(s) - \int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) f(s) d\mu_\alpha(s) \\ &= f(r) (1 - 1 *_\alpha q_{t,\alpha}(r)) + \int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) (f(r) - f(s)) d\mu_\alpha(s). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty (f(r) - f *_\alpha q_{t,\alpha}(r)) \frac{dt}{(\sinh 2t)^{\sigma+1}} &= f(r) \int_0^\infty (1 - 1 *_\alpha q_{t,\alpha}(r)) \frac{dt}{(\sinh 2t)^{\sigma+1}} \\ (3.11) \quad &+ \int_0^\infty \left[\int_0^\infty \mathcal{T}_r^\alpha q_{t,\alpha}(s) (f(r) - f(s)) d\mu_\alpha(s) \right] \frac{dt}{(\sinh 2t)^{\sigma+1}}. \end{aligned}$$

We have to justify the application of Fubini's theorem in the integral (3.11). In the following lines, we will denote by C_α (or simply C) a constant depending on α (or even independent of α) that may vary at any occurrence. By Lemma 3.2 and applying the change of variable due to S. Meda

$$t(\xi) = \frac{1}{2} \log \frac{1+\xi}{1-\xi}, \quad t \in (0, \infty), \quad \xi \in (0, 1),$$

we have, see [6, Proposition 4.2], that

$$\mathcal{T}_r^\alpha q_{t(\xi), \alpha}(s) \leq C(rs)^{-\alpha} \left(\frac{1-\xi^2}{2\xi} \right)^{1/2} e^{-\frac{1}{4} \left(\frac{|r-s|^2}{\xi} + \xi|r+s|^2 \right)},$$

and the expression in the right hand side of the inequality is precisely the heat kernel associated to the harmonic oscillator, see [29]. Moreover, such change of parameters also yields

$$\frac{dt}{(\sinh 2t)^{\sigma+1}} = \frac{(1-\xi^2)^\sigma}{(2\xi)^{\sigma+1}} d\xi, \quad t \in (0, \infty), \quad \xi \in (0, 1).$$

Notice that

$$\frac{(1-\xi^2)^\sigma}{(2\xi)^{\sigma+1}} d\xi \sim \frac{d\xi}{\xi^{\sigma+1}}, \quad \xi \sim 0, \quad \frac{(1-\xi^2)^\sigma}{(2\xi)^{\sigma+1}} d\xi \sim (1-\xi^2)^\sigma d\xi, \quad \xi \sim 1.$$

With the ingredients just collected, the absolute convergence of the integral in (3.11) can be concluded following exactly the same reasoning as in [28, Proof of Theorem 5.7, p. 2118], where the authors prove a pointwise formula for the fractional harmonic oscillator. Finally, we also deduce that $F_{\alpha, \sigma} \in C^\infty(\mathbb{R}^d)$ by following an analogous reasoning as in [28, Lemma 5.11]. \square

Remark 3.6. The constant $E_\sigma := \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (\cosh 2t - 1)(\sinh 2t)^{-\sigma-1} dt$ can be explicitly computed. Indeed, with Meda's change of variable, the integral boils down to

$$2^{-\sigma} \int_0^1 (1-\xi^2)^{\sigma-1} (\xi^2)^{(1-\sigma)/2} d\xi = 2^{-\sigma-1} \int_0^1 (1-w)^{\sigma-1} w^{-\sigma/2} dw = 2^{-\sigma-1} B(1-\sigma/2, \sigma),$$

where $B(x, y)$ is the Beta function. Then, by writing the latter in terms of Gamma functions, and in view of the duplication formula, we get

$$E_\sigma = \frac{2^\sigma}{|\Gamma(-\sigma)|} 2^{-\sigma-1} B(1-\sigma/2, \sigma) = \frac{\Gamma(1-\sigma/2)\Gamma(\sigma)}{2\Gamma(1+\sigma/2)|\Gamma(-\sigma)|} = \frac{(-\sigma/2)\Gamma(-\sigma/2)\Gamma(\sigma)}{2(\sigma/2)\Gamma(\sigma/2)|\Gamma(-\sigma)|} = 4^\sigma \frac{\Gamma(\frac{1+\sigma}{2})}{\Gamma(\frac{1-\sigma}{2})}.$$

Following the general procedure showed in Section 2, we continue with the integral representation for $\langle L_{\alpha, \sigma} f, f \rangle$.

Proposition 3.7. *Let $0 < \sigma < 1$ and $\alpha > -1$. Then, for all $f \in C_0^\infty(0, \infty)$*

$$\begin{aligned} \langle L_{\alpha, \sigma} f, f \rangle &= \frac{2^{\sigma-1}}{|\Gamma(-\sigma)|} \int_0^\infty \int_0^\infty |f(r) - f(s)|^2 \mathcal{K}_{\alpha, \sigma}(r, s) d\mu_\alpha(r) d\mu_\alpha(s) \\ &\quad + \frac{1}{2} \left(E_\sigma + \frac{2^{-\sigma}}{|\Gamma(-\sigma)|} \right) \int_0^\infty f(r)^2 F_{\alpha, \sigma}(r) d\mu_\alpha(r). \end{aligned}$$

Proof. Proceed as in Lemma 2.1, by using the integral representation in Proposition 3.5. \square

3.4. Spectral analysis for $L_{\alpha, \sigma}$ and the fundamental solution. In this subsection we figure out a suitable weight w_σ to be plugged in the computation of the ground state representation, and find out the action of the operator $L_{\alpha, \sigma}$ on such a weight. We may look at the situation in the Laguerre setting as a particular case of the Heisenberg setting, in which the functions considered are radial in the first variable, and the convolution is just the Laguerre convolution. So we find hints in the arguments developed in [24, Section 3].

For $\delta > 0$, let $u_{\sigma,\delta}(r, t)$ be the function defined on $(0, \infty) \times \mathbb{R}$ by

$$u_{\sigma,\delta}(r, t) := ((\delta + r^2)^2 + t^2)^{-(\alpha+2+\sigma)/2}$$

(observe that the function above is the one defined by [7, p. 530], see also [24, (3.1)], but radial in the first variable, and with suitable parameters).

Let f^λ stand for the inverse Fourier transform of a function $f \in L^1((0, \infty) \times \mathbb{R})$ in the second variable, i.e., for each $\lambda \in \mathbb{R}$, we set

$$f^\lambda(r) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} f(r, t) dt.$$

For $a, b \in \mathbb{R}^+$ and $c \in \mathbb{R}$ we define

$$L(a, b, c) = \int_0^\infty e^{-a(2x+1)} x^{b-1} (1+x)^{-c} dx.$$

In the following lemma we compute the Laguerre expansion of $u_{\sigma,\delta}^\lambda(r)$.

Lemma 3.8. *Let $0 < \sigma < 1$, $\delta > 0$, $\alpha > -1/2$, and $\lambda \in \mathbb{R} \setminus \{0\}$. We have*

$$u_{\sigma,\delta}^\lambda(r) = |\lambda|^{\alpha+1} \sum_{n=0}^{\infty} c_{n,\delta}^\lambda(\sigma) \varphi_n^\alpha((2|\lambda|)^{1/2}r),$$

where the coefficients $c_{n,\delta}^\lambda(\sigma)$ are given by

$$(3.12) \quad c_{n,\delta}^\lambda(\sigma) = \frac{2\pi|\lambda|^\sigma}{\Gamma((\alpha+2+\sigma)/2)^2} L\left(\delta|\lambda|, \frac{4n+2\alpha+2}{4} + \frac{1+\sigma}{2}, \frac{4n+2\alpha+2}{4} + \frac{1-\sigma}{2}\right).$$

Proof. The proof is already present implicitly in the proof of [24, Proposition 3.2]. \square

It happens that the Fourier transform of the function $u_{\sigma,\delta}(r, t)$ in the t -variable can be evaluated in terms of a Macdonald's function K_ν . Let us recall the definition of the weight $w_{\alpha,\sigma}(r)$, already given in (1.2)

$$w_{\alpha,\sigma}^\delta(r) := c_{\alpha,\sigma}(\delta + r^2)^{-(\alpha+1+\sigma)/2} K_{(\alpha+1+\sigma)/2}((\delta + r^2)/2),$$

where $c_{\alpha,\sigma}$ is the constant in (1.3). Indeed, we have the following.

Proposition 3.9. *Let $0 < \sigma < 1$, $\delta > 0$, and $\alpha > -1/2$. Then*

$$w_{\alpha,\sigma}^\delta(r) = \sum_{n=0}^{\infty} c_{n,\delta}^{1/2}(\sigma) \varphi_n^\alpha(r).$$

Proof. In [22, 2.5.6. 4., p. 390], we find the formula

$$\int_0^\infty \frac{\cos bx}{(x^2 + z^2)^\rho} dx = \left(\frac{2z}{b}\right)^{1/2-\rho} \frac{\sqrt{\pi}}{\Gamma(\rho)} K_{1/2-\rho}(bz),$$

which is valid for $b, \operatorname{Re} \rho, \operatorname{Re} z > 0$. This formula gives, assuming $\lambda > 0$,

$$\int_{-\infty}^{\infty} e^{i\lambda t} u_\sigma(r, t) dt = 2 \frac{\sqrt{\pi} 2^{-(\alpha+1+\sigma)/2}}{\Gamma[(\alpha+2+\sigma)/2]} \left(\frac{\delta + r^2}{\lambda}\right)^{-(\alpha+1+\sigma)/2} K_{-(\alpha+1+\sigma)/2}(\lambda(\delta + r^2)).$$

Thus, by Lemma 3.8 we have the following Laguerre expansion of $K_{-(\alpha+1+\sigma)/2}(\lambda(\delta + r^2))$:

$$2 \frac{\sqrt{\pi} 2^{-(\alpha+1+\sigma)/2}}{\Gamma[(\alpha+2+\sigma)/2]} \left(\frac{\delta + r^2}{\lambda}\right)^{-(\alpha+1+\sigma)/2} K_{-(\alpha+1+\sigma)/2}(\lambda(\delta + r^2)) = \lambda^{\alpha+1} \sum_{n=0}^{\infty} c_{n,\delta}^\lambda(\sigma) \varphi_n^\alpha((2\lambda)^{1/2}r).$$

By taking $\lambda = 1/2$ we get

$$\frac{\sqrt{\pi}2^{1-\sigma}}{\Gamma[(\alpha+2+\sigma)/2]}(\delta+r^2)^{-(\alpha+1+\sigma)/2}K_{-(\alpha+1+\sigma)/2}((\delta+r^2)/2) = \sum_{n=0}^{\infty} c_{n,\delta}^{1/2}(\sigma)\varphi_n^\alpha(r).$$

Finally, since $K_\nu = K_{-\nu}$ (see [19, (5.7.10)]), we obtain the desired result. \square

We establish now the relation between $c_{n,\delta}^{1/2}(\sigma)$ and $c_{n,\delta}^{1/2}(-\sigma)$. Actually, the result below follows from [24, Proposition 3.3]. Nevertheless, we will provide a different, self-contained proof.

Lemma 3.10. *For $0 < \sigma < 1$, $\delta > 0$, and $\alpha > -1/2$, we have*

$$c_{n,\delta}^{1/2}(-\sigma) = \delta^\sigma \frac{\Gamma(\frac{\alpha+2+\sigma}{2})^2}{\Gamma(\frac{\alpha+2-\sigma}{2})^2} S_n^{\alpha,-\sigma} c_{n,\delta}^{1/2}(\sigma),$$

where $S_n^{\alpha,\sigma}$ was defined in (1.6).

Proof. From (3.12), the identity to be proved is equivalent to

$$\begin{aligned} L\left(\frac{\delta}{2}, \frac{4n+2\alpha+2}{4} + \frac{1-\sigma}{2}, \frac{4n+2\alpha+2}{4} + \frac{1+\sigma}{2}\right) \\ = \delta^\sigma \frac{\Gamma(\frac{4n+2\alpha+2}{4} + \frac{1-\sigma}{2})}{\Gamma(\frac{4n+2\alpha+2}{4} + \frac{1+\sigma}{2})} L\left(\frac{\delta}{2}, \frac{4n+2\alpha+2}{4} + \frac{1+\sigma}{2}, \frac{4n+2\alpha+2}{4} + \frac{1-\sigma}{2}\right). \end{aligned}$$

Let us call $w := \frac{4n+2\alpha+2}{4}$, so that the identity above reads as

$$L\left(\frac{\delta}{2}, w + \frac{1-\sigma}{2}, w + \frac{1+\sigma}{2}\right) = \frac{\Gamma(w + \frac{1-\sigma}{2})}{\Gamma(w + \frac{1+\sigma}{2})} L\left(\frac{\delta}{2}, w + \frac{1+\sigma}{2}, w + \frac{1-\sigma}{2}\right).$$

The confluent hypergeometric function of second type is given by

$$U(a, b, x) := \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1} dt.$$

It is known that $U(a, b, x)$ is the solution of the differential equation $xy'' + (b-x)y' - ay = 0$. Moreover, it satisfies

$$(3.13) \quad U(a, b, x) = x^{1-b} U(a-b+1, 2-b, x).$$

Actually, both functions above are solutions to the differential equation and asymptotically behave like x^{-a} as $x \rightarrow \infty$ (for these properties see [21]). Then, it is clear that, by (3.13),

$$L(a, b, c) = e^{-a}\Gamma(b)U(b, b-c+1, 2a) = e^{-a}\Gamma(b)(2a)^{c-b}U(c, c-b+1, 2a) = \frac{\Gamma(b)}{\Gamma(c)}(2a)^{c-b}L(a, c, b).$$

Then, by taking $a = \delta/2$, $b = w + \frac{1-\sigma}{2}$, $c = w + \frac{1+\sigma}{2}$, we obtain the desired result. \square

We end up with the following results, that will motivate the definition of the ground state representation.

Theorem 3.11. *Let $0 < \sigma < 1$, $\delta > 0$, and $\alpha > -1/2$. Then, for any $f \in C_0^\infty(0, \infty)$ we have*

$$\int_0^\infty L_{\alpha,\sigma} f(r) w_{\alpha,-\sigma}^\delta(r) r^{2\alpha+1} dr = (4\delta)^\sigma \frac{\Gamma(\frac{\alpha+2+\sigma}{2})^2}{\Gamma(\frac{\alpha+2-\sigma}{2})^2} \int_0^\infty f(r) w_{\alpha,\sigma}^\delta(r) r^{2\alpha+1} dr.$$

Proof. By Proposition 3.9 we have

$$(3.14) \quad w_{\alpha, -\sigma}^\delta = \sum_{n=0}^{\infty} c_{n, \delta}^{1/2}(-\sigma) \varphi_n^\alpha.$$

Then, in view of the spectral decomposition of $L_{\alpha, \sigma}$ (3.5), the Laguerre decomposition of $w_{\alpha, -\sigma}$ in (3.14), (3.4), and Lemma 3.10 we have

$$L_{\alpha, \sigma} w_{\alpha, -\sigma}^\delta = (4\delta)^\sigma \frac{\Gamma(\frac{\alpha+2+\sigma}{2})^2}{\Gamma(\frac{\alpha+2-\sigma}{2})^2} \sum_{n=0}^{\infty} c_{n, \delta}^{1/2}(\sigma) \varphi_n^\alpha = (4\delta)^\sigma \frac{\Gamma(\frac{\alpha+2+\sigma}{2})^2}{\Gamma(\frac{\alpha+2-\sigma}{2})^2} w_{\alpha, \sigma}^\delta.$$

With this, we easily conclude the proof of the theorem. \square

Let $G_{\alpha, \sigma}$ be the function defined by

$$(3.15) \quad G_{\alpha, \sigma} := \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} S_n^{\alpha, \sigma} \varphi_n^\alpha,$$

where $S_n^{\alpha, \sigma}$ is the constant defined in (1.6). A direct computation, using (3.4) and Lemma 3.10, yields the following.

Proposition 3.12. *Let $0 < \sigma < 1$. Then,*

$$w_{\alpha, -\sigma}^\delta *_{\alpha} G_{\alpha, -\sigma} = 4^\sigma \frac{\Gamma(\frac{\alpha+2+\sigma}{2})^2}{\Gamma(\frac{\alpha+2-\sigma}{2})^2} w_{\alpha, \sigma}^\delta.$$

We finish this subsection by showing that the function $G_{\alpha, \sigma}$ given in (3.15) is indeed a fundamental solution for the operator $L_{\alpha, \sigma}$, and we obtain an explicit expression for it.

Theorem 3.13. *The function $H_{\alpha, \sigma} = \frac{\sqrt{2}\Gamma(\alpha+1/2)}{4^\sigma\Gamma(\alpha+1)} G_{\alpha, \sigma}$ verifies that $L_{\alpha, \sigma} H_{\alpha, \sigma} = \delta_0$ where δ_0 is the Dirac delta distribution with support at 0. Moreover,*

$$G_{\alpha, \sigma}(r) = \frac{2^{\sigma+\alpha}\Gamma(\frac{\alpha-\sigma}{2})}{\sqrt{\pi}\Gamma(\sigma)} r^{-(\alpha+1-\sigma)} K_{(\alpha+1-\sigma)/2}(r^2/2).$$

Proof. Since

$$\mathcal{T}_0^\alpha f(s) = \frac{\Gamma(\alpha+1/2)}{\sqrt{2}\Gamma(\alpha+1)} f(s),$$

we have

$$f *_{\alpha} \varphi_j^\alpha(0) = \frac{\Gamma(\alpha+1/2)}{\sqrt{2}\Gamma(\alpha+1)} \int_0^\infty f(s) \varphi_j^\alpha(s) s^{2\alpha+1} dx$$

and

$$\langle \varphi_k^\alpha, f *_{\alpha} \varphi_j^\alpha \rangle_{L^2(\mathbb{R}^+, d\mu_\alpha)} = \frac{(\Gamma(\alpha+1))^2}{\sqrt{2}\Gamma(\alpha+1/2)} \delta_{jk} f *_{\alpha} \varphi_j^\alpha(0).$$

Then the first statement of the theorem follows from the spectral decomposition of $L_{\alpha, \sigma}$ (3.5) together with the definition of $G_{\alpha, \sigma}$ in (3.15). Indeed,

$$\langle L_{\alpha, \sigma} H_{\alpha, \sigma}, f \rangle_{L^2(\mathbb{R}^+, d\mu_\alpha)} = \langle H_{\alpha, \sigma}, L_{\alpha, \sigma} f \rangle_{L^2(\mathbb{R}^+, d\mu_\alpha)} = \frac{2}{\Gamma(\alpha+1)} \sum_{j=0}^{\infty} f *_{\alpha} \varphi_j^\alpha(0) = f(0).$$

For the second one, we make use of the formula (3.9) with $\mu = (4n + 2\alpha + 2)$, $\nu = \sigma - 1$ and $\beta = 2$, so we have

$$G_{\alpha, \sigma}(r) = \frac{2^\sigma}{\Gamma(\sigma)} \int_0^\infty q_{t, \alpha}(r) (\sinh 2t)^{\sigma-1} dt,$$

where $q_{t,\alpha}$ is the heat kernel given in (3.7). Thus

$$G_{\alpha,\sigma}(r) = \frac{4^\sigma}{\Gamma(\sigma)} \int_0^\infty (\sinh 2t)^{-\alpha+\sigma-2} e^{-(\coth 2t)r^2/2} dt =: \frac{4^\sigma}{\Gamma(\sigma)} I(r).$$

Let us compute the integral above. The change of variable $\coth 2t = z + 1$ yields

$$I(r) = 2^{-1} e^{-r^2/2} \int_0^\infty e^{-zr^2/2} z^{(\alpha-\sigma)/2} (z+2)^{(\alpha-\sigma)/2} dz.$$

Now we use the identity (see [22, 2.3.6.10, p. 324])

$$\int_0^\infty e^{-px} x^{\mu-1} (x+z)^{\mu-1} dx = \frac{\Gamma(\mu)}{\sqrt{\pi}} (p/z)^{1/2-\mu} e^{pz/2} K_{\mu-1/2}(pz/2),$$

valid for $\operatorname{Re} \mu, \operatorname{Re} p > 0; |\arg z| < \pi$, with $\mu = (\alpha - \sigma)/2 + 1$, $p = r^2/2$ and $z = 2$, so that

$$I(r) = 2^{-1} \frac{\Gamma((\alpha - \sigma)/2) 2^{\alpha+1-\sigma}}{\sqrt{\pi}} r^{-(\alpha+1-\sigma)} K_{(\alpha+1-\sigma)/2}(r^2/2)$$

and we get the conclusion. \square

3.5. The ground state representation for $L_{\alpha,\sigma}$ and proof of Theorem 1.1. We proceed with the next step, namely, to get a ground state representation for the operator $L_{\alpha,\sigma}$. Let us set

$$H_{\alpha,\sigma}^\delta[f] := \langle L_{\alpha,\sigma} f, f \rangle - A_{\alpha,\sigma}^\delta \int_0^\infty |f(r)|^2 \frac{w_{\alpha,\sigma}^\delta(r)}{w_{\alpha,-\sigma}^\delta(r)} d\mu_\alpha(r),$$

where

$$(3.16) \quad A_{\alpha,\sigma}^\delta := (4\delta)^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)^2}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)^2}$$

(observe that $A_{\alpha,\sigma}^\delta = \frac{4^\sigma}{\delta^\sigma} (B_{\alpha,\sigma}^\delta)^2$, with $B_{\alpha,\sigma}^\delta$ defined in (1.4)).

Theorem 3.14. *Let $0 < \sigma < 1$. If $f \in C_0^\infty(0, \infty)$ and $g(r) = f(r)(w_{\alpha,-\sigma}^\delta(r))^{-1}$ then*

$$H_{\alpha,\sigma}^\delta[f] = \frac{2^{\sigma-1}}{|\Gamma(-\sigma)|} \int_0^\infty \int_0^\infty |g(r) - g(s)|^2 \mathcal{K}_{\alpha,\sigma}(r, s) w_{\alpha,-\sigma}^\delta(r) w_{\alpha,-\sigma}^\delta(s) d\mu_\alpha(r) d\mu_\alpha(s).$$

Proof. We proceed as in Theorem 2.2, by polarizing the integral representation in Proposition 3.7 and taking $G(x) = w_{\alpha,-\sigma}^\delta(r)$ and $F(x) = \frac{|f(x)|^2}{w_{\alpha,-\sigma}^\delta(r)}$. Moreover, Theorem 3.11 and (1.2) allow us to write

$$\langle L_{\alpha,\sigma} F, G \rangle = (4\delta)^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)^2}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)^2} \int_0^\infty |f(r)|^2 \frac{w_{\alpha,\sigma}^\delta(r)}{w_{\alpha,-\sigma}^\delta(r)} d\mu_\alpha(r).$$

The conclusion follows as in Theorem 2.2. \square

Observe that if we take $f = w_{\alpha,-\sigma}^\delta(r)$ in Theorem 1.1, both sides of the inequality reduce to

$$(4\delta)^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)^2}{\Gamma\left(\frac{2\alpha+2-\sigma}{2}\right)^2} \int_0^\infty w_{\alpha,-\sigma}^\delta(r) w_{\alpha,\sigma}^\delta(r) d\mu_\alpha(r),$$

so in that case the constant $(4\delta)^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)^2}{\Gamma\left(\frac{\alpha+3-\sigma}{2}\right)^2}$ is optimal in our inequality.

Proof of Theorem 1.1. From the ground state representation $H_{\alpha,\sigma}^\delta[f]$, it is clear that

$$\langle L_{\alpha,\sigma} f, f \rangle \geq (4\delta)^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)^2}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)^2} \int_0^\infty |f(r)|^2 \frac{w_{\alpha,\sigma}^\delta(r)}{w_{\alpha,-\sigma}^\delta(r)} d\mu_\alpha(r).$$

It is known that $K_\nu(x)$ is an increasing function of ν for $x > 0$ (see [20, p. 226]). With this, and recalling the definition of $w_{\alpha,\sigma}^\delta$ in (1.2), we have

$$\langle L_{\alpha,\sigma} f, f \rangle \geq (4\delta)^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)^2}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)^2} \int_0^\infty |f(r)|^2 \frac{w_{\alpha,\sigma}^\delta(r)}{w_{\alpha,-\sigma}^\delta(r)} d\mu_\alpha(r) \geq \delta^\sigma \frac{\Gamma\left(\frac{\alpha+2+\sigma}{2}\right)}{\Gamma\left(\frac{\alpha+2-\sigma}{2}\right)} \int_0^\infty \frac{|f(r)|^2}{(\delta+r^2)^\sigma} d\mu_\alpha(r),$$

which is the required inequality. \square

4. A HARDY INEQUALITY FOR THE FRACTIONAL DUNKL–HERMITE OPERATOR

4.1. The general Dunkl setting. We will introduce some basic facts concerning the general Dunkl setting. A complete picture of the Dunkl’s theory can be found in [9, 10, 25]. We also refer the reader to the survey article [27].

Let us use $\langle \cdot, \cdot \rangle$ for the standard inner product on \mathbb{R}^d . For $\nu \in \mathbb{R}^d \setminus \{0\}$, we denote by σ_ν the orthogonal reflection in the hyperplane perpendicular to ν , namely,

$$\sigma_\nu(x) = x - 2 \frac{\langle \nu, x \rangle}{|\nu|^2} \nu.$$

We say that a finite subset $R \subset \mathbb{R}^d \setminus \{0\}$ is a reduced root system if, for all $\nu \in R$, then $\sigma_\nu(R) = R$ and $\mathbb{R}\nu \cap R = \{\pm\nu\}$. Each root system can be written as a disjoint union $R = R_+ \cup (-R_+)$, where R_+ and $-R_+$ are separated by a hyperplane through the origin. Such R_+ is called the set of all positive roots in R . The group G generated by the reflections $\{\sigma_\nu : \nu \in R\}$ is called the reflection group or Coxeter group associated with R . A function

$$(4.1) \quad \kappa : R \rightarrow [0, \infty)$$

which is invariant under the action of G on the root system R is called a multiplicity function. Let T_j , $j = 1, 2, \dots, d$, be the difference -differential operators defined by

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\nu \in R_+} \kappa(\nu) \nu_j \frac{f(x) - f(\sigma_\nu x)}{\langle \nu, x \rangle}.$$

These operators, known as Dunkl operators, form a family of commuting operators. The Dunkl Laplacian Δ_κ is then defined to be the operator

$$(4.2) \quad \Delta_\kappa = \sum_{j=1}^d T_j^2$$

which can be explicitly calculated, see [10, Theorem 4.4.9]. It is known that the operators T_j have a joint eigenfunction $E_\kappa(x, y)$ satisfying

$$T_j E_\kappa(x, y) = y_j E_\kappa(x, y), \quad j = 1, \dots, d.$$

The function $(x, y) \mapsto E_\kappa(x, y)$ is called the *Dunkl kernel* or the *generalized exponential kernel* on $\mathbb{R}^d \times \mathbb{R}^d$, which is the generalization of the exponential function $e^{\langle x, y \rangle}$. Associated with the root system R and the multiplicity function κ , the weight function $h_\kappa(x)$ is defined by

$$h_\kappa(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{2\kappa(\nu)}.$$

The nonnegative real number

$$(4.3) \quad \gamma = \sum_{\nu \in R_+} \kappa(\nu)$$

defined in terms of the multiplicity function $\kappa(\nu)$ plays an important role in Dunkl theory. Note that $h_\kappa(x)$ is homogeneous of degree 2γ . For a radial function f in $L^1(\mathbb{R}^d, h_\kappa)$, there exists a function F on $[0, \infty)$ such that $f(x) = F(|x|)$, for all $x \in \mathbb{R}^d$. In view of the homogeneity of $h_\kappa(x)$, it follows that the function F is integrable with respect to the measure $r^{d+2\gamma-1} dr$ on $[0, \infty)$ and we have

$$\begin{aligned} \int_{\mathbb{R}^d} f(x) h_\kappa(x) dx &= \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} f(rx') h_\kappa(rx') d\sigma(x') \right) r^{d-1} dr \\ &= \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} h_\kappa(rx') d\sigma(x') \right) F(r) r^{d-1} dr \\ &= d_\kappa \int_0^\infty F(r) r^{d+2\gamma-1} dr, \end{aligned}$$

where

$$d_\kappa := \int_{\mathbb{S}^{d-1}} h_\kappa(x') d\sigma(x') = \frac{c_\kappa^{-1}}{2^{\frac{d}{2}+\gamma-1} \Gamma(\frac{n}{2} + \gamma)},$$

and c_κ is the Mehta-type constant

$$c_\kappa := \left(\int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2}} h_\kappa(x) dx \right).$$

4.2. h -harmonic expansions. The theory of spherical harmonics which deals with expansions of functions in $L^2(\mathbb{S}^{d-1}, d\sigma)$ in terms of spherical harmonics has an analogue for the space $L^2(\mathbb{S}^{d-1}, h_\kappa d\sigma)$. Given a function $f \in L^2(\mathbb{S}^{d-1}, h_\kappa d\sigma)$ we can expand it in terms of the so called *spherical h -harmonics* (or just *h -harmonics*). We refer the reader to [10, Chapter 5] concerning h -harmonics. These are the restrictions of solid h -harmonics to \mathbb{S}^{d-1} where by solid h -harmonics we mean homogeneous polynomials $P(x)$ satisfying $\Delta_\kappa P(x) = 0$. The h -harmonics are analogues of spherical harmonics and defined using Δ_κ in place of Δ . Let \mathcal{H}_m^d be the space of all h -harmonics of degree m . Then the space $L^2(\mathbb{S}^{d-1}, h_\kappa(x') d\sigma(x'))$ is the orthogonal direct sum of the finite dimensional spaces \mathcal{H}_m^d over $m = 0, 1, 2, \dots$. Thus there is an orthonormal basis $\{Y_{m,j}^h : j = 1, 2, \dots, d(m), m = 0, 1, 2, \dots\}$, where

$$d(m) = \dim(\mathcal{H}_m^d),$$

for $L^2(\mathbb{S}^{d-1}, h_\kappa(x') d\sigma(x'))$ so that for each m , $\{Y_{m,j}^h : j = 1, 2, \dots, d(m)\}$ is an orthonormal basis of h -harmonics for \mathcal{H}_m^d . For $x \in \mathbb{R}^d$, take $x = rx'$, $r \in \mathbb{R}^+$ and $x' \in \mathbb{S}^{d-1}$. The h -harmonic expansion of a function f on \mathbb{R}^d is given by

$$(4.4) \quad f(rx') = \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} f_{m,j}(r) Y_{m,j}^h(x') =: \sum_{m,j} f_{m,j}(r) Y_{m,j}^h(x'),$$

where the h -harmonic coefficients are

$$f_{m,j}(r) = \int_{\mathbb{S}^{d-1}} f(rx') Y_{m,j}^h(y') h_\kappa(y') d\sigma(y').$$

In the Dunkl setting we have a *Funk–Hecke formula for h -harmonics*. The classical Funk–Hecke formula for spherical harmonics states the following. For any continuous function f on $[-1, 1]$ and a (standard) spherical harmonic $Y_{m,j}$ of degree m we have

$$\int_{\mathbb{S}^{d-1}} f(\langle x', y' \rangle) Y_{m,j}(y') d\sigma(y') = \Lambda_m(f) Y_{m,j}(x')$$

where $\Lambda_m(f)$ is a constant defined by

$$\Lambda_m(f) = \frac{\omega_d \Gamma(d/2)}{\sqrt{\pi} \Gamma((d-1)/2)} \int_{-1}^1 f(u) P_m^{\frac{d}{2}-1}(u) (1-u^2)^{\frac{d-3}{2}} du.$$

Here P_m^λ stands for the normalized ultraspherical polynomials of type $\lambda > -\frac{1}{2}$ and degree m , and $\omega_d := \int_{\mathbb{S}^{d-1}} d\sigma(\omega)$. To state the Funk–Hecke formula for h -harmonics, we need to recall the intertwining operator in the Dunkl setting. It is known that there is an operator V_κ satisfying $T_j V_\kappa = V_\kappa \frac{\partial}{\partial x_j}$. However, the explicit form of V_κ is not known, except in a couple of simple cases. In particular, the Dunkl kernel is given by $E_\kappa(x, y) = V_\kappa e^{\langle \cdot, y \rangle}(x)$. For $d \geq 1$ and γ as in (4.3), let us recall the definition of the constant λ as

$$(4.5) \quad \lambda := \frac{d}{2} + \gamma - 1.$$

The Funk–Hecke formula for h -harmonics is as follows (see [8, Theorem 7.2.7] or [10, Theorem 5.3.4]).

Theorem 4.1. *Let f be a continuous function defined on $[-1, 1]$ and let λ be as in (4.5). Then for every $Y_{m,j}^h \in \mathcal{H}_m^d$,*

$$\int_{\mathbb{S}^{d-1}} V_\kappa f(\langle x', \cdot \rangle)(y') Y_{m,j}^h(y') h_\kappa(y') d\sigma(y') = \Lambda_m^\kappa(f) Y_{m,j}^h(x')$$

where $\Lambda_m^\kappa(f)$ is defined by

$$\Lambda_m^\kappa(f) = \frac{\omega_d^\kappa \Gamma(\lambda + 1)}{\sqrt{\pi} \Gamma(\lambda + 1/2)} \int_{-1}^1 f(u) P_m^\lambda(u) (1-u^2)^{\lambda-\frac{1}{2}} du$$

with

$$\omega_d^\kappa := \int_{\mathbb{S}^{d-1}} h_\kappa(\omega) d\sigma(\omega).$$

By applying Theorem 4.1 to the function $f(t) = e^{rst}$, $r, s \geq 0$, and using the fact $V_\kappa f(\langle x', y' \rangle) = E_\kappa(rx', sy')$, we immediately obtain the following.

Corollary 4.2 (Funk–Hecke for Dunkl kernel). *Let λ be as in (4.5). Then for every $Y_{m,j}^h \in \mathcal{H}_m^d$,*

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} E_\kappa(rx', sy') Y_{m,j}^h(y') h_\kappa(y') d\sigma(y') \\ = \frac{\omega_d^\kappa \Gamma(\frac{d}{2} + \gamma)}{\sqrt{\pi} \Gamma(\frac{d-1}{2} + \gamma)} \left(\int_{-1}^1 e^{rsu} P_m^\lambda(u) (1-u^2)^{\lambda-\frac{1}{2}} du \right) Y_{m,j}^h(x'). \end{aligned}$$

The following identity was proved in [3, Lemma 7.2].

Lemma 4.3 ([3], Lemma 7.2). *Let $z \in \mathbb{C}$ and $\lambda > -\frac{1}{2}$. Then the following holds*

$$\int_{-1}^1 e^{zu} P_m^\lambda(u) (1-u^2)^{\lambda-1/2} du = \sqrt{\pi} \Gamma(\lambda + 1/2) (z/2)^{-\lambda} I_{\lambda+m}(z), \quad m = 0, 1, \dots$$

4.3. The Dunkl–Hermite operator and the heat semigroup. Let us introduce the framework of the Dunkl–Hermite operator (which is also known as the Dunkl-harmonic oscillator)

$$(4.6) \quad H_\kappa := -\Delta_\kappa + |x|^2,$$

where Δ_κ stands for the Dunkl–Laplacian in \mathbb{R}^d (4.2). The parameter κ is the multiplicity function defined in (4.1). When $\kappa \equiv 0$, the operator H_κ becomes the classical harmonic oscillator $-\Delta + |x|^2$. The study of H_κ was initiated by M. Rösler [25, 26].

For each $\mu \in \mathbb{N}^d$, we consider the *generalized Hermite functions* (or *Dunkl–Hermite functions*) $\Phi_{\mu,\kappa}$. The system $\{\Phi_{\mu,\kappa} : \mu \in \mathbb{N}^d\}$ is orthonormal and complete in $L^2(\mathbb{R}^d, h_\kappa(x))$, cf. [25, Corollary

3.5 (ii)]. Furthermore, $\Phi_{\mu,\kappa}$ are eigenfunctions of the operator (4.6) with eigenvalues $(2|\mu| + d + 2\gamma)$, that is

$$(4.7) \quad H_\kappa \Phi_{\mu,\kappa} = (2|\mu| + d + 2\gamma) \Phi_{\mu,\kappa},$$

where γ is as in (4.3), and $|\mu| = \sum_{\ell=1}^d \mu_\ell$. For $\kappa \equiv 0$, $\Phi_{\mu,0}$ become the usual Hermite functions, see [25, p. 521]. Precise definitions and detailed description on results concerning generalized Dunkl–Hermite functions can be found in [25].

For a function $f \in L^2(\mathbb{R}^d, h_\kappa)$ we have the orthogonal expansion,

$$f = \sum_{n=0}^{\infty} P_{n,\kappa} f,$$

which converges in $L^2(\mathbb{R}^d, h_\kappa)$. Here the spectral projections are given by

$$P_{n,\kappa} f = \sum_{|\mu|=n} \langle f, \Phi_{\mu,\kappa} \rangle_\kappa \Phi_{\mu,\kappa}$$

with $\langle \cdot, \cdot \rangle_\kappa$ standing for the inner product in $L^2(\mathbb{R}^d, h_\kappa dx)$. More precisely, $P_{n,\kappa}$ is the orthogonal projection associated to the eigenspace corresponding to the eigenvalue $(2n + d + 2\gamma)$ of H_κ .

We state and prove the following theorem, that relates the projections $P_{n,\kappa}$ with the Laguerre convolution in Section 3. Such kind of an identity is called *Hecke–Bochner* identity.

Theorem 4.4 (Hecke–Bochner for Dunkl–Hermite projections). *Let $f(x) = f_0(|x|)Y_m^h(x')$ where $Y_m^h(x')$ is a h -harmonic of degree m . Then one has $P_{2n+m,\kappa} f(x) = F_n(|x|)Y_m^h(x')$ where*

$$F_n(r) = \frac{2}{\Gamma(\alpha + 1)} r^m [((\cdot)^{-m} f_0(\cdot)) *_{\lambda+m} \varphi_n^{\lambda+m}](r).$$

with λ as in (4.5). For other values of ℓ , $P_{\ell,\kappa} f = 0$.

Proof. The proof is similar to the proof of [29, Theorem 3.4.1]. Here we need to use the *Mehler’s formula for the Dunkl–Hermite functions* (see [25, Theorem 3.12]): for $|w| < 1$, one has

$$(4.8) \quad \sum_{\mu \in \mathbb{N}^d} \Phi_{\mu,\kappa}(x) \Phi_{\mu,\kappa}(y) w^{|\mu|} = \frac{2}{\omega_d^\kappa \Gamma(\lambda + 1)} (1-w^2)^{-(\lambda+1)} \exp \left\{ -\frac{1}{2} \left(\frac{1+w^2}{1-w^2} \right) (|x|^2 + |y|^2) \right\} E_\kappa \left(\frac{2wx}{1-w^2}, y \right).$$

We provide some details of the proof. Taking into account (4.8), it follows that

$$\begin{aligned} \sum_{n=0}^{\infty} P_{n,\kappa} f(x) w^n &= \frac{2}{\omega_d^\kappa \Gamma(\lambda + 1)} (1-w^2)^{-(\lambda+1)} \\ &\quad \times \int_{\mathbb{R}^d} \exp \left\{ -\frac{1}{2} \left(\frac{1+w^2}{1-w^2} \right) (|x|^2 + |y|^2) \right\} E_\kappa \left(\frac{2wx}{1-w^2}, y \right) f(y) h_\kappa(y) dy \\ &= \frac{2}{\omega_d^\kappa \Gamma(\lambda + 1)} (1-w^2)^{-(\lambda+1)} \exp \left\{ -\frac{1}{2} \left(\frac{1+w^2}{1-w^2} \right) r^2 \right\} \\ &\quad \times \int_0^\infty \left(\int_{\mathbb{S}^{d-1}} E_\kappa \left(\frac{2wr x'}{1-w^2}, s y' \right) Y_m^h(y') h_\kappa(y') d\sigma(y') \right) \exp \left\{ -\frac{1}{2} \left(\frac{1+w^2}{1-w^2} \right) s^2 \right\} f_0(s) s^{2\lambda+1} ds \\ &= \frac{2}{\sqrt{\pi} \Gamma(\lambda + 1/2)} (1-w^2)^{-(\lambda+1)} \exp \left\{ -\frac{1}{2} \left(\frac{1+w^2}{1-w^2} \right) r^2 \right\} Y_m^h(x') \\ &\quad \times \int_0^\infty \left(\int_{-1}^1 \exp \left\{ \frac{2wrs}{1-w^2} u \right\} P_m^\lambda(u) (1-u^2)^{\lambda-\frac{1}{2}} du \right) \exp \left\{ -\frac{1}{2} \left(\frac{1+w^2}{1-w^2} \right) s^2 \right\} f_0(s) s^{2\lambda+1} ds \end{aligned}$$

where the last equality is true in view of the Funk–Hecke formula in Corollary 4.2. In its turn, by Lemma 4.3, this last expression equals

$$\begin{aligned} & 2(1-w^2)^{-1}(wr)^{-\lambda} \exp\left\{-\frac{1}{2}\left(\frac{1+w^2}{1-w^2}\right)r^2\right\} Y_m^h(x') \\ & \quad \times \int_0^\infty \exp\left\{-\frac{1}{2}\left(\frac{1+w^2}{1-w^2}\right)s^2\right\} f_0(s) I_{\lambda+m}\left(\frac{2wrs}{1-w^2}\right) s^{\lambda+1} ds \\ & = 2(wr)^m Y_m^h(x') \\ & \quad \times \int_0^\infty (1-w^2)^{-1}(rsw)^{-\lambda-m} \exp\left\{-\frac{1}{2}\left(\frac{1+w^2}{1-w^2}\right)(r^2+s^2)\right\} I_{\lambda+m}\left(\frac{2wrs}{1-w^2}\right) f_0(s) s^{2\lambda+1+m} ds. \end{aligned}$$

Now, using the generating function identity for $\varphi_n^\delta(r)$ given in (3.8), we have that (by using also the identity [29, (6.1.28)] for the Laguerre translation in the second equality below)

$$\begin{aligned} \sum_{n=0}^\infty P_{n,\kappa} f(x) w^n & = 2 \sum_{n=0}^\infty w^{2n+m} r^m \int_0^\infty \frac{\Gamma(n+1)}{\Gamma(n+\lambda+m+1)} \varphi_n^{\lambda+m}(r) \varphi_n^{\lambda+m}(s) f_0(s) s^{2\lambda+1+m} ds Y_m^h(x') \\ & = \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^\infty w^{2n+m} r^m \int_0^\infty \mathcal{T}_r^{\lambda+m} \varphi_n^{\lambda+m}(s) f_0(s) s^{2\lambda+1+m} ds Y_m^h(x') \\ & = \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^\infty w^{2n+m} r^m \int_0^\infty \mathcal{T}_r^{\lambda+m} \varphi_n^{\lambda+m}(s) s^{-m} f_0(s) s^{2\lambda+2m+1} ds Y_m^h(x') \\ & = \frac{2}{\Gamma(\alpha+1)} \sum_{n=0}^\infty w^{2n+m} r^m [((\cdot)^{-m} f_0(\cdot)) *_{\lambda+m} \varphi_n^{\lambda+m}](r) Y_m^h(x'). \end{aligned}$$

The conclusion follows by comparing coefficients on both sides of the equality. \square

The solution to the heat equation associated to the Dunkl–Hermite operator, i.e.

$$\frac{\partial}{\partial t} u(x, t) = -H_\kappa u(x, t), \quad u(x, 0) = f(x), \quad t > 0, x \in \mathbb{R}^d,$$

is given by $u(x, t) =: e^{-tH_\kappa} f(x) =: T_t^\kappa f(x)$, where T_t^κ , $t \geq 0$ is the Dunkl–Hermite semigroup generated by H_κ , see [1, 25]. In terms of the spectral decomposition,

$$T_t^\kappa f = \sum_{n=0}^\infty e^{-t(2n+d+2\gamma)} P_{n,\kappa} f, \quad f \in L^2(\mathbb{R}^d, h_\kappa dx).$$

The following result relates the Dunkl–Hermite semigroup and the Laguerre heat semigroup.

Theorem 4.5. *Let λ be as in (4.5). For $x \in \mathbb{R}^d$, $x = rx'$, with $r \in \mathbb{R}^+$ and $x' \in \mathbb{S}^{d-1}$, then*

$$T_t^\kappa f(x) = \sum_{m=0}^\infty \sum_{j=1}^{d(m)} r^m T_{\lambda+m,t}((\cdot)^{-m} f_{m,j})(r) Y_{m,j}^h(x').$$

Proof. We have just to apply Theorem 4.4 to the spectral definition of $T_t^\kappa f(x)$. Indeed,

$$\begin{aligned} T_t^\kappa f(x) & = \sum_{n=0}^\infty e^{-t(2n+d+2\gamma)} P_{n,\kappa} f(x) \\ & = \sum_{n=0}^\infty e^{-t(2n+d+2\gamma)} P_{n,\kappa} \left(\sum_{m,j} f_{m,j}(r) Y_{m,j}^h(x') \right) \\ & = \frac{2}{\Gamma(\alpha+1)} \sum_{m,j} Y_{m,j}^h(x') \sum_{\ell=0}^\infty e^{-t(4\ell+2m+d+2\gamma)} r^m [((\cdot)^{-m} f_{m,j}(\cdot)) *_{\lambda+m} \varphi_\ell^{\lambda+m}](r) \end{aligned}$$

$$= \sum_{m,j} r^m T_{\lambda+m,t}((\cdot)^{-m} f_{m,j}(\cdot))(r) Y_{m,j}^h(x').$$

We have proved the theorem. \square

4.4. The Hardy inequality for $\mathbf{H}_{\kappa,\sigma}$: Proof of Theorem 1.2. Given $0 < \sigma < 1$, we define \mathbf{H}_σ to be the operator

$$\mathbf{H}_{\kappa,\sigma} := \frac{\Gamma\left(\frac{H_\kappa}{2} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{H_\kappa}{2} + \frac{1-\sigma}{2}\right)},$$

so, in view of (4.7), $\mathbf{H}_{\kappa,\sigma}$ corresponds to the spectral multiplier $\frac{\Gamma\left(\frac{2|\mu|+d+2\gamma}{2} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{2|\mu|+d+2\gamma}{2} + \frac{1-\sigma}{2}\right)}$. Then by Lemma 3.4 we have

$$\begin{aligned} \mathbf{H}_{\kappa,\sigma} f(x) &= \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{2j+d+2\gamma}{2} + \frac{1+\sigma}{2}\right)}{\Gamma\left(\frac{2j+d+2\gamma}{2} + \frac{1-\sigma}{2}\right)} P_{j,\kappa} f(x) \\ &= E_\sigma f(x) + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (f(x) - T_t^\kappa f(x)) (\sinh t)^{-\sigma-1} dt. \end{aligned}$$

By Theorem 4.5 and the expansion of f into h -harmonics (4.4), we can write

$$\begin{aligned} \mathbf{H}_{\kappa,\sigma} f(x) &= E_\sigma f(x) + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (f(x) - T_t^\kappa f(x)) (\sinh t)^{-\sigma-1} dt \\ &= \sum_{m,j} Y_{m,j}^h(x') \left[E_\sigma f_{m,j}(r) \right. \\ &\quad \left. + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (f_{m,j}(r) - r^m T_{\lambda+m,t}((\cdot)^{-m} f_{m,j})(r)) (\sinh t)^{-\sigma-1} dt \right] \\ &= \sum_{m,j} Y_{m,j}^h(x') r^m \left[E_\sigma r^{-m} f_{m,j}(r) \right. \\ &\quad \left. + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (r^{-m} f_{m,j}(r) - T_{\lambda+m,t}((\cdot)^{-m} f_{m,j})(r)) (\sinh t)^{-\sigma-1} dt \right] \\ &= \sum_{m,j} Y_{m,j}^h(x') r^m \left[E_\sigma g_{m,j}(r) + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (g_{m,j}(r) - T_{\lambda+m,t} g_{m,j}(r)) (\sinh t)^{-\sigma-1} dt \right] \\ &= \sum_{m,j} Y_{m,j}^h(x') r^m \left[g_{m,j}(r) \left(E_\sigma + \frac{2^\sigma}{|\Gamma(-\sigma)|} F_{\lambda+m,\sigma}(r) \right) \right. \\ &\quad \left. + \frac{2^\sigma}{|\Gamma(-\sigma)|} \int_0^\infty (g_{m,j}(r) - g_{m,j}(s)) \mathcal{K}_{\lambda+m,\sigma}(r,s) d\mu_{\lambda+m}(s) \right] \\ &= \sum_{m,j} Y_{m,j}^h(x') r^m L_{\lambda+m,\sigma} g_{m,j}(r), \end{aligned}$$

where we $g_{m,j}(r) = r^{-m} f_{m,j}(r)$ and $L_{\lambda+m,\sigma}$ is the modified fractional Laguerre operator defined spectrally in (3.5). With this, by Theorem 1.1, we have

$$\begin{aligned} \langle \mathbf{H}_{\kappa,\sigma} f, f \rangle_{L^2(\mathbb{R}^d, h_\kappa)} &= \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} \langle L_{\lambda+m,\sigma} g_{m,j}, g_{m,j} \rangle_{L^2((0,\infty), d\mu_{\lambda+m}(r))} \\ &\geq \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} A_{\lambda+m,\sigma}^\delta \int_0^\infty |g_{m,j}(r)|^2 \frac{w_{\lambda+m,\sigma}^\delta(r)}{w_{\lambda+m,-\sigma}^\delta(r)} d\mu_{\lambda+m}(r) \end{aligned}$$

$$= \sum_{m=0}^{\infty} \sum_{j=1}^{d(m)} A_{\lambda+m,\sigma}^{\delta} \int_0^{\infty} |f_{m,j}(r)|^2 \frac{w_{\lambda+m,\sigma}^{\delta}(r)}{w_{\lambda+m,-\sigma}^{\delta}(r)} d\mu_{\lambda}(r),$$

where $A_{\lambda+m,\sigma}^{\delta}$ is the constant in (3.16), i.e., $A_{\lambda+m,\sigma}^{\delta} = (4\delta)^{\sigma} \frac{\Gamma(\frac{\lambda+m+2+\sigma}{2})^2}{\Gamma(\frac{\lambda+m+2-\sigma}{2})}$. Taking into account the definition of $w_{\alpha,\sigma}^{\delta}(r)$ in (1.2) we get

$$(4.9) \quad A_{\lambda+m,\sigma}^{\delta} \frac{w_{\lambda+m,\sigma}^{\delta}(r)}{w_{\lambda+m,-\sigma}^{\delta}(r)} = \delta^{\sigma} \frac{\Gamma(\frac{\lambda+m+2+\sigma}{2})}{\Gamma(\frac{\lambda+m+2-\sigma}{2})} \frac{K_{(\lambda+m+2+\sigma)/2}((\delta+r^2)/2)}{K_{(\lambda+m+2-\sigma)/2}((\delta+r^2)/2)} (\delta+r^2)^{-\sigma}.$$

Now, as observed by D. Yafaev in [31], for $0 < x \leq y$ and $m \geq 0$ we have that $\frac{\Gamma(m+y)}{\Gamma(m+x)} \geq \frac{\Gamma(y)}{\Gamma(x)}$, so

$$(4.10) \quad \delta^{\sigma} \frac{\Gamma(\frac{\lambda+m+2+\sigma}{2})}{\Gamma(\frac{\lambda+m+2-\sigma}{2})} \geq \delta^{\sigma} \frac{\Gamma(\frac{\lambda+2+\sigma}{2})}{\Gamma(\frac{\lambda+2-\sigma}{2})} = B_{\lambda,\sigma}^{\delta},$$

where $B_{\lambda,\sigma}^{\delta}$ is the constant defined in (1.4). Recall again that $K_{\nu}(x)$ is an increasing function of ν for $x > 0$ (see [20, p. 226]). Therefore, with this, and putting together (4.9) and (4.10) we have

$$\begin{aligned} \langle \mathbf{H}_{\kappa,\sigma} f, f \rangle_{L^2(\mathbb{R}^d, h_{\kappa} dx)} &\geq \sum_{m,j} A_{\lambda+m,\sigma}^{\delta} \int_0^{\infty} |f_{m,j}(r)|^2 \frac{w_{\lambda+m,\sigma}^{\delta}(r)}{w_{\lambda+m,-\sigma}^{\delta}(r)} d\mu_{\lambda}(r) \\ &\geq B_{\lambda,\sigma}^{\delta} \sum_{m,j} \int_0^{\infty} |f_{m,j}(r)|^2 (\delta+r^2)^{-\sigma} d\mu_{\lambda}(r) \\ &= B_{\lambda,\sigma}^{\delta} \sum_{m,j} \langle (\delta+|\cdot|^2)^{-\sigma/2} f_{m,j}, (\delta+|\cdot|^2)^{-\sigma/2} f_{m,j} \rangle_{L^2((0,\infty), d\mu_{\lambda}(r))} \\ &= B_{\lambda,\sigma}^{\delta} \langle (\delta+|\cdot|^2)^{-\sigma/2} f, (\delta+|\cdot|^2)^{-\sigma/2} f \rangle_{L^2(\mathbb{R}^d, h_{\kappa})} \\ &= B_{\lambda,\sigma}^{\delta} \int_{\mathbb{R}^d} \frac{|f(x)|^2}{(\delta+|x|^2)^{\sigma}} h_{\kappa}(x) dx. \end{aligned}$$

This completes the proof of Theorem 1.2.

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