Some discrete maximum principles arising for nonlinear elliptic FEM problems

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Abstract: The discrete maximum principle (DMP) is an important measure of the qualitative reliability of the applied numerical scheme for elliptic problems. This paper starts with formulating simple sufficient conditions for the matrix case and for nonlinear forms in Banach space. Then a DMP is derived for FEM solutions for certain nonlinear PDEs: we address nonlinear elliptic problems with mixed boundary conditions and interface conditions, allowing possibly degenerate nonlinearities and thus extending our previous results.

1 Introduction

The maximum principle forms an important qualitative property of second order elliptic equations [20], therefore its discrete analogues, the so-called discrete maximum principles (DMPs) have drawn much attention. The DMP is in fact an important measure of the qualitative reliability of the numerical scheme, otherwise one could get unphysical numerical solutions like negative concentrations etc. Typical maximum principles arise either in the form

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u \tag{1}$$

(i.e. the solution u attains its maximum on the boundary), which occurs for proper elliptic operators with only principal part, or in the form

$$\max_{\overline{\Omega}} u \le \max\{0, \max_{\partial\Omega} u\} \tag{2}$$

(i.e. the solution u can attain a nonnegative maximum only on the boundary), which occurs for proper elliptic operators including lower order terms as well. We are interested for DMPs under finite element discretizations (FEM), in which case a DMP reproduces one of the above relations for the FEM solution u_h instead of u.

Various DMPs, including geometric conditions on the computational meshes for FEM solutions, have been given e.g. in [3, 6, 7, 9, 22, 25, 26]. The authors' previous work,

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arises in mean curvature and minimal surface equations, e.g. describing capillary surfaces, see, e.g., [21]; further,

$$k(|\nabla u|) := |\nabla u|^{p-2}$$

(for a given constant $p > 2$) leads to the p-Laplacian, which is a widespread model of nonlinear diffusion operator, arising e.g. for a compressible fluid in a homogeneous isotropic porous medium [27], and is degenerate, i.e. the coefficient $|\nabla u|^{p-2}$ may vanish inside the domain Ω.

Remark 2 Discrete maximum principles for similar problems have been considered in the earlier papers [14, 15, 17, 8, 19] as well. When compared to these, the main novelties in our work are as follows:

(i) For nonlinearities depending on $(x, |\nabla u|)$ and with potential structure, a generalization of the DMP to the so-called convex hull property is proved in [8]. Further, for similar nonlinearities as ours, without allowing degeneracy, a DMP in 3D is verified in [19]. In both papers, in addition to the above restrictions, the results only concern Dirichlet boundary conditions and do not include interface problems.

(ii) We have proved DMPs for nonlinear problems with mixed boundary conditions in [14, 17] and for interface problems in [15]. Those results assume lower and upper boundedness of the diffusion coefficients, hence do not allow degeneracy and thus are not applicable to such problems, illustrated above in the 'Examples' item.

Remark 3 Various practical and theoretical results related to generation of nonobtuse simplicial partitions are presented e.g. in the survey work [2].

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