EXPECTED VALUES OF THE SUB-FUNCTIONS

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1 Introduction

The goal of this supplementary document is to show how to compute the expected value of the sub-functions that arise from the decomposition of the elementary functions of the ELD of the QAP (see main paper). Given a symmetric instance of size $n$ with null main diagonals, the proposed sub-functions are defined as

\[
\begin{align*}
 f^m_{\chi}(\sigma) &= \frac{2\alpha^m + 2\beta^m}{r_m} \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \sum_{c=1}^{n-1} \sum_{d=c+1}^{n-1} \psi_{a,b,c,d} \chi_{(a,b)(c,d)}'(\sigma) \\
 f^m_{\omega}(\sigma) &= \frac{2\gamma^m + 2\epsilon^m}{r_m} \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \sum_{c=1}^{n-1} \sum_{d=c+1}^{n-1} \psi_{a,b,c,d} \omega_{(a,b)(c,d)}'(\sigma) \\
 f^m_{\tau}(\sigma) &= \frac{4\zeta^m}{r_m} \sum_{a=1}^{n-1} \sum_{b=a+1}^{n} \sum_{c=1}^{n-1} \sum_{d=c+1}^{n-1} \psi_{a,b,c,d} \tau_{(a,b)(c,d)}'(\sigma)
\end{align*}
\]

where $\sigma \in S_n$ is a permutation of size $n$ and $r_m, \alpha^m, \beta^m, \gamma^m, \epsilon^m, \zeta^m \in \mathbb{R}$ are parameters that depend on the corresponding elementary function (Table 1). In addition, $\psi_{a,b,c,d} = d_{a,b}$, where $d_{a,b}$ is the distance between the locations $a$ and $b$, and $h_{c,d}$ is the work flow between the facilities $c$ and $d$. The auxiliary functions $\chi', \omega'$ and $\tau'$ are defined as

\[
\begin{align*}
 \chi'_{(a,b)(c,d)}(\sigma) &= \begin{cases} 
 1 & \text{if } \sigma(a) = c \land \sigma(b) = d \lor \\
 0 & \text{otherwise} 
\end{cases} \\
 \omega'_{(a,b)(c,d)}(\sigma) &= \begin{cases} 
 1 & \text{if } \sigma(a) = c \oplus \sigma(b) = d \lor \\
 0 & \text{otherwise} 
\end{cases} \\
 \tau'_{(a,b)(c,d)}(\sigma) &= \begin{cases} 
 1 & \text{if } \sigma(a) \neq c, d \land \sigma(b) \neq c, d \\
 0 & \text{otherwise} 
\end{cases}
\end{align*}
\]

where $\oplus$ stands for the exclusive OR operator. Taking all this into account, we now have all the necessary information to compute the expected values of $f^m_{\chi}(\sigma)$, $f^m_{\omega}(\sigma)$ and $f^m_{\tau}(\sigma)$. 

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Therefore, the expected value equation can be rewritten as follows for any $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$:

$$
E[f^m_m(a, b, c, d)] = \frac{2\alpha^m + 2\beta^m}{r^m} E[\sum_{a=1}^{n} \sum_{b=a+1}^{n} \sum_{c=1}^{n} \sum_{d=c+1}^{n} \psi_{a,b,c,d} \chi(a,b)(c,d)(\sigma)]
$$

(7)

As the probabilities of $\chi(a,b)(c,d)(\sigma) = 0$ and $\chi'(a,b)(c,d)(\sigma) = 1$ are independent of the value of $\psi_{a,b,c,d}$,

$$
E[f^m_m(a, b, c, d)] = \frac{2\alpha^m + 2\beta^m}{r^m} E[\sum_{a=1}^{n} \sum_{b=a+1}^{n} \sum_{c=1}^{n} \sum_{d=c+1}^{n} E[\psi_{a,b,c,d}] E[\chi'(a,b)(c,d)(\sigma)]]
$$

(8)

Therefore, the expected value equation can be rewritten as follows for any $a, b, c$ and $d$ such that $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$:

$$
E[f^m_m(a, b, c, d)] = \frac{n^2(n-1)^2(2\alpha^m + 2\beta^m)}{4r^m} E[\psi_{a,b,c,d}] E[\chi'(a,b)(c,d)(\sigma)]
$$

(9)

For the sake of brevity, from now on $E[\psi_{a,b,c,d}]$ will be shortened to $\bar{\psi}$. For any fixed combination of $a, b, c$ and $d$ such that $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$, $\chi'(a,b)(c,d)(\sigma)$ only outputs 1 for $2(n-2)!$ out of $n!$ possible permutations $\sigma \in S_n$. Thus, the expected value of $\chi'(a,b)(c,d)(\sigma)$ is

$$
E[\chi'(a,b)(c,d)(\sigma)] = \frac{2(n-2)!}{n!} = \frac{2}{n(n-1)}
$$

(10)

Consequently, the expected value of $f^m_m(\sigma)$ can be calculated by combining Equations 9 and 10.

$$
E[f^m_m(\sigma)] = \frac{2n^2(n-1)^2(2\alpha^m + 2\beta^m)}{4n(n-1)r^m} \bar{\psi} = \frac{n(n-1)(\alpha^m + \beta^m)}{r^m} \bar{\psi}
$$

(11)
The same process can be used to compute the expected values of $f^m_\omega(\sigma)$ and $f^m_\tau(\sigma)$. To avoid repetition, we omit the mathematical details and only show the obtained final equations.

$$E[f^m_\omega(\sigma)] = \frac{2n(n-1)(n-2)(\gamma^m + \epsilon^m)}{r^m} \bar{\psi}$$ \hspace{1cm} (12)

$$E[f^m_\tau(\sigma)] = \frac{n(n-1)(n-2)(n-3)\zeta^m}{r^m} \bar{\psi}$$ \hspace{1cm} (13)

Finally, all that remains is to replace the parameters in Equations 11, 12 and 13 by their actual values according to Table 1. Taking this into account, the expected values of the sub-functions are shown in Table 2.

Table 2: Expected values of the sub-functions that arise from the decomposition of the elementary functions of the ELD.

<table>
<thead>
<tr>
<th></th>
<th>$\chi$</th>
<th>$\omega$</th>
<th>$\tau$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f^1$</td>
<td>$-(n-1)\bar{\psi}$</td>
<td>$-2(n-1)(n-2)\bar{\psi}$</td>
<td>$-(n-1)(n-2)(n-3)\frac{\bar{\psi}}{2}$</td>
</tr>
<tr>
<td>$f^2$</td>
<td>$\frac{n(n-1)(n-3)\bar{\psi}}{n-2}$</td>
<td>0</td>
<td>$\frac{n(n-1)(n-3)\bar{\psi}}{2}$</td>
</tr>
<tr>
<td>$f^3$</td>
<td>$\frac{2(n-1)^2\bar{\psi}}{n-2}$</td>
<td>$2(n-1)(n-2)\bar{\psi}$</td>
<td>$-(n-1)(n-3)\bar{\psi}$</td>
</tr>
</tbody>
</table>