A C^0 INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD FOR FOURTH ORDER TOTAL VARIATION FLOW. II: EXISTENCE AND UNIQUENESS

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Abstract. We prove the existence and uniqueness of a solution of a C^0 Interior Penalty Discontinuous Galerkin (C^0 IPDG) method for the numerical solution of a fourth order total variation flow problem that has been developed in part I of the paper. The proof relies on a nonlinear version of the Lax-Milgram Lemma. It requires to establish that the nonlinear operator associated with the C^0 IPDG approximation is Lipschitz continuous and strongly monotone on bounded sets of the underlying finite element space.

1. Introduction

We consider the following fourth order total variation flow (TVF) problem:

$$\frac{\hat{\partial} w}{\hat{\partial} \hat{t}} + \beta \hat{\Delta} \hat{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{\nabla}} w}{|\hat{\boldsymbol{\nabla}} w|} = 0 \quad \text{in } \hat{Q} := \hat{\Omega} \times (0, \hat{T}), \tag{1.1a}$$

1991 *Mathematics Subject Classification.* 35K35,35K55,65M60.

Key words and phrases. C^0 interior penalty discontinuous Galerkin method, fourth order total variation flow, existence and uniqueness.

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$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta \frac{\hat{\boldsymbol{\nabla}} w}{|\hat{\boldsymbol{\nabla}} w|} = \mathbf{n}_{\hat{\Gamma}} \cdot \hat{\boldsymbol{\nabla}} \hat{\boldsymbol{\nabla}} \Big(\hat{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{\nabla}} w}{|\hat{\boldsymbol{\nabla}} w|} \Big) = 0 \quad \text{on } \hat{\Sigma} := \hat{\Gamma} \times (0, \hat{T}), \tag{1.1b}$$

$$w(\cdot, 0) = w^0 \quad \text{in } \hat{\Omega}. \tag{1.1c}$$

Here, $\hat{\Omega} \subset \mathbb{R}^2$ is a bounded domain with boundary $\hat{\Gamma} = \partial\hat{\Omega}$, $\hat{T} > 0$ is the final time, $\beta > 0$ is some constant, $\mathbf{n}_{\hat{\Gamma}}$ stands for the exterior unit normal at $\hat{\Gamma}$, and $w^0 \in L^2(\hat{\Omega})$ is some given initial data.

The fourth order equation (1.1a) has to be understood as the flow problem

$$-\frac{\hat{\partial} w}{\hat{\partial} \hat{t}} \in \partial E_{H^{-1}}(w)$$

associated with the total variation-H^{-1} (TV-H^{-1}) minimization of the energy functional

$$E(w) = \beta \int_{\hat{\Omega}} |\hat{\boldsymbol{\nabla}} w| \, dx, \quad \beta > 0, \tag{1.2}$$

where $\partial_{H^{-1}} E(w)$ is the H^{-1} subdifferential of E.

In fact, if we introduce an inner product on $H^{-1}(\hat{\Omega}$ according to

$$(w, z)_{-1,\hat{\Omega}} := (\hat{\boldsymbol{\nabla}}(-\hat{\Delta}^{-1} w), \hat{\boldsymbol{\nabla}}(-\hat{\Delta}^{-1} z))_{0,\hat{\Omega}},$$

the subdifferential

$$\partial_{H^{-1}} E(w) = \{ v \in H^{-1}(\hat{\Omega}) \mid (v, z - w)_{-1,\Omega} \le E(z) - E(w) \text{ for all } z \in H^{-1}(\hat{\Omega}) \}$$

reads as follows (cf., e.g., [6]):

$$\partial_{H^{-1}} E(w) = \{ \hat{\Delta} \hat{\boldsymbol{\nabla}} \cdot \boldsymbol{\xi} \mid \boldsymbol{\xi}(\hat{x}) \in \partial \Phi(\hat{\boldsymbol{\nabla}} w(\hat{x})) \}.$$

Here, $\Phi(|\boldsymbol{\eta}|)$ and $\partial\Phi(|\boldsymbol{\eta}|)$ are given by

$$\Phi(\boldsymbol{\eta}) = \beta|\boldsymbol{\eta}|, \quad \partial\Phi(\boldsymbol{\eta}) = \begin{cases} \beta\boldsymbol{\eta}/|\boldsymbol{\eta}| \text{ , if } \boldsymbol{\eta} \neq \mathbf{0} \\ \{\boldsymbol{\tau} \in \mathbb{R}^2 \mid |\boldsymbol{\tau}| \leq \beta\} \text{ , if } \boldsymbol{\eta} = \mathbf{0} \end{cases} . \tag{1.3}$$

The fourth order total variation flow (TVF) problem (1.1a)-(1.1c) describes surface relaxation below the roughening temperature. We note that similar fourth order TVF problems occur in image recovery. For more details we refer to [2] and the references therein.

In the sequel, we consider the regularized fourth order TVF problem

$$\frac{\hat{\partial} w}{\hat{\partial} \hat{t}} + \beta\hat{\Delta}\hat{\boldsymbol{\nabla}} \cdot ((\delta^2 + |\hat{\boldsymbol{\nabla}} w|^2)^{-1/2}\hat{\boldsymbol{\nabla}} w) = 0 \quad \text{in } \hat{Q}, \tag{1.4a}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta(\delta^2 + |\hat{\boldsymbol{\nabla}} w|^2)^{-1/2}\hat{\boldsymbol{\nabla}} w = \ 0 \quad \text{on } \hat{\Sigma}, \tag{1.4b}$$

$$\mathbf{n}_{\hat{\Gamma}} \cdot \beta\hat{\boldsymbol{\nabla}}\Big(\hat{\boldsymbol{\nabla}} \cdot (\delta^2 + |\hat{\boldsymbol{\nabla}} w|^2)^{-1/2}\hat{\boldsymbol{\nabla}} w\Big) = 0 \quad \text{on } \hat{\Sigma},$$

$$w(\cdot, 0) = w^0 \quad \text{in } \hat{\Omega}, \tag{1.4c}$$

where $\delta > 0$ is a regularization parameter. We further consider a scaling in both the time variable and the spatial variables according to

$$t = \delta\hat{t}, \quad x_i = \delta\hat{x}_i, \ 1 \leq i \leq 2. \tag{1.5}$$

Setting $T := \delta\hat{T}, \Omega := \delta\hat{\Omega}, \Gamma := \partial\Omega, Q := \Omega \times (0,T), \Sigma := \Gamma \times (0,T)$, and $u^0(x) = w^0(\delta^{-1}x)$, as well as

$$\omega(\boldsymbol{\nabla} u) := 1 + |\boldsymbol{\nabla} u|^2, \tag{1.6}$$

the scaled and regularized fourth order TVF problem reads as follows

$$\frac{\partial u}{\partial t} + \beta\delta^2 \Delta \boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u)^{-1/2} \boldsymbol{\nabla} u) = 0 \quad \text{in } Q, \tag{1.7a}$$

$$\mathbf{n}_\Gamma \cdot \beta\delta^2 (\omega(\boldsymbol{\nabla} u)^{-1/2} \boldsymbol{\nabla} u) = \mathbf{n}_\Gamma \cdot \beta\delta^2 \boldsymbol{\nabla}\Big(\boldsymbol{\nabla} \cdot ((\omega(\boldsymbol{\nabla} u)^{-1/2} \boldsymbol{\nabla} u)\Big) = 0 \quad \text{on } \Sigma, \tag{1.7b}$$

$$u(\cdot, 0) = u^0 \quad \text{in } \Omega. \tag{1.7c}$$

The numerical solution of the regularized fourth order TVF problem with periodic boundary conditions has been considered in [7] based on a mixed formulation of the implicitly in time discretized problem. At each time-step, this amounts to the solution of two second order elliptic PDEs by standard Lagrangian finite elements with respect to a triangulation of the computational domain Ω. On the other hand, a C^0 Interior Penalty Discontinuous Galerkin (C^0IPDG) method has been developed and implemented in [2]. The advantage of the C^0IPDG approach is that it directly applies to the fourth order problem and thus only requires the numerical solution of one equation by using the same Lagrangian finite elements as in the mixed method.

The paper is organized as follows: After some basic notations from matrix analysis and Lebesgue and Sobolev spaces presented in section 2, in section 3 we recall the C^0IPDG approximation of the implicity in time discretized, regularized, and scaled fourth order TVF problem from [2]. Section 4 is devoted to a proof of the existence and uniqueness of a solution of the C^0IPDG approximation by an application of the nonlinear version of the Lax-Milgram Lemma. In particular, this requires to show that the nonlinear operator associated with the C^0IPDG approximation is Lipschitz continuous and strongly monotone on bounded subsets of the underlying function space.

2. Basic notations

For vectors $\underline{\mathbf{x}} = (x_1, \cdots, x_n)^T, \underline{\mathbf{y}} = (y_1, \cdots, y_n)^T \in \mathbb{R}^n$ and for matrices $\underline{\underline{\mathbf{A}}} = (a_{ij})_{i,j=1}^n, \underline{\underline{\mathbf{B}}} = (b_{ij})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ we denote by $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}}$ and $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}}$ the Euclidean inner product $\underline{\mathbf{x}} \cdot \underline{\mathbf{y}} = \sum_{i=1}^n x_i y_i$ and the Frobenius inner product $\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{B}}} = \sum_{i,j=1}^n a_{ij} b_{ij}$. In particular, $|\underline{\mathbf{x}}| := (\underline{\mathbf{x}} \cdot \underline{\mathbf{x}})^{1/2}$ and $|\underline{\underline{\mathbf{A}}}| := (\underline{\underline{\mathbf{A}}} : \underline{\underline{\mathbf{A}}})^{1/2}$ refer to the Euclidean norm and the Frobenius norm, respectively.

We will further use standard notation from Lebesgue and Sobolev space theory (cf., e.g., [9]). In particular, for a bounded domain $D \subset \mathbb{R}^d, d \in \mathbb{N}$, we refer to $L^p(D), 1 \leq p < \infty$, as the Banach space of p-th power Lebesgue integrable functions on D with norm $\| \cdot \|_{0,p,D}$ and to $L^\infty(D)$ as the Banach space of essentially bounded functions on D with norm $\| \cdot \|_{0,\infty,D}$. Moreover, we denote by $W^{s,p}(D), s \in \mathbb{R}_+, 1 \leq p \leq \infty$, the Sobolev spaces with norms $\| \cdot \|_{s,p,D}$. We note that for $p = 2$ the spaces $L^2(D)$ and $W^{s,2}(D) = H^s(D)$ are Hilbert spaces with inner products $(\cdot, \cdot)_{0,2,D}$ and $(\cdot, \cdot)_{s,2,D}$. In the sequel, we will suppress the subindex 2 and write $(\cdot, \cdot)_{0,D}, (\cdot, \cdot)_{s,D}$ and $\| \cdot \|_{0,D}, \| \cdot \|_{s,D}$ instead of $(\cdot, \cdot)_{0,2,D}, (\cdot, \cdot)_{s,2,D}$ and $\| \cdot \|_{0,2,D}, \| \cdot \|_{s,2,D}$. The space $W_0^{s,p}(D)$ is the closure of C_0^∞ with respect to the $\| \cdot \|_{s,p,D}$-norm. We refer to $W^{-s,p}(D), s \in \mathbb{R}_+, 1 \leq p \leq \infty$, as the dual of $W_0^{s,q}(D)$, where $1/p + 1/q = 1$. In particular, $H^{-s}(D) = (H_0^s(D))^*$.

3. C^0 Interior Penalty Discontinuous Galerkin approximation

We perform a discretization in time of (1.7) with respect to a partition of the time interval $[0, T]$ into subintervals $[t_{m-1}, t_m], 1 \leq m \leq M, M \in \mathbb{N}$, of length $\Delta t := t_m - t_{m-1} = T/M$. Denoting by u^m some approximation of u at time t_m, for $1 \leq m \leq M$ we have to solve the problems

$$u^m - u^{m-1} + \Delta t \beta \delta^2 \Delta \boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = 0 \text{ in } \Omega, \tag{3.1a}$$

$$\mathbf{n}_\Gamma \cdot \beta \delta^2 (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = 0 \text{ on } \Gamma, \tag{3.1b}$$

$$\mathbf{n}_\Gamma \cdot \beta \delta^2 \boldsymbol{\nabla} \Big(\boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) \Big) = 0 \text{ on } \Gamma. \tag{3.1c}$$

We reformulate the second term on the left-hand side of (3.1a) according to

$$\begin{aligned} &\Delta \boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \Big(\boldsymbol{\nabla} \cdot (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m) \Big) = \\ &\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} (\omega(\boldsymbol{\nabla} u^m)^{-1/2} \boldsymbol{\nabla} u^m). \end{aligned} \tag{3.2}$$

As has been shown in [2], we have

$$\boldsymbol{\nabla}(\omega(\boldsymbol{\nabla} u^m)^{-1/2} \nabla u^m) = \omega(\boldsymbol{\nabla} u^m)^{-3/2} \underline{\underline{\mathbf{M}}}(u^m) D^2 u^m, \tag{3.3}$$

where $D^2 u^m$ is the 2×2 matrix of second partial derivatives of u^m and the matrix $\underline{\underline{\mathbf{M}}}(u^m)$ is given by

$$\underline{\underline{\mathbf{M}}}(u^m) := \begin{pmatrix} 1 + (\frac{\partial u^m}{\partial x_2})^2 & -\frac{\partial u^m}{\partial x_1} \frac{\partial u^m}{\partial x_2} \\ & \\ -\frac{\partial u^m}{\partial x_1} \frac{\partial u^m}{\partial x_2} & 1 + (\frac{\partial u^m}{\partial x_2})^2 \end{pmatrix}. \tag{3.4}$$

We note that the matrix $\underline{\underline{\mathbf{M}}}(u^m)$ is symmetric positive definite with the eigenvalues

$$\lambda_{min}(\underline{\underline{\mathbf{M}}}(u^m)) = 1, \quad \lambda_{max}(\underline{\underline{\mathbf{M}}}(u^m)) = 1 + |\boldsymbol{\nabla} u^m|^2. \tag{3.5}$$

Setting

$$\underline{\underline{\mathbf{A}}}_1(v) := \omega(\boldsymbol{\nabla} v)^{-3/2} \underline{\underline{\mathbf{M}}}(v), \tag{3.6}$$

the weak formulation of the implicitly in time discretized regularized fourth order TVF problem (3.1a)-(3.1c) reads: Find

$$u^m \in V := \{v \in H^2(\Omega) \mid \mathbf{n}_\Gamma \cdot \beta\delta^2\omega(\boldsymbol{\nabla} v)^{-1/2}\boldsymbol{\nabla} v = 0 \text{ on } \Gamma\}$$

such that for all $v \in V$ it holds

$$(u^m - u^{m-1}, v)_{0,\Omega} + \Delta t\beta\delta^2 \int\limits_\Omega \left(\underline{\underline{\mathbf{A}}}_1(u^m) D^2 u^m\right) : D^2 v \; dx = 0. \tag{3.7}$$

For the discretization in space we assume $\mathcal{T}_h$ to be a geometrically conforming, simplicial triangulation of Ω. We denote by $\mathcal{E}_h(\Omega)$ and $\mathcal{E}_h(\Gamma)$ the set of edges of $\mathcal{T}_h$ in the interior of Ω and on the boundary Γ, respectively, and set $\mathcal{E}_h := \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma)$. For $K \in \mathcal{T}_h$ and $E \in \mathcal{E}_h$ we denote by h_K and h_E the diameter of K and the length of E, and we set $h := \max(h_K \mid K \in \mathcal{T}_h)$. Due to the assumptions on $\mathcal{T}_h$ there exist constants $0 < c_R \le C_R$, $0 < c_Q \le C_Q$, and $0 < c_S \le C_S$ such that for all $K \in \mathcal{T}_h$ it holds

$$c_R h_K \le h_E \le C_R h_K, \quad E \in \mathcal{E}_h(\partial K), \tag{3.8a}$$

$$c_Q h \le h_K \le C_Q h, \tag{3.8b}$$

$$c_S h_K^2 \le \text{meas}(K) \le C_S h_K^2. \tag{3.8c}$$

Denoting by $P_k(T), k \in \mathbb{N}$, the linear space of polynomials of degree $\le k$ on T, for $k \in \mathbb{N}$ we define

$$V_h := \{v_h \in C^0(\bar{\Omega}) \mid v_h|_T \in P_k(T),\ T \in \mathcal{T}_h\}, \tag{3.9}$$

and note that $V_h \subset H^1(\Omega)$, but $V_h \not\subset H^2(\Omega)$. Further, we introduce

$$\underline{\underline{\mathbf{M}}}_h := \{\underline{\underline{\mathbf{q}}}_h \in L^2(\Omega)^{2\times 2} \mid \underline{\underline{\mathbf{q}}}_h|_K \in P_k(K)^{2\times 2},\ K \in \mathcal{T}_h\} \tag{3.10}$$

as the space of element-wise polynomial moment tensors.

For interior edges $E \in \mathcal{E}_h(\Omega)$ such that $E = K_+ \cap K_-, K_\pm \in \mathcal{T}_h$ and boundary edges on Γ we introduce the average and jump of ∇v_h according to

$$\{\nabla v_h\}_E := \begin{cases} \frac{1}{2}\Big(\nabla v_h|_{E\cap K_+} + \nabla v_h|_{E\cap K_-}\Big) \ , \ E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E \ , \ E \in \mathcal{E}_h(\Gamma) \end{cases}, \tag{3.11a}$$

$$[\nabla v_h]_E := \begin{cases} \nabla v_h|_{E\cap K_+} - \nabla v_h|_{E\cap K_-} \ , \ E \in \mathcal{E}_h(\Omega) \\ \nabla v_h|_E \ , \ E \in \mathcal{E}_h(\Gamma) \end{cases}. \tag{3.11b}$$

The average $\{\Delta v_h\}_E$ and jump $[\Delta v_h]_E$ are defined analogously. We further denote by $\mathbf{n}_E$ the unit normal vector on E pointing in the direction from K_+ to K_-. In the sequel, for $E \in \mathcal{E}_h$ we will frequently use

$$|\{v_h w_h\}_E| \le 2\{|v_h|\}_E\{|w_h|\}_E, \tag{3.12a}$$

$$|[v_h w_h]_E| \le 4\{|v_h|\}_E\{|w_h|\}_E. \tag{3.12b}$$

In fact, for $E \in \mathcal{E}_h(\Omega)$ (3.12a) and (3.12b) follow from

$$|\{v_h w_h\}_E| \le \frac{1}{2}(|v_h|_{E_+}|w_h|_{E_+} + |v_h|_{E_-}|w_h|_{E_-}) \le 2\{|v_h|\}_E\{|w_h|\}_E,$$

$$|[v_h w_h]_E| \le (|v_h|_{E_+}|w_h|_{E_+} + |v_h|_{E_-}|w_h|_{E_-}) \le 4\{|v_h|\}_E\{|w_h|\}_E,$$

whereas it is obvious for $E \in \mathcal{E}_h(\Gamma)$. We will also use

$$\sum_{E\in\mathcal{E}_h} [v_h w_h]_E = \sum_{E\in\mathcal{E}_h} \{v_h\}_E [w_h]_E + \sum_{E\in\mathcal{E}_h(\Omega)} [v_h]_E \{w_h\}_E. \tag{3.13}$$

Following the general approach [1] for DG approximations of second order elliptic boundary value problems, in [2] we have derived the following C^{0}IPDG approximation of (3.7): Find $u_h^m \in V_h$ such that for all $v_h \in V_h$ it holds

$$(u_h^m, v_h)_{0,\Omega} + \Delta t \beta \delta^2 a_h^{IP}(u_h^m, v_h; u_h^m) = (u_h^{m-1}, v_h)_{0,\Omega}, \quad v_h \in V_h. \tag{3.14}$$

Here, for $z_h \in V_h$ the mesh-dependent semilinear C^0IPDG form $a_h^{IP}(\cdot,\cdot;z_h) : V_h \times V_h \to \mathbb{R}$ is given by

$$\begin{aligned} a_h^{IP}(u_h, v_h; z_h) &:= \sum_{K \in \mathcal{T}_h} (\underline{\underline{\mathbf{A}}}_1(z_h) D^2 u_h, D^2 v_h)_{0,K} - \\ &\sum_{E \in \mathcal{E}_h} (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(z_h) D^2 u_h\}_E \mathbf{n}_E, \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} z_h)^{-1/4} \boldsymbol{\nabla} v_h]_E)_{0,E} - \\ &\sum_{E \in \mathcal{E}_h} (\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(z_h) D^2 v_h\}_E \mathbf{n}_E, \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} z_h)^{-1/4} \boldsymbol{\nabla} u_h]_E)_{0,E} + \\ &\alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} (\mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} z_h)^{-1/4} \boldsymbol{\nabla} u_h]_E, \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} z_h)^{-1/4} \boldsymbol{\nabla} v_h]_E)_{0,E}, \end{aligned} \tag{3.15}$$

where $\alpha > 0$ is a penalty parameter and

$$\underline{\underline{\mathbf{A}}}_2(z_h) := \omega(\boldsymbol{\nabla} z_h)^{-5/4} \underline{\underline{\mathbf{M}}}(z_h). \tag{3.16}$$

4. Existence and uniqueness of a solution of the C^{0}IPDG approximation

The existence and uniqueness of a solution of the C^{0}IPDG approximation (3.14) can be shown using the following nonlinear analogue of the Lax-Milgram Lemma.

Theorem 4.1. *Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$ and associated norm $\|\cdot\|_V$ and let V^* be the dual space with norm $\|\cdot\|_{V^*}$. We denote by $\langle\cdot,\cdot\rangle_{V^*,V}$ the dual pairing between V^* and V. Let $A : V \to V^*$ be a nonlinear operator with*

$A(0)=0$ that is Lipschitz continuous on $B(0,R) := \{v \in V \mid \|v\|_V \le R\}, R>0$, i.e., there exists a constant $\Gamma(R)>0$ such that for all $v,w \in V$ it holds

$$\|A(v)-A(w)\|_{V_h^*} \le \Gamma(R)\ \|v-w\|_V. \tag{4.1}$$

Moreover, assume that $A : V \to V^*$ is strongly monotone on $B(0,R)$, i.e., there exists a constant $\gamma(R)>0$ such that for all $v,w \in B(0,R)$ it holds

$$\langle A(v)-A(w), v-w\rangle_{V^*,V} \ge \gamma(R)\ \|v-w\|_V^2. \tag{4.2}$$

Then, for any $\ell \in V^*$ with

$$\|\ell\|_{V^*} \le \frac{\Gamma(R)^2}{\gamma(R)}\Big(1-\sqrt{1-\frac{\gamma(R)^2}{\Gamma(R)^2}}\Big)R, \tag{4.3}$$

the nonlinear equation

$$Au = \ell \tag{4.4}$$

has a unique solution $u \in B(0,R)$.

Proof. We refer to $\tau : V^* \to V$ as the Riesz mapping, i.e.,

$$\langle \ell, v\rangle_{V^*,V} = (\tau\ell, v)_V, \quad \ell \in V^*,\ v \in V. \tag{4.5}$$

Then, $u \in B(0,R)$ is a solution of (4.4) if and only if u is a fixed point of the nonlinear map $T : V \to V$ defined by means of

$$T(v) := v - \rho(\tau A(v) - \tau\ell), \quad v \in V,\ \rho > 0.$$

Due to (4.5) we have

$$\begin{aligned} \|T(v)-T(w)\|_V^2 \ =& \\ \|v-w\|_V^2 - 2\rho\langle A(v)-A(w), v-w\rangle_{V^*,V} &+ \rho^2\|A(v)-A(w)\|_{V^*}^2. \end{aligned} \tag{4.6}$$

Now, using (4.1) and (4.2) it follows that

$$\|T(v)-T(w)\|_V^2 \le q\|v-w\|_V^2, \quad q := 1-2\rho\gamma(R)+\rho^2\Gamma(R)^2.$$

For $\rho \in (0, 2\gamma(R)/\Gamma(R)^2)$ we have $q<1$ and hence, T is a contraction on $B(0,R)$. We note that q attains its minimum $q_{min} = 1-\gamma(R)^2/\Gamma(R)^2$ for $\rho_{min} = \gamma(R)/\Gamma(R)^2$. Moreover, choosing $w=0$ in (4.6) and observing $A(0)=0$, we have

$$\|T(v)-T(0)\|_V^2 \le q_{min}\|v\|_V^2,$$

and hence, for $v \in B(0,R)$ it holds

$$\|T(v)\|_V \le \|T(v)-T(0)\|_V + \|T(0)\|_V \le \sqrt{q_{min}}R + \rho\|\ell\|_{V^*}.$$

Consequently, we have

$$\|T(v)\|_V \le R, \tag{4.7}$$

if $\ell \in V^*$ satisfies (4.3). We deduce from (4.7) that $T(B(0,R)) \subset B(0,R)$. The Banach fixed point theorem asserts the existence and uniqueness of a fixed point in $B(0,R)$. □

In order to apply the previous result to the C^0IPDG method (3.14), we introduce a mesh-dependent semi-norm $|\cdot|_{2,h,\Omega}$ and weighted norm $\|\cdot\|_{2,h,\Omega}$ on V_h according to

$$|v_h|_{2,h,\Omega} := \Big(\sum_{K\in\mathcal{T}_h} \int_K D^2 v_h : D^2 v_h \, dx + \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_{E\in\mathcal{E}_h} |\mathbf{n}_E \cdot [\boldsymbol{\nabla} v_h]_E|^2 \, ds\Big)^{1/2}, \tag{4.8a}$$

$$\|v_h\|_{2,h,\Omega} := \Big(\|v_h\|_{0,\Omega}^2 + |v_h|_{2,h,\Omega}^2\Big)^{1/2}. \tag{4.8b}$$

We further note that (3.14) can be written as the nonlinear equation

$$A_h^{DG} u_h^m = \ell_h, \tag{4.9}$$

where the nonlinear operator $A_h^{DG} : V_h \to V_h^*$ and the linear functional $\ell_h \in V_h^*$ are given by

$$\langle A_h^{DG} v_h, w_h \rangle_{V_h^*, V_h} := (v_h, w_h)_{0,\Omega} + \Delta t \beta \delta^2 \ a_h^{DG}(v_h, w_h; v_h), \quad v_h, w_h \in V_h, \tag{4.10}$$

$$\ell_h(v_h) := (u_h^{m-1}, v_h)_{0,\Omega}, \quad v_h \in V_h. \tag{4.11}$$

For the proof of Lipschitz continuity on bounded sets and strong monotonicity of the nonlinear operator A_h^{DG} we need the inverse estimates (cf., e.g., [3, 5]):
For $p \in [1, \infty]$ and $\ell, m \in \mathbb{N}_0$ it holds

$$\|v_h\|_{m,p,K} \leq \frac{C_{inv}}{\text{meas}(K)^{max(0, \frac{1}{2} - \frac{1}{p})} \ h_K^{m-\ell}} \|v_h\|_{\ell,K}, \quad v_h \in V_h, \tag{4.12}$$

where C_{inv} is a positive constant that only depends on k, ℓ, m, p and the shape regularity of the triangulation. We further need the trace inequalities (cf., e.g., [8, 10]): For $p \in [1, \infty]$, $m \in \mathbb{N}_0$, and $K \in \mathcal{T}_h$ it holds

$$\|\boldsymbol{\nabla} v_h\|_{m,p,\partial K} \leq C_T h_K^{-1/p} \|\boldsymbol{\nabla} v_h\|_{m,p,K}, \quad v_h \in V_h, \tag{4.13a}$$

$$\|D^2 v_h\|_{m,p,\partial K} \leq C_T h_K^{-1/p} \|D^2 v_h\|_{m,p,K}, \quad v_h \in V_h, \tag{4.13b}$$

where C_T is a positive constant that only depends on k, m, p and the shape regularity of the triangulation. Moreover, we will frequently use the following Poincaré-Friedrichs inequality for piecewise H^2-functions (cf., e.g., [4])

$$\|\boldsymbol{\nabla} v_h\|_{0,\Omega} \leq C_{PF} |v_h|_{2,h,\Omega}, \quad v_h \in V_h, \tag{4.14}$$

where $C_{PF} > 0$ is a constant that only depends on Ω and the shape regularity of the triangulation.

In the sequel, we will frequently use some basic estimates for the weight function $\omega(\boldsymbol{\nabla} v_h)$. In particular, for $\beta > 0$ and $v \in V_h$ it holds

$$\omega(\boldsymbol{\nabla} v)^{-\beta} = (1 + |\boldsymbol{\nabla} v|^2)^{-\beta} \le 1, \tag{4.15a}$$

$$\begin{aligned} \omega(\boldsymbol{\nabla} v)^{-(\beta+1)} |\boldsymbol{\nabla} v| &\le \omega(\boldsymbol{\nabla} v)^{-(\beta+1)} (1 + |\boldsymbol{\nabla} v|^2)^{1/2} \\ &\le \omega(\boldsymbol{\nabla} v)^{-(\beta+1/2)} \le 1. \end{aligned} \tag{4.15b}$$

Moreover, for $v, w \in V_h$ and $\xi(s) := w + s(v - w), s \in [0, 1]$, it holds

$$\omega(\boldsymbol{\nabla} v)^{-\beta} - \omega(\boldsymbol{\nabla} w)^{-\beta} = -2\beta \int_0^1 \omega(\boldsymbol{\nabla} \xi(s))^{-\beta-1} \boldsymbol{\nabla} \xi(s) \cdot \boldsymbol{\nabla}(v - w) \; ds, \tag{4.16a}$$

$$\begin{aligned} \omega(\boldsymbol{\nabla} v)^{-\beta} \underline{\underline{\mathbf{M}}}(v) - \omega(\boldsymbol{\nabla} w)^{-\beta} \underline{\underline{\mathbf{M}}}(w) = \int_0^1 \omega(\boldsymbol{\nabla} \xi(s))^{-\beta} \underline{\underline{\mathbf{F}}}(\xi(s); v - w) \; ds \; - \\ 2\beta \int_0^1 \omega(\boldsymbol{\nabla} \xi(s))^{-\beta-1} \boldsymbol{\nabla} \xi(s) \cdot \boldsymbol{\nabla}(v - w) \underline{\underline{\mathbf{M}}}(\xi(s)) \; ds, \end{aligned} \tag{4.16b}$$

where the matrix $\underline{\underline{\mathbf{F}}}(v; w), v, w \in V_h$ is given by

$$\underline{\underline{\mathbf{F}}}(v; w) := \begin{pmatrix} 2 \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_2} & \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} \\ \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_2} + \frac{\partial w}{\partial x_2} \frac{\partial v}{\partial x_1} & 2 \frac{\partial w}{\partial x_1} \frac{\partial v}{\partial x_1} \end{pmatrix}, \quad v, w \in V_h. \tag{4.17}$$

An easy computation yields

$$|\underline{\underline{\mathbf{F}}}(v; w)|^2 \le 5 \; |\boldsymbol{\nabla} v|^2 \; |\boldsymbol{\nabla} w|^2. \tag{4.18}$$

It follows from (4.15b) and (4.16a) that

$$|\omega(\boldsymbol{\nabla} v)^{-\beta} - \omega(\boldsymbol{\nabla} w)^{-\beta}| \le 2\beta |\boldsymbol{\nabla}(v - w)|, \tag{4.19a}$$

whereas in view of (3.5),(4.15b),(4.16b), and (4.18) we have

$$|\omega(\boldsymbol{\nabla} v)^{-\beta}\underline{\underline{\mathbf{M}}}(v) - \omega(\boldsymbol{\nabla} w)^{-\beta}\underline{\underline{\mathbf{M}}}(w)| \le (2\beta + \sqrt{5})|\boldsymbol{\nabla}(v-w)|, \tag{4.19b}$$

We will first show that the nonlinear operator A_h^{DG} is Lipschitz continuous on the ball

$$B_h(0,R) := \{v_h \in V_h \mid \|v_h\|_{2,h,\Omega} \le R\}. \tag{4.20}$$

Theorem 4.2. *The nonlinear operator A_h^{DG} is Lipschitz continuous on the ball $B_h(0,R)$. In particular, there exists $\Gamma(h,R) > 0$ such that*

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} \le \Gamma(h,R)\ \|v_h - w_h\|_{2,h,\Omega}, \quad v_h, w_h \in B_h(0,R). \tag{4.21}$$

Proof. For $v_h, w_h \in B_h(0,R)$ we set $\xi_h := v_h - w_h$. In view of the definition (4.10) of the nonlinear operator A_h^{DG} we have

$$\|A_h^{DG} v_h - A_h^{DG} w_h\|_{V_h^*} = \sup_{\|z_h\|_{2,h,\Omega}\le 1} |\langle A_h^{DG} v_h - A_h^{DG} w_h, z_h\rangle_{V_h^*,V_h}| \ = \tag{4.22}$$

$$\sup_{\|z_h\|_{2,h,\Omega}\le 1} |(\xi_h, z_h)_{0,\Omega} + \Delta t\beta\delta^2\ \Big(a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h)\Big)|.$$

According to the definition (3.15) of the semilinear form $a_h^{DG}(\cdot,\cdot;\cdot)$ we find

$$a_h^{DG}(v_h, z_h; v_h) - a_h^{DG}(w_h, z_h; w_h) \ = \tag{4.23}$$

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h)D^2v_h - \underline{\underline{\mathbf{A}}}_1(w_h)D^2w_h\Big) : D^2z_h\ dx\\
&-\sum_{E\in\mathcal{E}_h}\int_E \Big(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2v_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} z_h]_E\ -\\
&\qquad\qquad \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2w_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} z_h]_E\Big)\ ds\\
&-\sum_{E\in\mathcal{E}_h}\int_E \Big(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2z_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E\ -\\
&\qquad\qquad \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2z_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E\Big)\ ds\\
&+\alpha\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E \Big(\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} z_h]_E\ -\\
&\qquad\qquad \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} z_h]_E\Big)\ ds.
\end{aligned}
$$

We will estimate the four terms on the right-hand side of (4.23) separately.

(i) For the first term on the right-hand side of (4.23) we obtain

$$
\begin{aligned}
&\sum_{K\in\mathcal{T}_h}\int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h)D^2v_h - \underline{\underline{\mathbf{A}}}_1(w_h)D^2w_h\Big) : D^2z_h\ dx\ =\\
&\underbrace{\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h : D^2z_h\ dx}_{=\ I_1} + \underbrace{\sum_{K\in\mathcal{T}_h}\int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)\Big)D^2w_h : D^2z_h\ dx}_{=\ I_2}\,.
\end{aligned}
$$

In view of (3.5),(3.6), and (4.15a) and using Hölder's inequality as well as the Cauchy-Schwarz inequality, we get the following upper bound for I_1:

$$
\begin{aligned}
(4.24)\qquad |I_1| \le &\sum_{K\in\mathcal{T}_h}\int_K |D^2\xi_h||D^2z_h|\ dx\ \le\\
&\sum_{K\in\mathcal{T}_h}\Big(\int_K |D^2\xi_h|^2\ dx\Big)^{1/2}\Big(\int_K |D^2z_h|^2\ dx\Big)^{1/2}\ \le\\
&\Big(\sum_{K\in\mathcal{T}_h}\|D^2\xi_h\|_{0,K}^2\ dx\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\|D^2z_h\|_{0,K}^2\ dx\Big)^{1/2}.
\end{aligned}
$$

Likewise, using (3.8b),(3.8c),(4.16b), the inverse inequality (4.12), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), and observing $\|D^2 w_h\|_{0,K} \leq \|w_h\|_{2,h,\Omega} \leq R, K \in \mathcal{T}_h$, we can estimate I_2 from above as follows:

$$
\begin{aligned}
|I_2| \leq & \sum_{K\in\mathcal{T}_h} \int_K |\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)||D^2 w_h||D^2 z_h| \ dx \ \leq \\
& (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \int_K |\nabla \xi_h||D^2 w_h||D^2 z_h| \ dx \ \leq \\
& (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \|\nabla \xi_h\|_{0,\infty,K} \Big(\int_K |D^2 w_h|^2 \ dx \Big)^{1/2} \Big(\int_K |D^2 z_h|^2 \ dx \Big)^{1/2} \ \leq \\
& (3+\sqrt{5}) c_S^{-1/2} C_{inv} R \sum_{K\in\mathcal{T}_h} h_K^{-1} \|\nabla \xi_h\|_{0,K} \|D^2 z_h\|_{0,K} \ \leq \\
& (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} R h^{-1} \Big(\sum_{K\in\mathcal{T}_h} \|\nabla \xi_h\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \Big)^{1/2} \\
\leq & (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R h^{-1} |\xi_h|_{2,h,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \Big)^{1/2}.
\end{aligned}
$$

Hence, setting $C_A^{(1)} := (3+\sqrt{5}) c_Q^{-1} c_S^{-1/2} C_{inv} C_{PF} R$, we thus have

$$
|I_2| \leq C_A^{(1)} h^{-1} |\xi_h|_{2,h,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|D^2 z_h\|_{0,K}^2 \Big)^{1/2}. \tag{4.25}
$$

(ii) Setting $\tilde{\omega}(\nabla v_h, \nabla w_h) := \omega(\nabla v_h)^{-1/4} - \omega(\nabla w_h)^{-1/4}$, the second term on the right-hand side of (4.23) can be written as

$$
\begin{aligned}
\sum_{E\in\mathcal{E}_h} \int_E \Big(& \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla v_h)^{-1/4} \nabla z_h]_E \ - \\
& \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\nabla w_h)^{-1/4} \nabla z_h]_E \Big) \ ds \ =
\end{aligned}
$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2\xi_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}z_h]_E\ ds}_{=\,II_1}\ +$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{(\underline{\underline{\mathbf{A}}}_2(v_h)-\underline{\underline{\mathbf{A}}}_2(w_h))D^2w_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}z_h]_E\ ds}_{=\,II_2}\ +$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2w_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\tilde{\omega}(\boldsymbol{\nabla}v_h,\boldsymbol{\nabla}w_h)\boldsymbol{\nabla}z_h]_E\ ds}_{=\,II_3}.$$

Setting $E_1 := E_+, E_2 := E_-$, for $E\in\mathcal{E}_h(\Omega)$, and using (3.5),(3.8a),(3.16),(4.15a), and the trace inequality (4.13b), for the first term II_1 we find

$$|II_1| \le \sum_{E\in\mathcal{E}_h}\int_E |\{D^2\xi_h\}_E||\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|\ ds\ \le$$

$$\frac{1}{2}\sum_{E\in\mathcal{E}_h(\Omega)}\int_E\sum_{i=1}^2|D^2\xi_h|_{E_i}||\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|\ ds+\sum_{E\in\mathcal{E}_h(\Gamma)}\int_E|D^2\xi_h|||\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|\ ds\ \le$$

$$\sum_{E\in\mathcal{E}_h(\Omega)}\sum_{i=1}^2 h_E^{1/2}\Big(\int_E|D^2\xi_h|_{E_i}|^2\ ds\Big)^{1/2}h_E^{-1/2}\Big(\int_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2}+$$

$$\sum_{E\in\mathcal{E}_h(\Gamma)}\Big(h_E^{1/2}\int_E|D^2\xi_h|^2\ ds\Big)^{1/2}h_E^{-1/2}\Big(\int_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2}\ \le$$

$$C_R^{1/2}\Big(\sum_{K\in\mathcal{T}_h}h_K\|D^2\xi_h\|_{0,\partial K}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2}\ \le$$

$$C_R^{1/2}C_T\Big(\sum_{K\in\mathcal{T}_h}\|D^2\xi_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2}.$$

We thus have

$$|II_1|\le C_A^{(2)}\Big(\sum_{K\in\mathcal{T}_h}\|D^2\xi_h\|_{0,K}^2\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2}, \tag{4.26}$$

where $C_A^{(2)} := C_R^{1/2}C_T$. In a similar way, using (3.5),(3.8a)-(3.8c),(3.16),(4.16b), the inverse inequality (4.12), the trace inequality (4.13a), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), and observing $\|D^2w_h\|_{0,K}\le\|w_h\|_{2,h,\Omega}\le$

$R, K \in \mathcal{T}_h$, the second term II_2 can be estimated from above according to

$$|II_2| \le (\frac{5}{2}+\sqrt{5})C_R^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K}^2 h_K \int\limits_{\partial K}|D^2 w_h|^2\ ds\Big)^{1/2}$$

$$\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int\limits_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2} \le$$

$$(\frac{5}{2}+\sqrt{5})C_R^{1/2}C_T\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K}^2 \int\limits_{K}|D^2 w_h|^2\ ds\Big)^{1/2}$$

$$\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int\limits_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2} \le$$

$$(\frac{5}{2}+\sqrt{5})c_Q^{-1}c_S^{-1}C_{inv}C_R^{1/2}C_T R h^{-1}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,K}^2\Big)^{1/2}$$

$$\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int\limits_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2} \le$$

$$\le C_A^{(3)}h^{-1}|\xi_h|_{2,h,\Omega}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int\limits_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}z_h]_E|^2\ ds\Big)^{1/2}, \tag{4.27}$$

where $C_A^{(3)} := (\frac{5}{2}+\sqrt{5})c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_T R$. In a similar way, for II_3 we obtain

$$|II_3| \le$$

$$C_R^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}z_h\|_{0,\infty,K}^2 h_K\int\limits_{\partial K}|D^2w_h|^2 ds\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int\limits_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds\Big)^{1/2} \le$$

$$C_R^{1/2}C_T\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}z_h\|_{0,\infty,K}^2 \int\limits_{K}|D^2w_h|^2 dx\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int\limits_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds\Big)^{1/2} \le$$

$$c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_T R h^{-1}|z_h|_{2,h,\Omega}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int\limits_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2\ ds\Big)^{1/2}.$$

and hence,

$$|II_3| \le C_A^{(4)}h^{-1}|z_h|_{2,h,\Omega}\Big(\sum_{E\in\mathcal{E}_h}h_E^{-1}\int\limits_E|\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2\ ds\Big)^{1/2}, \tag{4.28}$$

where $C_A^{(4)} := c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_T R$.

(iii) For the third term on the right-hand side of (4.23) we have

$$\sum_{E\in\mathcal{E}_h}\int_E \Big(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2z_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E -$$

$$\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2z_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E\Big)\ ds\ =$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{(\underline{\underline{\mathbf{A}}}_2(v_h)-\underline{\underline{\mathbf{A}}}_2(w_h))D^2z_h\}_E\mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E\ ds}_{=\ III_1}\ +$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2z_h\}_E\mathbf{n}_E\ \mathbf{n}_E\cdot[\tilde{\omega}(\boldsymbol{\nabla} v_h,\boldsymbol{\nabla} w_h)\boldsymbol{\nabla} v_h]_E\ ds}_{=\ III_2}\ +$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2z_h\}_E\mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\ ds}_{=\ III_3}\,.$$

The terms III_1, III_2, and III_3 can be estimated from above in much the same way as the corresponding terms for II. We obtain

$$|III_1| \le C_A^{(5)}h^{-1}|\xi_h|_{2,h,\Omega}\Big(\sum_{K\in\mathcal{T}_h}\int_K |D^2z_h|^2\ ds\Big)^{1/2}, \tag{4.29}$$

where $C_A^{(5)} := (\frac{5}{2}+\sqrt{5})c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_TR$, and

$$|III_2| \le C_A^{(6)}h^{-1}|\xi_h|_{2,h,\Omega}\Big(\sum_{K\in\mathcal{T}_h}\int_K |D^2z_h|^2\ ds\Big)^{1/2}, \tag{4.30a}$$

$$|III_3| \le C_A^{(7)}h^{-1}|\xi_h|_{2,h,\Omega}\Big(\sum_{K\in\mathcal{T}_h}\int_K |D^2z_h|^2\ ds\Big)^{1/2}, \tag{4.30b}$$

where $C_A^{(6)} = C_A^{(7)} := c_Q^{-1}c_S^{-1}C_{inv}C_{PF}C_R^{1/2}C_TR$.

(iv) Finally, for the fourth term on the right-hand side of (4.23) we get

$$\begin{aligned}(4.31)\quad & \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \Big(\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} v_h]_E \ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} z_h]_E \ - \\ & \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E \ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} z_h]_E\Big)\, ds \ = \\ & \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} \xi_h]_E \ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}\boldsymbol{\nabla} z_h]_E \ ds}_{=\ IV_1} \ + \\ & \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} w_h]_E \ \mathbf{n}_E\cdot[\tilde\omega(\boldsymbol{\nabla} v_h,\boldsymbol{\nabla} w_h)\boldsymbol{\nabla} z_h]_E \ ds}_{=\ IV_2} \ + \\ & \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E \mathbf{n}_E\cdot[\tilde\omega(\boldsymbol{\nabla} v_h,\boldsymbol{\nabla} w_h)\boldsymbol{\nabla} w_h]_E \ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} w_h)^{-1/4}\boldsymbol{\nabla} z_h]_E \ ds}_{=\ IV_3}\,.\end{aligned}$$

Using (3.8a),(4.15a), the trace inequality (4.13a), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), for IV_1 we obtain

$$\begin{aligned}&|IV_1| \le \alpha \sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E||\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E| \ ds \ \le \\ &\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1/2}\Big(\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds\Big)^{1/2} h_E^{-1/2}\Big(\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 ds\Big)^{1/2} \le \\ &C_A^{(8)}\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1}\int_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds\Big)^{1/2}\Big(\sum_{E\in\mathcal{E}_h}\int_E h_E^{-1}|\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 ds\Big)^{1/2},\end{aligned}$$

where $C_A^{(8)} := \alpha$. Setting $K_1 := K_+$ and $K_2 := K_-$ for $E \in \mathcal{E}_h(\Omega), E = K_+ \cap K_-$, and $K_1 = K_2 = K$ for $E \in \mathcal{E}_h(\Gamma), E \in \mathcal{E}_h(K \cap \Gamma)$, the term IV_2 can be estimated from above as follows:

$$\begin{aligned}|IV_2| \le \alpha \sum_{E\in\mathcal{E}_h}\sum_{i=1}^{2} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K_i}&\Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\boldsymbol{\nabla} w_h]_E|^2 ds\Big)^{1/2} \\ &\Big(\int_E h_E^{-1}|\mathbf{n}_E\cdot[\boldsymbol{\nabla} z_h]_E|^2 ds\Big)^{1/2}.\end{aligned}$$

Using (3.8b),(3.8c), the inverse inequality (4.12), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), for IV_1, we have

$$\sum_{i=1}^{2} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K_i} \leq c_R^{-1} c_S^{-1} C_{inv} h^{-1} \sum_{i=1}^{2} \|\boldsymbol{\nabla}\xi_h\|_{0,K_i} \leq$$

$$2c_R^{-1} c_S^{-1} C_{inv} h^{-1} \|\boldsymbol{\nabla}\xi_h\|_{0,\Omega} \leq 2c_R^{-1} c_S^{-1} C_{inv} C_{PF} h^{-1} |\xi_h|_{2,h,\Omega}.$$

Hence, observing $\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla} w_h]_E|^2 ds\Big)^{1/2} \leq \|w_h\|_{2,h,\Omega} \leq R$, we obtain

$$|IV_2| \leq C_A^{(9)} h^{-1} |\xi_h|_{2,h,\Omega} \Big(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\boldsymbol{\nabla} z_h]_E|^2 ds\Big)^{1/2}. \tag{4.32}$$

where $C_A^{(9)} := 2\alpha c_R^{-1} c_S^{-1} C_{inv} C_{PF} R$. In the same way we get

$$|IV_3| \leq C_A^{(10)} h^{-1} |\xi_h|_{2,h,\Omega} \Big(\int_E h_E^{-1} |\mathbf{n}_E \cdot [\boldsymbol{\nabla} z_h]_E|^2 ds\Big)^{1/2}. \tag{4.33}$$

where $C_A^{(10)} := C_A^{(9)}$.

Setting $C_A := \sum_{i=1}^{10} C_A^{(i)}$, it follows from (4.22)-(4.33) that

$$|\langle A_h^{DG} v_h - A_h^{DG} w_h, z_h\rangle_{V_h^*,V_h}| \leq \max(1, \beta\Delta t\delta^2 C_A h^{-1}) \; \|\xi_h\|_{2,h,\Omega} \|z_h\|_{2,h,\Omega},$$

which implies (4.21) with $\Gamma(h,R) := \max(1, \beta\Delta t\delta^2 C_A h^{-1})$. $\square$

Theorem 4.3. *Under the assumption that there exist constants* $0 < \kappa \ll 1$ *and* $C_\Delta > 0$ *such that*

$$\beta\Delta t\delta^2 \leq C_\Delta h^{4+\kappa}, \tag{4.34}$$

for sufficiently small $0 < h < 1$ *there exists* $\gamma(h,R) > 0$ *such that for* $v_h, w_h \in B_h(0,R)$ *it holds*

$$\langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h\rangle_{V_h^*,V_h} \geq \gamma(h,R) \|v_h - w_h\|_{2,h,\Omega}^2. \tag{4.35}$$

Proof. For $v_h, w_h \in B_h(0,R)$ we set $\xi_h := v_h - w_h$. Taking the definition (4.10) of the nonlinear operator A_h^{DG} into account, we have

$$\begin{aligned}
(4.36) \qquad & \langle A_H^{DG} v_h - A_h^{DG} w_h, \xi_h \rangle_{V_h^*, V_h} \; = \\
& \|\xi_h\|_{0,\Omega}^2 + \beta \Delta t \delta^2 \Big(a_h^{DG}(v_h, \xi_h; v_h) - a_h^{DG}(w_h, \xi_h; w_h) \Big).
\end{aligned}$$

Recalling the definitions (3.6),(3.16) of $\underline{\underline{\mathbf{A}}}_1$ and $\underline{\underline{\mathbf{A}}}_2$, for the second term on the right-hand side of (4.36) it follows that

$$\begin{aligned}
(4.37) \quad & a_h^{DG}(v_h, \xi_h; v_h) - a_h^{DG}(w_h, \xi_h; w_h) \; = \\
& \sum_{K \in \mathcal{T}_h} \int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) D^2 v_h - \underline{\underline{\mathbf{A}}}_1(w_h)) D^2 w_h \Big) : D^2 \xi_h \; dx \\
& - \sum_{E \in \mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h \}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} \xi_h]_E \; - \\
& \qquad \mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h \}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} \xi_h]_E \Big) \; ds \\
& - \sum_{E \in \mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(v_h) D^2 \xi_h \}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} v_h]_E \; - \\
& \qquad \mathbf{n}_E \cdot \{ \underline{\underline{\mathbf{A}}}_2(w_h) D^2 \xi_h \}_E \; \mathbf{n}_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} w_h]_E \Big) \; ds \\
& + \alpha \sum_{E \in \mathcal{E}_h} h_E^{-1} \int_E \Big(\mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} v_h]_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla} \xi_h]_E \; - \\
& \qquad \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} w_h]_E \; \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla} \xi_h]_E \Big) \; ds.
\end{aligned}$$

As in the previous theorem, we will estimate the four terms on the right-hand side of (4.37) separately.

(i) For the first term we obtain

$$\sum_{K\in\mathcal{T}_h}\int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h)D^2v_h - \underline{\underline{\mathbf{A}}}_1(w_h)D^2w_h\Big) : D^2\xi_h\ dx \ = $$

$$\underbrace{\sum_{K\in\mathcal{T}_h}\int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h : D^2\xi_h\ dx}_{=\ I_1} + \underbrace{\sum_{K\in\mathcal{T}_h}\int_K \Big(\underline{\underline{\mathbf{A}}}_1(v_h) - \underline{\underline{\mathbf{A}}}_1(w_h)\Big)D^2w_h : D^2\xi_h\ dx}_{=\ I_2}.$$

As far as I_1 is concerned, due to (3.5) and (3.6) we have

$$\int_K \underline{\underline{\mathbf{A}}}_1(v_h)D^2\xi_h : D^2\xi_h\ dx \ge (1+\|\boldsymbol{\nabla} v_h\|^2_{0,\infty,K})^{-3/2}\|D^2\xi_h\|^2_{0,K}.$$

Using (3.8b),(3.8c), the inverse inequality (4.12), the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), and observing $\|v_h\|_{2,h,\Omega} \le R$, we get

$$\|\boldsymbol{\nabla} v_h\|^2_{0,\infty,K} \le c_S^{-2}C_{inv}^2 h_K^{-2}\|\boldsymbol{\nabla} v_h\|^2_{0,K} \le c_Q^{-2}c_S^{-2}C_{inv}^2 h^{-2}\|\boldsymbol{\nabla} v_h\|^2_{0,\Omega} \ \le$$

$$c_Q^{-2}c_S^{-2}C_{inv}^2 C_{PF}^2 h^{-2}\|v_h\|^2_{2,h,\Omega} \le \gamma_M^{(0)} h^{-2},$$

where $\gamma_M^{(0)} := c_Q^{-2}c_S^{-2}C_{inv}^2 C_{PF}^2 R^2$. Observing $h \le 1$, it follows that

$$(1+\|\boldsymbol{\nabla} v_h\|^2_{0,\infty,K})^{-3/2} \ge h^3(h^2+\gamma_M^{(0)})^{-3/2} \ge h^3(1+\gamma_M^{(0)})^{-3/2} = \gamma_M^{(1)} h^3,$$

where $\gamma_M^{(1)} := (1+\gamma_M^{(0)})^{-3/2}$. Hence, we obtain the following lower bound for I_1:

$$I_1 \ge \gamma_M^{(1)} h^3 \sum_{K\in\mathcal{T}_h} \|D^2\xi_h\|^2_{0,K}. \tag{4.38}$$

In order to estimate I_2 from above, we use (3.8b),(3.8c),(4.19b), Hölder's inequality, the inverse inequality (4.12), the Cauchy-Schwarz inequality, and observe $\|D^2w_h\|_{0,K}$

$\leq \|w_h\|_{2,h,\Omega} \leq R, K \in \mathcal{T}_h$:

$$|I_2| \leq (3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \int_K |\boldsymbol{\nabla}\xi_h||D^2 w_h||D^2\xi_h| \ dx \ \leq$$

$$(3+\sqrt{5}) \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,\infty,K} \Big(\int_K |D^2 w_h|^2 \ dx\Big)^{1/2} \Big(\int_K |D^2\xi_h|^2 \ dx\Big)^{1/2} \ \leq$$

$$(3+\sqrt{5}) c_S^{-1} C_{inv} \sum_{K\in\mathcal{T}_h} h_K^{-1} \|\xi_h\|_{0,K} \|D^2 w_h\|_{0,K} \|D^2\xi_h\|_{0,K} \ \leq$$

$$(3+\sqrt{5}) c_S^{-1} C_{inv}^2 R \sum_{K\in\mathcal{T}_h} h_K^{-1} \|\xi_h\|_{0,K} h_K^{-2} \|\xi_h\|_{0,K} \ \leq$$

$$(3+\sqrt{5}) c_Q^{-3} c_S^{-1} C_{inv}^2 R h^{-3} \sum_{K\in\mathcal{T}_h} \|\xi_h\|_{0,K}^2 .$$

Hence, it follows that

$$|I_2| \leq C_B^{(1)} h^{-3} \|\xi_h\|_{0,\Omega}^2, \tag{4.39}$$

where $C_B^{(1)} := (3+\sqrt{5}) c_Q^{-3} c_S^{-1} C_{inv}^2 R$.

(ii) We now deal with the second term on the right-hand side of (4.37) which we rewrite as follows:

$$\sum_{E\in\mathcal{E}_h} \int_E \Big(\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 v_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \ -$$

$$\mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} w_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \Big) \ ds \ =$$

$$\underbrace{\sum_{E\in\mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(v_h) D^2 \xi_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \ ds}_{= II_1} \ +$$

$$\underbrace{\sum_{E\in\mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{(\underline{\underline{\mathbf{A}}}_2(v_h) - \underline{\underline{\mathbf{A}}}_2(w_h)) D^2 w_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\omega(\boldsymbol{\nabla} v_h)^{-1/4} \boldsymbol{\nabla}\xi_h]_E \ ds}_{= II_2}$$

$$+ \underbrace{\sum_{E\in\mathcal{E}_h} \int_E \mathbf{n}_E \cdot \{\underline{\underline{\mathbf{A}}}_2(w_h) D^2 w_h\}_E \ \mathbf{n}_E \ \mathbf{n}_E \cdot [\tilde{\omega}(\boldsymbol{\nabla} v_h, \boldsymbol{\nabla} w_h) \boldsymbol{\nabla}\xi_h]_E \ ds,}_{= II_3}$$

where $\tilde{\omega}(\nabla v_h, \nabla w_h) := \omega(\nabla v_h)^{-1/4} - \omega(\nabla w_h)^{-1/4}$. In view of (3.5),(3.8b),(3.12), (3.16),(4.15), Hölder's inequality, the Cauchy-Schwarz inequality, the inverse inequality (4.12), and the trace inequality (4.13b) we can estimate II_1 from above as follows:

$$
\begin{aligned}
&|II_1| \leq 8 \sum_{E \in \mathcal{E}_h} \int_E \{|D^2 \xi_h|\}_E \{|\nabla \xi_h|\}_E \; ds \;\leq \\
&4 \sum_{E \in \mathcal{E}_h} \Big(\int_E \{|D^2 \xi_h|^2\}_E \; ds \Big)^{1/2} \Big(\int_E \{|\nabla \xi_h|^2\}_E \; ds \Big)^{1/2} \;\leq \\
&4 \sum_{K \in \mathcal{T}_h} \Big(\int_{\partial K} |D^2 \xi_h|^2 \; ds \Big)^{1/2} \Big(\int_{\partial K} |\nabla \xi_h|^2 \; ds \Big)^{1/2} \;\leq \\
&4 c_Q^{-1} h^{-1} \sum_{K \in \mathcal{T}_h} h_K^{1/2} \|D^2 \xi_h\|_{0,\partial K} h_K^{1/2} \|\nabla \xi_h\|_{0,\partial K} \leq \\
&4 c_Q^{-1} C_T^2 h^{-1} \Big(\sum_{K \in \mathcal{T}_h} \|D^2 \xi_h\|_{0,K}^2 \Big)^{1/2} \Big(\sum_{K \in \mathcal{T}_h} \|\nabla \xi_h\|_{0,K}^2 \Big)^{1/2} \;\leq \\
&4 c_Q^{-4} C_{inv}^2 C_T^2 h^{-4} \sum_{K \in \mathcal{T}_h} \|\xi_h\|_{0,K}^2 \;\leq \\
&4 c_Q^{-4} C_{inv}^2 C_T^2 h^{-4} \|\xi_h\|_{0,\Omega}^2 .
\end{aligned}
$$

Hence, we obtain

$$
|II_1| \leq C_B^{(2)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \tag{4.40}
$$

where $C_B^{(2)} := 4 c_Q^{-4} C_{inv}^2 C_T^2$. Likewise, for II_2 we have

$$
\begin{aligned}
&|II_2| \leq 4(\frac{5}{2} + \sqrt{5}) \sum_{E \in \mathcal{E}_h} \int_E \{|\nabla \xi_h|^2\}_E \{|D^2 w_h|\}_E \; ds \;\leq \\
&2(\frac{5}{2} + \sqrt{5}) \sum_{E \in \mathcal{E}_h} \Big(\int_E \{|\nabla \xi_h|^4\}_E \; ds \Big)^{1/2} \Big(\int_E \{|D^2 w_h|^2\}_E \; ds \Big)^{1/2} \;\leq
\end{aligned}
$$

$$2(\frac{5}{2}+\sqrt{5})\sum_{K\in\mathcal{T}_h}\Big(\int\limits_{\partial K}|\boldsymbol{\nabla}\xi_h|^4\ ds\Big)^{1/2}\Big(\int\limits_{\partial K}|D^2w_h|^2\ ds\Big)^{1/2} =$$

$$2(\frac{5}{2}+\sqrt{5})c_Q^{-1}h^{-1}\sum_{K\in\mathcal{T}_h}h_K^{1/2}\|\boldsymbol{\nabla}\xi_h\|_{0,4,\partial K}^2 h_K^{1/2}\|D^2w_h\|_{0,\partial K} \le$$

$$2(\frac{5}{2}+\sqrt{5})c_Q^{-1}C_T^2h^{-1}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,4,K}^4\Big)^{1/2}\Big(\sum_{K\in\mathcal{T}_h}\|D^2w_h\|_{0,K}^2\Big)^{1/2} \le$$

$$2(\frac{5}{2}+\sqrt{5})c_Q^{-1}c_S^{-1/2}C_{inv}C_T^2Rh^{-2}\Big(\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\nabla}\xi_h\|_{0,K}^4\Big)^{1/2} \le$$

$$2(\frac{5}{2}+\sqrt{5})c_Q^{-3}c_S^{-1/2}C_{inv}^3C_T^2Rh^{-4}\sum_{K\in\mathcal{T}_h}\|\xi_h\|_{0,K}^2.$$

It follows that

$$|II_2| \le C_B^{(3)}h^{-4}\|\xi_h\|_{0,\Omega}^2, \tag{4.41}$$

where $C_B^{(3)} := 2(\frac{5}{2}+\sqrt{5})c_Q^{-3}c_S^{-1/2}C_{inv}^3C_T^2R$. Finally, II_3 can be bounded from above in much the same way as II_2. We get

$$|II_3| \le C_B^{(4)}h^{-4}\|\xi_h\|^2, \tag{4.42}$$

where $C_B^{(4)} := 2c_Q^{-3}c_S^{-1/2}C_{inv}^3C_T^2R$.

(iii) For the third term on the right-hand side of (4.37) we have

$$\sum_{E\in\mathcal{E}_h}\int\limits_E\Big(\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(v_h)D^2\xi_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}v_h]_E\ -$$

$$\mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2\xi_h\}_E\ \mathbf{n}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}w_h)^{-1/4}\boldsymbol{\nabla}w_h]_E\Big)\ ds\ =$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{(\underline{\underline{\mathbf{A}}}_2(v_h)-\underline{\underline{\mathbf{A}}}_2(w_h))D^2\xi_h\}_E\mathbf{n}_E\;\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}v_h]_E ds}_{=\,III_1}\;+$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2\xi_h\}_E\mathbf{n}_E\;\mathbf{n}_E\cdot[\tilde{\omega}(\boldsymbol{\nabla}v_h,\boldsymbol{\nabla}w_h)\boldsymbol{\nabla}v_h]_E ds}_{=\,III_2}\;+$$

$$\underbrace{\sum_{E\in\mathcal{E}_h}\int_E \mathbf{n}_E\cdot\{\underline{\underline{\mathbf{A}}}_2(w_h)D^2\xi_h\}_E\mathbf{n}_E\;\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}v_h]_E ds}_{=\,III_3}.$$

The three terms can be estimated from above in a similar way as the corresponding terms in II. We obtain

(4.43)

$$|III_1|\le C_B^{(5)}h^{-4}\|\xi_h\|_{0,\Omega}^2,\quad |III_2|\le C_B^{(6)}h^{-4}\|\xi_h\|_{0,\Omega}^2,\quad |III_3|\le C_B^{(7)}h^{-4}\|\xi_h\|_{0,\Omega}^2,$$

where $C_B^{(5)}:=2(\frac{5}{2}+\sqrt{5})c_Q^{-3}c_S^{-1/2}C_{inv}^3C_T^2R$, $C_B^{(6)}:=2c_Q^{-3}c_S^{-1/2}C_{inv}^3C_T^2R$, and $C_B^{(7)}:=C_B^{(6)}$.

(iv) For the fourth term on the right-hand side of (4.37) we obtain

$$\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E\Big(\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}v_h]_E\;\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\;- \tag{4.44}$$
$$\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}w_h)^{-1/4}\boldsymbol{\nabla}w_h]_E\;\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}w_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\Big)\,ds\;=$$

$$\underbrace{\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\;\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}v_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\;ds}_{=\,IV_1}\;+$$

$$\underbrace{\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}w_h)^{-1/4}\boldsymbol{\nabla}w_h]_E\;\mathbf{n}_E\cdot[\tilde{\omega}(\boldsymbol{\nabla}v_h,\boldsymbol{\nabla}w_h)\boldsymbol{\nabla}\xi_h]_E\;ds}_{=\,IV_2}\;+$$

$$\underbrace{\alpha\sum_{E\in\mathcal{E}_h}h_E^{-1}\int_E \mathbf{n}_E\cdot[\tilde{\omega}(\boldsymbol{\nabla}v_h,\boldsymbol{\nabla}w_h)\boldsymbol{\nabla}w_h]_E\;\mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla}w_h)^{-1/4}\boldsymbol{\nabla}\xi_h]_E\;ds}_{=\,IV_3}.$$

In view of (3.13), the first term IV_1 can be further split according to

$$IV_1 = \underbrace{\alpha \sum_{E\in\mathcal{E}_h} h_E^{-1} \int\limits_E \mathbf{n}_E\cdot\{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E[\boldsymbol{\nabla}\xi_h]_E\ \mathbf{n}_E\cdot\{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E[\boldsymbol{\nabla}\xi_h]_E\ ds}_{=\ IV_{11}} +$$

$$\underbrace{\alpha \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1} \int\limits_E \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E\{\boldsymbol{\nabla}\xi_h\}_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E\{\boldsymbol{\nabla}\xi_h\}_E\ ds}_{=\ IV_{12}}\ +$$

$$\underbrace{\alpha \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1} \int\limits_E \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E\{\boldsymbol{\nabla}\xi_h\}_E\ \mathbf{n}_E\cdot\{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E[\boldsymbol{\nabla}\xi_h]_E\ ds}_{=\ IV_{13}}\ +$$

$$\underbrace{\alpha \sum_{E\in\mathcal{E}_h(\Omega)} h_E^{-1} \int\limits_E \mathbf{n}_E\cdot\{\omega(\boldsymbol{\nabla} v_h)^{-1/4}\}_E[\boldsymbol{\nabla}\xi_h]_E\ \mathbf{n}_E\cdot[\omega(\boldsymbol{\nabla} v_h)^{-1/4}]_E\{\boldsymbol{\nabla}\xi_h\}_E\ ds}_{=\ IV_{14}}.$$

For IV_{11}, setting $E_1 := E_+$ and $E_2 := E_-$ for $E\in\mathcal{E}_h(\Omega)$, we have

$$IV_{11} \ge \alpha \sum_{E\in\mathcal{E}_h(\Omega)} (1+\frac{1}{2}\sum_{i=1}^{2}\|\boldsymbol{\nabla} v_h\|_{0,\infty,E_i}^2)^{-1/2} h_E^{-1}\int\limits_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds\ +$$

$$\alpha \sum_{E\in\mathcal{E}_h(\Gamma)} (1+\|\boldsymbol{\nabla} v_h\|_{0,\infty,E}^2)^{-1/2} h_E^{-1}\int\limits_E |\mathbf{n}_E\cdot[\boldsymbol{\nabla}\xi_h]_E|^2 ds.$$

Taking advantage of (3.8b),(3.8c), the inverse inequality (4.12), and the Poincaré-Friedrichs inequality for piecewise H^2-functions (4.14), it follows that for $E\in\mathcal{E}_h(\partial K)$ it holds

$$\|\boldsymbol{\nabla} v_h\|_{0,\infty,E} \le \|\boldsymbol{\nabla} v_h\|_{0,\infty,K} \le c_S^{-1/2} C_{inv} h_K^{-1}\|\boldsymbol{\nabla} v_h\|_{0,K}\ \le$$

$$c_Q^{-1}c_S^{-1/2}C_{inv}h^{-1}\|\boldsymbol{\nabla} v_h\|_{0,\Omega} \le c_Q^{-1}c_S^{-1/2}C_{inv}C_{PF}h^{-1}\|v_h\|_{2,h,\Omega} \le c_Q^{-1}c_S^{-1/2}C_{inv}C_{PF}h^{-1}R,$$

and hence, observing $h<1$, we get

$$(1+\|\boldsymbol{\nabla} v_h\|_{0,\infty,E}^2)^{-1/2} \ge (1+c_Q^{-2}c_S^{-1}C_{inv}^2C_{PF}^2R^2h^{-2})^{-1/2}\ =$$

$$(h^2+c_Q^{-2}c_S^{-1}C_{inv}^2C_{PF}^2R^2)^{-1/2}h \ge (1+c_Q^{-2}c_S^{-1}C_{inv}^2C_{PF}^2R^2)^{-1/2}h.$$

Consequently, we obtain

$$IV_{11} \geq \alpha\gamma_M^{(2)} h \sum_{E\in\mathcal{E}_h} h_E^{-1} \int\limits_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla}\xi_h]_E|^2 ds, \tag{4.45}$$

where $\gamma_M^{(2)} := \alpha(1 + c_Q^{-2} c_S^{-1} C_{inv}^2 C_{PF}^2 R^2)^{-1/2}$.

The remaining terms $IV_{1i}, 2 \leq i \leq 4$, can be estimated from above similarly as the corresponding terms in Theorem 4.2:

(4.46)

$$|IV_{12}| \leq C_B^{(8)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad |IV_{13}| \leq C_B^{(9)} h^{-4} \|\xi_h\|_{0,\Omega}^2, \quad |IV_{14}| \leq C_B^{(10)} h^{-4} \|\xi_h\|_{0,\Omega}^2,$$

where $C_B^{(8)} := 2\alpha c_Q^{-4} c_R^{-1} C_{inv}^2 C_T^2$ and $C_B^{(9)} = C_B^{(10)} := 2C_B^{(8)}$. The remaining two terms IV_2 and IV_3 can be estimated from above in the same way. Using (3.8a),(3.8b),(4.19a), the inverse inequality (4.12), the trace inequality (4.13a), the Cauchy-Schwarz inequality, and observing

$$\Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int\limits_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla} w_h]_E|^2 \; ds \Big)^{1/2} \leq \|w_h\|_{2,h,\Omega} \leq R,$$

we obtain

$$|IV_2| \leq 4\alpha c_Q^{-1/2} c_R^{-1/2} h^{-1/2} \sum_{E\in\mathcal{E}_h} h_E^{-1/2} \int\limits_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla} w_h]_E| \{|\boldsymbol{\nabla}\xi_h|\}_E^2 \; ds \; \leq$$

$$4\alpha c_Q^{-1/2} c_R^{-1/2} h^{-1/2} \sum_{E\in\mathcal{E}_h} h_E^{-1/2} \Big(\int\limits_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla} w_h]_E|^2 \; ds \Big)^{1/2} \Big(\int\limits_E \{|\boldsymbol{\nabla}\xi_h|\}_E^4 \; ds \Big)^{1/2} \; \leq$$

$$2\alpha c_Q^{-3/2} c_R^{-1/2} h^{-3/2} \Big(\sum_{E\in\mathcal{E}_h} h_E^{-1} \int\limits_E |\mathbf{n}_E \cdot [\boldsymbol{\nabla} w_h]_E|^2 \; ds \Big)^{1/2} \Big(\sum_{K\in\mathcal{T}_h} h_K^2 \|\boldsymbol{\nabla}\xi_h\|_{0,4,\partial K}^4 \Big)^{1/2}$$

$$\leq 2\alpha c_Q^{-3/2} c_R^{-1/2} C_T R h^{-3/2} \Big(\sum_{K\in\mathcal{T}_h} \|\boldsymbol{\nabla}\xi_h\|_{0,4,K}^4 \Big)^{1/2} \; \leq$$

$$2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R h^{-7/2} \Big(\sum_{K\in\mathcal{T}_h} \|\xi_h\|_{0,K}^4 \Big)^{1/2} \; \leq$$

$$2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R h^{-7/2} \sum_{K\in\mathcal{T}_h} \|\xi_h\|_{0,K}^2.$$

Hence, it follows that

$$|IV_2| \le C_B^{(11)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \tag{4.47}$$

where $C_B^{(11)} := 2\alpha c_Q^{-7/2} c_R^{-1/2} C_{inv}^2 C_T R$. Moreover, we get

$$|IV_3| \le C_B^{(12)} h^{-7/2} \|\xi_h\|_{0,\Omega}^2, \tag{4.48}$$

where $C_B^{(12)} := C_B^{(11)}$.

Setting $C_B := \sum_{i=1}^{12} C_B^{(i)}$ and observing (4.34) as well as $h < 1$, it follows from (4.36)-(4.48) that

$$\begin{aligned} &\langle A_H^{DG} v_h - A_h^{DG} w_h, v_h - w_h \rangle_{V_h^*, V_h} \ \ge \\ &(1 - C_\Delta C_B h^\kappa)\ \|\xi_h\|_{0,\Omega}^2 + \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h^3\ |\xi_h|_{2,h,\Omega}^2. \end{aligned} \tag{4.49}$$

We choose $h_{min} > 0$ such that

$$q := C_\Delta C_B h_{min}^\kappa < 1 \quad \text{and} \quad \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h_{min}^3 < 1 - q. \tag{4.50}$$

Then, for $h \le h_{min}$ (4.35) follows from (4.49),(4.50) with

$$\gamma(h,R) := \min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)}) h^3. \tag{4.51}$$

□

Corollary 4.1. *Assume that u_h^{m-1} satisfies*

$$\|u_h^{m-1}\|_{0,\Omega} \le \frac{\Gamma(R)^2}{\gamma(R)} \Big(1 - \sqrt{1 - \frac{\gamma(R)^2}{\Gamma(R)^2}}\Big) R$$

for some $R > 0$ and that (4.34) holds true. Then, for sufficiently small grid size h, the C^0IPDG approximation (3.14) has a unique solution $u_h^m \in B_h(0,R)$.

Proof. Using the Lipschitz continuity (4.22) and the strong monotonicity (4.35) of the nonlinear operator A_h^{DG}, the result follows from the nonlinear analogue of the Lax-Milgram Lemma (Theorem 4.1). □

Remark 4.1. *If we choose $h_{min} > 0$ such that (4.49) is satisfied as well as $h_{min} < \beta C_\Delta C_A$, for $h \le h_{min}$ we have $\Gamma(h,R) = \beta C_\Delta C_A h^{-1}$ in Theorem 4.2 and the application of Theorem 4.1 for $V = V_h$ and $A = A_h^{DG}$ implies that the fixed point operator T is a contraction as long as*

$$\rho < 2\ \frac{\gamma(h,R)}{\Gamma(h,R)^2} = 2\ \frac{min(\gamma_M^{(1)}, \alpha\gamma_M^{(2)})}{C_\Delta^2 C_A^2}\ h^5. \tag{4.52}$$

In other words, the contraction property degenerates for $h \to 0$. This reflects the very singular character of the fourth order total variation flow.

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