

ON EXTENSION PROBLEM, TRACE HARDY AND HARDY'S INEQUALITIES FOR SOME FRACTIONAL LAPLACIANS

PRADEEP BOGGARAPU

Department of Mathematics
BITS Pilani K K Birla Goa Campus
Zuarinagar, South Goa 403 726, Goa, India

LUZ RONCAL*

BCAM - Basque Center for Applied Mathematics 48009 Bilbao, Spain
and Ikerbasque, Basque Foundation for Science, 48011 Bilbao, Spain

SUNDARAM THANGAVELU

Department of Mathematics
Indian Institute of Science
560 012 Bangalore, India

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ABSTRACT. We obtain generalised trace Hardy inequalities for fractional powers of general operators given by sums of squares of vector fields. Such inequalities are derived by means of particular solutions of an extended equation associated to the above-mentioned operators. As a consequence, Hardy inequalities are also deduced. Particular cases include Laplacians on stratified groups, Euclidean motion groups and special Hermite operators. Fairly explicit expressions for the constants are provided. Moreover, we show several characterisations of the solutions of the extension problems associated to operators with discrete spectrum, namely Laplacians on compact Lie groups, Hermite and special Hermite operators.

1. Introduction and main results. The aim of this paper is two-fold. On the one hand, we prove trace Hardy and Hardy's inequalities for fractional powers of operators that are given as sums of squares of vector fields on a Lie group. In some particular cases, the results will be more precise, showing explicit constants. On the other hand, we obtain several characterisations of solutions of the extension problem when the operator involved has discrete spectrum.

Hardy's inequality for fractional powers of the Laplacian Δ on \mathbb{R}^n has been investigated by many authors, see [21, 40, 16, 4, 5] and there is a vast literature on this topic. There are several ways of proving such a Hardy's inequality, which reads as

$$((-\Delta)^{s/2} f, f) \geq 2^s \frac{\Gamma(\frac{n+s}{4})^2}{\Gamma(\frac{n-s}{4})^2} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^s} dx, \quad (1.1)$$

valid for $0 < s < 2$ and suitable functions. Though the constant is known to be sharp, it is never achieved. On the other hand, there is another version of Hardy's inequality where the homogeneous weight function $|x|^{-s}$ is replaced by a non-homogeneous one:

$$((-\Delta)^{s/2} f, f) \geq 2^s \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n-s}{2})} \delta^s \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(\delta^2 + |x|^2)^s} dx, \quad \delta > 0, \quad (1.2)$$

where again the constant is sharp and equality is achieved for the functions $(\delta^2 + |x|^2)^{-(n-s)/2}$ and their translates. Though (1.2) is known to the experts in the field, we are not able to locate a reference where it is actually proved (a proof will be given in Remark 2.6).

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* Corresponding author: Luz Roncal.

Ramifications of Hardy's inequality (1.1) have been investigated in the literature and in [38] Tzirakis has improved (1.1) with sharp error terms. The main idea used in [38] is the use of trace Hardy inequality. For suitable functions $u(x, \rho)$ on $\mathbb{R}_+^{n+1} := \mathbb{R}^n \times \mathbb{R}_+$, where $\mathbb{R}_+ := [0, \infty)$, the trace Hardy inequality states that

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2 \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \left(\frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)} \right)^2 \int_{\mathbb{R}^n} \frac{|u(x, 0)|^2}{|x|^s} dx. \quad (1.3)$$

On the other hand, inequalities of the following form are also of interest:

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2 \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \frac{\Gamma((n+s)/2)}{\Gamma((n-s)/2)} \delta^s \int_{\mathbb{R}^n} \frac{|u(x, 0)|^2}{(\delta^2 + |x|^2)^s} dx, \quad \delta > 0. \quad (1.4)$$

Such inequalities are known as trace Hardy inequalities with non-homogeneous weights. As we see later, trace Hardy inequalities lead to Hardy inequalities and such inequalities with non-homogeneous weights lead to uncertainty principles for fractional powers (see e.g. Garofalo-Lanconelli [18] and Roncal-Thangavelu [27]). In this paper we will see that a more general version of the inequalities (1.3) and (1.4) can be proved by means of the use of solutions of the initial value problem

$$(\Delta + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) v(x, \rho) = 0, \quad x \in \mathbb{R}^n, \quad \rho > 0; \quad v(x, 0) = \varphi(x), \quad x \in \mathbb{R}^n. \quad (1.5)$$

In the above initial value problem we take $\varphi \in L^2(\mathbb{R}^n)$ and the limit is taken in the L^2 sense. Let us explain more. The problem (1.5) is known as the extension problem for the Laplacian in the literature and has been studied e.g. by Caffarelli and Silvestre [7] (see also, for instance, [25, 9]). One of the most interesting facts shown in [7] about solutions of the above problem is that when both φ and $(-\Delta)^{s/2} \varphi$ are in $L^2(\mathbb{R}^n)$,

$$-\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} (-\Delta)^{s/2} \varphi(x),$$

for $0 < s < 2$. Here $(-\Delta)^{s/2}$ is the fractional power of the Laplacian and the limit is understood in the L^2 sense. Indeed, it can be seen that, by taking into account the solutions of (1.5), we can obtain the generalised trace Hardy inequality

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^n} |u(x, 0)|^2 \frac{(-\Delta)^{s/2} \varphi(x)}{\varphi(x)} dx. \quad (1.6)$$

One of our main goals in this paper is to prove an analogue of the trace Hardy inequality (1.6) for very general operators of the form

$$L = \sum_{j=1}^m X_j^2 \quad (1.7)$$

and deduce a version of Hardy's inequality for fractional powers of L . Here we assume that $X_j, j = 1, 2, \dots, m$ are left invariant vector fields on a Lie group G which satisfies Hörmander's condition, that is, the vector fields are smooth and their commutators up to certain order span the tangent space at each point. Then it is known (see [23]) that the second order differential operator L is essentially selfadjoint and hypoelliptic i.e., whenever $Lf = g$ and $g \in C^\infty$, then f is in C^∞ . Some examples for these operators are the Euclidean Laplacian, the sublaplacian on the Heisenberg group (or more generally the sublaplacian on stratified Lie groups), Laplacian on motion groups, or Laplacian on compact Lie groups. In order to prove a trace Hardy inequality for L we need to find solutions of the extension problem

$$(L + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) v(x, \rho) = 0, \quad x \in G, \quad \rho > 0; \quad v(x, 0) = f(x), \quad x \in G. \quad (1.8)$$

The extension problem for general second order partial differential operators has been extensively studied by Stinga-Torrea [31], Galé et al [17] and Banica et al [3]. Once we have a solution of the above extension problem we will be able to obtain an analogue of the inequality (1.6), see Theorem 1.1. Let us introduce the gradient

$$\nabla = \left(X_1, \dots, X_m, \frac{\partial}{\partial \rho} \right)$$

on $G \times \mathbb{R}_+$. We define $|\nabla u(x, \rho)|^2 = |X_1 u(x, \rho)|^2 + \dots + |X_m u(x, \rho)|^2 + |\frac{\partial}{\partial \rho} u(x, \rho)|^2$. For $0 < s < 2$, let $W_0^s(G \times \mathbb{R}_+)$ be the completion of $C_0^1([0, \infty), C_0^1(G))$ with respect to the norm

$$\|u\|_{(s)}^2 = \int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho.$$

In the above $C_0^1([0, \infty), C_0^1(G))$ stands for the space of all compactly supported C^1 functions on $[0, \infty)$ taking values in $C_0^1(G)$. The following theorem is our first main result: a trace Hardy inequality concerning operators of the form (1.7) given by sums of Hörmander vector fields on general Lie groups G .

Theorem 1.1 (General trace Hardy inequality). *Let $0 < s < 2$ and let φ be a real valued function in the domain of $(-L)^{s/2}$. Assume also that $\varphi^{-1}(-L)^{s/2}\varphi$ is locally integrable. Then for any real valued function $u \in W_0^s(G \times \mathbb{R}_+)$, we have the inequality*

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \frac{(-L)^{s/2}\varphi(x)}{\varphi(x)} dx.$$

It is enough to prove the inequality in Theorem 1.1 for functions from $C_0^1([0, \infty), C_0^1(G))$. Then standard density arguments guarantee the validity for $u \in W_0^s(G \times \mathbb{R}_+)$. Equality is attained when u is a solution of the extension problem, under an extra assumption on the heat kernel associated to the operator L , see Proposition 2.5. We will consider several particular cases (Laplacians on stratified groups, Euclidean motion groups and special Hermite operators) in which this extra assumption is easily seen to be satisfied.

In order to prove Theorem 1.1 we will strongly use the solutions of the extension problem (1.8) and the characterisation of the fractional powers of the operators L by means of this extension problem due to Stinga and Torrea, see [31]. This technique was recently developed in the Euclidean setting (by using the characterisation for the Euclidean fractional Laplacian in [7]) to get improved trace Hardy inequalities on bounded domains [38], or domains satisfying suitable geometric assumptions [12, 13], see also [26].

From Theorem 1.1, we can prove the following Hardy type inequality for $(-L)^{s/2}$.

Corollary 1.2. *Let $0 < s < 2$. Let $f \in L^2(G)$ be such that $(-L)^{s/2}f \in L^2(G)$. If u is a solution of the extension problem (1.8) with initial condition f such that $u \in W_0^s(G \times \mathbb{R}_+)$ then*

$$((-L)^{s/2}f, f) \geq \int_G f^2(x) \frac{(-L)^{s/2}\varphi(x)}{\varphi(x)} dx,$$

for any real valued function φ in the domain of $(-L)^{s/2}$ such that the right hand side is finite.

We will study several particular settings throughout the paper to get more precise inequalities. In order to obtain these inequalities, we will start from the general forms obtained in Theorem 1.1 and Corollary 1.2. Then, the ideas to follow will be different from those used in [12, 13, 38, 26]. Instead, we will make use of the solutions of the corresponding extensions problems.

First, we will consider the case of the sublaplacian L on stratified groups (see for instance [15, 6] for definitions and facts on stratified Lie groups). In this case, good Gaussian bounds for the associated heat kernel q_t are known (see e.g. Theorem 3.4 in [10]) and we can obtain more explicit results. Indeed, by means of Theorem 1.1, we can prove an analogue of (1.3), i.e., a trace Hardy inequality with a homogeneous weight function for L . Let Q stand for the homogeneous dimension of the stratified group G . Using properties of the kernels \mathcal{R}_α of the Riesz potentials $(-L)^{-\alpha/2}$ (see Subsection 2.4) we will prove the following result.

Theorem 1.3 (Trace Hardy inequality for stratified groups). *There exists a positive weight function $w_s(x)$ which is homogenous of degree s such that*

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2 \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \frac{\Gamma((Q+s)/4)^2}{\Gamma((Q-s)/4)^2} \int_G \frac{u^2(x, 0)}{w_s(x)} dx \quad (1.9)$$

is valid for real valued functions $u \in W_0^s(G \times \mathbb{R})$ and $0 < s < 2$.

As a corollary of Theorem 1.3, it can be proved (after an application of Corollary 1.2) that, when u satisfies the extension problem with initial condition f , the left hand side of (1.9) reduces to $((-L)^{s/2}f, f)$, leading to the following Hardy inequality.

Corollary 1.4. *Let $0 < s < 2$. Let $f \in L^2(G)$ be such that $(-L)^{s/2}f \in L^2(G)$. Then, there exists a positive weight function $w_s(x)$ which is homogenous of degree s such that*

$$((-L)^{s/2}f, f) \geq 2^s \left(\frac{\Gamma((Q+s)/4)}{\Gamma((Q-s)/4)} \right)^2 \int_G \frac{f(x)^2}{w_s(x)} dx.$$

Though the function w_s has the right homogeneity we do not have any way of computing it explicitly. However, it is possible to estimate w_s in terms of a fixed homogeneous norm from above and below. It is also possible to write w_s as a ratio of two different homogeneous norms.

We point out that in [8] the authors already obtained Hardy-type inequalities on stratified Lie groups. The results are slightly different and they estimate the constants arising.

We will also consider trace Hardy inequalities associated to the Euclidean motion group $M(n)$, which is the semidirect product of \mathbb{R}^n with the special orthogonal group $SO(n)$ with its natural action on \mathbb{R}^n . In the case of $M(2)$ our results are more explicit. Identifying \mathbb{R}^2 with \mathbb{C} and $SO(2)$ with the circle group S^1 we write the coordinates on $M(2)$ as $(z, e^{i\theta})$ and let $\Delta_G = \Delta + \partial_\theta^2$, where Δ is the Laplacian on \mathbb{R}^2 , and define $\nabla = (\partial_x, \partial_y, \partial_\theta)$, with $z = x + iy$.

The trace Hardy inequality for the motion group $M(2)$ can now be stated in the following form. It is also possible to prove a similar inequality for $M(n)$.

Theorem 1.5. *Let $0 < s < 2$. Let $u \in W_0^s(M(2) \times \mathbb{R}_+)$ be a real valued function on $M(2) \times \mathbb{R}^+$. Fix $\delta > 0$. Then*

$$\begin{aligned} \int_0^\infty \int_{\mathbb{C}} \int_0^{2\pi} |\nabla u(z, e^{i\theta}, \rho)|^2 \rho^{1-s} d\theta dz d\rho \\ \geq \delta^s 2 \cdot 4^s \frac{\Gamma(\frac{3+s}{2}) \Gamma(1-s/2)}{\Gamma(\frac{3-s}{2}) \Gamma(s/2)} \int_{\mathbb{C}} \int_{-\infty}^\infty \frac{u^2(z, e^{i\theta}, 0)}{(\delta^2 + |z|^2 + |\theta|^2)^{(3+s)/2}} w_s(z, \theta) d\theta dz \end{aligned}$$

where $w_s(z, \theta)^{-1} = \sum_{k \in \mathbb{Z}} (\delta^2 + |z|^2 + (\theta + 2k\pi)^2)^{-\frac{(3-s)}{2}}$.

Another operator that we will consider is the special Hermite operator L_1 . We will exploit the relation between L_1 and the sublaplacian \mathcal{L}_{red} on the reduced Heisenberg group \mathbb{H}_{red}^n , namely, the quotient group \mathbb{H}^n/Γ , where \mathbb{H}^n is the Heisenberg group and the central subgroup Γ is given by $\Gamma = \{(0, 2k\pi) : k \in \mathbb{Z}\}$ (see [34, Chapter 4]). The reduced Heisenberg group can be identified with $\mathbb{C}^n \times \mathbb{T}$, where \mathbb{T} is the torus. It happens that when considering functions independent of $\tau \in \mathbb{T}$, the sublaplacian in \mathbb{H}_{red}^n coincides with the Euclidean Laplacian on \mathbb{R}^{2n} . Then, we can obtain a trace Hardy inequality in \mathbb{H}_{red}^n , and also Hardy's inequality for the special Hermite operator. The latter reads as follows.

Theorem 1.6. *Let $0 < s < 2$. Let f be a real valued function on \mathbb{C}^n such that f and $L_1^{s/2}f$ are both in $L^2(\mathbb{C}^n)$. Then we have*

$$\operatorname{Re}(L_1^{s/2}f, f) \geq 2^s \frac{\Gamma(\frac{2n+s}{4})^2}{\Gamma(\frac{2n-s}{4})^2} \int_{\mathbb{C}^n} \frac{f(z)^2}{|z|^s} dz.$$

In [1] the authors proved a sharp Hardy–Sobolev inequality for the twisted Laplacian on \mathbb{C}^n using the fundamental solutions, see Corollary 1.2 therein. This technique was already used in [2] to obtain Hardy–Sobolev inequalities for the Laplace–Beltrami operator on certain manifolds and for the sublaplacian on the Heisenberg group. Observe that the weight function involved in [2, Corollary 1.2] is different from the one obtained in Theorem 1.6.

Remark 1.7. We do not state the optimality of the constants obtained in Theorems 1.3, 1.6 and Corollary 1.4, since we do not know about that. Anyway, intuitively, the constants in Theorem 1.3 and Corollary 1.4 seem to be the right ones, because they match with the optimal constants in the Euclidean setting. Moreover, in the statement we are taking general homogeneous weights, and new constants. However, the constant in Theorem 1.5 is sharp. Actually, for general inequalities of the form

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq s \delta^{-s} \int_G u^2(x, 0) \frac{u_{s,\delta}(x)}{u_{-s,\delta}(x)} dx,$$

where $u_{s,\delta}$ are defined in (2.2), it can be easily checked that the function $u_{-s,\delta}$ optimize the above inequality.

We emphasize that, in the results above, a precise knowledge on the solution of the extension problem is crucial. We will take the chance to further study several characterisations of solutions of the extension problem (1.8) in the case of compact Lie groups, Hermite operator and special Hermite operator. This will be the second important part of our paper. Characterisations of L^p solutions of the extension problem were already carried out in [31]. We try to go beyond and, in the case of compact Lie groups and the Hermite operator, we find all the solutions when the initial data is a tempered distribution and, in the case of special Hermite operator, when the initial data belongs to a suitable Sobolev space. Moreover, we will characterise all the solutions of the extension problem which are tempered distributions. Our characterisations concern operators with discrete spectrum. Characterisations of solutions of the extension problem when the operator involved has continuous spectrum are more involved and they will appear elsewhere. We do not state the results on characterisations here, for the sake of easy reading. Instead, we defer all the related set-up, results and the proofs to Section 3.

The structure of the paper is as follows. Trace Hardy and Hardy's inequalities, and the proofs of the theorems stated in the introduction are given in Section 2. In Section 3 we deal with the characterisations of all solutions of the extension problem in the case in which the associated operators have discrete spectrum.

2. An extension problem, trace Hardy and Hardy's inequalities for fractional Laplacians. In this section we prove the results stated in the introduction related to trace Hardy and Hardy's inequalities.

2.1. A basic lemma. The proof of the trace Hardy inequality in Theorem 1.1 depends on the following result, which is well known to experts in Partial Differential Equations. We include it here for the convenience of the readers. We introduce the gradient

$$\nabla = \left(X_1, \dots, X_m, \frac{\partial}{\partial \rho} \right)$$

on $G \times \mathbb{R}_+$. Recall that we are dealing with left invariant vector fields satisfying Hörmander's condition, and they are skew-symmetric:

$$\int_G X_j f(x) g(x) dx = - \int_G f(x) X_j g(x) dx \quad (2.1)$$

where dx stands for the left Haar measure on G . Denote by L_s the operator

$$L_s := L + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho.$$

With these notations we have the following lemma. It is initially stated for C_0^∞ functions, but it remains valid for functions u coming from a suitable Sobolev space.

Lemma 2.1. *Let $0 < s < 2$. Let $u(x, \rho)$ be a real valued function from $C_0^1([0, \infty), C_0^1(G))$ and let $v(x, \rho)$ be another real valued function for which $\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho)$ exists and $\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) v^{-1}(x, 0) \in L_{\text{loc}}^1(G)$. We have*

$$\begin{aligned} \int_0^\infty \int_G \left| \nabla u(x, \rho) - \frac{u(x, \rho)}{v(x, \rho)} \nabla v(x, \rho) \right|^2 \rho^{1-s} dx d\rho &= \int_0^\infty \int_G \left| \nabla u(x, \rho) \right|^2 \rho^{1-s} dx d\rho \\ &+ \int_0^\infty \int_G \frac{u(x, \rho)^2}{v(x, \rho)} (L_s v(x, \rho)) \rho^{1-s} dx d\rho + \int_G \frac{u(x, 0)^2}{v(x, 0)} \lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) dx. \end{aligned}$$

Proof. Consider the following integral:

$$\int_G \left(X_j u - \frac{u}{v} X_j v \right)^2 dx = \int_G \left((X_j u)^2 - 2 \frac{u}{v} X_j u X_j v + \left(\frac{u}{v} X_j v \right)^2 \right) dx.$$

Integrating by parts and using (2.1), we get

$$\begin{aligned} \int_G \frac{u}{v} X_j u X_j v dx &= - \int_G u X_j \left(\frac{u}{v} X_j v \right) dx \\ &= - \int_G \frac{u}{v} X_j u X_j v dx - \int_G u^2 X_j \left(\frac{1}{v} X_j v \right) dx. \end{aligned}$$

Since $\int_G u^2 X_j (\frac{1}{v} X_j v) dx = -\int_G \frac{u^2}{v^2} (X_j v)^2 dx + \int_G \frac{u^2}{v} X_j^2 v dx$, the above gives

$$\int_G \left(\frac{u^2}{v^2} (X_j v)^2 - 2 \frac{u}{v} X_j u X_j v \right) dx = \int_G \frac{u^2}{v} X_j^2 v dx.$$

On the other hand, a similar calculation with the ρ -derivative gives

$$\begin{aligned} \int_0^\infty \left(\frac{u^2}{v^2} (\partial_\rho v)^2 - 2 \frac{u}{v} \partial_\rho u \partial_\rho v \right) \rho^{1-s} d\rho \\ = \int_0^\infty \frac{u^2}{v} \partial_\rho (\rho^{1-s} \partial_\rho v) d\rho + \frac{u(x,0)^2}{v(x,0)} \lim_{\rho \rightarrow 0} (\rho^{1-s} \partial_\rho v)(x, \rho). \end{aligned}$$

Adding and then taking all integrations into account we get our result, in view of (2.1). \square

Remark 2.2. The assumption in Lemma 2.1 that u is from $C_0^1([0, \infty), C_0^1(G))$, i.e. it is a compactly supported C^1 function on $[0, \infty)$ taking values in $C_0^1(G)$ allows to guarantee that integrations by parts are justified leaving no boundary terms.

In Lemma 2.1, if v satisfies the extension problem (1.8), i.e., the equation $L_s v = 0$ on $G \times \mathbb{R}_+$ with a given initial condition $v(x, 0) = \varphi(x)$, then we get the inequality

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq - \int_G \frac{u^2(x, 0)}{v(x, 0)} \lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho) dx.$$

In view of the above, we need to solve the extension problem for L with a given initial condition φ . We also need to compute $\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho v(x, \rho)$ in terms of L and φ .

2.2. An extension problem for sum of squares of vector fields. We will look at the extension problem (1.8) more carefully. A good reference for this subsection is the article by Stinga-Torrea [31].

Let X_j , $j = 1, 2, \dots, m$, be left invariant vector fields on a Lie group G satisfying the Hörmander's condition. Let q_t be the heat kernel associated to the Laplacian $L = \sum_{j=1}^m X_j^2$. We assume that $\int_G q_t(x) dx = 1$. For $-2 < s < 2$ we define

$$u_{s, \rho}(x) = \frac{\rho^s}{2^s |\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} q_t(x) t^{-s/2-1} dt. \quad (2.2)$$

For $s > 0$ the integral defines an L^1 function and $\int_G u_{s, \rho}(x) dx = 1$. But for $-2 < s < 0$ the function need not be integrable. However, if we assume that $\|q_t\|_2 \leq C t^{-\gamma}$, $\gamma > 1$, the integral defining $u_{s, \rho}$ converges even for $-2 < s < 0$ and defines an L^2 function. Indeed,

$$\|u_{s, \rho}\|_2 \leq C_s \rho^s \int_0^\infty e^{-\frac{\rho^2}{4t}} \|q_t\|_2 t^{-s/2-1} dt \leq C_s \rho^{-2\gamma}. \quad (2.3)$$

In Theorem 2.4 below a formula for the solution to the extension problem is given, see [31, Theorem 1.1] (see also [17] and [11]). We sketch a proof of the theorem for the sake of completeness, and because some parts of the proof will be referred later. First of all, we have the following lemma.

Lemma 2.3. *For $-2 < s < 2$, the functions $u_{s, \rho}$ and $u_{-s, \rho}$ are related via the equation*

$$\rho^s (-L)^{s/2} u_{-s, \rho} = \frac{2^s |\Gamma(s/2)|}{|\Gamma(-s/2)|} u_{s, \rho}. \quad (2.4)$$

Proof. This can be proved, as in [31], by considering $((-L)^{s/2} u_{-s, \rho}, g)$ and using the spectral definition of $(-L)^{s/2}$. In fact, the proof depends on the numerical identity

$$\frac{\rho^s}{2^s} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{-t\lambda} t^{-s/2-1} dt = \lambda^{s/2} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{-t\lambda} t^{s/2-1} dt \quad (2.5)$$

valid for $\lambda > 0$ (which is true by a simple change of variables). \square

Theorem 2.4 (Stinga-Torrea). *For $f \in L^p(G)$, $1 \leq p \leq \infty$, the function $u(x, \rho) = f * u_{s, \rho}(x)$ solves the extension problem (1.8). Moreover, for $0 < s < 2$,*

$$\lim_{\rho \rightarrow 0} \rho^{1-s} \partial_\rho (f * u_{s, \rho}) = -2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} (-L)^{s/2} f$$

where the convergence is understood in the L^p sense, under the extra assumption that $(-L)^{s/2}f \in L^p(G)$, $1 \leq p < \infty$.

Proof. We first show that $u_{s,\rho}$ solves the extension problem:

$$(L + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)u_{s,\rho} = 0.$$

To see this, note that $u_{s,\rho} = \rho^s v_{s,\rho}$, with $v_{s,\rho} = \frac{1}{2^s|\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} q_t(x) t^{-s/2-1} dt$. Hence by a simple calculation we get

$$(\partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)u_{s,\rho} = (1+s)\rho^{s-1}\partial_\rho v_{s,\rho} + \rho^s \partial_\rho^2 v_{s,\rho}. \quad (2.6)$$

On the other hand, since $Lq_t(x) = \partial_t q_t(x)$ we see that

$$Lu_{s,\rho}(x) = \frac{\rho^s}{2^s|\Gamma(s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} \partial_t q_t(x) t^{-s/2-1} dt.$$

Writing

$$e^{-\frac{\rho^2}{4t}} t^{-s/2-1} = (4\pi t)^{-1/2} e^{-\frac{\rho^2}{4t}} (4\pi t)^{1/2} t^{-s/2-1}$$

integrating by parts, using the fact that

$$\partial_t((4\pi t)^{-1/2} e^{-\frac{\rho^2}{4t}}) = \partial_\rho^2((4\pi t)^{-1/2} e^{-\frac{\rho^2}{4t}})$$

and simplifying we get

$$Lu_{s,\rho} = -(1+s)\rho^{s-1}\partial_\rho v_{s,\rho} - \rho^s \partial_\rho^2 v_{s,\rho}. \quad (2.7)$$

Combining (2.6) and (2.7) we finish the proof of the first claim.

On the other hand, by a simple change of variables, we have

$$u_{s,\rho}(x) = \frac{1}{2^s|\Gamma(s/2)|} \int_0^\infty e^{-\frac{1}{4t}} q_{\rho^2 t}(x) t^{-s/2-1} dt.$$

As $f * q_t$ converges to f in L^p as t tends to 0 it follows that

$$f * u_{s,\rho}(x) = \frac{1}{2^s|\Gamma(s/2)|} \int_0^\infty e^{-\frac{1}{4t}} f * q_{\rho^2 t}(x) t^{-s/2-1} dt$$

converges to f in L^p , and therefore it solves the extension problem with initial condition f (in other words, $u_{s,\rho}$ is an approximate identity). With the relation (2.4) at hand, it is easy to calculate the limit of $\rho^{1-s}\partial_\rho(f * u_{s,\rho})$ as ρ goes to zero. Indeed, by the relation (2.5) we have

$$\rho^s(-L)^{s/2} f * u_{-s,\rho} = \frac{2^s|\Gamma(s/2)|}{|\Gamma(-s/2)|} f * u_{s,\rho}.$$

Since

$$\rho^s u_{-s,\rho}(g) = \frac{2^s}{|\Gamma(-s/2)|} \int_0^\infty e^{-\frac{\rho^2}{4t}} q_t(g) t^{s/2-1} dt$$

it is easy to see that

$$\rho^{1-s}\partial_\rho(f * u_{s,\rho}) = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} (-L)^{s/2} f * u_{2-s,\rho}.$$

Consequently, $\rho^{1-s}\partial_\rho(f * u_{s,\rho})$ converges to $-2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} (-L)^{s/2} f$ as $\rho \rightarrow 0$. \square

2.3. Proofs of Theorem 1.1 and Corollary 1.2.

Proof of Theorem 1.1. As remarked earlier, it is enough to prove the result when $u \in C_0^1([0, \infty), C_0^1(G))$. We take $v = \varphi * u_{s,\rho}$ and observe that, by Theorem 2.4, v solves the equation $(L + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)v = 0$, with $v(x, 0) = \varphi(x)$. Then, by taking this v in Lemma 2.1 and taking into account Theorem 2.4, we obtain the inequality

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \frac{(-L)^{s/2} \varphi(x)}{\varphi(x)} dx,$$

as desired. \square

Moreover, we claim the equality in Theorem 1.1 under certain assumption on the heat kernel associated to L .

Proposition 2.5. *Under the same hypotheses as in Theorem 1.1, let q_t be the heat kernel associated to the Laplacian $L = \sum_{j=1}^m X_j^2$ and assume that $\|q_t\|_2 \leq C t^{-\gamma}$, $\gamma > 1$. Let $\varphi \in L^2(G)$ be such that $(-L)^{s/2}\varphi \in L^2(G)$ and let $u = \varphi * u_{s,\rho}$ be the solution of the extension problem (1.8) with initial condition φ . Then*

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho = 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G \varphi(x) (-L)^{s/2} \varphi(x) dx.$$

Proof. Note that if f and g are both from $L^2(G)$, then their convolution is uniformly continuous and vanishes at infinity. This can be proved by approximating f and g by a sequence of compactly supported smooth functions. Since φ and $u_{s,\rho}$ are from $L^2(G)$, due to (2.3), it follows that the solution u of the extension problem given by $u = \varphi * u_{s,\rho}$ vanishes at infinity as a function of x for any fixed ρ . Moreover, $X_j u_{s,\rho} \in L^2(G)$, and the same is true for $\partial_\rho[\rho^{-s} u_{s,\rho}]$. Integrating by parts and using the fact that u vanishes at infinity, we have

$$\int_G |X_j u(x, \rho)|^2 dx = \int_G u(x, \rho) X_j^2 u(x, \rho) dx.$$

Furthermore, by (2.3), $|u(x, \rho)| \leq C \|\varphi\|_2 \|u_{s,\rho}\|_2 \leq C_s \rho^{-2\gamma} \|\varphi\|_2$ which goes to 0 as ρ tends to infinity. The same is true for $\partial_\rho u(x, \rho)$. A similar computation with the ρ -derivative yields

$$\int_0^\infty (\partial_\rho u(x, \rho))^2 \rho^{1-s} d\rho = \int_0^\infty u(x, \rho) \partial_\rho (\rho^{1-s} \partial_\rho u(x, \rho)) d\rho - u(x, 0) \lim_{\rho \rightarrow 0} (\rho^{1-s} \partial_\rho u)(x, \rho).$$

Now, we sum up and use the fact that u solves the extension problem with initial condition φ . The result follows. \square

Proof of Corollary 1.2. Let $u(x, \rho) = f * u_{s,\rho}(x, \rho)$. By Theorem 2.4, u solves the equation $(L + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho)u = 0$, with $u(x, 0) = f(x)$. By Lemma 2.1 with $v(x, \rho) = u(x, \rho)$, and taking into account that $u = f * u_{s,\rho}$ solves the differential equation, we have that

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho = - \int_G u(x, 0) \lim_{\rho \rightarrow 0} (\rho^{1-s} \partial_\rho u)(x, \rho) dx.$$

Then, by Theorem 2.4, the right hand side of the above identity reduces to

$$2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G f(x) (-L)^{s/2} f(x) dx.$$

On the other hand, by Theorem 1.1, we have that

$$\int_0^\infty \int_G |\nabla u(x, \rho)|^2 \rho^{1-s} dx d\rho \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \frac{(-L)^{s/2} \varphi(x)}{\varphi(x)} dx.$$

Combining all these facts, we conclude the result. \square

2.4. Sublaplacian on stratified groups and proof of Theorem 1.3. In this subsection we consider $L = \sum_{j=1}^n X_j^2$ to be the hypoelliptic sublaplacian on a stratified Lie group G . A Lie group G is called *stratified* if it is nilpotent and simply connected and its Lie algebra \mathfrak{g} admits a vector space direct sum decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ such that $[\mathfrak{g}_1, \mathfrak{g}_k] = \mathfrak{g}_{k+1}$ for $1 \leq k < m$ and $[\mathfrak{g}_1, \mathfrak{g}_m] = \{0\}$, where $\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}]$. We refer to Folland-Stein [15] for more about stratified groups and sublaplacians.

Theorem 1.1 says that

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \frac{(-L)^{s/2} \varphi(x)}{\varphi(x)} dx. \quad (2.8)$$

Let q_t be the associated heat kernel of L . Using the functions $u_{s,\rho}$ defined in the previous subsection in (2.2) and Lemma 2.3 we could prove the following trace Hardy inequality:

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq s \delta^{-s} \int_G u^2(x, 0) \frac{u_{s,\delta}(x)}{u_{-s,\delta}(x)} dx.$$

Remark 2.6. In the case of the Laplacian Δ on \mathbb{R}^n the heat kernel is explicitly given by

$$q_t(x) = (4\pi t)^{-n/2} e^{-\frac{1}{4t}|x|^2}$$

and hence the expression in (2.2) can be explicitly computed to be

$$u_{s,\delta}(x) = \frac{\delta^s \Gamma((n+s)/2)}{\pi^{n/2} |\Gamma(s/2)|} (\delta^2 + |x|^2)^{-(n+s)/2}$$

for $-2 < s < 2$. Then observe that inequality (1.2) follows from Corollary 1.2 (indeed, if $f \in L^2$ and $(-\Delta)^{s/2} \in L^2$, the solution of the extension problem given by $u(x, \rho) = f * u_{s,\rho}(x)$ belongs to $\widetilde{W}_0^{s,2}(\mathbb{R}^{n+1})$, see [28, Proposition 3.13]), after choosing $\varphi = u_{-s,\delta}$ as above, and taking into account Lemma 2.3 (with L to be the Euclidean Laplacian Δ).

In the case of stratified Lie groups, we do not have an explicit expression for the heat kernel, so we are not able to arrive at a sharp constant. The situation is the same even in the case of the Heisenberg group \mathbb{H}^n where the heat kernel is almost explicit.

Now, consider the following functions which are kernels of the Riesz potentials: for $\alpha < Q$, where Q is the homogeneous dimension of G ,

$$\mathcal{R}_\alpha(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty q_t(x) t^{\alpha/2-1} dt.$$

Then it is known ([14] or [8]) that they are the kernels associated to $(-L)^{-\alpha/2}$. They can be explicitly calculated (e.g. in the case of Δ on \mathbb{R}^n) once we have a good expression for the heat kernel. Moreover, they are homogeneous of degree $\alpha - Q$ and satisfy $\mathcal{R}_\alpha * \mathcal{R}_\beta = \mathcal{R}_{\alpha+\beta}$.

Proof of Theorem 1.3. The proof is very similar to the one given in the context of H -type groups, in [28, Proof of Theorem 1.5]. We sketch here the main ideas.

We fix a homogeneous norm $|\cdot|$ on G and choose a smooth function $0 \leq \psi_\epsilon \leq 1$ which is supported on $\frac{1}{2}\epsilon \leq |x| \leq 2\epsilon^{-1}$ and is identically one on $\epsilon \leq |x| \leq \epsilon^{-1}$. We consider the inequality (2.8) with $\varphi(x) = \varphi_\delta(x) := (\psi_\epsilon \mathcal{R}_\alpha) * u_{-s,\delta}(x)$. In view of the relation in Lemma 2.3 we obtain that

$$\frac{(-L)^{s/2} \varphi_\delta(x)}{\varphi_\delta(x)} = \delta^{-s} \frac{2^s \Gamma(s/2)}{|\Gamma(-s/2)|} \frac{(\psi_\epsilon \mathcal{R}_\alpha) * u_{s,\delta}(x)}{(\psi_\epsilon \mathcal{R}_\alpha) * u_{-s,\delta}(x)}. \quad (2.9)$$

Recalling the definition of $u_{-s,\delta}$ in (2.2) we see that $\delta^s u_{-s,\delta}$ converges pointwise to $2^s \frac{|\Gamma(s/2)|}{|\Gamma(-s/2)|} \mathcal{R}_s$ as δ tends to zero (it actually converges in the sense of distributions. Both convergences are easy to see from the definition). Since $(\psi_\epsilon \mathcal{R}_\alpha) * u_{s,\delta}$ converges to $\psi_\epsilon \mathcal{R}_\alpha$ (because $u_{s,\delta}$ is an approximate identity, see the proof of Theorem 2.4) we obtain the following inequality, from (2.8), by letting $\delta \rightarrow 0$:

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \frac{\psi_\epsilon(x) \mathcal{R}_\alpha(x)}{\psi_\epsilon \mathcal{R}_\alpha * \mathcal{R}_s(x)} dx.$$

As $0 \leq \psi_\epsilon \leq 1$ and $\psi_\epsilon \mathcal{R}_\alpha * \mathcal{R}_s \leq \mathcal{R}_\alpha * \mathcal{R}_s = \mathcal{R}_{\alpha+s}$, the latter reads as

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \psi_\epsilon(x) \frac{\mathcal{R}_\alpha(x)}{\mathcal{R}_{\alpha+s}(x)} dx. \quad (2.10)$$

By letting ϵ tend to 0 we obtain the inequality

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_G u^2(x, 0) \frac{\mathcal{R}_\alpha(x)}{\mathcal{R}_{\alpha+s}(x)} dx.$$

Let us define

$$F_\alpha(x) := \frac{1}{\Gamma((Q-\alpha)/2)} \int_0^\infty q_t(x) t^{\alpha/2-1} dt = \frac{\Gamma(\alpha/2)}{\Gamma((Q-\alpha)/2)} \mathcal{R}_\alpha(x).$$

The choice $\alpha = \frac{Q-s}{2}$ leads to the inequality

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \frac{\Gamma((Q+s)/4)^2}{\Gamma((Q-s)/4)^2} \int_G \frac{u^2(x, 0)}{w_s(x)} dx$$

where $w_s(x) = \frac{F_{(Q+s)/2}(x)}{F_{(Q-s)/2}(x)}$ is homogeneous of degree s . \square

Remark 2.7. For each γ we define the function $|x|_{(\gamma)}$ by the equation

$$\int_0^\infty q_t(x) t^{\frac{Q+\gamma}{4}-1} dt = \Gamma((Q-\gamma)/4) |x|_{(\gamma)}^{-(Q-\gamma)/2}.$$

Then it is clear that $|x|_{(\gamma)}$ is a homogeneous norm on G . With this definition, we see that

$$w_s(x) = |x|_{(s)}^{-(Q-s)/2} |x|_{(-s)}^{(Q+s)/2}$$

and the trace Hardy inequality reads as

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \left(\frac{\Gamma((Q+s)/4)}{\Gamma((Q-s)/4)} \right)^2 \int_G u^2(x, 0) \frac{|x|_{(s)}^{(Q-s)/2}}{|x|_{(-s)}^{(Q+s)/2}} dx.$$

Since all the homogeneous norms on G are equivalent we have $C_1|x|_{(-s)} \leq |x| \leq C_2|x|_{(s)}$ for some constants where $|x|$ is a standard homogeneous norm. With this we can replace the weight $w_s(x)$ by $C_s^{-1}|x|^s$ and get the inequality

$$\int_G \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq C_s 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \left(\frac{\Gamma((Q+s)/4)}{\Gamma((Q-s)/4)} \right)^2 \int_G \frac{u^2(x, 0)}{|x|^s} dx.$$

The values of the constant depend on good estimates of the heat kernel in terms of Gaussian from above and below.

Proof of Corollary 1.4. By Corollary 1.2 and the computations in the proof of Theorem 1.3, one immediately obtains

$$((-L)^{s/2} f, f) \geq \left(\frac{\Gamma((Q+s)/4)}{\Gamma((Q-s)/4)} \right)^2 \int_G \frac{|f(x)|^2}{w_s(x)} dx.$$

□

Remark 2.8. Observe that we could get also the following Hardy's inequality for stratified Lie groups

$$((-L)^{s/2} f, f) \geq C_s \left(\frac{\Gamma((Q+s)/4)}{\Gamma((Q-s)/4)} \right)^2 \int_G \frac{|f(x)|^2}{|x|^s} dx.$$

In [8, Section 3], Ciatti et al obtain a family of Hardy-type inequalities on stratified Lie groups. Their result is slightly different in the sense that they studied L^p boundedness of a Hardy operator associated to the corresponding homogeneous norm. They provide estimates for the constants involved.

Remark 2.9. Consider the case $G = \mathbb{R}^n$. It can be easily checked (see for instance [27, Lemma A.1]) that

$$\mathcal{R}_\alpha(x) = \frac{\Gamma((n-\alpha)/2)}{\Gamma(\alpha/2) 2^\alpha \pi^{n/2}} |x|^{\alpha-n}.$$

Consequently, by choosing $\alpha = (n-s)/2$ in (2.10), the trace Hardy inequality for \mathbb{R}^n takes the form

$$\int_{\mathbb{R}^n} \int_0^\infty |\nabla u(x, \rho)|^2 \rho^{1-s} d\rho dx \geq 2 \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \left(\frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)} \right)^2 \int_{\mathbb{R}^n} \frac{u^2(x, 0)}{|x|^s} dx.$$

This inequality is known to be sharp, though equality is never attained (see [38]). Similarly, Hardy inequality (1.1) on \mathbb{R}^n follows from Corollary 1.2 and the computation starting from (2.9), choosing again $\alpha = (n-s)/2$:

$$((-\Delta)^{s/2} f, f) \geq 2^s \left(\frac{\Gamma((n+s)/4)}{\Gamma((n-s)/4)} \right)^2 \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^s} dx,$$

which coincides with (1.1).

2.5. Euclidean motion groups and proof of Theorem 1.5. In this subsection we consider the Euclidean motion group $M(n)$ which is the semidirect product of \mathbb{R}^n with the orthogonal group $\text{SO}(n)$ acting on \mathbb{R}^n . Then the Laplacian on $G = M(n)$ is given by $\Delta_G = \Delta_{\mathbb{R}^n} + \Delta_K$ where Δ_K is the Laplacian on $K = \text{SO}(n)$. By a suitable choice of the inner product on the Lie algebra of K we can assume that $\Delta_K = \sum_{j=1}^m X_j^2$ where X_j are vector fields on K forming an orthonormal basis for its Lie algebra. Thus Δ_G is a sum of squares of vector fields and the results proved in Subsection 2.2 apply to the case of $M(n)$. In particular, the solution of the extension problem

$$(\Delta_G + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho)u(x, k, \rho) = 0, \quad \rho > 0; \quad u(x, k, 0) = f(x, k), \quad x \in \mathbb{R}^n, \quad k \in K,$$

is explicitly given by $f * u_{s,\rho}(x, k)$, where

$$u_{s,\rho}(x, k) = \frac{\rho^s}{|\Gamma(s/2)|} \int_0^\infty e^{-\frac{1}{4t}\rho^2} q_t(x, k) t^{-s/2-1} dt,$$

for $-2 < s < 2$. In the above formula, q_t is the heat kernel associated to Δ_G , which is just the product of the Euclidean heat kernel $p_t(x)$ with the heat kernel $h_t(k)$ associated to Δ_K . Thus we have

$$u_{s,\rho}(x, k) = \frac{\rho^s}{(4\pi)^{n/2} |\Gamma(s/2)|} \int_0^\infty e^{-\frac{1}{4t}(\rho^2 + |x|^2)} h_t(k) t^{-(n+s)/2-1} dt.$$

The kernels $u_{s,\rho}(x, k)$ can be expressed in terms of known functions as follows. We need to set up some notation here; we refer the reader to Subsection 3.2 for more details.

Given a compact Lie group K we let \widehat{K} stand for the unitary dual consisting of equivalence classes of irreducible unitary representations of K . If $\pi \in \widehat{K}$ then χ_π will stand for the character associated to π . Then it is known that χ_π are eigenfunctions of the Laplacian Δ_K and the eigenvalues are denoted by $-\lambda_\pi^2$. The heat kernel h_t is given by the expansion

$$h_t(k) = \sum_{\pi \in \widehat{K}} d_\pi e^{-t\lambda_\pi^2} \chi_\pi(k),$$

where d_π is the dimension of π . In view of this, the function $u_{s,\rho}$ on the motion group is given by

$$u_{s,\rho}(x, k) = \sum_{\pi \in \widehat{K}} d_\pi u_{s,\rho}(x; \pi) \chi_\pi(k)$$

where the functions $u_{s,\rho}(x; \pi)$ are

$$u_{s,\rho}(x; \pi) = \frac{\rho^s}{(4\pi)^{n/2} |\Gamma(s/2)|} \int_0^\infty e^{-\frac{1}{4t}(\rho^2 + |x|^2)} e^{-t\lambda_\pi^2} t^{-(n+s)/2-1} dt.$$

We can easily see that $u_{s,\rho}(x; \pi) = \rho^{-n} \varphi_{(n+s)/2}(\rho^{-1}x, \rho\lambda_\pi)$ where

$$\varphi_{(n+s)/2}(x, \lambda) = \frac{1}{(4\pi)^{n/2} |\Gamma(s/2)|} \int_0^\infty e^{-\frac{1}{4t}(1+|x|^2)} e^{-t\lambda^2} t^{-(n+s)/2-1} dt.$$

By recalling the integral representation of the Macdonald functions $K_{(n+s)/2}$ (see (3.8) for the definition) we obtain

$$\varphi_{(n+s)/2}(x, \lambda) = \frac{2^{-\frac{n-s}{2}+1}}{\pi^{n/2} |\Gamma(s/2)|} \lambda^{(n+s)/2} (1+|x|^2)^{-(n+s)/4} K_{(n+s)/2}(\lambda(1+|x|^2)^{1/2}).$$

Thus we have the following formula for the kernel $u_{s,\rho}(x, k)$:

$$u_{s,\rho}(x, k) = \rho^{-n} \sum_{\pi \in \widehat{K}} d_\pi \varphi_{(n+s)/2}(\rho^{-1}x, \rho\lambda_\pi) \chi_\pi(k). \quad (2.11)$$

Then, by choosing $\varphi(x, k) = u_{-s,\delta}(x, k)$ in Theorem 1.1, we obtain the following.

Theorem 2.10. *Let $0 < s < 2$ and fix $\delta > 0$. Then for any real valued function $u(x, k, \rho)$, $(x, k, \rho) \in G \times \mathbb{R}^+$, we have the inequality*

$$\int_0^\infty \int_G \left| \nabla u(x, k, \rho) \right|^2 \rho^{1-s} dx dk d\rho \geq s \delta^{-s} \int_G u^2(x, k, 0) \frac{u_{s,\delta}(x, k)}{u_{-s,\delta}(x, k)} dx dk$$

where the functions $u_{\pm s,\delta}$ are the ones in (2.11).

In the case of $M(2)$, we can get a more precise expression for the kernel $u_{s,\rho}(x, k)$ using Poisson summation formula for the heat kernel $h_t(k)$. We identify $K = \text{SO}(2)$ with the circle group S^1 and \mathbb{R}^2 with \mathbb{C} so that $M(2)$ is the semidirect product of \mathbb{C} with S^1 . The elements of $G = M(2)$ will be written as $(z, e^{i\theta})$ rather than (x, k) .

Proposition 2.11. *For $-2 < s < 2$, we have*

$$u_{s,\rho}(z, e^{i\theta}) = \frac{2^s \Gamma((3+s)/2) \rho^s}{\pi^{3/2} |\Gamma(s/2)|} \sum_{k \in \mathbb{Z}} (\rho^2 + |z|^2 + (\theta + 2k\pi)^2)^{-\frac{(3+s)}{2}}.$$

Proof. In the case of $M(2)$ the heat kernel is given by

$$h_t(\theta) = \sum_{k=-\infty}^{\infty} e^{-tk^2} e^{ik\theta}.$$

In view of the Poisson summation formula (see [30, Chapter VII]) we also have the representation

$$h_t(\theta) = (4\pi t)^{-1/2} \sum_{k=-\infty}^{\infty} e^{-\frac{1}{4t}(\theta+2\pi k)^2}.$$

Thus, the integral representation for $u_{s,\rho}(z, e^{i\theta})$ gives

$$u_{s,\rho}(z, e^{i\theta}) = \frac{\rho^s}{(4\pi)^{3/2} |\Gamma(s/2)|} \sum_{k=-\infty}^{\infty} \int_0^{\infty} e^{-\frac{1}{4t}(\rho^2+|z|^2+(\theta+2\pi k)^2)} t^{-(3+s)/2-1} dt.$$

By evaluating the integral the proposition is proved. \square

Proof of Theorem 1.5. For functions $f \in L^1(\mathbb{R})$ we have

$$\int_0^{2\pi} \sum_{k \in \mathbb{Z}} f(\theta + 2k\pi) d\theta = \sum_{k \in \mathbb{Z}} \int_{2k\pi}^{2(k+1)\pi} f(\theta) d\theta = \int_{-\infty}^{\infty} f(\theta) d\theta.$$

Therefore, taking into account of this with $f(\theta) = (\delta^2 + |z|^2 + |\theta|^2)^{-\frac{(3+s)}{2}}$, by Theorem 2.10 and Proposition 2.11, we obtain

$$\begin{aligned} & s \delta^{-s} \int_0^{2\pi} u^2(z, e^{i\theta}, 0) \frac{u_{s,\delta}(z, e^{i\theta})}{u_{-s,\delta}(z, e^{i\theta})} d\theta \\ &= \delta^s 2 \cdot 4^s \frac{\Gamma(\frac{3+s}{2}) \Gamma(1-s/2)}{\Gamma(\frac{3-s}{2}) \Gamma(s/2)} \int_{-\infty}^{\infty} \frac{u^2(z, e^{i\theta}, 0)}{(\delta^2 + |z|^2 + |\theta|^2)^{\frac{3+s}{2}}} \left[\sum_{k \in \mathbb{Z}} (\delta^2 + |z|^2 + (\theta + 2k\pi)^2)^{-\frac{(3-s)}{2}} \right]^{-1} d\theta, \end{aligned}$$

which completes the proof. \square

Remark 2.12. There is also Poisson summation formula available for $\text{SO}(n)$ (see [39, Theorem 3]). Thus, we could obtain analogous results as in Proposition 2.11 and Theorem 1.5 in higher dimensions. In order to avoid additional notation, we leave the details to the reader.

2.6. The special Hermite operator and proof of Theorem 1.6. In this section we prove trace Hardy and Hardy's inequalities for fractional powers of the special Hermite operator (also known as twisted Laplacian) on \mathbb{C}^n . This operator is related to the sublaplacian on the Heisenberg group, and so we will define it starting from the Heisenberg group setting. Let us set up the notation.

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ denote the $(2n+1)$ -dimensional Heisenberg group with the group operation $(z, t)(w, s) = (z + w, t + s + \frac{1}{2} \text{Im}(z \cdot \bar{w}))$. Its Lie algebra \mathfrak{h}_n is generated by the $(2n+1)$ left invariant vector fields

$$T = \frac{\partial}{\partial t}, \quad X_j = \left(\frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t} \right), \quad Y_j = \left(\frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t} \right), \quad j = 1, 2, \dots, n.$$

The operator $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$ is called the sublaplacian on \mathbb{H}^n . Written explicitly

$$\mathcal{L} = -\Delta_{\mathbb{C}^n} - \frac{1}{4} |z|^2 \frac{\partial^2}{\partial t^2} + N \frac{\partial}{\partial t}$$

where $\Delta_{\mathbb{C}^n}$ is the ordinary Laplacian on \mathbb{C}^n , $z = x + iy \in \mathbb{C}^n$ and

$$N = \sum_{j=1}^n \left(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).$$

We can also write \mathcal{L} as

$$\mathcal{L} = -2 \sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j)$$

where $Z_j = \frac{1}{2}(X_j - iY_j)$ and $\bar{Z}_j = \frac{1}{2}(X_j + iY_j)$. Observe that

$$Z_j = \frac{\partial}{\partial z_j} + \frac{i}{4} \bar{z}_j \frac{\partial}{\partial t}, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{i}{4} z_j \frac{\partial}{\partial t}$$

where $\frac{\partial}{\partial z_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right)$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)$ are the Cauchy-Riemann operators.

For each nonzero $\lambda \in \mathbb{R}$, let the special Hermite operators L_λ (or twisted Laplacian) be defined by the equation

$$\mathcal{L}(f(z)e^{i\lambda t}) = e^{i\lambda t} L_\lambda f(z). \quad (2.12)$$

Then it is clear from the definition that

$$L_\lambda = -\Delta_{\mathbb{C}^n} + \frac{\lambda^2}{4} |z|^2 + i\lambda N$$

and we can also write L_λ as

$$L_\lambda = -2 \sum_{j=1}^n (Z_j(\lambda) \bar{Z}_j(\lambda) + \bar{Z}_j(\lambda) Z_j(\lambda))$$

where $Z_j(e^{i\lambda t} f(z)) = e^{i\lambda t} Z_j(\lambda) f(z)$ and $\bar{Z}_j(e^{i\lambda t} f(z)) = e^{i\lambda t} \bar{Z}_j(\lambda) f(z)$. Each of these L_λ is an elliptic differential operator on \mathbb{C}^n whose spectral decomposition is known explicitly and is given by special Hermite functions. We refer to [33] for the theory of special Hermite expansions. Though this operator is not a sum of squares of real vector fields, it is related to the sublaplacian \mathcal{L} on \mathbb{H}^n (which is a sum of squares) via the relation (2.12) and we make use of this connection.

There is an orthonormal basis for $L^2(\mathbb{C}^n)$ consisting of the ‘‘special Hermite functions’’ $\Phi_{\alpha,\beta}$, $\alpha, \beta \in \mathbb{N}^n$, which are eigenfunctions of L_1 , more precisely, $L_1 \Phi_{\alpha,\beta} = (2|\beta| + n) \Phi_{\alpha,\beta}$. Note that the spectrum of L_1 consists of $(2k + n)$, $k \in \mathbb{N}$ and each eigenspace is infinite dimensional. The projection onto the corresponding eigenspace can be written in a compact form. If

$$\varphi_k(z) = L_k^{n-1} \left(\frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2}, \quad k \in \mathbb{N},$$

stand for Laguerre functions of type $(n-1)$ then we have

$$(2\pi)^{-n} \varphi \times \varphi_k(z) = \sum_{|\beta|=k} \sum_{\alpha \in \mathbb{N}^n} (\varphi, \Phi_{\alpha,\beta}) \Phi_{\alpha,\beta}(z), \quad \varphi \in L^2(\mathbb{C}^n)$$

where the twisted convolution $\varphi \times \psi$ of two functions on \mathbb{C}^n is defined by

$$\varphi \times \psi(z) = \int_{\mathbb{C}^n} \varphi(z-w) \psi(w) e^{\frac{i}{2} \text{Im}(z \cdot \bar{w})}.$$

More generally, the λ -twisted convolution is defined by

$$\varphi *_\lambda \psi(z) = \int_{\mathbb{C}^n} \varphi(z-w) \psi(w) e^{\frac{i}{2} \lambda \text{Im}(z \cdot \bar{w})}$$

which is related to the convolution on the Heisenberg group via the identity $(f * g)^\lambda(z) = f^\lambda *_\lambda g^\lambda(z)$ where $f * g$ is the convolution of f and g on the Heisenberg group \mathbb{H}^n and

$$f^\lambda(z) = \int_{-\infty}^{\infty} f(z, t) e^{i\lambda t} dt.$$

Actually, it is convenient to work with the reduced Heisenberg group $\mathbb{H}_{\text{red}}^n$ rather than the Heisenberg group. This group is defined as the quotient group \mathbb{H}^n / Γ where \mathbb{H}^n is the Heisenberg group and the central

subgroup Γ is given by $\Gamma = \{(0, 2k\pi) : k \in \mathbb{Z}\}$ (see [34, Chapter 4]). The reduced Heisenberg group can be identified with $\mathbb{C}^n \times \mathbb{T}$ where \mathbb{T} is the torus with group law written as

$$(z, e^{it})(w, e^{i\tau}) = (z + w, e^{i(t+\tau + \frac{1}{2} \operatorname{Im} z \cdot \bar{w})}).$$

We let \mathcal{L}_{red} stand for the sublaplacian as $\mathbb{H}_{\text{red}}^n$ which is simply the sublaplacian on \mathbb{H}^n acting on functions $f(z, w)$ which are 2π -periodic in the last variable.

We need also to relate the heat semigroups associated to the sublaplacian and to the special Hermite operator. The sublaplacian is a self-adjoint, non-negative, hypoelliptic operator, and it generates a contraction semigroup which we denote by $e^{-t\mathcal{L}}$. It is known (and it can be deduced from the facts shown above) that $(e^{-t\mathcal{L}}f)^\lambda(z) = e^{-tL\lambda} f^\lambda(z) = f^\lambda *_\lambda q_t^\lambda(z)$, where

$$q_t^\lambda(z) = (4\pi)^{-n} \left(\frac{\lambda}{\sinh t\lambda} \right)^n e^{-\frac{1}{4}\lambda(\coth t\lambda)|z|^2}$$

see [35, Theorem 2.8.1].

The operator \mathcal{L}_{red} generates a diffusion semigroup $e^{-t\mathcal{L}_{\text{red}}}$ whose heat kernel \tilde{q}_t is given by

$$\tilde{q}_t(z, \tau) = \sum_{k=-\infty}^{\infty} q_t(z, \tau + 2k\pi)$$

where q_t is the heat kernel on \mathbb{H}^n . We then have, for $k \in \mathbb{Z}, k \neq 0$

$$\int_0^{2\pi} \tilde{q}_t(z, \tau) e^{ik\tau} d\tau = q_t^k(z)$$

and when $k = 0$ we also have

$$\int_0^{2\pi} \tilde{q}_t(z, \tau) d\tau = (4\pi t)^{-n} e^{-\frac{1}{4t}|z|^2}$$

which is just the Euclidean heat kernel on \mathbb{C}^n .

Consider the extension problem

$$(\Delta_{\mathbb{C}^n} + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) v(z, \rho) = 0, \quad z \in \mathbb{C}^n, \rho > 0; \quad v(z, 0) = \varphi(z), \quad z \in \mathbb{C}^n,$$

for the Laplacian $\Delta_{\mathbb{C}^n}$. We can also think of this v as a solution of the extension problem

$$(-\mathcal{L}_{\text{red}} + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) u(z, \tau, \rho) = 0, \quad z \in \mathbb{C}^n, \tau \in \mathbb{R}, \rho > 0; \quad u(z, \tau, 0) = \varphi(z), \quad z \in \mathbb{C}^n \quad (2.13)$$

on the reduced Heisenberg group. This is so simply because $\mathcal{L}_{\text{red}} v(z, \rho) = \Delta_{\mathbb{C}^n} v(z, \rho)$ for functions independent of τ . Theorem 1.1 reads in this context as follows.

Theorem 2.13. *For real valued functions φ on \mathbb{C}^n and u on $\mathbb{H}_{\text{red}}^n$ with mild decay conditions we have the inequality*

$$\int_{\mathbb{H}_{\text{red}}^n} \int_0^\infty |\nabla u(z, \tau, \rho)|^2 \rho^{1-s} d\rho dz d\tau \geq 2^{1-s} \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \int_{\mathbb{H}_{\text{red}}^n} u^2(z, \tau, 0) \frac{(-\Delta_{\mathbb{C}^n})^{s/2} \varphi(z)}{\varphi(z)} dz d\tau.$$

From Theorem 2.13 we can also prove the following trace Hardy inequality for functions on $\mathbb{H}_{\text{red}}^n$:

Theorem 2.14. *Let $0 < s < 2$. Then, for real valued functions u on $\mathbb{H}_{\text{red}}^n \times \mathbb{R}_+$*

$$\int_{\mathbb{H}_{\text{red}}^n} \int_0^\infty |\nabla u(z, \tau, \rho)|^2 \rho^{1-s} d\rho dz d\tau \geq 2 \frac{\Gamma(1-s/2)}{\Gamma(s/2)} \frac{\Gamma(\frac{2n+s}{4})^2}{\Gamma(\frac{2n-s}{4})^2} \int_{\mathbb{H}_{\text{red}}^n} \frac{u(z, \tau, 0)^2}{|z|^s} dz d\tau.$$

Proof. The proof is analogous to that of Theorem 1.3 for stratified Lie groups or more precisely, as in the Euclidean case as explained in Remark 2.9. Indeed, it is enough to observe that $\mathbb{C}^n = \mathbb{R}^{2n}$. \square

We know that when u satisfies the extension problem (2.13) with the initial condition f the left hand side of the above integral reduces to $(\mathcal{L}_{\text{red}}^{s/2} f, f)$ (i.e., we have Corollary 1.2 in this context). By taking f in the particular form $g(z) \cos(\tau)$, where g is a real valued function we can obtain the Hardy's inequality for fractional powers of the special Hermite operator L_1 stated in Theorem 1.6. Now we show the proof.

Proof of Theorem 1.6. For the sublaplacian in the reduced Heisenberg group, Corollary 1.2 reads as

$$(\mathcal{L}_{\text{red}}^{s/2} f, f) \geq \int_{\mathbb{H}_{\text{red}}^n} f^2(z, t) \frac{(-\Delta_{\mathbb{C}^n})^{s/2} \varphi(z)}{\varphi(z)} dz d\tau. \quad (2.14)$$

Take now $f(z, \tau) = g(z) \cos(\tau)$, where g is a real valued function. On one hand we have

$$(\mathcal{L}_{\text{red}}^{s/2} f, f) = \pi \operatorname{Re} (L_1^{s/2} g, g).$$

Indeed, we have the integral representation (see [31])

$$\mathcal{L}_{\text{red}}^{s/2} f = \frac{1}{\Gamma(-s/2)} \int_0^\infty (e^{-r\mathcal{L}_{\text{red}}} f - f) r^{-\frac{s}{2}-1} dr.$$

Recalling that $e^{-r\mathcal{L}_{\text{red}}} f(z, \tau) = f * \tilde{q}_r(z, \tau)$ and $f(z, t) = \frac{1}{2}g(z)(e^{i\tau} + e^{-i\tau})$, we have

$$f * \tilde{q}_r(z, \tau) = \operatorname{Re}(e^{-i\tau} e^{-rL_1} g(z)).$$

From this we get

$$e^{-r\mathcal{L}_{\text{red}}} f(z, \tau) = \operatorname{Re}(e^{-i\tau} e^{-rL_1} g(z))$$

and consequently, $(\mathcal{L}_{\text{red}}^{s/2} f, f) = \pi \operatorname{Re} (L_1^{s/2} g, g)$ as claimed. On the other hand, by following the analogous reasoning as for the Euclidean case in Remark 2.9, we obtain that the expression on the right hand side of (2.14) is precisely

$$\pi 2^s \frac{\Gamma(\frac{2n+s}{4})^2}{\Gamma(\frac{2n-s}{4})^2} \int_{\mathbb{C}^n} \frac{g(z)^2}{|z|^s} dz.$$

This completes the proof. \square

3. Operators with discrete spectrum: characterisations of solutions to the extension problem.

Our aim in this section is to characterise all the solutions of the extension problem when the associated operator has discrete spectrum. We remark that the initial considerations in the present section have been already presented in the literature (see [31, 3]). The starting point will be the most general framework, and then we will study several particular cases.

Let us consider a Riemannian manifold (M, g) , with or without boundary. On M we have a natural second-order partial differential operator, namely the Laplace–Beltrami operator Δ_g , defined by $\Delta_g f = -\operatorname{div}(\nabla f)$. We assume that the spectrum of Δ_g is discrete. This is the case e.g. when M is compact. Let $\{\lambda_k^2\}$ be the spectrum. Let φ_k be the associated eigenfunctions

$$\Delta_g \varphi_k = -\lambda_k^2 \varphi_k$$

normalised so that $\{\varphi_k\}$ forms an orthonormal basis of $L^2(M)$.

We consider the associated extension problem

$$(\Delta_g + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) u(x, \rho) = 0, \quad x \in M, \quad \rho > 0; \quad u(x, 0) = f(x), \quad x \in M. \quad (3.1)$$

Let us initially impose mild conditions on the initial data, say, f be a distribution. Then we have the equation

$$\langle (\Delta_g + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) u(\cdot, \rho), \varphi_k \rangle_M = 0$$

which shows that the Fourier coefficients $u_k(\rho) := \langle u(\cdot, \rho), \varphi_k \rangle_M = \int_M u(x, \rho) \overline{\varphi_k(x)} dx$ satisfy the equation

$$(-\lambda_k^2 + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) u_k(\rho) = 0. \quad (3.2)$$

It is well known (see [24, Chapter 5, Section 5.7]) that the functions $(\rho\lambda_k)^{s/2} I_{s/2}(\rho\lambda_k)$ and $(\rho\lambda_k)^{s/2} K_{s/2}(\rho\lambda_k)$, where I_ν is the modified Bessel function of first kind and K_ν is the Macdonald's function, cf. [24, Chapter 5, Section 5.7] (see (3.3) and (3.4) for the definitions), are two linearly independent solutions of the equation (3.2), i.e.,

$$u_k(\rho) = c_k (\rho\lambda_k)^{s/2} K_{s/2}(\rho\lambda_k) + d_k (\rho\lambda_k)^{s/2} I_{s/2}(\rho\lambda_k).$$

Then, for the time being formally, we have

$$u(x, \rho) = \sum_k u_k(\rho) \varphi_k(x) = \sum_k c_k (\rho\lambda_k)^{s/2} K_{s/2}(\rho\lambda_k) \varphi_k(x) + \sum_k d_k (\rho\lambda_k)^{s/2} I_{s/2}(\rho\lambda_k) \varphi_k(x).$$

We also know that $u(x, \rho)$ converges to the distribution $f(x)$ as ρ goes to zero. Therefore, $u_k(\rho)$ converges to $\langle f, \varphi_k \rangle_M$ as ρ goes to zero. This implies that (the constant in the first summand is justified by Remark 3.1)

$$u_k(\rho) = 2^{1-s/2} \Gamma(s/2)^{-1} \langle f, \varphi_k \rangle_M (\rho \lambda_k)^{s/2} K_{s/2}(\rho \lambda_k) + d_k (\rho \lambda_k)^{s/2} I_{s/2}(\rho \lambda_k).$$

Then, in view of the asymptotics (3.5), (3.6) of the function $I_{s/2}$, the coefficient d_k should be chosen suitably. Thus, taking into account the above considerations, our aim will be to give characterisations for all possible solutions of the extension problem (3.1) in several particular cases. Such characterisations will go beyond [31, 3].

Before going through this issue, we will first collect several definitions and properties of Bessel functions.

3.1. Some facts on Bessel functions. Let $I_\nu(z)$ be the modified Bessel function of first kind given by the formula (see [24, Chapter 5, Section 5.7])

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}, \quad |z| < \infty, \quad |\arg z| < \pi \quad (3.3)$$

and let K_ν be the Macdonald's function of order ν defined by (see also [24, Chapter 5, Section 5.7])

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \quad |\arg z| < \pi, \quad \nu \neq 0, \pm 1, \pm 2, \dots \quad (3.4)$$

and, for integral $\nu = n$, $K_n(z) = \lim_{\nu \rightarrow n} K_\nu(z)$, $n = 0, \pm 1, \pm 2, \dots$. From (3.3) and (3.4) it is clear that there exist constants $C_1, C_2, c_1, c_2 > 0$ such that

$$c_1 z^\nu \leq I_\nu(z) \leq C_1 z^\nu, \quad cz^{-\nu} \leq K_\nu(z) \leq Cz^{-\nu}, \quad \text{for } z \rightarrow 0^+. \quad (3.5)$$

Moreover, it is well known (see [24, Chapter 5, Section 5.11]) that

$$I_\nu(z) = Ce^z z^{-1/2} + R_\nu(z), \quad |R_\nu(z)| \leq C_\nu e^z z^{-3/2}, \quad |\arg z| \leq \pi - \delta \quad (3.6)$$

and

$$K_\nu(z) = Ce^{-z} z^{-1/2} + \tilde{R}_\nu(z), \quad |\tilde{R}_\nu(z)| \leq C_\nu e^{-z} z^{-3/2}, \quad |\arg z| \leq \pi - \delta. \quad (3.7)$$

We have the integral representation for the Macdonald's functions, see for instance [24, Chapter 5, (5.10.25)]

$$K_\nu(z) = 2^{-\nu-1} z^\nu \int_0^\infty e^{-(t+\frac{1}{4t}z^2)} t^{-\nu-1} dt. \quad (3.8)$$

From (3.8) it can be seen that $z^{s/2} K_{s/2}(z)$ is actually a function of z^2 and hence it is even in the z variable.

Remark 3.1. Even more, observe that, after a change of variables,

$$z^\nu K_\nu(z) = 2^{-\nu-1} z^{2\nu} \int_0^\infty e^{-(t+\frac{1}{4t}z^2)} t^{-\nu-1} dt = 2^{\nu-1} \int_0^\infty e^{-\frac{z^2}{4u}} e^{-u} u^{\nu-1} du.$$

Then, letting z tend to 0 above, we have that $z^\nu K_\nu(z)$ tends to the constant $2^{\nu-1} \Gamma(\nu)$.

3.2. Compact Lie groups. We start with some definitions and notations. Some of them were already introduced in Subsection 2.5, but we collect them also here for the sake of the reading. Let K be a connected, simply connected compact Lie group. Let \hat{K} stand for its unitary dual, viz. the set of all equivalence classes of irreducible unitary representations of K . For each $\pi \in \hat{K}$ we let d_π stand for the dimension of π and χ_π the character. For $f \in L^2(K)$, we define its (operator valued) Fourier transform as

$$\pi(f) = \int_K f(k) \pi(k)^* dk,$$

and we have the Plancherel formula

$$\int_K |f(k)|^2 dk = \sum_{k \in \hat{K}} d_\pi \|\pi(f)\|_{\text{HS}}^2$$

where $\|\pi(f)\|_{\text{HS}}^2 = \text{tr}(\pi(f)\pi(f)^*)$ is the Hilbert-Schmidt norm of $\pi(f)$. We can write the inversion in two different ways

$$f(k) = \sum_{\pi \in \hat{K}} d_\pi \text{tr}(\pi(f)\pi(k)), \quad \text{or} \quad f(k) = \sum_{\pi \in \hat{K}} d_\pi f * \chi_\pi(k),$$

so that $f * \chi_\pi(k) = \text{tr}(\pi(f)\pi(k))$. We have

$$\int_K |f * \chi_\pi(k)|^2 dk = d_\pi \|\pi(f)\|_{\text{HS}}^2,$$

and consequently the Plancherel theorem can also be written as

$$\int_K |f(k)|^2 dk = \sum_{\pi \in \hat{K}} \|f * \chi_\pi\|_2^2.$$

For basic facts about representation theory of compact Lie groups we refer to Simon [29] and Hall [20].

Let Δ_K be the Laplacian on K . In this case, we are interested in the extension problem

$$(\Delta_K + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho)u(k, \rho) = 0, \quad k \in K, \quad \rho > 0; \quad u(k, 0) = f(k), \quad k \in K$$

for the Laplacian Δ_K . Let us start with some known facts. If $f \in L^p(K)$, in view of Theorem 2.4, we can write down a solution of the above problem as

$$u(k, \rho) = \frac{\rho^s}{2^s \Gamma(s/2)} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{t\Delta_K} f(k) t^{-s/2-1} dt, \quad (3.9)$$

where $e^{t\Delta_K}$ stands for the heat semigroup generated by Δ_K . Indeed, it is known that to each π there corresponds a real number $\lambda_\pi > 0$ such that

$$\Delta_K \chi_\pi = -\lambda_\pi^2 \chi_\pi. \quad (3.10)$$

By defining the heat kernel

$$h_t(k) = \sum_{\pi \in \hat{K}} d_\pi e^{-t\lambda_\pi^2} \chi_\pi(k),$$

the solution of the heat equation associated to Δ_K is given by

$$e^{t\Delta_K} f(k) = f * h_t(k).$$

It is well known that $h_t(k) > 0$, $\int_K h_t(k) dk = 1$ and for $f \in L^p(K)$, $1 \leq p < \infty$, $f * h_t \rightarrow f$ in $L^p(K)$ as $t \rightarrow 0$. Then the solution u defined in (3.9) takes the form

$$u(k, \rho) = \frac{\rho^s}{2^s \Gamma(s/2)} \int_0^\infty e^{-\frac{\rho^2}{4t}} f * h_t(k) t^{-s/2-1} dt,$$

which can be written as the convolution $f * u_{s,\rho}(k)$, where the kernel $u_{s,\rho}$ is defined by

$$u_{s,\rho}(k) = \frac{\rho^s}{2^s \Gamma(s/2)} \int_0^\infty e^{-\frac{\rho^2}{4t}} h_t(k) t^{-s/2-1} dt.$$

Recalling the expansion of $h_t(k)$ in terms of the character χ_π we see that

$$u_{s,\rho}(k) = \sum_{\pi \in \hat{K}} d_\pi \varphi_s(\rho, \pi) \chi_\pi(k)$$

where

$$\varphi_s(\rho, \pi) = \frac{\rho^s}{2^s \Gamma(s/2)} \int_0^\infty e^{-\frac{\rho^2}{4t}} e^{-t\lambda_\pi^2} t^{-s/2-1} dt.$$

The above is precisely the integral representation of the Macdonald function $(\rho\lambda_\pi)^{s/2} K_{s/2}(\rho\lambda_\pi)$, see (3.8). Thus

$$u_{s,\rho}(k) = \sum_{\pi \in \hat{K}} d_\pi (\rho\lambda_\pi)^{s/2} K_{s/2}(\rho\lambda_\pi) \chi_\pi(k)$$

and the function

$$u_1(k, \rho) = \sum_{\pi \in \hat{K}} d_\pi (\rho\lambda_\pi)^{s/2} K_{s/2}(\rho\lambda_\pi) f * \chi_\pi(k) \quad (3.11)$$

solves the extension problem.

In view of the asymptotic properties (3.5) and (3.7) of the Macdonald function $K_{s/2}$, the series (3.11) converges uniformly as long as $\|f * \chi_\pi\|_\infty$ has a polynomial growth in terms of λ_π . This is certainly the case when, say, $f \in L^p(K)$, or even when f is a distribution.

Nevertheless, if we do not care about the initial condition, the extension problem admits another solution u_2 (as explained at the beginning of the section) which can be represented as

$$u_2(k, \rho) = \sum_{\pi \in \hat{K}} d_\pi (\rho \lambda_\pi)^{s/2} I_{s/2}(\rho \lambda_\pi) g * \chi_\pi(k) \quad (3.12)$$

where $I_{s/2}$ is the Bessel function of second kind. To see this, let u satisfy the equation

$$(\Delta_K + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho) u(k, \rho) = 0.$$

Then, in view of (3.10), it follows that $u(\cdot, \rho) * \chi_\pi(k)$ satisfies the equation

$$\left(-\lambda_\pi^2 + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho\right) u(\cdot, \rho) * \chi_\pi(k) = 0.$$

As we already explained, for each fixed k , this equation admits two linearly independent solutions, viz. $(\rho \lambda_\pi)^{s/2} K_{s/2}(\rho \lambda_\pi)$ and $(\rho \lambda_\pi)^{s/2} I_{s/2}(\rho \lambda_\pi)$. The first choice leads to solutions of the form u_1 whereas the second choice leads to u_2 .

However, as the modified Bessel functions $I_{s/2}(t)$ have exponential growth as $t \rightarrow \infty$, the series (3.12) defining the function u_2 will not converge unless $g * \chi_\pi$ has enough decay as $\lambda_\pi \rightarrow \infty$. It is certainly well defined if g has only finitely many non zero components in its Peter–Weyl expansion: that is, $g * \chi_\pi = 0$ for all but finitely many $\pi \in \hat{K}$. Note also that when $u_2(k, \rho)$ is well defined, it converges to 0 as $\rho \rightarrow 0$.

We are thus led to find conditions on the function g so that $g * \chi_\pi$ has enough decay as $\lambda_\pi \rightarrow \infty$. The answer to this question lies in the holomorphic extendability of g to the complexification of the group K .

In order to explain this, we closely follow the notations used in [22, Section 9.2]. Let $K_{\mathbb{C}}$ stand for the universal complexification of K , which is a complex Lie group. Let T be a maximal torus in K with Lie algebra \mathfrak{t} and let W stand for the Weyl group of K with respect to T . Then, we have the classical Cartan decomposition $K_{\mathbb{C}} = K \cdot \exp i\mathfrak{t} \cdot K$. In view of this decomposition, domains $\mathcal{D} \subset K_{\mathbb{C}}$ which are bi-invariant under the action of K are given by W -invariant closed sets $B \subset \mathfrak{t}$. Thus, all such domains, called Reinhardt domains, are of the form $\mathcal{D}_B = K \cdot \exp iB \cdot K$. As $K_{\mathbb{C}}$ is a complex Lie group we can talk about holomorphic functions on such Reinhardt domains.

Let $\pi \in \hat{K}$ be an irreducible representation of K on a Hilbert space H_π . Then it has a unique extension, denoted by the same symbol, to the complex group $K_{\mathbb{C}}$. Thus the characters χ_π of the representation π extend to the whole of $K_{\mathbb{C}}$ as holomorphic functions. Hardy spaces associated to Reinhardt domains \mathcal{D}_B in $K_{\mathbb{C}}$ have been studied by Lassalle in [22]. Assuming that B is convex, we introduce the norm

$$\|f\|_B^2 = \sup_{H \in B} \int_{K \times K} |f(k_1 \cdot \exp iH \cdot k_2)|^2 dk_1 dk_2$$

for functions $f \in \mathcal{O}(\mathcal{D}_B)$, the space of all holomorphic functions on \mathcal{D}_B . The Hardy space $H^2(\mathcal{D}_B)$ is then defined by

$$H^2(\mathcal{D}_B) = \{f \in \mathcal{O}(\mathcal{D}_B) : \|f\|_B < \infty\}.$$

In [22, Theorem 2 bis], the following characterisation of $H^2(\mathcal{D}_B)$ has been proved.

Theorem 3.2 (Lassalle). *Let B be any W -invariant closed convex subset of \mathfrak{t} . Then a function f belongs to $H^2(\mathcal{D}_B)$ if and only if it has the expansion*

$$f(g) = \sum_{\pi \in \hat{K}} d_\pi \operatorname{tr}(\pi(f)\pi(g)), \quad g \in \mathcal{D}_B$$

where the operators $\pi(f), \pi \in \hat{K}$ satisfy the following condition:

$$\sup_{H \in B} \sum_{\pi \in \hat{K}} \|\pi(f)\|_{\text{HS}}^2 \chi_\pi(\exp 2iH) < \infty.$$

In fact the above expression is precisely $\|f\|_B^2$ as shown by Lassalle. It easily follows from the following result known as Lassalle–Gutzmer formula.

Theorem 3.3 (Lassalle). *Suppose $f \in \mathcal{O}(\mathcal{D}_B)$. Then for every $H \in B$ one has*

$$\int_{K \times K} |f(k_1 \cdot \exp iH \cdot k_2)|^2 dk_1 dk_2 = \sum_{\pi \in \hat{K}} \|\pi(f)\|_{\text{HS}}^2 \chi_\pi(\exp 2iH).$$

At this point we would like to know how $\chi_\pi(\exp 2iH)$ grows as a function of H and d_π . To see this we need to recall some results from the representation theory of compact Lie groups, see [19, 20, 29]. Without getting into details, let us recall that there is a parametrisation of \hat{K} in terms of the so called highest weight vectors in the dual of a Cartan subalgebra. Thus there is a one to one correspondence between $\pi \in \hat{K}$ and such highest weight vectors λ . Let the representation π be associated to the weight vector λ . Then it is known that the eigenvalue λ_π corresponding to π grows as a constant multiple of $|\lambda|$ and that the dimension d_π of the representation grows polynomially in $|\lambda|$. Thus there are constants a, b and C such that: (i) $\lambda_\pi \geq a|\lambda|$; (ii) $d_\pi \leq C(1 + |\lambda|)^b$, see [19, Lemma 6]. Moreover, there exists a constant $\gamma > 0$ such that

$$|\chi_\pi(\exp iH)| \leq d_\pi e^{\gamma \lambda_\pi |H|}, \quad H \in \mathfrak{t}.$$

This can be easily proved using the Weyl character formula for χ_π , (see [19, Lemma 7]).

From Theorem 3.2 and the above estimate for the characters $\chi_\pi(\exp iH)$, we infer that whenever $f \in H^2(\mathcal{D}_B)$, $\|\pi(f)\|_{\text{HS}}^2$ and hence $\|f * \chi_\pi\|_2^2$ have certain exponential decay (depending on B). In particular, when $f \in H^2(K_{\mathbb{C}})$ it follows that $\|f * \chi_\pi\|_2^2$ decays faster than $e^{-\delta \lambda_\pi}$ for any $\delta > 0$. With this information at our disposal, we can prove a characterisation of the solutions of the extension problem.

We consider the extension problem (3.2) for Δ_K with initial condition f , where f is a distribution on K . Note that the initial condition is interpreted in the distribution sense: for every $\varphi \in C^\infty(K)$ we have

$$\lim_{\rho \rightarrow 0} \int_K u(k, \rho) \varphi(k) dk = (f, \varphi).$$

Given distributions f, g , let us define

$$P_\rho f(k) = \frac{2^{1-s/2}}{\Gamma(s/2)} \sum_{\pi \in \hat{K}} d_\pi (\lambda_\pi \rho)^{s/2} K_{s/2}(\lambda_\pi \rho) f * \chi_\pi(k) \quad (3.13)$$

and

$$Q_\rho g(k) = \sum_{\pi \in \hat{K}} d_\pi (\lambda_\pi \rho)^{s/2} I_{s/2}(\lambda_\pi \rho) g * \chi_\pi(k).$$

We call $P_\rho f$ the generalised Poisson integral of f . For $t > 0$, let $B_t := \{H \in \mathfrak{t} : |H| < t\}$. We prove the following.

Theorem 3.4. *A smooth function $u(k, \rho)$ is a solution of the extension problem (3.2) with initial condition f being a distribution if and only if $u(k, \rho) = P_\rho f(k) + Q_\rho g(k)$ for some $g \in \bigcap_{t>0} H^2(\mathcal{D}_{B_t})$.*

Proof. For any distribution f on K , we note that $\|f * \chi_\pi\|_\infty \leq c(1 + \lambda_\pi)^n$ for a fixed integer n . This follows from the fact that every distribution on K is of finite order. Consequently the series (3.13) defining $P_\rho f$ converges uniformly and solves the extension problem. It is also clear that for $\varphi \in C^\infty(K)$ we have

$$\lim_{\rho \rightarrow 0} \int_K P_\rho f(k) \varphi(k) dk = (f, \varphi).$$

To see this observe that

$$\int_K P_\rho f(k) \varphi(k) dk = \frac{2^{1-s/2}}{\Gamma(s/2)} \sum_{\pi \in \hat{K}} d_\pi (\lambda_\pi \rho)^{s/2} K_{s/2}(\lambda_\pi \rho) (f * \chi_\pi, \varphi)$$

which is the same as $(f, P_\rho \varphi)$. As $P_\rho \varphi \rightarrow \varphi$ in $C^\infty(K)$ as $\rho \rightarrow 0$, it follows that $(f, P_\rho \varphi)$ converges to (f, φ) as $\rho \rightarrow 0$.

On the other hand, since we take g from $\bigcap_{t>0} H^2(\mathcal{D}_{B_t})$, we know from the remarks previous to the statement of the theorem, that $Q_\rho g$ is well defined as $\|g * \chi_\pi\|_2 \leq C e^{-\delta \lambda_\pi}$ for any $\delta > 0$. Since each of the components $(\lambda_\pi \rho)^{s/2} I_{s/2}(\lambda_\pi \rho) g * \chi_\pi$ satisfies the extension problem, so does their sum $Q_\rho g$. Moreover, as $(\lambda_\pi \rho)^{s/2} I_{s/2}(\lambda_\pi \rho) \rightarrow 0$ as $\rho \rightarrow 0$, it follows that $Q_\rho g \rightarrow 0$ as $\rho \rightarrow 0$, again in the sense of distributions. Thus $u = P_\rho f + Q_\rho g$ solves the extension problem with initial condition f .

Conversely, if $u(k, \rho)$ is a solution of the extension problem, then for any $\pi \in \hat{K}$, $u(\cdot, \rho) * \chi_\pi(k)$, $k \in K$ fixed, will be a solution of

$$\left(-\lambda_\pi^2 + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho\right) v(k, \rho) = 0, \quad v(k, 0) = f * \chi_\pi(k).$$

Solving this ordinary differential equation, in view of (3.5) and (3.1), we are led to

$$v(k, \rho) = \frac{2^{1-s/2}}{\Gamma(s/2)} (\rho\lambda_\pi)^{s/2} K_{s/2}(\rho\lambda_\pi) f * \chi_\pi(k) + c_\pi(k) (\rho\lambda_\pi)^{s/2} I_{s/2}(\rho\lambda_\pi).$$

As f is given to be a distribution, $\|f * \chi_\pi\|_2$ has a polynomial growth in λ_π . Hence the first term goes to 0 as $\lambda_\pi \rightarrow \infty$. Consequently, as $\|u(\cdot, \rho) * \chi_\pi\|_2$ has a polynomial growth, it follows that $\|c_\pi\|_2 (\rho\lambda_\pi)^{s/2} I_{s/2}(\rho\lambda_\pi)$ remains bounded as $\lambda_\pi \rightarrow \infty$ for every $\rho > 0$. We also note that $v(\cdot, \rho) * \chi_\pi = v(\cdot, \rho)$ which leads to $c_\pi = c_\pi * \chi_\pi$. Therefore, the function

$$g(k) = \sum_{\pi} c_\pi * \chi_\pi(k)$$

satisfies $\|g * \chi_\pi\|_2 \leq C e^{-\delta\lambda_\pi}$ for every $\delta > 0$. This means that g extends to $K_{\mathbb{C}}$ as a holomorphic function and $g \in \bigcap_{t>0} H^2(\mathcal{D}_{B_t})$. And we have the representation

$$u(k, \rho) = P_\rho f(k) + Q_\rho g(k).$$

□

In Theorem 3.4, if we assume that $u(k, \rho)$ has tempered growth in both variables, then so does $u(\cdot, \rho) * \chi_\pi$ and hence from the equation

$$u(\cdot, \rho) * \chi_\pi = \frac{2^{1-s/2}}{\Gamma(s/2)} f * \chi_\pi (\rho\lambda_\pi)^{s/2} K_{s/2}(\rho\lambda_\pi) + g * \chi_\pi (\rho\lambda_\pi)^{s/2} I_{s/2}(\rho\lambda_\pi)$$

we can conclude that $g * \chi_\pi = 0$ for all π and $u = P_\rho f$. We can now characterise all such solutions of tempered growth with $f \in L^p(K)$.

Theorem 3.5. *Let $u(k, \rho)$ be a solution of the extension problem (3.2) which is of tempered growth (in both variables). Then $u = P_\rho f$ for some $f \in L^p(K)$, $1 < p \leq \infty$ if and only if there exists a constant $C > 0$ such that*

$$\sup_{\rho>0} \int_K |u(k, \rho)|^p dk \leq C. \quad (3.14)$$

When $p = 1$, the same happens if and only if f is a complex Borel measure.

Proof. When $f \in L^p(K)$, $1 \leq p \leq \infty$, the estimate (3.14) is immediate from Theorem 2.4 since the heat semigroup $e^{t\Delta_K}$ is a contraction on L^p spaces.

Conversely, if (3.14) is assumed, then there exists a subsequence $\rho_j \rightarrow 0$ and $f \in L^p(K)$ ($f = \mu$, with μ a complex Borel measure, when $p = 1$), such that $u(k, \rho_j) \rightarrow f(k)$ weakly in $L^p(K)$, $1 < p < \infty$ ($u(\cdot, \rho_j) \rightarrow \mu$, for $p = 1$). But then, as we remarked earlier, u has to be of the form $P_\rho f$ with the same f . □

Remark 3.6. When $f \in L^2(K)$ we can say something more about the solution $u(k, \rho) = P_\rho f(k)$. In view of Lassalle-Gutzmer formula in Theorem 3.3 it follows that for each ρ the function $u(\cdot, \rho)$ belongs to $H^2(\mathcal{D}_{B_{\delta\rho}})$, where $B_{\delta\rho}$ denotes the ball of radius $\delta\rho$, for any $\delta < \gamma$ where γ is the constant appearing in the estimate $|\chi_\pi(\exp iH)| \leq d_\pi e^{\gamma\lambda_\pi|H|}$.

3.3. Hermite operator. Now we will characterise solutions of the extension problem for the Hermite operator on \mathbb{R}^n . For $\alpha \in \mathbb{N}^n$, let Φ_α be the normalised Hermite functions on \mathbb{R}^n which are eigenfunctions of the Hermite operator $H := -\Delta + |x|^2$ with eigenvalues $(2|\alpha| + n)$. Here Δ denotes the Euclidean Laplacian and $|\alpha| = \alpha_1 + \dots + \alpha_n$. Thus,

$$H\Phi_\alpha = (2|\alpha| + n)\Phi_\alpha. \quad (3.15)$$

Let us look at the corresponding extension problem

$$\left(-H + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho\right)u(x, \rho) = 0, \quad x \in \mathbb{R}^n, \quad \rho > 0; \quad u(x, 0) = f(x), \quad x \in \mathbb{R}^n. \quad (3.16)$$

In view of (3.15), the coefficient $\widehat{u}(\alpha, \rho) = \int_{\mathbb{R}^n} u(x, \rho)\Phi_\alpha(x) dx$ satisfies the equation

$$\left(- (2|\alpha| + n) + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho\right)\widehat{u}(\alpha, \rho) = 0, \quad \widehat{u}(\alpha, 0) = (f, \Phi_\alpha)$$

and the two linearly independent solutions are the functions

$$(\rho\sqrt{2|\alpha| + n})^{s/2} K_{s/2}(\rho\sqrt{2|\alpha| + n})$$

and

$$(\rho\sqrt{2|\alpha|+n})^{s/2}I_{s/2}(\rho\sqrt{2|\alpha|+n}).$$

For each $t > 0$, let us define the tube domain

$$\Omega_t = \{z = x + iy \in \mathbb{C}^n : |y| < t\}.$$

Fixed $t > 0$, take $\mathcal{O}(\Omega_t)$ to be the space of all holomorphic functions on Ω_t . For any $F \in \mathcal{O}(\Omega_t)$ let us introduce the norm

$$\|F\|_{H_t}^2 := \int_{\mathbb{R}^n} \int_{|y| < t} |F(x + iy)|^2 (t^2 - |y|^2)^{n/2} \frac{J_{n/2-1}(2i(t^2 - |y|^2)^{1/2}|x|)}{(2i(t^2 - |y|^2)|x|)^{n/2-1}} dy dx,$$

where $J_\alpha(z)$ is the Bessel function of the first kind of type α , and define the Hardy space $H^2(\Omega_t)$ as

$$H^2(\Omega_t) = \{F \in \mathcal{O}(\Omega_t) : \|F\|_{H_t} < \infty\}.$$

Let P_k be the spectral projections of the Hermite operator onto the k th eigenspace

$$P_k f = \sum_{|\alpha|=k} (f, \Phi_\alpha) \Phi_\alpha.$$

The Hermite functions $\Phi_\alpha(x)$ have extensions to \mathbb{C}^n as entire functions, and so $P_k f$ has a holomorphic extension. The following characterisation for these spaces is proved in [37].

Theorem 3.7 (Thangavelu). *For any holomorphic function F in the tube domain Ω_t , we have the identity*

$$\|F\|_{H_t}^2 = c_n \sum_{k=0}^{\infty} \|P_k f\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} L_k^{n-1}(-2t^2) e^{t^2},$$

where f is the restriction of F to \mathbb{R}^n .

In Theorem 3.7, $L_k^{n-1}(-2t^2)e^{t^2}$ stands for Laguerre functions of type $(n-1)$. Given a tempered distribution f , we define

$$P_\rho^H f(x) = \frac{2^{1-s/2}}{\Gamma(s/2)} \sum_{k=0}^{\infty} (\rho\sqrt{2k+n})^{s/2} K_{s/2}(\rho\sqrt{2k+n}) P_k f(x) \quad (3.17)$$

and if the function g is such that $P_k g$ has enough decay we also define

$$Q_\rho^H g(x) = \sum_{k=0}^{\infty} (\rho\sqrt{2k+n})^{s/2} I_{s/2}(\rho\sqrt{2k+n}) P_k g(x). \quad (3.18)$$

We will prove the following characterisation of solutions to the extension problem (3.16).

Theorem 3.8. *A smooth function $u(x, \rho)$ is a solution of the extension problem (3.16) with initial condition a tempered distribution f if and only if $u(x, \rho) = P_\rho^H f(x) + Q_\rho^H g(x)$, for some $g \in \cap_{t>0} H^2(\Omega_t)$.*

Proof. Let us suppose that the function $u(x, \rho)$ can be written as $P_\rho^H f(x) + Q_\rho^H g(x)$, with the assumptions on f, g as in the statement. If f is a tempered distribution on \mathbb{R}^n , its Fourier–Hermite coefficients satisfy

$$|(f, \Phi_\alpha)| \leq C(2|\alpha|+n)^m,$$

for some integer m . As a consequence, the expression (3.17) is well defined and it solves the extension problem. On the other hand, from Theorem 3.7, we infer that $\|P_k g\|_2^2$ has to decay as $(L_k^{n-1}(-2t^2))^{-1}$, asymptotically in k . But it is known that $L_k^{n-1}(-2t^2) \approx C e^{ct(2k+n)^{1/2}}$ as $k \rightarrow \infty$, for n and t fixed (see [32]), thus $\|P_k g\|_2$ decays as $C e^{-ct(2k+n)^{1/2}}$, for k large enough. Then, the expression (3.18) is also well defined and solves the extension problem. By following analogous reasoning as in the proof of Theorem 3.4, we conclude that $u = P_\rho^H f + Q_\rho^H g$ solves the extension problem with initial condition f .

Conversely, suppose that $u(x, \rho)$ is a solution of (3.16). Then for every $\alpha \in \mathbb{N}^n$, the function $\hat{u}(\alpha, \rho)$ is given by the following sum:

$$\frac{2^{1-s/2}}{\Gamma(s/2)} (f, \Phi_\alpha) (\rho\sqrt{2|\alpha|+n})^{s/2} K_{s/2}(\rho\sqrt{2|\alpha|+n}) + c_\alpha (\rho\sqrt{2|\alpha|+n})^{s/2} I_{s/2}(\rho\sqrt{2|\alpha|+n}),$$

as explained at the beginning of the present section. As $u(x, \rho)$ is tempered, $\hat{u}(\alpha, \rho)$ has at most polynomial growth in $|\alpha|$ and consequently c_α will have exponential decay $e^{-\rho\sqrt{2|\alpha|+n}}$ for every $\rho > 0$. But then

the function g defined by $g(x) = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \Phi_\alpha(x)$ satisfies $\|P_k g\|_2 \leq C e^{-t\sqrt{2k+n}}$ for every $t > 0$. As a consequence, it holomorphically extends and belongs to the space $H^2(\Omega_t)$ for every $t > 0$. This proves the theorem. \square

As in the case of compact Lie groups, we have also the following characterisation. The proof is completely analogous, since the heat semigroup e^{-tH} is a contraction on L^p spaces (see [33]), so we omit it.

Theorem 3.9. *Let $u(x, \rho)$ be a solution of the extension problem (3.16) which is of tempered growth (in both variables). Then $u = P_\rho^H f$ for some $f \in L^p(\mathbb{R}^n)$, $1 < p \leq \infty$ if and only if there exists a constant $C > 0$ such that*

$$\sup_{\rho > 0} \int_{\mathbb{R}^n} |u(x, \rho)|^p dx \leq C.$$

When $p = 1$, the same happens if and only if f is a complex Borel measure.

Remark 3.10. For the corresponding (generalised) Poisson integrals P_ρ^H of L^2 functions, an analogous remark to the one for compact Lie groups in Remark 3.6 can be made. Indeed, there is Gutzmer's formula for Hermite expansions available (see [37, Theorem 6.2]). Actually, such Gutzmer's formula is used to prove Theorem 3.7 in [37].

3.4. Special Hermite operator. In this section we study characterisations of solutions of the extension problem for the special Hermite operator. We borrow the notations, facts and definitions from Subsection 2.6.

As in the case of the Hermite operator, we would like to study the most general solutions of the extension problem with the initial condition f a tempered distribution. It is known that a distribution f on \mathbb{R}^n is tempered if and only if it belongs to the Hermite-Sobolev space $W_H^m(\mathbb{R}^n)$ for some integer m . These spaces are defined by the condition $H^m f \in L^2(\mathbb{R}^n)$ where H^m is defined via spectral theorem when m is negative. The above result simply means that f is tempered if and only if its Hermite coefficients satisfy the estimate $|(f, \Phi_\alpha)| \leq C(2|\alpha| + n)^m$ for some non-negative integer m . As there is no such characterisation of tempered distributions in terms of their special Hermite coefficients we start with initial conditions f coming from certain analogues of the Sobolev spaces known as twisted Sobolev spaces. For any integer m the space $W_{L_1}^m(\mathbb{C}^n)$ is defined by the requirement $L_1^m f \in L^2(\mathbb{C}^n)$. Again, for m negative, the operator L_1^m is defined via spectral theorem and the condition translates into

$$\sum_{\alpha, \beta \in \mathbb{N}^n} (2|\beta| + m)^{2m} |(f, \Phi_{\alpha, \beta})|^2 < \infty.$$

Note that when $f \in W_{L_1}^m(\mathbb{C}^n)$ we have the estimates

$$\sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_{\alpha, \beta})|^2 \leq C(2|\beta| + n)^{-2m} \quad (3.19)$$

for any β . Observe that when m is negative, the above is a growth restriction on the special Hermite coefficients of the distribution f .

Again we consider the corresponding extension problem

$$\left(-L_1 + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho\right) u(z, \rho) = 0, \quad z \in \mathbb{C}^n, \quad \rho > 0; \quad u(z, 0) = f(z), \quad z \in \mathbb{C}^n. \quad (3.20)$$

Let $u(z, \rho)$ be a solution of the initial value problem (3.20). Assume that both f and $u(\cdot, \rho)$ belong to $W_{L_1}^m(\mathbb{C}^n)$ for some m . Then for each pair $\alpha, \beta \in \mathbb{N}^n$ the function $\hat{u}_{\alpha, \beta}(\rho) = (u(\cdot, \rho), \Phi_{\alpha, \beta})$ satisfies the equation

$$\left(- (2|\beta| + n) + \partial_\rho^2 + \frac{1-s}{\rho} \partial_\rho\right) \hat{u}_{\alpha, \beta}(\rho) = 0, \quad \hat{u}_{\alpha, \beta}(0) = (f, \Phi_{\alpha, \beta}).$$

Again, two linearly independent solutions are given by

$$(\rho\sqrt{2|\beta| + n})^{s/2} K_{s/2}(\rho\sqrt{2|\beta| + n})$$

and

$$(\rho\sqrt{2|\beta| + n})^{s/2} I_{s/2}(\rho\sqrt{2|\beta| + n}).$$

For each $t > 0$, let us define the tube domain

$$\Omega_t = \{(x, u) + i(y, v) \in \mathbb{C}^{2n} = \mathbb{R}^{2n} \times \mathbb{R}^{2n} : |y|^2 + |v|^2 < t\}.$$

We consider, for a fixed $t > 0$, the space of all holomorphic functions on Ω_t , that we denote by $\mathcal{O}(\Omega_t)$. Given $F \in \mathcal{O}(\Omega_t)$, we introduce the norm

$$\|F\|_{H_t}^2 = \int_{\Omega_t} |F(z, w)|^2 e^{u \cdot y - v \cdot x} dz dw$$

where $z = x + iy$, $w = u + iv$. We define the following Hardy space

$$H^2(\Omega_t) = \{F \in \mathcal{O}(\Omega_t) : \|F\|_{H_t} < \infty\}$$

(observe that we are using the same notation as in the case of the Hermite operator). The special Hermite functions $\Phi_{\alpha, \beta}$ have extensions to \mathbb{C}^{2n} , and so the projections $f \times \varphi_k$ can be holomorphically extended.

Let us consider the generalised Poisson integral $P_\rho^{L_1} f$ given by

$$P_\rho^{L_1} f(z) = \frac{2^{1-s/2}}{\Gamma(s/2)} \sum_{k=0}^{\infty} (\rho\sqrt{2k+n})^{s/2} K_{s/2}(\rho\sqrt{2k+n}) f \times \varphi_k(z).$$

Note that the Poisson integral is also given by the sum

$$\frac{2^{1-s/2}}{\Gamma(s/2)} \sum_{\alpha, \beta \in \mathbb{N}^n} (f, \Phi_{\alpha, \beta}) (\rho\sqrt{2|\beta|+n})^{s/2} K_{s/2}(\rho\sqrt{2|\beta|+n}) \Phi_{\alpha, \beta}(z).$$

Similarly, for functions g with enough decay, we define $Q_\rho^{L_1} g$ by

$$Q_\rho^{L_1} g(z) = \sum_{k=0}^{\infty} (\rho\sqrt{2k+n})^{s/2} I_{s/2}(\rho\sqrt{2k+n}) g \times \varphi_k(z).$$

Theorem 3.11. *A function $u(\cdot, \rho) \in W_{L_1}^m(\mathbb{C}^n)$, for some integer m , is a solution of the extension problem (3.20) with initial condition $f \in W_{L_1}^m(\mathbb{C}^n)$ if and only if $u(z, \rho) = P_\rho^{L_1} f(z) + Q_\rho^{L_1} g(z)$, for some $g \in \cap_{t>0} H^2(\Omega_t)$.*

Proof. Again, let us suppose that the function $u(z, \rho)$ can be written as $P_\rho^{L_1} f(z) + Q_\rho^{L_1} g(z)$, with the assumptions on f, g as in the statement. By the explanations previous to the statement of the theorem, we have that $\sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_{\alpha, \beta})|^2 \leq C(2|\beta|+n)^{-2m}$ for any β . This implies that

$$\|f \times \varphi_k\|_2^2 = \sum_{|\beta|=k} \sum_{\alpha \in \mathbb{N}^n} |(f, \Phi_{\alpha, \beta})|^2 \leq C \sum_{|\beta|=k} (2|\beta|+n)^{-2m} = C(2k+n)^{-2m+n-1}$$

since $\sum_{|\beta|=k} 1 = \frac{(k+n-1)!}{k!(n-1)!} = O((2k+n)^{n-1})$. Consequently,

$$\sum_{k=0}^{\infty} \|f \times \varphi_k\|_2^2 \left((\rho\sqrt{2k+n})^{s/2} K_{s/2}(\rho\sqrt{2k+n}) \right)^2 < \infty,$$

and the expression for $P_\rho^{L_1} f(z)$ is well defined and it solves the extension problem. On the other hand, by the Gutzmer's formula for special Hermite expansions (see [36, Section 6]), we have

$$\int_{\mathbb{C}^n} |g(x + iy, u + iv)|^2 e^{(u \cdot y - v \cdot x)} dx du = c_n \sum_{k=0}^{\infty} \|g \times \varphi_k\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k(2iy, 2iv)$$

where $\varphi_k(2iy, 2iv) = L_k^{n-1}(-2(|y|^2 + |v|^2))e^{(|y|^2 + |v|^2)}$. As $g \in \cap_{t>0} H^2(\Omega_t)$ we deduce that $\|g \times \varphi_k\|_2$ has to decay as $e^{-t\sqrt{2k+n}}$ for large k , by the same reasoning as in the proof of Theorem 3.8. Then, the expression $Q_\rho^{L_1} g(z)$ is well defined and solves the extension problem. By following analogous reasoning as in the proof of Theorem 3.4, we conclude that $u = P_\rho^{L_1} f + Q_\rho^{L_1} g$ solves the extension problem with initial condition f .

Conversely, suppose that $u(z, \rho)$ is a solution of (3.20). Then for every $\alpha, \beta \in \mathbb{N}^n$

$$\hat{u}_{\alpha, \beta}(\rho) = \frac{2^{1-s/2}}{\Gamma(s/2)} (f, \Phi_{\alpha, \beta}) (\rho\sqrt{2|\beta|+n})^{s/2} K_{s/2}(\rho\sqrt{2|\beta|+n}) + c_{\alpha, \beta} (\rho\sqrt{2|\beta|+n})^{s/2} I_{s/2}(\rho\sqrt{2|\beta|+n}).$$

Since $u(\cdot, \rho) \in W_{L^1}^m(\mathbb{C}^n)$, we have the polynomial growth (3.19) in $|\beta|$ for the infinite sum $\sum_{\alpha \in \mathbb{N}^n} |\widehat{u}_{\alpha, \beta}(\rho)|^2$. Thus, $\sum_{\alpha \in \mathbb{N}^n} |c_{\alpha, \beta}|^2$ has exponential decay $e^{-\rho\sqrt{2|\beta|+n}}$ for every $\rho > 0$. Then the function

$$g(z) = \sum_{\beta, \alpha \in \mathbb{N}^n} c_{\alpha, \beta} \Phi_{\alpha, \beta}(z)$$

satisfies $\|g \times \varphi_k\|_2 \leq C e^{-t\sqrt{2k+n}}$ for every $t > 0$. This implies that it belongs to the space $H^2(\Omega_t)$ for every $t > 0$. \square

Remark 3.12. As in the case of compact Lie groups and the Hermite operator, we have also another characterisation as in Theorem 3.5. The proof is completely analogous, and we will omit it. Moreover, an analogous remark to Remark 3.6 also holds in this setting.

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REFERENCES

- [1] Adimurthi, P. K. Ratnakumar and V. K. Sohani, A Hardy–Sobolev inequality for the twisted Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A* **147** (2017), 1–23.
- [2] Adimurthi and A. Sekar, Role of the fundamental solution in Hardy–Sobolev-type inequalities, *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), 1111–1130.
- [3] V. Banica, M.d.M. González and M. Sáez, Some constructions for the fractional Laplacian on noncompact manifolds, *Rev. Mat. Iberoam.* **31** (2015), 681–712.
- [4] W. Beckner, Pitt’s inequality and the fractional Laplacian: sharp error estimates, *Forum Math.* **24** (2012), 177–209.
- [5] K. Bogdan, B. Dyda and P. Kim, Hardy inequalities and non-explosion results for semigroups, *Potential Anal.* **44** (2016), 229–247.
- [6] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, 2007.
- [7] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations* **32** (2007), no. 7-9, 1245–1260.
- [8] P. Ciatti, M. G. Cowling, and F. Ricci, Hardy and uncertainty inequalities on stratified Lie groups, *Adv. Math.* **277** (2015), 365–387.
- [9] R. D. DeBlassie, The first exit time of a two-dimensional symmetric stable process from a wedge, *Ann. Probab.* **18** (1990), no. 3, 1034–1070.
- [10] B. Driver, L. Gross and L. Saloff-Coste, Holomorphic functions and subelliptic heat kernels over Lie groups, *J. Eur. Math. Soc. (JEMS)* **11** (2009), 941–978.
- [11] F. Ferrari and B. Franchi, Harnack inequality for fractional sub-Laplacians in Carnot groups, *Math. Z.* **279** (2015), 435–458.
- [12] S. Filippas, L. Moschini and A. Tertikas, Sharp trace Hardy–Sobolev–Maz’ya inequalities and the fractional Laplacian, *Arch. Ration. Mech. Anal.* **208** (2013), 109–161.
- [13] S. Filippas, L. Moschini and A. Tertikas, Trace Hardy–Sobolev–Maz’ya inequalities for the half fractional Laplacian, *Commun. Pure Appl. Anal.* **14** (2) (2015), 373–382.
- [14] G. B. Folland, Subelliptic estimates and function spaces on Nilpotent Lie groups, *Ark. Math.* **13** (1975), 161–207.
- [15] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, **28**. Princeton University Press, N.J.; University of Tokyo Press, Tokyo, 1982.
- [16] R. L. Frank, E. H. Lieb and R. Seiringer, Hardy–Lieb–Thirring inequalities for fractional Schrödinger operators, *J. Amer. Math. Soc.* **21** (2008), 925–950.
- [17] J. E. Galé, P. J. Miana and P. R. Stinga, Extension problem and fractional operators: semigroups and wave equations, *J. Evol. Equ.* **13** (2013), 343–368.
- [18] N. Garofalo and E. Lanconelli, Frequency functions on the Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier (Grenoble)*, 40 (1990), 313–356.

- [19] B. C. Hall, The Segal–Bargmann “coherent state” transform for compact Lie groups, *J. Funct. Anal.* **122** (1994), 103–151.
- [20] B. C. Hall, *Lie groups, Lie algebras, and representations: an elementary introduction, 2nd ed.*, Graduate Texts in Mathematics, **222**. Springer, 2015.
- [21] I. W. Herbst, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, *Commun. Math. Phys.* **53** (1977), 285–294.
- [22] M. Lassalle, L’espace de Hardy d’un domaine de Reinhardt généralisé, *J. Funct. Anal.* **60** (1985), 309–340.
- [23] L. Hörmander, Hypoelliptic second order differential equations, *Acta Math.* **119** (1967), 147–171.
- [24] N. N. Lebedev, *Special functions and its applications*, Dover, New York, 1972.
- [25] S. A. Molchanov and E. Ostrovskii, Symmetric stable processes as traces of degenerate diffusion processes, *Theor. Probab. Appl.* **14** (1969): 128–131.
- [26] V. H. Nguyen, Some trace Hardy type inequalities and trace Hardy-Sobolev-Maz’ya type inequalities, *J. Funct. Anal.* **270** (2016), 4117–4151.
- [27] L. Roncal and S. Thangavelu, Hardy’s inequality for fractional powers of the sublaplacian on the Heisenberg group, *Adv. Math.* **302** (2016), 106–158.
- [28] L. Roncal and S. Thangavelu, An extension problem and trace Hardy inequality for the sublaplacian on H , arXiv:1708.09258, to appear in *Int. Math. Res. Not. IMRN*.
- [29] B. Simon, *Representations of finite and compact groups. Graduate Studies in Mathematics*, 10. American Mathematical Society, Providence, RI, 1996. xii+266 pp.
- [30] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton Mathematical Series **32**, Princeton University Press, Princeton, NJ, 1971.
- [31] P. R. Stinga and J. L. Torrea, Extension problem and Harnack’s inequality for some fractional operators, *Comm. Partial Differential Equations* **35** (2010), 2092–2122.
- [32] G. Szegő, *Orthogonal polynomials*, Fourth Edition, Amer. Math. Soc. Colloq. Publ. 23, Amer. Math. Soc., Providence, R. I., 1975.
- [33] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes **42**. Princeton University Press, Princeton, NJ, 1993.
- [34] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Progress in Mathematics **159**. Birkhäuser, Boston, MA, 1998.
- [35] S. Thangavelu, *An introduction to the uncertainty principle. Hardy’s theorem on Lie groups. With a foreword by Gerald B. Folland*, Progress in Mathematics **217**. Birkhäuser, Boston, MA, 2004.
- [36] S. Thangavelu, Gutzmer’s formula and Poisson integrals on the Heisenberg group, *Pacific J. Math.* **231** (2007), no. 1, 217–237.
- [37] S. Thangavelu, On the unreasonable effectiveness of Gutzmer’s formula. Harmonic analysis and partial differential equations, 199–217, *Contemp. Math.*, **505**, Amer. Math. Soc., Providence, RI, 2010.
- [38] K. Tzirakis, Improving interpolated Hardy and trace Hardy inequalities on bounded domains, *Nonlinear Anal.* **127** (2015), 17–34.
- [39] H. Urakawa, The heat equation on compact Lie group, *Osaka J. Math.* **12** (1975), no. 2, 285–297.
- [40] D. Yafaev, Sharp constants in the Hardy-Rellich inequalities, *J. Funct. Anal.* **168** (1999), 121–144.

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E-mail address: pradeepb@goa.bits-pilani.ac.in

E-mail address: lroncal@bcamath.org

E-mail address: veluma@iisc.ac.in